Aspects of Quantum Field Theory in curved spacetime

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Abstract

This doctoral thesis explores the intersection of Quantum Field Theory and General Relativity within the framework of Quantum Field Theory in curved spacetimes. The research is divided into two primary sections: the development of mathematical tools regarding the effective action and heat kernel methods, and the application of these tools to two-dimensional black hole models.

In the first part, we address the definition and calculation of the 1-loop effective action, a crucial object for studying quantum corrections to classical field theories. While the standard Gilkey-Seeley-DeWitt expansion provides a method for regularizing divergencies, it is computationally costly and perturbative in nature. This document presents and expands upon a resummation scheme for the heat kernel that captures some non-perturbative information and simplifies the calculation of the leading order terms. This resummation formula is successfully applied to various systems, including scalar fields with Yukawa couplings, Quantum Electrodynamics, and inhomogeneous field configurations. These methods allow for a more precise analysis of phenomena such as the Schwinger effect and vacuum polarization.

The second part focuses on the semiclassical evolution of black holes, utilizing 2D dilaton gravity models (specifically the Callan-Giddings-Harvey-Strominger model) as a testing ground to address conceptual issues like information loss and spacetime singularities. We examine the Russo-Susskind-Thorlacius model, which solves the backreaction equations but typically predicts the formation of naked singularities, threatening the unitarity of the theory. To resolve these issues, we propose a modification to the standard semiclassical approach by incorporating fields with negative central charges (such as Faddeev-Popov ghosts) coupled to an auxiliary conformally flat metric. The results demonstrate that, under specific conditions involving sufficiently large negative central charges, the naked singularity is removed entirely from the spacetime evolution. Instead, the system evolves with an apparent horizon that hides a regular region of spacetime, suggesting a pathway for black hole evolution that avoids catastrophic breakdowns of predictability and offers new perspectives on the role of matter content in gravitational collapse.

Sommario

Questa tesi di dottorato esplora l'intersezione tra la Teoria Quantistica dei Campi e la Relatività Generale nel quadro della Teoria Quantistica dei Campi in spaziotempo curvo. La ricerca è suddivisa in due sezioni principali: lo sviluppo di strumenti matematici riguardanti l'azione efficace e i metodi del nucleo del calore (*heat kernel*), e l'applicazione di questi strumenti a modelli di buchi neri bidimensionali.

Nella prima parte, affrontiamo la definizione e il calcolo dell'azione efficace a 1-loop, un oggetto cruciale per lo studio delle correzioni quantistiche alle teorie di campo classiche. Sebbene l'espansione standard di Gilkey-Seeley-DeWitt fornisca un metodo per regolarizzare le divergenze, essa risulta computazionalmente costosa e di natura perturbativa. Questo documento presenta e sviluppa uno schema di risommazione per il nucleo del calore che cattura alcune informazioni non perturbative e semplifica il calcolo dei termini di ordine principale. Questa formula di risommazione viene applicata con successo a vari sistemi, inclusi campi scalari con accoppiamenti di Yukawa, l'Elettrodinamica Quantistica e configurazioni di campo disomogenee. Tali metodi consentono un'analisi più precisa di fenomeni come l'effetto Schwinger e la polarizzazione del vuoto.

La seconda parte si concentra sull'evoluzione semiclassica dei buchi neri, utilizzando modelli di gravità dilatonica in 2D (in particolare il modello Callan-Giddings-Harvey-Strominger) come banco di prova per affrontare questioni concettuali come la perdita di informazione e le singolarità spaziotemporali. Esaminiamo il modello Russo-Susskind-Thorlacius, che risolve analiticamente le equazioni di retroreazione (*backreaction*), ma che tipicamente prevede la formazione di singolarità nude, minacciando l'unitarietà. Per risolvere tali problemi, proponiamo una modifica all'approccio semiclassico standard incorporando campi con cariche centrali negative (come i fantasmi di Faddeev-Popov) accoppiati a una metrica ausiliaria conformemente piatta. I risultati mostrano che, sotto specifiche condizioni che coinvolgono cariche centrali negative sufficientemente grandi, la singolarità nuda viene completamente rimossa dall'evoluzione dello spaziotempo. Al suo posto, il sistema evolve con un orizzonte apparente che nasconde una regione regolare dello spaziotempo, suggerendo un percorso per l'evoluzione dei buchi neri che evita rotture catastrofiche della predicibilità e offre nuove prospettive sul ruolo del contenuto di materia nel collasso gravitazionale.

Sumario

Esta tesis doctoral explora la intersección de la Teoría Cuántica de Campos y la Relatividad General en el marco de la Teoría Cuántica de Campos en espaciotiempos curvos. La investigación se divide en dos secciones principales: el desarrollo de herramientas matemáticas relacionadas con la acción efectiva y el núcleo de calor (*heat kernel*), y la aplicación de estas herramientas a modelos de agujeros negros bidimensionales.

En la primera parte, abordamos la definición y el cálculo de la acción efectiva a 1-bucle, un objeto crucial para el estudio de las correcciones cuánticas en teorías de campos clásicas. Si bien la expansión estándar de Gilkey-Seeley-DeWitt proporciona un método para regularizar divergencias, resulta computacionalmente costosa y es de naturaleza perturbativa. Este documento presenta y desarrolla un esquema de resumación para el núcleo de calor que captura cierta información no perturbativa y simplifica el cálculo de los términos de orden principal. Esta fórmula de resumación se aplica con éxito a diversos sistemas, incluyendo campos escalares con acoplamientos de Yukawa, Electrodinámica Cuántica, y configuraciones de campos inhomogéneos. Estos métodos permiten un análisis más preciso de fenómenos como el efecto Schwinger y la polarización del vacío.

La segunda parte se centra en la evolución semiclásica de los agujeros negros, utilizando modelos de gravedad dilatónica en 2D (específicamente el modelo Callan-Giddings-Harvey-Strominger) como campo de prueba para abordar problemas conceptuales como la pérdida de información y las singularidades espaciotemporales. Examinamos el modelo Russo-Susskind-Thorlacius, el cual resuelve analíticamente las ecuaciones de reacción inversa (*backreaction*), pero que típicamente predice la formación de singularidades desnudas, amenazando la unitariedad. Para resolver estos problemas, proponemos una modificación al enfoque semiclásico estándar incorporando campos con cargas centrales negativas (como los fantasmas de Faddeev-Popov) acoplados a una métrica auxiliar conforme plana. Los resultados muestran que, al considerar la inclusión de cargas centrales negativas suficientemente grandes, la singularidad desnuda se elimina por completo de la evolución del espaciotiempo. En su lugar, el sistema evoluciona con un horizonte aparente que oculta una región regular del espaciotiempo, sugiriendo una vía para la evolución de los agujeros negros que evita rupturas catastróficas de la predictibilidad y ofrece nuevas perspectivas sobre el papel del contenido material en el colapso gravitacional.

Resumen de la tesis

Durante las últimas décadas, uno de los campos de estudio más explorados en el ámbito de la Física Teórica es el comportamiento de la materia a nivel cuántico, especialmente cuando dicha materia se somete a altos niveles de energía. El estudio de las fuerzas fundamentales a nivel microscópico ha permitido el desarrollo de teorías tan poderosas como la Electrodinámica y Cromodinámica Cuánticas, culminando en la formulación del Modelo Estándar de la Física de Partículas, que a día de hoy sigue representando la piedra angular de nuestro entendimiento de las leyes físicas en general, y de la Teoría Cuántica de Campos, en particular.

Sin embargo, el Modelo Estándar no se encuentra libre de imperfecciones. A niveles bajos de energía todas las interacciones fundamentales se pueden explicar, rudimentariamente, empleando cuatro fuerzas independientes (a saber, electromagnética, nuclear fuerte y débil, y gravitatoria). Las diferencias de intensidad de estas interacciones permiten crear una jerarquía mediante la cual los efectos de las interacciones más débiles en un sistema pueden ser desestimados en favor de las más fuertes. En particular, la Teoría Cuántica de Campos tradicionalmente desestima la interacción gravitatoria entre las partículas pues, en situaciones cotidianas o alcanzables por nuestros experimentos actuales, su contribución resulta despreciable en comparación con las demás.

Sin embargo, existen situaciones en las que los niveles de energía se vuelven tan elevados que resulta imposible seguir ignorando las contribuciones de las interacciones cuánticas por separado. Es en estos regímenes donde surgen las llamadas teorías de unificación, revelando que lo que *a priori* parecieran construcciones muy diferentes pueden en realidad englobarse en formalismos teóricos más potentes y de mayor rango de aplicación. La teoría electrodébil y las Teorías de Gran Unificación (GUTs por sus siglas en inglés) son ejemplos de este fenómeno, permitiendo englobar tres de las interacciones fundamentales en un solo marco común. Es por ello que no es de extrañar que la atención de muchos investigadores se haya dirigido a intentar encontrar una teoría unificada o "Teoría del Todo", que permita explicar todas las interacciones de la Física mediante un único modelo global.

Precisamente la interacción gravitatoria presenta uno de los mayores escollos en esta búsqueda. Aunque la descripción clásica de la gravedad lleva más de un siglo teóricamente afianzada y experimentalmente corroborada, a día de hoy no podemos decir que hayamos podido describir la forma en que los sistemas cuánticos interactúan gravitacionalmente.

Son incontables las propuestas que a lo largo de los años han surgido al respecto, pero por el momento ninguna parece haberse impuesto definitivamente como la ruta a seguir. En no poca medida, esto se debe a que la comprobación experimental de las teorías de gravedad cuántica no es técnicamente posible en el estado actual de nuestros experimentos. Se requiere la observación de sistemas en interacción a niveles de energía tan elevados que es simplemente imposible diseñar un experimento de laboratorio. Nuestra única baza por el momento es la observación de sistemas reales en los que estas condiciones se den de forma natural, lo cual limita sobremanera nuestras posibilidades para realizar comprobaciones. Los únicos fenómenos gravitatorios suficientemente energéticos (como para que los efectos cuánticos sean claramente visibles) de los que tenemos constancia son aquellos asociados a la presencia de agujeros negros u otro tipo de singularidades espaciotemporales, como el Big Bang.

Así pues, en el momento actual resulta imposible determinar si una teoría completa de gravedad cuántica es correcta, o si es siquiera factible cuantizar las interacciones gravitatorias sin introducir nuevos fenómenos que por el momento nos son desconocidos. Muchas de las teorías más populares que intentan explicar la gravedad a nivel cuántico, como la Teoría de Cuerdas o la Gravedad Cuántica de Bucles (LQG por sus siglas en inglés) se basan en la introducción de nuevos sistemas de partículas fundamentales y/o de la redefinición del espaciotiempo en el que nuestro universo se encuentra, añadiendo dimensiones adicionales o discretizando la propia estructura del mismo. Siendo esta la situación, es posible que la presentación de una teoría completa que involucre la gravedad y las demás interacciones a nivel cuántico sea un proyecto demasiado ambicioso por el momento.

El planteamiento principal detrás de la Teoría Cuántica de Campos en espaciotiempos curvos (QFTCS por sus siglas en inglés) podría resumirse en la necesidad de dar un paso atrás y enfocarse en modelos “de transición” que permitan vislumbrar algunas de las propiedades que una teoría de gravedad cuántica debería poseer para poder explicar debidamente fenómenos más accesibles, de la misma forma que la teoría de la Electrodinámica Cuántica debe ser capaz de explicar los fenómenos clásicos de la electricidad y el magnetismo que experimentamos cotidianamente. En este sentido, QFTCS propone un planteamiento semiclásico en el que los sistemas cuánticos se encuentran inmersos en una interacción gravitatoria lo suficientemente potente como para que sus efectos afecten de alguna forma a las interacciones cuánticas, pero no lo suficiente como para que la aproximación clásica dada por la Relatividad General de Einstein deje de ser efectiva. Así pues, el campo gravitatorio se muestra como un "escenario" de fondo que puede modificar el comportamiento de los campos cuánticos que se propagan por él, pero que no se ve afectado a su vez por estos como sí lo haría en un modelo completo de gravedad cuántica.

El objetivo de esta tesis es profundizar en una de las herramientas más potentes en la formulación de QFTCS, conocida como la acción efectiva. Para este propósito, el trabajo se divide en dos grandes bloques. Mientras que en la primera parte del documento se explora la definición y cálculo de este objeto, la segunda se enfocará en su aplicación a un modelo particular de agujeros negros, obteniendo así una serie de predicciones y resultados que no serían posibles de un tratamiento meramente clásico del sistema. Así pues, debería quedar demostrada la necesidad de seguir impulsando la investigación en modelos de QFTCS como manera de ahondar en la naturaleza cuántica de la gravedad, al menos mientras seguimos a la espera de un modelo completo que pueda ser comprobado experimentalmente.

En términos generales, cuando hablamos de "cuantizar" una teoría, solemos verlo como un proceso activo a través del cual una serie de modelos y variables que conforman una teoría física clásica son sustituidas por sus equivalentes cuánticos. Sin embargo, esta definición ignora la naturaleza y peculiaridades de los dos formalismos (clásico y cuántico), que son muchas y de gran calado incluso si nos centramos exclusivamente en el aspecto matemático, sin profundizar en las posibles interpretaciones físicas. Las teorías físicas clásicas se construyen sobre una filosofía base de determinismo puro, según la cual el único impedimento a la hora de realizar mediciones y predicciones (hacia el futuro o hacia el pasado) infinitamente precisas es el error experimental dado, por ejemplo, por el desconocimiento de algunas variables iniciales o por la imprecision de los detectores empleados. Los modelos matemáticos que describen estas teorías, en cambio, carecen de esta limitación al estar formulados en términos de funciones reales y conmutativas, permitiendo a priori realizar un número ilimitado de predicciones de infinita exactitud. Las teorías cuánticas, por su parte, solo se pueden describir en términos de operadores cuánticos, objetos que, en general, están sujetos a reglas de (anti)conmutación no-triviales; así pues, la propia formulación matemática empleada acarrea inherentemente una incertidumbre asociada a estas reglas. Adicionalmente, el mero hecho de "medir" pasa a ser una fuente de dudas, puesto que supone el colapso instantáneo e irreversible de la función de onda del sistema, el objeto fundamental en teorías cuánticas que describe todas las "posibilidades" del mismo, a otra forma completamente distinta (aquella compatible con el resultado de la medida) a través de un proceso indeterminado. Estas y otras cuestiones apuntan al hecho de que tender un puente entre la Física clásica y la cuántica no puede ser un proceso sencillo.

Consecuentemente, existen diferentes esquemas de cuantización que han surgido con el paso del tiempo, cada uno con sus propias reglas, suposiciones y justificaciones. La idea implícita detrás de esta variedad es que, aunque recorran diferentes caminos, todos ellos deberían dar lugar a las mismas predicciones y por tanto ser simplemente "diferentes formas de ver el mismo problema". Sin embargo, aunque esto es demostrable para sistemas con un número finito de grados de libertad, no está garantizado en el momento en el que pasamos a un número infinito de ellos (por ejemplo, cuando consideramos un campo definido sobre todo el espaciotiempo). Esta ambigüedad en los métodos de cuantización es lo que da lugar a fenómenos conocidos, como el efecto Unruh, por el cual dos observadores distintos pueden llegar a tener dos definiciones no-equivalentes del vacío y de lo que es una partícula. El proceso más directo de cuantización, empleado en sistemas discretos y generalizable a sistemas continuos, es la llamada cuantización canónica, que sustituye todas las coordenadas y momentos asociados a estas por operadores que conmutan siguiendo una regla equivalente a la de los corchetes de Poisson en dinámica clásica. Existen otros métodos de cuantización que emplean prescripciones diferentes, como la cuantización geométrica o la cuantización BRST.

Sin embargo, para sistemas continuos en los que haya varios campos cuánticos interactuando entre sí, el método más comúnmente empleado es sin duda el formalismo de la integral de camino de Feynman. En lugar de intentar describir el sistema en términos de una funcionalidad de onda que evoluciona en el tiempo, este procedimiento permite el cálculo directo de la probabilidad de transición entre un estado inicial y uno final mediante la consideración simultánea y ponderación de todas las posibles evoluciones que el sistema haya podido seguir para llegar del uno al otro; en pocas palabras, representa un análogo cuántico a las ideas de la mecánica Lagrangiana y el principio de acción estacionaria. El cálculo explicito de la integral de Feynman suele realizarse mediante una expansion en serie perturbativa, donde cada término se asocia a un diagrama de Feynman cuya estructura interna determina el peso relativo de dicho término en el resultado final; visualmente, esto se manifiesta en la presencia de bucles internos dentro del diagrama, distinguiendo el orden principal (sin bucles) de las correcciones a 1, 2, etc. bucles.

El formalismo de Feynman representa sin duda uno de los pilares fundamentales en nuestro entendimiento actual de las interacciones cuánticas, proporcionando una forma relativamente concisa y directa para su estudio. Sin embargo, aplicar directamente el formalismo de Feynman a la teoría gravitatoria resulta en en un modelo no-renormalizable que se vuelve inutilizable a altas energías. Específicamente, la expansión en diagramas de Feynman presenta un número infinito de términos divergentes que no pueden ser reabsorbidos de ninguna manera empleando un número finito de contratérminos, por lo que se vuelve necesario cortar el cálculo de manera artificial y solo considerar aproximaciones hasta cierto punto. Se hace evidente que es necesario explorar alternativas para tratar la gravedad en este tipo de sistemas.

La acción efectiva se presenta como una de estas alternativas, siendo esencialmente una adaptación del concepto clásico de la acción de un sistema al formalismo cuántico, generalmente a través del principio de Schwinger. Hacer esto representa un salto conceptual que no puede desestimarse: ser capaz de construir un perfecto análogo de la acción clásica en el formalismo cuántico implicaría poder condensar toda la información de un sistema cuántico en un único objeto, y realizar cálculos y predicciones sin necesidad de recurrir a expansiones perturbativas infinitas. Por desgracia, encontrar una expresión cerrada y explícita de la acción efectiva de un sistema solo es posible en situaciones muy concretas y para sistemas sencillos, por lo que un tratamiento general completamente no-perturbativo de los campos cuánticos aún queda lejos de ser una realidad.

Sin embargo, es posible encontrar una expresión exacta de la acción efectiva para cualquier sistema cuya acción clásica sea, como mucho, cuadrática en los campos o, siendo más genéricos, para cualquier sistema cuyos campos se puedan considerar "casi clásicos", es decir, donde se puedan tomar como la suma de un término clásico y una corrección cuántica desestimable en una expansión perturbativa de la acción para términos de orden mayor a 2. Haciendo esto, se puede encontrar que la acción efectiva consta de dos términos

$$\Gamma = S + i \log \det \mathcal{Q}$$

donde el primero es la acción clásica, y el segundo es conocido como la acción efectiva a 1-bucle. En próximos párrafos entraremos en la definición y cálculo de este término, pero es importante destacar que esta expansion ya resulta de utilidad para explorar efectos semiclásicos en teorías de campos, como la producción de partículas en el vacío.

La expresión anterior puede parecer sorprendentemente sencilla; sin embargo, el objeto $\mathcal{Q}$ descrito en ella no es una simple matriz finita que permita hacer cálculos explícitos, sino un operador diferencial asociado a la acción clásica del sistema en cuestión (y que, por tanto, será generalmente de dimension infinita). Debido a esto, esta expresión solo puede tomarse como una definición formal, y solo podrá adquirir un sentido práctico al encontrar un método para regularizar y acotar un valor para "$\log\det\mathcal{Q}$". Esto suele realizarse mediante la definición de un operador de "núcleo de calor" (*heat kernel*) $K(\tau) = \exp(i\tau\mathcal{Q})$. En términos de este operador, la acción efectiva a 1-bucle puede pasar a redefinirse como una integral sobre la traza del mismo. La ventaja de este operador radica en que, tal y como su nombre indica, sus elementos de matriz satisfacen ecuaciones diferenciales de difusión de segundo orden. Si es posible encontrar una solución exacta de dichas ecuaciones, entonces se puede hacer un cálculo explicito de la acción efectiva del sistema. En este documento se presentan algunos ejemplos de esto, mostrando resultados ya conocidos, como el Lagrangiano efectivo de Euler-Heisenberg.

Para aquellos casos en los que sea imposible encontrar una solución exacta de la ecuación de difusión, se vuelve a hacer imperativo emplear una expansión perturbativa. A efectos prácticos, el método más comúnmente empleado se conoce como expansión de Gilkey-Seeley-DeWitt (GSDW), donde el orden de los términos viene dictado por el exponente del tiempo propio τ. La justificación de este método se basa en que solo los primeros términos de la expansión darán lugar a divergencias en la expresión de la acción efectiva, siendo estos los que deberán regularizarse para que el resultado final no sea divergente. Dado que los coeficientes de la expansión son calculables iterativamente empezando desde el primero, esto implica que solo un número finito de ellos serán realmente relevantes.

La expansión GSDW resulta ser una herramienta muy poderosa para el cálculo de las divergencias en la acción efectiva a 1-bucle, permitiendo determinar correcciones cuánticas a la acción del sistema con relativa simpleza. Sin embargo, el proceso iterativo de cálculo de los coeficientes de la serie se vuelve complejo rápidamente. Adicionalmente, uno de los motivos principales para utilizar la acción efectiva en lugar del formalismo de Feynman era la posibilidad de encontrar un método lo más exacto posible para tratar sistemas cuánticos; sería por tanto preferible ser capaz de encontrar una formulación alternativa que permitiera extraer algo más de información no-perturbativa del sistema y pudiese simplificar los cálculos en el proceso. A lo largo de los años ha habido numerosos trabajos en esta línea, proponiendo diferentes métodos de "resumación" del heat kernel; sin embargo, hasta el momento no se ha podido encontrar un modelo que funcione en todo tipo de sistemas siendo cada uno restringido a diferentes familias de teorías.

La propuesta principal de la primera parte de la tesis es, precisamente, una de estas fórmulas de resumación. Basada en trabajos previos de Brown y Duff, en este documento mostramos que es posible encontrar una resumación que funciona para cualquier sistema consistente en un campo cuántico inmerso en un potencial clásico en espaciotiempo plano. La fórmula obtenida permite extraer toda la información que depende exclusivamente del potencial y sus primeras dos derivadas de la expansión perturbativa, colocándola en su lugar en un prefactor global multiplicativo. En otras palabras, para cualquier sistema con un potencial cuadrático, la expresión del heat kernel se vuelve exacta, mientras que para sistemas con un potencial de mayor rango el proceso de cálculo iterativo se vuelve considerablemente más sencillo. Esta fórmula se desarrolla y comprueba para sistemas escalares con interacción de Yukawa y electromagnética y, tras algunos ajustes, es posible generalizarla a un sistema de Electrodinámica Cuántica sencillo. Este último caso permite resumar las contribuciones que no dependen de las derivadas del campo electromagnético, simplificando los cálculos asociados; adicionalmente, el prefactor global obtenido puede relacionarse directamente con el Lagrangiano de Euler-Heisenberg, presentándose como una versión local expandida del mismo.

De la misma forma, el mismo método empleado para obtener esta fórmula parece poder adaptarse a sistemas donde la ecuación de difusión del heat kernel presenta un término inhomogéneo lineal en derivadas. La argumentación se basa, en primer lugar, en un pequeño modelo teórico con un campo vectorial genérico, para luego pasar a discutir el caso de una interacción entre un campo espinorial cuántico interactuando con un campo axial clásico; estos modelos pueden utilizarse para describir la transmisión de un campo espinorial en un espaciotiempo con torsión. Aunque no ha sido posible extraer resultados concluyentes, existen argumentos para creer que este método es efectivamente utilizable en este caso, lo cual apunta a posibles generalizaciones a espaciotiempos curvos. Esto es especialmente emocionante porque, aunque es probable que el caso de un campo gravitatorio genérico sea demasiado complejo, la aplicación de este método a campos sencillos podría ser de utilidad en la construcción de modelos análogos en el campo de Estructura de la Materia. Estos modelos, hoy en día, son lo más cercano a experimentos en QFTCS que se puede realizar en un laboratorio, por lo que podrían suponer una puerta a nuevas observaciones en ambos campos.

La segunda parte de este documento se enfoca en las aplicaciones de la acción efectiva en teorías gravitatorias. Tal y como se introdujo anteriormente, la teoría clásica de la gravitación más completa hoy en día es la Relatividad General de Einstein, cuya principal característica (y uno de los mayores impedimentos a la hora de intentar cuantizarla) es la forma en que se trata la gravedad. En lugar de interpretar la gravedad como una fuerza que interactúa con los campos de un sistema, esta se entiende como una característica geométrica del propio espaciotiempo en el que los campos se propagan, representado matemáticamente en forma de una variedad diferenciable, con un tensor métrico que define su curvatura. Dado que este espaciotiempo considera las dimensiones espaciales y temporales al mismo nivel, conceptos como los vectores, las distancias y tamaños o las velocidades deben asimismo generalizarse adecuadamente.

Uno de los elementos principales de la Relatividad Especial es la noción de que la velocidad de la luz en el vacío es un máximo inalcanzable por ningún sistema material (al menos, sin introducir nociones exóticas como los taquiones). En Relatividad General este concepto se vuelve local, por lo que es posible que en ciertos sistemas de referencia haya objetos que aparenten desplazarse más rápido que la luz; un ejemplo paradigmático de esto es la expansion cosmológica. Sin embargo, la estructura de causalidad entre diferentes eventos en el espaciotiempo siempre debe mantenerse; así, en cada punto del espaciotiempo se define un cono de luz que separa aquellos eventos que se encuentren causalmente conectados con él (es decir, para los que es posible mandar una señal de uno a otro sin superar la velocidad de la luz) de aquellos que no. Estas nociones se extienden en general a trayectorias y superficies sobre todo el espaciotiempo, pudiendo clasificarlas en tipo tiempo, tipo luz (en el caso extremo en que solo un rayo de luz u otra posible partícula sin masa pueda recorrerla) y tipo espacio.

Si es posible construir una superficie tipo espacio que intersecciona con todas las trayectorias tipo luz y tiempo del espaciotiempo una única vez (lo que se conoce como superficie de Cauchy), entonces se dice que el espaciotiempo es globalmente hiperbólico. La existencia de esta superficie garantiza que la teoría es completamente determinista; conociendo el estado de un sistema en una superficie de Cauchy, es posible determinar la evolución tanto hacia el futuro como el pasado con total precisión; algunos ejemplos de espaciotiempos globalmente hiperbólicos incluyen el espaciotiempo plano de Minkowski, y el modelo FLRW de expansion cosmológica. La hiperbolicidad global de un espactiotiempo implica necesariamente que ningún objeto puede viajar a su propio pasado.

Como todas las teorías clásicas de campos, la dinámica de la Relatividad General puede obtenerse de una acción o un Lagrangiano. El Lagrangiano más común utilizado en este caso es el Einstein-Hilbert, que puede utilizarse para derivar las ecuaciones de campo de Einstein. Estas ecuaciones relacionan directamente el contenido de materia de un sistema físico (parametrizado por el tensor de energía-momento) con la curvatura del espaciotiempo en el que se encuentra (parametrizada por invariantes como el escalar de Ricci); popularmente hablando, "la curvatura le indica a la materia cómo moverse, y la materia le indica al espaciotiempo cómo curvarse". Esta interconexión presenta una de las mayores dificultades a la hora de desarrollar un modelo semiclásico o cuántico de la gravedad; al introducir correcciones cuánticas en la curvatura, estas modifican los valores del tensor de energía-momento, que a su vez modifican de nuevo la gravedad en un proceso denominado reacción inversa (*backreaction*), entrando en un proceso retroalimentado infinito. Por otra parte, y aunque existen fuertes indicaciones de lo contrario, no está demostrado a día de hoy que el proceso de cuantización no rompa la estructura globalmente hiperbólica del espaciotiempo; así pues, en el proceso de cuantizar una teoría de gravedad es necesario asegurarse de que la estructura causal no se vea alterada.

Volviendo al formalismo clásico, un conjunto de soluciones de las ecuaciones de Einstein de particular interés son las llamadas soluciones de agujero negro. Aunque originalmente se consideraban un mero artificio matemático, su existencia ha sido probada en numerosas ocasiones desde entonces. En términos populares, los agujeros negros son regiones del espaciotiempo de las cuales es imposible escapar sin superar la velocidad de la luz. Su característica más conocida es la presencia de una singularidad, una región en la que la curvatura del espaciotiempo se vuelve infinita, en su interior. Las singularidades son el indicador más estridente de que la Relatividad General por sí misma no puede ser la explicación definitiva de la gravedad, puesto que la teoría pierde su capacidad predictiva en ellas. En otras palabras, para obtener un conocimiento completo de los agujeros negros, resulta imprescindible extender la teoría de la gravedad, seguramente incluyendo nuevos fenómenos semiclásicos y cuánticos que surjan al aumentar la curvatura del espaciotiempo.

El agujero negro más simple, y a la vez más conocido, es el modelo de agujero negro de Schwarzschild. Representa la curvatura generada por una distribución esférica de masa concentrada en un único punto (o al menos, en un volumen menor al determinado por el radio de Schwarzschild, que indica la posición del horizonte de eventos del agujero negro).

Se trata de una solución estática de las ecuaciones de Einstein, en la que el agujero negro existe en todo momento desde el pasado infinito hasta el futuro infinito. Es, por tanto, una solución que no esperamos encontrar en el mundo real, pero que puede servir para comprender algunos de los elementos básicos del estudio de agujeros negros. Entre ellos, quizás la herramienta más potente sea el desarrollo del diagrama de Penrose asociado al espaciotiempo. Aunque esta representación de las coordenadas distorsiona completamente las distancias que separan los puntos, también otorga una visualización clara de la estructura causal del espaciotiempo, permitiendo distinguir las regiones asociadas al agujero negro de aquellas libres del mismo, donde las trayectorias de tipo luz pueden escapar a las conocidas como regiones asintóticamente planas (tan alejadas del agujero negro que la curvatura es virtualmente indistinguible de la del espaciotiempo plano).

Una idea más realista de un agujero negro es considerar que su formación se debe al colapso gravitacional de un objeto masivo, como una estrella. Desde el punto de vista puramente clásico, una vez formados los agujeros negros se vuelven elementos estáticos y eternos, cuya información puede determinarse mediante solo tres parámetros: su masa, su carga eléctrica y su momento angular. Cualquier otra información acerca de la materia que formó el agujero negro se pierde por completo, lo cual supone un conflicto directo con las leyes de la Termodinámica pues el simple hecho de arrojar materia al interior de un agujero negro permitiría disminuir la entropía del universo. Bekenstein propuso que la entropía del agujero negro debía aumentar para compensar esta pérdida, argumentando que esta debía ser proporcional a su superficie externa. El hecho de que los agujeros negros tengan una entropía naturalmente implica que también deben tener una temperatura, que podría detectarse en forma de radiación emitida. En efecto, este fenómeno fue descrito por Hawking poco después; los agujeros negros deberían comportarse como un cuerpo negro que emite radiación a una temperatura concreta, que depende de su masa. Este resultado constituye el primer efecto inherentemente semiclásico o cuántico asociado a los agujeros negros. No debemos asumir que esto resuelve por completo las dudas en torno a la evolución de los agujeros negros; presumiblemente, el proceso de radiación Hawking continúa indefinidamente hasta que el agujero negro pierda toda su masa y se evapore, pero la naturaleza exacta de este fenómeno no está clara, especialmente en lo que concierne a sus últimas fases. Por otra parte, sigue habiendo un conflicto en lo que a la información de la materia que formó el agujero negro se refiere; aunque esta pudo haber sido de cualquier tipo, el agujero negro solo devuelve radiación de cuerpo negro.

Entender estas particularidades de los agujeros negros seguramente requiera de la elaboración de un modelo cuántico completo de la gravedad. Sin embargo, a la complejidad conceptual de estos modelos se suma la dificultad mecánica de tratar con estos sistemas en cuatro dimensiones espacio-temporales. Es por ello que en los últimos años ha habido cierto interés por la construcción y estudio de sistemas de agujeros negros en dos dimensiones, simplificando enormemente el componente matemático mientras se conserva la mayor cantidad de elementos esenciales posibles de los modelos en dimensionalidades superiores. Sin embargo, esta decisión no está exenta de problemas; uno de los más alarmantes es que el Lagrangiano de Einstein-Hilbert en dos dimensiones se vuelve un invariante topológico, perdiendo su capacidad para generar ecuaciones dinámicas para los campos. Es por ello que se vuelve imperativo buscar modelos alternativos de gravedad en dos dimensiones que sí puedan generar dichas ecuaciones. Una familia de sistemas de especial interés en este sentido son los modelos de campos dilatónicos, como el modelo Callan-Giddings-Harvey-Strominger (CGHS).

Uno de los puntos clave para trabajar en 2D es que, en estas condiciones, cualquier métrica puede escribirse localmente en una forma conforme plana. Clásicamente, la simetría conforme se asocia con una ligadura de traza del tensor energía-momento de la materia presente en el sistema. Sin embargo, al intentar calcular la acción efectiva asociada a este, se puede demostrar que uno de los contratérminos necesarios para regularizar las divergencias de la misma añade una modificación a la traza del tensor energía-momento, rompiendo esta ligadura del sistema en un proceso llamado "anomalía de traza". El valor de la traza puede relacionarse de forma directa con la carga central del sistema estudiado, que en cierta forma da cuenta del número de grados de libertad del mismo.

Así pues, podemos considerar el modelo CGHS clásico y tratar de expandirlo para incluir fenómenos semiclásicos. Al hacerlo, la acción clásica se ve modificada por un término extra denominado acción de Polyakov. Desafortunadamente, el sistema formado por estos dos términos no es analíticamente resoluble, por lo que es necesario introducir contratérminos adicionales que recuperen la analiticidad sin modificar los resultados físicos. Existen diferentes acercamientos a este problema, siendo dos de ellos los modelos de Russo-Susskind-Thorlacius (RST) y Bose-Parker-Peleg (BPP). Es posible demostrar que estos dos modelos forman en realidad parte de una familia uniparamétrica de teorías, por lo que las conclusiones obtenidas por ambos deberían ser equivalentes. Utilizando estas teorías, es posible construir un modelo de agujeros negros en dos dimensiones que se forma por el colapso de materia, empieza a emitir radiación Hawking mientras se evapora, hasta el punto en el que la singularidad y el horizonte del agujero negro se encuentran. Es en este punto donde deja de ser posible estudiar la evolución del sistema sin introducir nueva información *ad hoc*; en el caso de RST, una solución habitual es considerar que el espaciotiempo vuelve a ser plano tras la desaparición del agujero negro. Esto, sin embargo, trae ciertos problemas conceptuales, asociados principalmente a la presencia de dos fenómenos conocidos como *thunderpop* (una pequeña emisión de energía negativa) y *thunderbolt* (una singularidad desnuda que rompe la unitariedad de la teoría).

Hay otro punto importante a destacar en este proceso, que merece atención. El modelo CGHS, como se ha introducido arriba, tiene un alto grado de simetría en su métrica. Esta libertad gauge debe ser considerada en el proceso de cuantización; habitualmente, el procedimiento empleado para ello es la introducción de campos fantasma de Faddev-Popov. Sin embargo, esto modifica la carga central del sistema; particularmente, este conjunto de campos presenta una carga central negativa. Si simplemente modificamos el sistema para incluir estos campos, entonces es necesario incluirlos en el proceso de radiación Hawking, lo que haría visibles estos campos *a priori* auxiliares. Generalmente, el modelo RST considera solo casos en los que la cantidad de campos físicos es tan elevada que esta contribución es despreciable en comparación, pero en sistemas menos poblados es imposible eludirla. En esta tesis proponemos una alternativa a esta construcción en la que los campos de carga central negativa no interaccionan con la métrica general del sistema, sino con su equivalente (a través de transformaciones de simetría) conforme plana. Esto debería asegurar que su contribución a la radiación Hawking sea nula.

El modelo resultante presenta una serie de elementos clave que inmediatamente llaman la atención. En primer lugar, aunque existe una región que podríamos asociar a un horizonte aparente de agujero negro, el espaciotiempo nunca desarrolla una singularidad en su interior. En su lugar, se desarrollan tres regiones asintóticamente planas distintas, cuya radiación Hawking puede ser estudiada. Los resultados parecen indicar que, conforme nos acercamos a tiempos infinitos, el flujo de energía se vuelve similar al del sistema CGHS semiclásico en la zona exterior, mientras que la radiación en el interior parece colapsar en torno al horizonte y permanecer relativamente nula lejos de este. Se requiere de cálculos más precisos para completar el dibujo de este modelo, y especialmente se vuelve imprescindible comprobar el balance de energía del mismo, puesto que de él depende si este tipo de construcción es capaz de evitar fenómenos como el del *thunderpop/thunderbolt*.

En cualquier caso, resulta evidente que el campo de estudios en los modelos de agujeros negros en particular, y en efectos semiclásicos en sistemas gravitatorios en general, se encuentra en un momento muy activo. Herramientas como la acción efectiva permiten indagar en correcciones cuánticas causadas por la interacción a altas energías, desarrollando aproximaciones que quizás puedan servir para guiar nuestra búsqueda de modelos más potentes en el futuro.

List of publications

The following thesis is based on the following publications by the candidate.

1. *Resummed heat kernel and effective action for Yukawa and QED*
   S. A. Franchino-Viñas, C. García-Pérez, F. D. Mazzitelli, V. Vitagliano & U. Wainstein-Haimovichi
   Physics Letters B, vol. 854 (2024), p.138684

   Primarily discussed in Section 2.2.4

2. *Strong-field resummed heat kernels and effective actions: inhomogeneous fields*
   S. A. Franchino-Viñas, C. García-Pérez, F. D. Mazzitelli, S. Pla, V. Vitagliano & U. Wainstein-Haimovichi
   (arXiv) Contribution to: 17th Marcel Grossmann Meeting Proceedings (2025)

   Primarily discussed in Section 2.2.6

3. *Heat kernels and resummations: the spinor case*
   S. A. Franchino-Viñas, C. García-Pérez, F. D. Mazzitelli, S. Pla & V. Vitagliano
   Phys. Rev. D, vol.113 (2026), p.025001

   Primarily discussed in Sections 2.2.5 and 2.3.

4. *Singularity resolution and unitarity in two-dimensional dilaton black holes with negative central charge*
   C. García-Pérez, F. J. Marañón-González, J. Navarro-Salas & S. Pla
   To appear

   Primarily discussed in Sections 3.2.2, 3.2.3 and 3.2.4.

Other articles that have been published during the thesis but are not directly related to its results are:

1. *An agent based simulation of COVID-19 history in Catalonia using extensive real datasets*
   M. Bosman, Y. Cordón, M. Durán-Sala, L. Gabbanelli, C. García-Pérez, X. Jordán, M. Manera, P. Masjuan, A. Medina, Ll. M. Mir, A. Oròs & V. Vitagliano
   Scientific Reports, vol. 14 (2024), p.31858

List of Notations

Symbol	Description
General objects	
Roman indices	Internal bundle of a field / a set of fields
Greek indices	Spacetime components of a field
d	Spacetime dimensionality
$g_{\mu\nu}$	Spacetime metric
$\eta_{\mu\nu}$	Flat (Minkowski) spacetime metric, given by $\mathrm{diag}(-1, +1, ..., +1)$
γ^μ, γ_5	Dirac matrices, chiral element of the Clifford algebra
Constants	
i	Imaginary unit number
$\hbar$	Planck's constant
G	Newton's constant
c	Speed of light in vacuum
e	Charge of the electron
k_B	Boltzmann constant
δ_i^j	Kronecker δ symbol
$\epsilon_{\mu\nu\rho\sigma}$	Levi-Civita symbol
$B_{2j},\ j \geq 0$	Bernoulli numbers
Classical and Quantum Field Theory	
$\mathcal{L}$	Lagrangian density
S	Classical action
Γ	(1-loop) Effective action
K	Heat kernel
$A^\mu, F_{\mu\nu}$	Electromagnetic field vector, field strength tensor
$\mathcal{F}, \mathcal{G}$	Electromagnetic invariants in $d = 4$
$c_j(x, x'),\ j \geq 0$	Gilkey-Seeley-DeWitt coefficients

Symbol	Description
Differential Geometry and General Relativity	
$\mathcal{M}$	Manifold
$T_P\mathcal{M}$	Tangent space to $\mathcal{M}$ at a point $P \in \mathcal{M}$
∂	Partial derivative
D, ∇	Covariant derivative
$\Box, \nabla^2$	D'Alembertian
$\sigma(x, x')$	Synge world function
$\Delta_{\mathrm{VVM}}^{1/2}$	Van Vleck-Morette determinant
$\Gamma^{\rho}{}_{\mu\nu}$	Christoffel symbols
R	Ricci curvature scalar
$R_{\mu\nu}$	Ricci tensor
$R^{\sigma}{}_{\mu\nu\rho}$	Riemann curvature tensor
$C_{\mu\nu\rho\sigma}$	Weyl tensor
Λ, λ	Cosmological constant
ϕ	Dilaton field
$c, c^{\pm}$	Central charge of a system
Special functions and operators	
$\delta(x, x'), \delta(x - x')$	Dirac δ distribution (function)
$\Theta(x - x')$	Heaviside step function
$T(...)$	Chronological ordering operator
$\mathfrak{Re}(...), \mathfrak{Im}(...)$	Real, Imaginary part of a function / operator
$\mathbf{divp}(...)$	Divergent part of a function / operator
$[F]$	Coincidence limit $x' \to x$ of $F(x, x')$
$W(z), W_k(z)$	Lambert W function and its real branches
$\mathrm{Ei}(x)$	Exponential integral function

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Chapter 1

Introduction

Over the past century, there has been an increasing interest in the idea of unification of Theoretical and Mathematical Physics models. Motivated by successful stories like the unification of electricity and magnetism back in the 1800s, as well as the deep connections between symmetries and physics that lie at the heart of fields like Particle Physics, many physicists have favored the idea of a grand model of the universe that encompasses every possible phenomena, a so-called "Theory of Everything". This aspiration is deeply seeded in a combination of logical necessity and, most prominently, an idealistic aspiration that nature follows a simple, elegant blueprint. While the existence of such a theory, at the time of writing, remains a distant chimera, it is still of great relevance to gain a better understanding of the possible commonalities and even explicit interconnections between *a priori* very different systems.

One such particular connection that lies at the near top of the "priority list" for modern researchers is the one between gravity and the rest of the fundamental forces. At the mathematical level, this implies the formulation of a model that ultimately encompasses both General Relativity and Quantum Field Theory. Many attempts at a theory of “Quantum Gravity” have been made over the years; to name a few, String Theory, Loop Quantum Gravity, asymptotic safety, CDTs (Causal Dynamical Triangulations) or emergent gravity all pose alternative constructions that could theoretically accomplish the goal [1–5]. However, the energy scales at which all of these operate make it impossible to experimentally discern which of them, if any, may be correct.

Taking a step back, we may reason that while a full theory of gravity at the quantum level may be out of reach for now, we can still look for its remnants in slightly more achievable regimes. Such is the reasoning for constructing a theory of quantum fields in curved spacetimes (QFTCS), where the gravity is still treated as a classical background that follows the rules of Einstein’s relativity, while any other field that interacts with it is taken as a quantum object obeying its particular rules as such. The energy scales at which QFTCS becomes relevant are still too high for its results to be determined by a regular experiment; the only “available” places where gravity becomes strong enough to be comparable to the rest of the forces in order to allow us to distinguish its effects are the proximities of spacetime singularities, namely black holes and the Big Bang singularity. Nevertheless, analogue experiments can be set up in other fields of research; for example, in Condensed Matter models it is possible to consider the material itself as the generator of a curved background on which different kinds of excitation (modeled as quasi-particles like phonons or excitons) move and interact [6, 7].

In any case, it remains painfully clear that we currently lack a good enough (conceptual and experimental) understanding of both Quantum Physics and gravitational interactions in order to build a completely satisfying description of their connections. The main purpose of this document is to expand upon our current knowledge of the tools commonly used in QFTCS in order to understand loop-order corrections due to quantum effects to classical interactions, namely the effective action and heat kernel methods, and apply these to the study of gravitational systems, most prominently black hole evolution.

Broadly speaking, the process of quantizing a classical field theory is taken as an active procedure in which a set of classical variables is promoted into quantum operators. In doing so, however, one must be aware of the fundamental differences between both formalisms, not the least of which being that operators are not, in general, (anti)commutative objects, unlike classical functions. While classical theories can give theoretically infinitely accurate predictions on many relevant observables at once, only being limited in real life by the precision of our measurements, quantum theories have an inherent unremovable uncertainty that stems from this noncommutativity of the observables. But even before that, quantum theories find themselves against a wall when describing the mere act of "measuring" an observable, since our current understanding implies that the deterministic in nature wave function that describes the system instantly collapses to another completely different form (that compatible only with the result of the measurement) through some undetermined process; classical theories, on the other hand, do not have any of these quirks with regards to measuring. Thus, it is not unreasonable to conclude that bridging the gap between the classical and quantum realms is a highly nontrivial endeavor.

Many quantization procedures have been devised over the years, each with different reasonings, assumptions and procedures attached to them. The implicit understanding is that they should give rise to the same predictions, eventually conflating to "different views of the same problem"; however, while that is indeed the case for finite systems, it is not guaranteed for systems with infinitely many degrees of freedom [8]. In fact, one can have the vacuum state defined in two unitarily inequivalent ways, leading to phenomena like the Unruh effect [9, 10].

The most evident and direct quantization method, known as canonical quantization, stablishes that we promote all positions and momenta of a system of particles to operators, while enforcing the commutation relations,

$$[\hat{x}, \hat{p}] = i\hbar, \tag{1.1}$$

as a quantum equivalent to the classical Poisson brackets. Second quantization generalizes the idea to fields defined over spacetime, with the understanding that their excitations can be seen as a process of particle creation and annihilation. Other proposals like geometric quantization try to turn this process from a nearly *ad hoc* imposition into a rigorous geometric process using the symplectic structure of classical spacetime [11].

When treating complicated systems with different quantum fields interacting in specific regions of spacetime, the most commonly employed procedure is by far the Feynman, or path integral formalism. Instead of producing a wavefunction for the quantum system to follow, this process allows for a direct calculation of transition amplitudes from one initial state to another, by simultaneously considering all the possible evolutions in-between, essentially expanding on the ideas of classical Lagrangian mechanics and the principle of stationary action [12]. Nevertheless, trying to apply Feynman's formalism directly to gravity theories results in a nonrenormalizable model that quickly becomes unusable at higher energies. Specifically, after expanding the integral using the standard Feynman diagram perturbative expansion, it is possible to see that the energy associated with the diagrams at every loop level grows larger and larger, needing infinitely many counterterms in order to get rid of the infinities that arise [13, 14]. It becomes necessary to introduce a specific cutoff point and consider only quantum corrections up to a specific level. One such way to study these effects is through the definition of the effective action, which is the primary object of interest of the first half of this document.

Our best current theory for gravitation comes in the form of General Relativity, which is a completely classical formalism. Its main claim to fame (and one of the bigger impediments in trying to quantize it) is the way in which gravity is interpreted; rather than being a force that acts on the fields that conform any system we study, gravity is understood as a geometric feature of the spacetime itself in which the fields live. This spacetime is formalized as a d-dimensional manifold that encompasses both spatial and temporal directions, with an associated metric tensor

g defining its curvature. The interval that separates two infinitesimally close events in spacetime, characterized by their coordinates $\{x^\mu\}_{\mu=0,\ldots,d-1}$, is given by

$$ds^2 = g_{\mu\nu}(x)\; dx^\mu dx^\nu, \tag{1.2}$$

where repeated indices, with one as a subscript and one as a superscript, are implicitly summed over. Flat spacetime will be taken to satisfy the specific relation $(g_{\mu\nu}) = (\eta_{\mu\nu}) = \mathrm{diag}(-1, +1, ..., +1)$, where the -1 is assigned to the single time coordinate. Throughout this document we will always use this mostly-plus convention, as well as Einstein's implicit summation notation, unless otherwise noted.

In General Relativity, the global notion from Special Relativity that the speed of light is a theoretical maximum that cannot be exceeded without introducing some strange element like an imaginary mass field becomes a local statement (meaning there might be some reference frames where objects seem to be moving faster than light, like with cosmological expansion [15]). Nevertheless, the structure of causality is always preserved; information can only travel between points whose spacetime interval (1.2) satisfies $ds^2 \leq 0$, which is referred to as timelike (or, in the edge case $ds^2 = 0$ where only light and other massless particles can connect them, lightlike or null) separation. Events with a spacelike separation between them, $ds^2 > 0$, are causally disconnected and thus independent of each other. At every point in time one can thus project a future-bound and past-bound light cone that separates all events that are causally connected to it from those that are not.

If we are able to define a spacelike surface that intersects every inextendible time- and lightlike trajectory only once (what is known as a Cauchy surface), then the spacetime is said to be globally hyperbolic. This constitutes the gold standard with regards to spacetime causality considerations, since the existence of a Cauchy surface guarantees that the theory is fully deterministic. Given that finding this surface is usually not a trivial task, the alternative way of determining global hyperbolicity is by simultaneously satisfying [16]:

i) There are no timelike closed loops; this prevents the possibility of "time travel" of any object to its own past.

ii) The intersection of the future lightcone of a particular point x_i and the past lightcone of another point x_f that lies to its future is a closed and bounded set of points. This condition prevents the existence of "naked singularities", which may introduce information that cannot be derived from General Relativity, thus breaking the predictive power of the theory.

In a globally hyperbolic spacetime, the information contained in a Cauchy surface is enough to determine the state of spacetime at any other point in time before or after it; examples of this include flat spacetime and the FLRW model of cosmological expansion. Additional information, like boundary conditions, is needed in order to fully characterize the trajectories in those spacetimes that do not satisfy this condition; one such case is given by the anti-deSitter spacetime model.

Like other classical field theories, the dynamics of General Relativity can be derived from a Lagrangian

$$\mathcal{L}_{E-H} \sim \sqrt{-\det g} R, \tag{1.3}$$

where R is the Ricci scalar of curvature, built from a specific combination of the metric and its first two derivatives; it is the only nontrivial scalar object that can be constructed in such a way, which is what allows the resulting equations of motion to be of second order in derivatives of the metric. Coupling this Lagrangian to some matter fields and following the usual procedures of variational calculus we arrive at Einstein's equations of motion. This set of differential equations show the deep connection between the curvature of spacetime, parametrized by invariants like R, and the matter content of the theory, parametrized by its energy-momentum tensor. As is commonly said in popular terms, "curvature tells matter how to move, and matter tells spacetime how to curve". This mutual feedback is one of the other main difficulties in developing a consistent semiclassical or quantum model for gravity; when introducing quantum corrections to the gravity sector, this modifies the expected values of the energy-momentum tensor, which in turn modify the gravity (a process known as backreaction), and so on in an infinite loop. Additionally, there is still the question of whether the quantization process may or may not break the globally hyperbolic structure, since at the quantum level it is technically possible to have local violations of the weak energy condition, allowing for negative energy values. These negative energy values could theoretically lead to closed timelike curves, breaking causality. Fortunately, so far these effects seem to be heavily subdued for all the semiclassical models that have been studied, to the point that there seems to be some hidden mechanism that prevents them from happening [17]. Nevertheless, the "chronology protection conjecture" is by no means a proven statement, and causality should always be a major concern that needs to be checked in trying to semiclassically extend gravity theories. A definitive statement on this matter may only be possible to achieve with a fully realized quantum theory of gravity [18].

Going back to the classical formalism, a particular set of solutions to these equations that will be of special interest to us gives rise to what we know as black hole solutions. Originally thought of as a mere mathematical artifact, their existence has since then been proven time and time again, both indirectly and directly [19]. In layman's terms, a black hole is a region of spacetime from which no element is able to escape without exceeding the speed of light. By far their most characteristic feature is the existence of a singularity in their interior, a region of spacetime where curvature becomes infinite and General Relativity loses its predictive value. They are the most blatant indication that General Relativity as it is cannot fully explain all gravitational phenomena, and needs some way to deal with these objects. In other words, a full understanding of singularities and the structure of black holes as a whole will need a more powerful theory, most likely one that incorporates quantum effects that may arise when curvatures (and thus energies) reach high enough values. The second part of this document will deal with the definition and study of black holes in 2 spacetime dimensions, which will allow us to simplify the mathematical apparatus of the theory without losing on any of the relevant phenomena that we need to address, like the process of Hawking radiation (evidently, from this construction it is not possible to address any of the black hole features that may be related to its angular momentum).

As stated above, we may think of black holes (and large-curvature systems as a whole) as a giant "lab" in which we want to study the interplay between gravity and quantum effects, at least at first order. The "tools" needed for such a project will be given by the effective action. However, the derivation of effective actions is a whole field of study in itself; while the original formulation for calculating 1-loop corrections via heat kernel expansions has been established for decades, its practical usefulness can get quickly overshadowed by its computational complexity. It is of great interest to look for alternative derivations that may lead to simplified calculations.

Therefore, the document will be structured as follows. Chapter 2 will be devoted to covering everything from the basic definition of the effective action to its relevance in studying some important phenomena like the Schwinger effect, before delving into the question of how we can actually compute the 1-loop effective action of a system. Over the course of this research project, we were able to consider several different systems of interest, arriving at a fairly general resummation scheme that allows for some great simplifications in the calculation of their effective actions with respect to the usual procedure already established in the literature. These simplifications become really powerful when studying the divergencies of the effective action, which are usually given by the first few terms in the so-called heat kernel expansion and help identify semiclassical effects that may appear at the first order in quantum corrections. In particular, the divergencies of the effective action associated with the classical Einstein-Hilbert Lagrangian can be closely related to the trace anomaly of the energy-momentum tensor, which can then in turn be seen as the source of Hawking radiation.

Chapter 3 will thus present an example of the use of effective actions with potentially important implications for modeling black hole evaporation in simplified settings. In particular, we will discuss the CGHS model for a 2D black hole solution and couple it to some conformal matter fields. This model may be viewed as a slight modification of a 2D model obtained via the spherical reduction of General Relativity. By introducing specific forms of "exotic" fields with negative central charge (most prominently, Faddeev-Popov ghosts) in an appropriate and consistent manner, we were able to see that there exists a possibility for the singularity associated with this black hole to fully disappear (again, with the caveats of introducing these extra fields and working on an effective semiclassical framework), hinting at the idea of black hole evolution being dependent on its matter content (contrary to previous belief). This opens the door to further considerations with regards to the structure and dynamical behaviour of black holes, which will be touched upon in the conclusions.

Chapter 2

Effective actions and heat kernel resummations

Effective actions come about as an alternative way of dealing with scattering amplitudes, most commonly associated with the path integral formalism [20]. In contrast to the fundamentally perturbative nature of the latter, the main objective of the effective action formalism is to look for an exact solution that can then be used in further calculations. While obviously not achievable in general, we will see that for a wide variety of systems it is possible to find a good approximation that fully encapsulates the first-order effects introduced by quantum corrections to the classical field theory. Solving this expression will require the introduction of some regularization schemes to account for the infinite-dimensionality of the relevant operators, which will lead to the definition of heat kernel operators, as well as some resummation formulas whose aim will be to expand on the traditional methods employed in their derivation.

2.1 The effective action

2.1.1 The Schwinger action principle

Over the following paragraphs we will present a simple derivation of the basic formulation of the Schwinger action principle. In particular, our main object of interest will be the transition amplitude between two different quantum states of a generic local field theory. In doing so, we will set up a path to define the effective action of such a system in later Sections.

Classically, a local field will be denoted as a function $\varphi^k(x)$ that depends only on the spacetime point x^μ at which the field is defined, rather than an extended section of the manifold. The index k is left completely generic; at this point in time, it could stand for any spacetime, spinor or internal bundle index that the particular theory demands (for the time being, though, we will assume the fields are Grassmann-even variables; the case of spinors will be discussed in Section 2.1.4). In the process of quantization, as explained before, our aim is to replace all classical fields with their corresponding field operator equivalents $\phi^k(x)$.

Let us define Σ to be a spacelike hypersurface in spacetime, that is to say, a $(d-1)$-dimensional submanifold embedded in spacetime, in which any two points have a spacelike separation. By our definition in the previous chapter, this means that all the points in Σ are causally disconnected, and therefore the values of ϕ^k at different points in Σ must be independent to each other. Thus their commutator must satisfy

$$[\phi^k(x_1), \phi^{k'}(x_2)] = 0 \qquad \forall x_1, x_2 \in \Sigma, \tag{2.1}$$

since the order in which we measure the values cannot influence the result; this relation can be extended to any other operator constructed using only the fields and their associated momenta. The first assumption is that we can construct a complete set of commuting observables (from here on CSCO) on Σ by using only these operators. This lets us fully parametrize and "label" the state of the system at any point

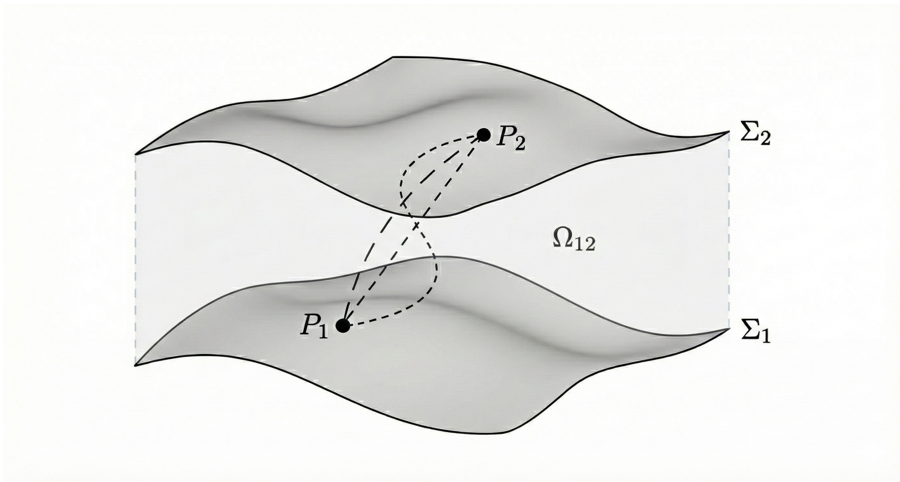

Figure 2.1: Possible evolutions between two arbitrary points $P_i = \{x_i^\mu\}_{\mu=0,\dots,d-1}$, each of them defined on a different spacelike hypersurface Σ_i, with Σ_2 to the future of Σ_1. The spacetime region between the surfaces is denoted by Ω_{12}.

by the eigenvalues of all observables in the CSCO at that point. Consequently, our states will be denoted by $|\zeta', \Sigma\rangle$, where ζ' denotes the eigenvalues of the CSCO ζ at the examined point in Σ.

Now suppose we have two different spacelike hypersurfaces Σ_1, Σ_2, each with their own different observables, such that all the points in Σ_2 lie to the future of Σ_1. Assuming the eigenvalue spectrum $\sigma(\zeta)$ of both CSCOs remains the same, then the two of them must be related by a unitary transformation

$$\zeta_2 = U_{12}\zeta_1(U_{12})^{-1}, \tag{2.2}$$

with their respective eigenvectors related by

$$|\zeta_2', \Sigma_2\rangle = U_{12}\,|\zeta_1', \Sigma_1\rangle\,. \tag{2.3}$$

Notice that the states themselves are kept in a time-independent manner, leaving all the time evolution dependency on the unitary operator U_{12}. Figure 2.1 shows a schematic diagram of the construction. Our main object of interest, the transition amplitude between two states, will then be

$$\langle \zeta_2', \Sigma_2 | \zeta_1', \Sigma_1 \rangle = \langle \zeta_1', \Sigma_1 | \left(U_{12} \right)^{-1} | \zeta_1', \Sigma_1 \rangle \, . \tag{2.4}$$

By construction, the operator U_{12} will depend on our choices for the hypersurfaces and the CSCOs defined on them, as well as the particular details of the system considered for the time evolution between the initial and final states. Any modification that occurs to any of these will induce a change in the transition amplitude (2.4). It proves convenient to write U_{12} as

$$U_{12} = \exp\left(-\frac{i}{\hbar} S_{12} \right) , \tag{2.5}$$

where S_{12} is a self-adjoint operator; using this definition it is possible to write

$$\delta \langle \zeta_2', \Sigma_2 | \zeta_1', \Sigma_1 \rangle = \frac{i}{\hbar} \langle \zeta_2', \Sigma_2 | \, \delta S_{12} \, | \zeta_1', \Sigma_1 \rangle \, . \tag{2.6}$$

By introducing the identity operator

$$\mathbb{I} = \sum_{\zeta' \in \sigma(\zeta)} | \zeta', \Sigma \rangle \langle \zeta', \Sigma | \tag{2.7}$$

and selecting appropriate start- and endpoints or adding more spacelike hypersurfaces to extend the time evolution to, it is possible to prove

i) $\delta S_{12} \to 0$ as Σ_2 approaches Σ_1.

ii) $\delta S_{12} = -\delta S_{21}$.

iii) $\delta S_{13} = \delta S_{12} + \delta S_{23}$ for any three spacelike hypersurfaces defined as above.

Equation (2.6) is the main statement of the Schwinger action principle. As we shall promptly see, we will identify the S_{12} operator as a quantum action operator; thus, all the information needed to understand any corrections to our transition amplitudes due to perturbations in the system is encoded in a single object. Property iii) stated above shows that it is possible to study the time evolution between an initial and final state by "bridging the gap" between them in smaller steps and adding them up; this heavily mirrors the traditional construction of the Feynman path integral approach to Quantum Field Theory. We will study the connection between both approaches shortly but, as a hand-wavy comparison, we can think of the Schwinger action principle as a differential version of the path integral formulation.

Without modifying the chosen hypersurfaces themselves, the most general change we can perform on the system is one that will affect the operators defined on Σ_1 and Σ_2, as well as those defined over the spacetime region confined between them, which we will denote Ω_{12}. Focusing first on the former, assume the operators undergo infinitesimal unitary transformations that induce a change in the states given by the expression

$$\begin{aligned} |\zeta_i', \Sigma_i\rangle \mapsto U_i\, |\zeta_i', \Sigma_i\rangle = \exp\left(-\frac{i}{\hbar} F_i\right) |\zeta_i', \Sigma_i\rangle \approx \left(1 - \frac{i}{\hbar} F_i\right) |\zeta_i', \Sigma_i\rangle \Rightarrow \\ \Rightarrow \delta\, |\zeta_i', \Sigma_i\rangle = -\frac{i}{\hbar} F_i\, |\zeta_i', \Sigma_i\rangle\,, \end{aligned} \tag{2.8}$$

where F_i are self-adjoint operators. Plugging this into (2.6) leads to

$$\delta S_{12} = F_2 - F_1. \tag{2.9}$$

The F_i operators can be expressed as a surface integral

$$F_i = \int_{\Sigma_i} d\sigma_i\; n_i^\mu F_\mu(x), \tag{2.10}$$

where $d\sigma_i$ is the element of area in Σ_i, n_i^μ is the (outward-pointing) normal vector to Σ_i, and $F_\mu(x)$ is an as of yet undetermined operator. Assuming $F_\mu(x)$ is defined not only on the hypersurfaces but also in the interior region, we make use of Gauss-Ostrogradski's theorem to rewrite the variation of S_{12} as

$$\delta S_{12} = \int_{\Omega_{12}} dv_x \; \nabla^{\mu} F_{\mu}(x), \tag{2.11}$$

where dv_x is the element of volume defined in the interior region Ω_{12}, which serves as the integration measure. Inspired by this result, we may also imagine that for any perturbation that only affects the interior region we can define

$$\delta S_{12} = \int_{\Omega_{12}} dv_x \; \delta\mathcal{L}(x), \tag{2.12}$$

with $\delta\mathcal{L}(x)$ encapsulating the infinitesimal perturbations that are introduced at every point in the region, in the same way the F_i encapsulate the perturbations at the two boundaries. The full variation of the action will then be

$$\delta S_{12} = \int_{\Omega_{12}} dv_x \; [\delta\mathcal{L}(x) + \nabla^{\mu} F_{\mu}]. \tag{2.13}$$

The key assumption needed in order to continue is that δS_{12} can be obtained from a variation of

$$S_{12} = \int_{\Omega_{12}} dv_x \; \mathcal{L}(x), \tag{2.14}$$

where $\mathcal{L}(x)$ is a Lagrangian density, and thus S_{12} can be read as the action of the system.

In Quantum Field Theory, the transition amplitude between two states of the same system can only be altered if the initial and final states themselves are. Thus, for any perturbation that only affects the fields in the region Ω_{12}, but not on the boundaries Σ_1, Σ_2, we get the result

$$\delta S_{12} = 0, \tag{2.15}$$

which is the operator version of the principle of stationary action. If we only vary the fields and leave the hypersurfaces and states fixed, and assuming $\mathcal{L}$ depends only on the fields themselves and their first derivatives, we may find δS_{12} by taking functional derivatives on (2.14)

$$\begin{aligned}\delta S_{12} &= \int_{\Omega_{12}} dv_x \left[\frac{\partial\mathcal{L}}{\partial\phi^i}\delta\phi^i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\delta(\partial_\mu\phi^i)\right] \\ &= \int_{\Omega_{12}} dv_x \left[\frac{\partial\mathcal{L}}{\partial\phi^i} - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\right)\right]\delta\phi^i + \int_{\Omega_{12}} dv_x \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\delta\phi^i\right),\end{aligned} \tag{2.16}$$

where in order to go from the first to the second line we integrate by parts the second term inside the integral (which requires $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$). First of all, by comparison with (2.13), we may set

$$F^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\delta\phi^i. \tag{2.17}$$

If the variation on the fields vanishes on the boundary hypersurfaces Σ_1, Σ_2, then the second term vanishes and, by imposing (2.15), we get the equations of motion

$$\frac{\partial\mathcal{L}}{\partial\phi^i} - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)}\right) = 0, \tag{2.18}$$

which look exactly like the classical Euler-Lagrange equations, with classical fields substituted by their quantum operator counterparts.

If we now consider that the variation on the fields only happens at the boundary, then the first term vanishes and only the second remains. Out of convenience, let us say that the fields only change in one of the hypersurfaces Σ; if we define

$$\pi_i^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^i)} \tag{2.19}$$

and undo the change from a volume integral back into a surface integral, then

$$F = \int_\Sigma d\sigma\; n_\mu\pi_i^\mu\,\delta\phi^i. \tag{2.20}$$

Given that an infinitesimal transformation of this kind is given by (2.8), and choosing the states $|\zeta', \Sigma\rangle$ to be the eigenstates of ϕ^i so that the fields transform as (2.2), then

$$\delta\phi^k = -\frac{i}{\hbar}[F, \phi^k] = -\frac{i}{\hbar}\int_\Sigma d\sigma\ n_\mu[\pi_i^\mu, \phi^k]\delta\phi^i. \tag{2.21}$$

Finally, if we consider that the spacetime is globally hyperbolic so that the spacelike hypersurface can be additionally defined to be a Cauchy surface, and thus can be identified as the surface given by a constant value of x^0, then $n^\mu = \delta_0^\mu$ and we may conclude

$$[\pi_i^0(x^0, x^j), \phi^k(x^0, x'^j)] = i\hbar\, \delta_i^k \delta(x, x'), \tag{2.22}$$

where δ_i^k and $\delta(x, x')$ are the Kronecker δ and Dirac distribution, respectively. In other words, the Schwinger action principle can be used to derive the canonical commutation relations without needing to introduce them explicitly as an extra element.

Notice that, unlike similar definitions that are used in classical field theory, or even in the path integral formulation, here the action functional and Lagrangian density are operator objects. Therefore, we need to tackle the possible issues that may arise from the ordering of operators inside S_{12}, which have so far been ignored in the previous expressions (and thus these should only be taken as formal expressions at this point). In particular, any term in the action that takes the form of a product $\zeta^1(x)\zeta^2(x)$, where $\zeta^i(x)$ are noncommuting operators, will be ill-defined and divergent as both operators are evaluated at the same exact point in spacetime. In order to solve this, we need to utilize a regularization scheme that allows for these divergencies to be studied and dealt with. The method originally devised by Schwinger roughly involves introducing an infinitesimal separation $\zeta^1(x-\epsilon)\zeta^2(x)$ (with additional terms needing to be introduced in order to preserve gauge symmetries, if the theory had them), performing the derivations that would lead to the equations of motion, and only at the end taking the limit $\epsilon \to 0$. The resulting equations of motion will be modified to include new terms which, at the level of the action, imply the necessity of introducing some counterterms that originate them, alongside the breaking of classical symmetries when introducing the quantum formalism [21], a process commonly known as a quantum anomaly. While this particular formulation lies outside the range of this document, we will revisit the need of a regularization in future Sections when considering the usual formulation of the effective action. Anomalies will be further discussed in the next Chapter, particularly in relation to the energy-momentum trace anomaly. It is important to note, however, that anomalies are not a simple mathematical artifact that has to be dealt with; they carry actual physical meaning, particularly in regards to quantum effects that have no classical analogues.

2.1.2 Connection to the Feynman path integral

As briefly stated in the previous Section and as a consistency check, the following pages provide a detailed technical derivation of the formal connection between the Schwinger action principle and the path integral formalism most commonly used in Quantum Field Theory. Consider now the classical action $S[\varphi]$ and let us modify it by coupling the system to an external source (meaning, that it is completely independent of the fields). With no specific theory in mind, the modified action will take the form

$$S[\varphi, J] = S[\varphi] + J_k\varphi^k \tag{2.23}$$

where, as explained above, k stands for any indices of the fields; in addition, for ease of notation in the following sections the scalar product of spacetime functions will be defined such that

$$J_k\varphi^k := \int dv_x \ J_k(x)\varphi^k(x), \tag{2.24}$$

i.e. we will be conflating the spacetime arguments into the fields unless explicitly stated. The quantum states will now also be abbreviated from here on, in the simple manner

$$|\zeta_i', \Sigma_i\rangle \equiv |i\rangle \,. \tag{2.25}$$

The transition amplitude for this system can then be understood as a functional of J_k, meaning the equivalent of (2.6) will be

$$\delta \langle 2|1\rangle [J] = \frac{i}{\hbar}\langle 2|\delta S_J|1\rangle[J], \tag{2.26}$$

with the operator for the quantum action (S_J for convenience) can be obtained from (2.23) by substituting φ^k with their corresponding field operators ϕ^k. If we take a variation with respect to the fields that keeps the endpoints unchanged, then applying (2.15) to S_J will lead to

$$\frac{\delta S[\phi]}{\delta \phi^k} = -J_k. \tag{2.27}$$

If we assume now that the perturbation of the system is only due to changes in the source J, then

$$\delta \langle 2|1\rangle [J] = \frac{i}{\hbar}\delta J_k \left\langle 2|\phi^k|1\right\rangle [J] \Longleftrightarrow \frac{\delta\langle 2|1\rangle[J]}{\delta J_k} = \frac{i}{\hbar}\langle 2|\phi^k|1\rangle[J]. \tag{2.28}$$

In order to study this relation, let us consider a new spacelike hypersurface Σ_α lying somewhere between Σ_1 and Σ_2 without intersecting either of them. Any change in the system can be decomposed into the sum of one that is nonvanishing in the past of Σ_α (i.e. between Σ_1 and Σ_α) but vanishes in its future (between Σ_α and Σ_2), and one that exhibits the opposite behaviour. Considering first the former case, and once again introducing the identity operator as defined in (2.7), we can derive

$$\left\langle 2|\phi^k|1\right\rangle [J] = \sum_{\zeta'_\alpha} \left\langle 2|\phi^k|\alpha\right\rangle [J] \left\langle \alpha|1\right\rangle [J] \tag{2.29}$$

and operate with the right hand side to get

$$
\begin{aligned}
\delta \left\langle 2|\phi^k|1\right\rangle [J] &= \sum_{\zeta'_\alpha} \left\langle 2|\phi^k|\alpha\right\rangle [J] \; \delta \left\langle \alpha|1\right\rangle [J] \\
&= \frac{i}{\hbar}\delta J_j \sum_{\zeta'_\alpha} \left\langle 2|\phi^k|\alpha\right\rangle [J] \; \left\langle \alpha|\phi^j|1\right\rangle [J] \\
&= \frac{i}{\hbar}\delta J_j \left\langle 2|\phi^k\phi^j|1\right\rangle [J],
\end{aligned} \tag{2.30}
$$

where in the first line we note that only the region contained to the past of Σ_α is changed, in the second we apply the Schwinger action principle (notice that, due to the construction employed, ϕ^j must be defined at an earlier time than ϕ^k), and in the third we retrieve the identity. Following the same steps for the other case, we arrive at the same expression with ϕ^k and ϕ^j swapped, as well as their temporal ordering. Adding both results then gives

$$
\delta \left\langle 2|\phi^k|1\right\rangle [J] = \frac{i}{\hbar}\delta J_j \langle 2|T(\phi^k\phi^j)|1\rangle[J], \tag{2.31}
$$

where T is the chronological ordering operator. Combining (2.28) with (2.31), it is straightforward to conclude

$$
\frac{\delta^2 \left\langle 2|1\right\rangle [J]}{\delta J_k \; \delta J_j} = \left(\frac{i}{\hbar}\right)^2 \langle 2|T(\phi^k\phi^j)|1\rangle[J], \tag{2.32}
$$

which can further be generalized via induction to

$$
\frac{\delta^n \left\langle 2|1\right\rangle [J]}{\delta J_{k_1}...\delta J_{k_n}} = \left(\frac{i}{\hbar}\right)^n \langle 2|T(\phi^{k_1}...\phi^{k_n})|1\rangle[J]. \tag{2.33}
$$

Since $\left\langle 2|1\right\rangle [J]$ is effectively a functional of J_k, and our objective at the end of the day will be to take the limit where this external source current vanishes, let us expand $\langle 2|1\rangle [J]$ in terms of its Taylor series about $J_k = 0$:

$$\begin{aligned}\langle 2|1\rangle [J] &= \sum_{n=0}^{\infty} \frac{1}{n!} J_{k_1}...J_{k_n} \quad \left(\left. \frac{\delta^n \langle 2|1\rangle [J]}{\delta J_{k_1}...\delta J_{k_n}} \right|_{J=0} \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{\hbar}\right)^n J_{k_1}...J_{k_n} \langle 2|T(\phi^{k_1}...\phi^{k_n})|1\rangle [J=0] \\ &= \langle 2|T\left(\exp\left(\frac{i}{\hbar} J_k \phi^k\right)\right)|1\rangle, \end{aligned} \tag{2.34}$$

where the exponential on the last line is taken to be defined by its Taylor expansion around $J = 0$.

The action $S[\phi]$ can also be taken as a Taylor expansion around $\phi = 0$ (in fact, for many theories of interest it will already be given in the form of a polynomial in ϕ, such as the Klein-Gordon or Dirac actions). Doing so, one can also define

$$S_{,i}[\phi] = \frac{\delta S}{\delta \phi^i}[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \phi^{k_1}...\phi^{k_n} \quad \left(\left. \frac{\delta^{n+1} S[\phi]}{\delta \phi^i \delta \phi^{k_1}...\delta \phi^{k_n}} \right|_{\phi=0} \right). \tag{2.35}$$

We can then further replace ϕ^{k_i} with $\frac{\hbar}{i}\frac{\delta}{\delta J_{k_i}}$ in the previous expression. After applying the resulting definition for the operator $S_{,i}\left[\frac{\hbar}{i}\frac{\delta}{\delta J}\right]$ on (2.34), we obtain

$$S_{,i}\left[\frac{\hbar}{i}\frac{\delta}{\delta J}\right] \langle 2|1\rangle [J] = \langle 2|T\left(S_{,i}[\phi] \exp\left(\frac{i}{\hbar} J_k \phi^k\right)\right)|1\rangle, \tag{2.36}$$

where once again the exponential is defined through its expansion around $J = 0$. Given the result in (2.27), we can further simplify

$$S_{,i}\left[\frac{\hbar}{i}\frac{\delta}{\delta J}\right] \langle 2|1\rangle [J] = -J_i \langle 2|1\rangle [J], \tag{2.37}$$

which is a differential equation for the transition amplitude. To solve it, we may perform a "functional Fourier transform"

$$\langle 2|1\rangle\,[J] = \int d\mu[\varphi]\;\, F[\varphi]\exp\left(\frac{i}{\hbar}J_k\varphi^k\right), \tag{2.38}$$

where $d\mu[\varphi]$ is the integration measure that depends on the fields involved and establishes the domain of integration; the only fields φ^k that will contribute to the integral will be those for which their values at Σ_1, Σ_2 are compatible with the states $|1\rangle, |2\rangle$ respectively. The basic path integral derivation for a set of scalar fields leads to $d\mu[\varphi] = \prod_{i=1}^{d} \dfrac{d\varphi^i}{\sqrt{2i\pi\hbar}}$, but for the most part we will keep the measure unspecified in case more general field theories are considered (such as those with gauge symmetries that would require the inclusion of ghost field terms). By imposing that (2.38) satisfies (2.37) and after some amount of computation, we arrive at an equation of the form

$$0 = \int d\mu[\varphi]\;\left[S_{,i}[\varphi]F[\varphi] - \frac{\hbar}{i}\frac{\delta F[\varphi]}{\delta\varphi^i}\right]\exp\left(\frac{i}{\hbar}J_k\varphi^k\right), \tag{2.39}$$

assuming that the fields vanish sufficiently rapidly at infinity (for the entirety of this document we will assume we work on a spacetime background with no boundaries; additional terms and specific considerations are needed in the case of spacetimes with boundaries [22, 23]). Therefore, up to a multiplicative constant, we see that

$$F[\varphi] = f\exp\left(\frac{i}{\hbar}S[\varphi]\right) \tag{2.40}$$

and it immediately follows from (2.38) that

$$\langle 2|1\rangle\,[J] = f\int d\mu[\varphi]\;\,\exp\left(\frac{i}{\hbar}\left(S[\varphi] + J_k\varphi^k\right)\right). \tag{2.41}$$

The constant f is usually chosen as $(\langle 2|1\rangle[0])^{-1}$ in order to normalize $\langle 2|1\rangle\,[J]$ to be 1 in the absence of J; for the most part, we will consider the constant implicit. This is, indeed, the Feynman path integral formulation for the transition amplitude between the states $|1\rangle$ and $|2\rangle$. Equation (2.41) serves as a formal, definitive proof of the vague statement that was given in the last section: that the Schwinger action principle is intrinsically tied to the path integral formalism, and they both must give rise to the same results in studying a particular quantum system; one from a local, differential point of view and the other from a global, integral one.

2.1.3 The 1-loop effective action

Equation (2.41) is, arguably, one of the most important objects in the field of Particle Physics. Usually denoted by $Z[J]$, it is used as the generating functional for all n-point correlation functions if $|1\rangle$ and $|2\rangle$ are both taken to be vacuum states at the infinite past and future, respectively (this choice is encapsulated in the measure at the level of the integral on the right hand side of (2.41)). From the diagrammatic point of view, it encompasses all contributions from connected and disconnected Feynman diagrams to the transition amplitudes. It proves useful to introduce a related object $W[J]$, defined via

$$\langle 2|1\rangle [J] := \exp\left(\frac{i}{\hbar} W[J]\right), \tag{2.42}$$

which in the Feynman diagram formalism corresponds to the generator of only fully connected diagrams between the initial and final states. Differentiating with respect to the source and using the expression (2.28), it is possible to show that

$$\frac{\delta W[J]}{\delta J_k} = \frac{\left\langle 2|\phi^k|1\right\rangle [J]}{\langle 2|1\rangle [J]}. \tag{2.43}$$

The right-hand side of this expression, which corresponds to the expectation value of the field in the presence of a source J, will be abbreviated as $\bar{\varphi}^k$ for convenience from now on.

The effective action is defined as a Legendre transformation in order to get rid of the dependency of the external source current J

$$\Gamma[\bar{\varphi}] := W[J] - J_k \bar{\varphi}^k. \tag{2.44}$$

Once again, by selecting the initial and final states in $\langle 2|1\rangle\,[J]$ to be vacuum states in the infinite past and future, the effective action becomes of interest in the Feynman diagrammatic construction of scattering amplitudes, this time giving the generating functional for all so-called 1PI (one-particle irreducible) correlation functions.

The focus of this and the following Sections will be to understand the effective action in a nonperturbative context, instead. Exponentiating both sides of (2.44) and using (2.41), we can easily see

$$\begin{aligned}
\exp\left(\frac{i}{\hbar}\Gamma[\bar{\varphi}]\right) &= \langle 2|1\rangle\,[J]\exp\left(-\frac{i}{\hbar}J_k\bar{\varphi}^k\right) \\
&= \int d\mu[\varphi]\;\exp\left(\frac{i}{\hbar}S[\varphi] + \frac{i}{\hbar}J_k\left(\varphi^k - \bar{\varphi}^k\right)\right) \\
&= \int d\mu[\varphi]\;\exp\left(\frac{i}{\hbar}S[\varphi] - \frac{i}{\hbar}\frac{\delta\Gamma}{\delta\bar{\varphi}^k}\left(\varphi^k - \bar{\varphi}^k\right)\right),
\end{aligned} \tag{2.45}$$

where the change in the last line can be seen by differentiating (2.44). Equation (2.45) gives an implicit, integro-differential equation for $\Gamma[\bar{\varphi}]$. It is also, unfortunately, impossible to evaluate $\Gamma[\bar{\varphi}]$ from this expression in general, though it is iteratively solvable.

Nevertheless, it is possible to work out an exact solution of (2.45) if the action is at most quadratic in the fields φ or, more generally [24], by performing a perturbative expansion on the action functional and dropping all cubic and upwards terms (in this case the following results would serve as a first order approximation). Let us consider then that our fields are of the form

$$\varphi^k = \bar{\varphi}^k + \sqrt{\hbar}\,\xi^k + O(\xi^2), \tag{2.46}$$

where ξ denotes a small correction of the field that deviates it from the expectation value $\bar{\varphi}$ (which may be interpreted as a "background" field) due to quantum effects; by definition, their expectation value must be $\langle \xi^k \rangle = 0$. The classical action can then be expanded up to second order in ξ around $\varphi = \bar{\varphi}$ to obtain

$$S[\varphi] = S[\bar{\varphi}] + \sqrt{\hbar}\ \xi^k \left.\frac{\delta S}{\delta \varphi^k}\right|_{\bar{\varphi}} + \frac{\hbar}{2} \xi^i \left.\frac{\delta^2 S}{\delta \varphi^i \delta \varphi^j}\right|_{\bar{\varphi}} \xi^j. \tag{2.47}$$

It is simpler to derive an explicit solution for the effective action associated with (2.47) by looking at (2.42) in conjunction with (2.41):

$$\begin{aligned} \exp\left(\frac{i}{\hbar} W[J]\right) = \int d\mu[\varphi]\ \exp\Bigg(&\frac{i}{\hbar}\left(S[\bar{\varphi}] + J_k \bar{\varphi}^k\right) \\ &+ \frac{i}{\sqrt{\hbar}}\left(\left.\frac{\delta S}{\delta \varphi^k}\right|_{\bar{\varphi}} + J_k\right)\xi^k + \frac{i}{2}\xi^i \left.\frac{\delta^2 S}{\delta \varphi^i \delta \varphi^j}\right|_{\bar{\varphi}} \xi^j\Bigg). \end{aligned} \tag{2.48}$$

The first term in the exponential is just a constant with respect to the path integral, while the term linear in ξ will be responsible of the cancellation of all non-1PI contributions that may appear on the cubic or higher order terms (essentially preventing the expected value of the corrections to deviate from $\langle \xi \rangle = 0$). It is of no relevance to the discussion of the following Sections, and as such shall be ignored from this point forward. As for the quadratic term, after defining

$$\mathcal{Q}_{ij} := \left.\frac{\delta^2 S}{\delta \varphi^i \delta \varphi^j}\right|_{\bar{\varphi}} \tag{2.49}$$

for ease of notation and rescaling $d\varphi = \sqrt{\hbar}\ d\xi$, we notice that the path integral becomes a Gaussian-like integral

$$\int d\mu[\xi] \exp\left(\frac{i}{2}\xi^i \mathcal{Q}_{ij} \xi^j\right). \tag{2.50}$$

In order to solve this, we may formally generalize the equivalent expression where $\mathcal{Q}$

is replaced by a finite matrix A (we will come back to this topic in a following Section, in order to properly treat this extension). In the finite case, the expression above can be solved exactly by diagonalizing A in order to separate the full integral into a product of one-dimensional Fresnel integrals. Defining an adequate integration contour in the complex plane, these can be solved explicitly

$$\int_{-\infty}^{\infty} dx \ \exp(iax^2) = \sqrt{\frac{i\pi}{a}}. \tag{2.51}$$

The constant a will be given by a different eigenvalue λ_n of A for each integration, meaning that the final result will be proportional to $\prod \lambda_n^{-\frac{1}{2}} = (\det A)^{-\frac{1}{2}}$. All in all, the explicit solution of this integral leads us to

$$\begin{aligned} \exp\left(\frac{i}{\hbar}W[J]\right) &= \exp\left(\frac{i}{\hbar}\left(S[\bar{\varphi}] + J_k\bar{\varphi}^k\right)\right)\det{}^{-\frac{1}{2}}\left(l^2\mathcal{Q}\right) \Longrightarrow \\ &\Longrightarrow W[J] = S[\bar{\varphi}] + J_k\bar{\varphi}^k + \frac{i}{2}\hbar\log\det(l^2\mathcal{Q}), \end{aligned} \tag{2.52}$$

where l is an arbitrary dimensionful constant introduced to keep the argument of the determinant dimensionless; as it will not play any relevant role in the following Sections, we will drop it from here on to maintain ease of notation. Finally, the effective action takes the form

$$\Gamma[\bar{\varphi}] = S[\bar{\varphi}] + \frac{i}{2}\hbar\log\det\mathcal{Q}. \tag{2.53}$$

The first order correction in $\hbar$ is called the 1-loop effective action. If the action of the system is given by (2.47) in its entirety, i.e. if there are no cubic or higher order terms, then (2.53) becomes the complete effective action, giving us a fully nonperturbative expression to be used in further calculations. It should be reiterated that at this point the expression above is merely a formal definition; following Sections will be entirely devoted to the study and calculation of the 1-loop effective action for a handful of different systems of interest, specifically via the use of the heat kernel operator.

Before that, though, it should also be pointed out that the above derivation implicitly assumes that A and by extension $\mathcal{Q}$ are invertible and thus have a non-vanishing determinant. For theories with gauge symmetries that will no longer be the case since the action will remain invariant under some field redefinitions; given (2.49), this means $\mathcal{Q}_{ij}$ will vanish for variations taken along the orbits of gauge transformations. One must then consider a gauge-fixed action in order to get rid of these vanishing terms; in doing so, additional gauge-fixing and ghost field terms must be considered to account for this redefinition, leading to a slightly modified expression of the effective action that will include these terms. A review on how we may tackle these redefinitions is given in [25]. Throughout this part of the document we will not be quantizing fields that present a gauge symmetry, instead leaving them as classical background terms; thus we will not need such considerations. We will postpone our discussion on gauge-fixing and ghost fields for the next Chapter, when we consider semiclassical gravity models.

2.1.4 The effective action for spinor fields

The preceding paragraphs have presented a full discussion of the effective action of a quantum field system, and how this effective action can be explicitly derived for the special case where the action is at most quadratic in the fields. Up until (2.49), the procedure is completely agnostic to the nature of the fields involved; however, when performing the integration in (2.52), the previous Section implicitly assumed that the variables involved (namely, the fields ϕ and specifically the corrections ξ) where of bosonic nature. As such, it is immediately self-evident that the operator in (2.49) is symmetric in its indices, allowing for the path integral to be taken as a Gaussian.

This is no longer the case for spinor fields. While a full discussion of the mathematical formulation of spinor fields and actions (specially in general curved spacetimes) is beyond the scope of this work, we may exemplify the situation for the well-known Dirac action for a free spinor field Ψ in flat spacetime

$$S[\Psi, \bar{\Psi}] = -\int dv_x \; \bar{\Psi}(\gamma^\mu \partial_\mu + m)\Psi, \tag{2.54}$$

where the γ-matrices satisfy the Clifford algebra relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}$ (we will always assume an even value of d in order to have a unique representation of the γ-matrices) and $\bar{\Psi}$ is the Dirac adjoint of Ψ (in a mostly-plus metric, the matrices are defined so that γ^0 is anti-Hermitian while γ^i are Hermitian, and $\bar{\Psi} = \Psi^\dagger(i\gamma^0)$ so that $\bar{\Psi}\Psi$ is a real scalar). The components of the Dirac spinor Ψ are Grassmann-odd variables ψ^a, meaning they anticommute with each other:

$$\psi^a \psi^b = -\psi^b \psi^a. \tag{2.55}$$

We can immediately notice that the operator $\mathcal{Q}_{ij}$ in (2.49) is now antisymmetric in its indices. The Gaussian integration method is not applicable in this case. Instead, by implementing the rules of Berezin integration of Grassmann-odd variables, we eventually see

$$\begin{aligned} \exp\left(\frac{i}{\hbar}W[J]\right) &= \exp\left(\frac{i}{\hbar}\left(S[\bar{\Psi}, \Psi] + \bar{J}\,\Psi + \bar{\Psi}\,J\right)\right) \int d\mu[\bar{\Psi}, \Psi] \; \exp\left(\frac{i}{\hbar}S[\bar{\Psi}, \Psi]\right) \\ &= \exp\left(\frac{i}{\hbar}\left(S[\bar{\Psi}, \Psi] + (J, \Psi)\right)\right) \det(l\mathcal{Q}), \end{aligned} \tag{2.56}$$

where the terms outside of the integral are all evaluated on the uncorrected fields and the current J is now understood as a Dirac spinor in itself. Again, we will drop the dimensionful constant l from here on for notational convenience. We can then conclude that the effective action will be

$$\Gamma[\bar{\Psi}, \Psi] = S[\bar{\Psi}, \Psi] - i\hbar \log\det \mathcal{Q}, \tag{2.57}$$

where once again the fields $\bar{\Psi}, \Psi$ should be taken on both sides in their uncorrected states.

In following Sections, it will prove to be beneficial to study the cases where $\mathcal{Q}$ is a differential operator of second-order in spacetime derivatives, such as the Klein-Gordon operator. This is not the case for the Dirac action, given that

$$\mathcal{Q} = -(\gamma^\mu \partial_\mu + m). \tag{2.58}$$

In general, spinors theories will always be accompanied by first-order differential operators. In even dimensions, however, we can use a different representation of the γ-matrices, namely $\{\gamma^\mu\} \mapsto \{-\gamma^\mu\}$, to define a different operator

$$\mathcal{Q}' = -(-\gamma^\mu \partial_\mu + m). \tag{2.59}$$

Since both representations should be equivalent to each other, it is to be expected that $\det \mathcal{Q} = \det \mathcal{Q}'$. Therefore, we can use the formal properties of the logarithm to rewrite

$$\log \det \mathcal{Q} = \frac{1}{2} \log(\det \mathcal{Q})^2 = \frac{1}{2} \log \det(\mathcal{Q}\mathcal{Q}'), \tag{2.60}$$

and recover an expression that more closely resembles the one obtained in the previous Section

$$\Gamma[\bar{\Psi}, \Psi] = S[\bar{\Psi}, \Psi] - \frac{i}{2} \hbar \log \det(\mathcal{Q}\mathcal{Q}'). \tag{2.61}$$

The sign change ultimately stems from the anticommutative nature of spinor variables. As a side note, the reasoning behind this particular definition comes from the fact that for the basic Dirac operator we have been discussing,

$$\mathcal{Q}\mathcal{Q}' = -\partial^2 + m^2, \tag{2.62}$$

reminiscent of the connection between the Klein-Gordon and Dirac equations.

In the following, when studying the 1-loop effective action for spinor fields, it should always be understood (unless otherwise stated) that there is an extra sign change with respect to bosonic systems, and that $\mathcal{Q}\mathcal{Q}'$ is the operator of interest in these cases, even if the text only explicitly mentions $\mathcal{Q}$ for the sake of brevity. However, it should be pointed that this process of "squaring" the operator also erases some relevant information about the studied fields. Specifically, it gets rid of any complex phases that $\mathcal{Q}$ may carry. These phases become important when studying discrete symmetries (like those related to charge conjugations, parity transformations and time reversals), as well as some topological properties like the Chern-Simons term in 3-dimensional theories. One of the most well-known effects that cannot be properly studied from this approach is the chiral anomaly of electromagnetic theories; in order to properly characterize this phenomenon we need to consider the original operator and effective action [26], although it is possible to find links between the results obtained by the two approaches [27], specially when considering heat kernel expansions as the ones in Section 2.2.3 and onward. A different case of anomalies arising from the effective action will be studied in the next Chapter.

2.1.5 Particle production

As stated in Section 2.1.3, the generating functional for n-point functions is defined in Quantum Field Theory as

$$Z[J] = \langle 0|0\rangle\,[J] = \exp\left(\frac{i}{\hbar}(\Gamma[\bar{\varphi}] + J\bar{\varphi})\right), \tag{2.63}$$

where $|0\rangle$ is the vacuum state of the theory. In particular, $Z[J]$ returns the scattering amplitude for a system that goes from its free vacuum state in the distant past, interacts with the current J (a process which, realistically, should only last for a finite amount of time), and returns to the vacuum at the infinite future. Taking the square of this expression in order to find the probability of this transition, we obtain

$$\left|\,\langle 0|0\rangle\,[J]\,\right|^2 = \left|\exp\left(\frac{i}{\hbar}(\Gamma[\bar{\varphi}] + J\bar{\varphi})\right)\right|^2 = \exp\left(-2\hbar^{-1}\,\mathfrak{Im}\,\Gamma[\bar{\varphi}]\right), \tag{2.64}$$

where $\mathfrak{Im}\,\Gamma[\bar{\varphi}]$ denotes the imaginary part of $\Gamma[\bar{\varphi}]$. Interpreting this result leads us to the first well-known quantum behaviour that serves as a correction to classical field theory. The presence of a current makes it so that the vacuum is not a completely stable state; instead, with a probability of

$$P = 1 - \exp\left(-2\hbar^{-1}\ \mathfrak{Im}\,\Gamma[\bar{\varphi}]\right) \approx 2\hbar^{-1}\ \mathfrak{Im}\,\Gamma[\bar{\varphi}], \tag{2.65}$$

there will be an excitation of the vacuum state, meaning that there will necessarily be a process of particle production. In terms of the 1-loop effective action, this phenomenon will directly depend on $\mathfrak{Re}(\log\det\mathcal{Q})$. This same process can be seen when there is no external current J but the field is not free from the start, and is instead immersed in a (curved) classical background. We will cover this case in a following Section, once we discuss the heat kernel methods used in calculating the 1-loop effective action.

It is important to note, however, that at this point the effective action $\Gamma[\bar{\varphi}]$ will be divergent; this ultimately comes down to the operator ordering issues discussed in Section 2.1.1. Therefore, the derivations done above will lack any actual physical meaning until we properly regularize the divergencies in $\Gamma[\bar{\varphi}]$ and introduce the necessary counterterms. We will take on this regularization process in the next Section.

2.2 Heat kernels

2.2.1 The heat kernel operator

In a previous Section we derived the 1-loop effective action expression for a system whose action is quadratic in the quantum fields

$$\Gamma = \frac{i}{2}\hbar\log\det\mathcal{Q}, \tag{2.66}$$

where $\mathcal{Q}_{ij}$ is the quadratic form that is coupled to the fields in the action. For most systems of interest we will want to consider spacetime as an infinite, boundary-less manifold. Unfortunately, for such cases $\mathcal{Q}$ will be an infinite-dimensional differential operator with a continuous spectrum (since it will always at least include some kinetic term), rather than a finite matrix. This implies that, at this point in time, (2.66) only works as a formal definition.

Nevertheless, let us momentarily suppose that we are working with a matrix A; in that case, the most simple definition of the determinant is given by the product of its eigenvalues

$$\det A = \prod_n \lambda_n, \tag{2.67}$$

and then the 1-loop effective action associated with this matrix would become

$$\Gamma = \frac{i}{2}\hbar \log \det A = \frac{i}{2}\hbar \log\left(\prod_n \lambda_n\right) = \frac{i}{2}\hbar \sum_n \log \lambda_n = \frac{i}{2}\hbar \, \mathrm{Tr}(\log A). \tag{2.68}$$

These expressions can be extended to general operators (assuming that $\mathcal{Q}$ is a formally self-adjoint operator, which will be the case for the systems studied in this document; a general review of operator determinants for more general theories may be found in [28]). In doing so, the sum over eigenvalues will naturally translate into an integral. Strictly speaking, when dealing with operators the expression in terms of the trace should be taken as the starting point in order to define the determinant, and not the other way around. As we shall momentarily see, this reasoning stems from the fact that the trace-dependent expression is regularisable into a well-defined problem.

To better illustrate this, let us consider the simple case of a flat spacetime Klein-Gordon operator

$$\mathcal{Q} = -\partial^2 + m^2. \tag{2.69}$$

In this setup, we can solve the eigenvalue problem by using periodic conditions on a spacetime box of volume $V = L_0 L_1 ... L_{n-1}$, with L_μ the "length" of the box in each spacetime direction. The resulting solutions are plane waves of the form $e^{ik_\mu x^\mu}$, as expected, where the momentum k_μ becomes quantized due to the boundary conditions. The associated eigenvalues are given by

$$\lambda_k = k^2 + m^2, \tag{2.70}$$

and the 1-loop effective action then reads as

$$\Gamma = \frac{i\hbar}{2} V \int \frac{d^d k}{(2\pi)^d} \log(k^2 + m^2). \tag{2.71}$$

The resulting expression has two major sources of divergence. The first one comes from taking the limit where the box used for the boundary condition, and thus its volume $V \sim \int dv_x$, becomes infinite. This can be dealt with by simply defining a 1-loop effective "potential density" through

$$\Gamma = -\int dv_x \; \gamma \tag{2.72}$$

and working with γ instead (from here on, we will still mostly refer to Γ and keep the treatment of this volume integral implicit). The second one stems from the fact that the integral itself is divergent, and must be dealt with via regularization.

The best way to define the regularization scheme is to first rewrite the definition of the 1-loop effective action

$$\Gamma = \frac{i}{2} \hbar \, \mathrm{Tr} \log \mathcal{Q} \tag{2.73}$$

by extending to general operators the following integral expression for the logarithm of a number / matrix [29]

$$\log A = -\int_0^\infty \frac{d\tau}{\tau} e^{-i\tau A}, \tag{2.74}$$

which is valid up to an (infinitely large) integration constant that for our purposes could be later reabsorbed in the definition of the fields or interaction constants; the limits of integration will be omitted from here on. The parameter τ is oftentimes called the proper time (not to be confused with the spacetime coordinate). Using this expression into (2.73) results in

$$\Gamma = -\frac{i}{2}\hbar \int_0^\infty \frac{d\tau}{\tau} \mathrm{Tr}\; e^{-i\tau \mathcal{Q}}. \tag{2.75}$$

From this, it becomes convenient to define the heat kernel operator

$$K(\tau) := \exp(-i\tau\mathcal{Q}), \tag{2.76}$$

as well as its matrix elements (in position space)

$$K(\tau; x, x') := \langle x | K(\tau) | x' \rangle, \tag{2.77}$$

which can then be used to expand the trace into

$$\Gamma = -\frac{i}{2}\hbar \int dv_x \int_0^\infty \frac{d\tau}{\tau} K(\tau; x, x). \tag{2.78}$$

The point x'^{μ} is a reference point that will be arbitrarily chosen; if the system is well-behaved, the choice will ultimately not matter given that the trace over spacetime implies taking a limit $x' \to x$ at a later point. There is still a trace over internal indices (those associated with the structure of the fields) that will be left implicit.

The main advantage of the heat kernel operator is that, by definition, its matrix elements satisfy an equation of the form

$$(-i\partial_\tau + \mathcal{Q})K(\tau; x, x') = 0, \qquad K(0; x, x') = \delta(x, x'), \tag{2.79}$$

where $\delta(x, x')$ is the Dirac delta distribution. For the vast majority of systems of interest (and all the ones covered in this document), $\mathcal{Q}$ is a second-order differential (in spacetime) operator, meaning (2.79) becomes a diffusion-like differential equation.

Before moving on to discussing the solutions of Equation (2.79), it will prove convenient to redefine all our expressions in a more appropriate form. We do so by performing a Wick rotation to imaginary proper time which, for our purposes, will be equivalent to performing a change $\tau \mapsto i\tau$ (it should be reiterated that τ is not a spacetime coordinate, but an auxiliary variable; this allows us to perform this trick even in curved backgrounds, where true spacetime Wick rotation becomes a very subtle and often times impossible ordeal [30]). Additionally, from now on we will continue using units in which $\hbar \mapsto 1$ for convenience. Under this reparametrization, the 1-loop effective action becomes

$$\Gamma = \frac{1}{2}\mathrm{Tr}\log\mathcal{Q} = -\frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\mathrm{Tr}\; e^{-\tau\mathcal{Q}} = -\frac{1}{2}\int dv_x \int_0^\infty \frac{d\tau}{\tau}K(\tau;x,x), \tag{2.80}$$

which motivates the definition of the heat kernel operator as

$$K(\tau) := \exp(-\tau\mathcal{Q}). \tag{2.81}$$

The heat kernel equation will also see some slight modification and read as

$$(\partial_\tau + \mathcal{Q})K(\tau;x,x') = 0, \qquad K(0;x,x') = \delta(x,x'), \tag{2.82}$$

which is now fully equivalent to the heat diffusion equation (hence its name).

2.2.2 Exact solutions of the heat kernel equation

A quick look at the integrand in (2.80) shows that the behaviour at large proper times (the IR limit) is fully dominated by the exponential, which for a Euclidean signature tends rapidly to 0 and makes the whole integral regular. On the other hand, the presence of a τ^{-1} term will potentially lead to some divergencies in the small proper time (or UV) limit, as described in the previous Section. Therefore, in order to fully characterize and counteract these divergencies, all it matters is that we gain a better understanding of the behaviour of the (diagonal elements of the) heat kernel at small proper times.

Evidently, deriving an explicit solution to (2.82) would be the best case scenario, and it is indeed possible to do so for some of the more simple systems. For instance, let us take the case of a free massless scalar field on a flat spacetime, characterized by $\mathcal{Q} = -\partial^2$. The heat kernel then becomes a simple Gaussian of the form

$$K_0(\tau; x, x') = \frac{1}{(4\pi\tau)^{d/2}} \exp\left(-\frac{(x-x')^2}{4\tau}\right). \tag{2.83}$$

Adding a mass term, $\mathcal{Q} = -\partial^2 + m^2$, just modifies the above result by a multiplicative term of the form $e^{-\tau m^2}$. For a more generic curved spacetime, the Euclidean distance is replaced with the geodesic distance $d^2(x, x')$ or, more commonly, with the Synge world function

$$\sigma(x, x') := \frac{1}{2} d^2(x, x'), \tag{2.84}$$

which satisfies the following useful properties (when both derivatives are taken with respect to x^μ or x'^μ):

i) $\partial_\mu \sigma \; \partial^\mu \sigma = 2\sigma$

ii) $\partial_\mu \partial_\nu \sigma = g_{\mu\nu}$, and therefore $\partial^2 \sigma = d$.

However, when curvature is introduced, it is also necessary to account for the density of geodesics in every direction, which will affect the diffusion. This is done in the form of an extra term, called the Van Vleck-Morette determinant, which is related to the cross-derivatives of the Synge world function via

$$\Delta_{_{VVM}}^{^{1/2}} = -\frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}} \det\left(\frac{\partial^2 \sigma}{\partial x \, \partial x'}\right). \tag{2.85}$$

All in all, the heat kernel for a free scalar field in a general background is [31]

$$K(\tau; x, x') = \frac{1}{(4\pi\tau)^{d/2}} \Delta_{_{VVM}}^{^{1/2}}(x, x') \exp\left(-m^2\tau - \frac{\sigma(x, x')}{2\tau}\right). \tag{2.86}$$

For the purposes of this chapter, however, the most relevant case for which exact solutions may be found will be a system consisting of a quantum scalar field interacting with a constant electromagnetic background (meaning, with a constant $F_{\mu\nu}$). In this case, the operator in question is of the form

$$\mathcal{Q} = -(\partial_\mu + ieA_\mu)(\partial^\mu + ieA^\mu) + m^2 \equiv Q + m^2, \tag{2.87}$$

where A^μ is linearly dependent on $F_{\mu\nu}$ (as will be expanded upon in Section 2.2.5), and we have once again limited ourselves to a flat spacetime. Ignoring the mass term again as per the argument presented above, we are now interested in finding the trace of the heat kernel associated with Q. Considering even values of d, the antisymmetry of F allows us to find a block-diagonal form

$$F = \begin{pmatrix} 0 & f_1 & 0 & 0 & \dots & 0 & 0 \\ -f_1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & f_2 & \dots & 0 & 0 \\ 0 & 0 & -f_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & f_{d/2} \\ 0 & 0 & 0 & 0 & \dots & -f_{d/2} & 0 \end{pmatrix}, \tag{2.88}$$

where $\pm i f_j$ are the (purely imaginary, therefore making f_j real) eigenvalues of F, via orthogonal transformations. Doing so, we can separate

$$Q = \sum_{j=1}^{d/2} Q_j = \sum_{j=1}^{d/2} -(\partial_\mu^{(j)} + ieA_\mu^{(j)})(\partial^\mu_{(j)} + ieA^\mu_{(j)}), \tag{2.89}$$

where each Q_j only acts on the subspace generated by the corresponding eigenvectors of $\pm i f_j$. Finding the eigenvalues of these Q_j is equivalent to the Landau level problem of a particle moving on a plane while subjected to a perpendicular magnetic interaction of magnitude $B_j = f_j$. They are given by [32]

$$\lambda_{n_j}^{(j)} = (2n_j + 1)ef_j, \qquad \text{with } n_j = 0, 1, 2, ... \tag{2.90}$$

The heat kernel diagonal elements (minus the mass term) will then be

$$K(\tau; x, x) = \langle x|e^{-\tau Q}|x\rangle = \prod_{j=1}^{d/2} \langle x|e^{-\tau Q_j}|x\rangle = \prod_{j=1}^{d/2} \sum_{n_j=0}^{\infty} \frac{ef_j}{2\pi} e^{-\tau\lambda_{n_j}^{(j)}}, \tag{2.91}$$

where the prefactors $\frac{ef_j}{2\pi}$ account for the degeneracy of each eigenvalue $\lambda_{n_j}^{(j)}$. For every Q_j, the sum in (2.91) is a geometric series which converges to

$$\langle x|e^{-\tau Q_j}|x\rangle = \frac{ef_j}{2\pi} \frac{e^{-\tau ef_j}}{1 - e^{-2\tau ef_j}} = \frac{ef_j}{4\pi \sinh(\tau ef_j)}. \tag{2.92}$$

Therefore, the diagonal elements of the heat kernel will be given by

$$K(\tau; x, x) = \prod_{j=1}^{d/2} \frac{ef_j}{4\pi \sinh(\tau ef_j)} = \frac{1}{(4\pi\tau)^{d/2}} \left[\det\left(\frac{\tau eF}{\sinh(\tau eF)}\right)\right]^{\frac{1}{2}}, \tag{2.93}$$

on account of each $i\, f_j$ appearing twice in F. The 1-loop effective action can be finally written as

$$\Gamma = -\int dv_x \int_0^{\infty} \frac{d\tau}{\tau} \frac{e^{-\tau m^2}}{(4\pi\tau)^{d/2}} \left[\det\left(\frac{\tau eF}{\sinh(\tau eF)}\right)\right]^{\frac{1}{2}}. \tag{2.94}$$

Notice there is an additional factor of 2 with respect to (2.80); this is due to the scalar field in this derivation being a complex field, instead of a real one like in our previous derivation.

If we now restrict ourselves to the specific case $d = 4$, we may define the electromagnetic invariants

$$\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.95}$$

$$\mathcal{G} = \frac{1}{4} \tilde{F}_{\mu\nu} F^{\mu\nu} = \frac{1}{8} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} F^{\mu\nu}, \tag{2.96}$$

in terms of which the determinant in (2.94) can be rewritten as

$$\left[\det\left(\frac{\tau e F}{\sinh(\tau e F)}\right)\right]^{\frac{1}{2}} = \frac{(e\tau)^2 \mathcal{G}}{\mathfrak{Im}\left(\cosh\left(e\tau\sqrt{2(\mathcal{F} + i\mathcal{G})}\right)\right)}. \tag{2.97}$$

To study the divergencies of the effective action, we may look at the first few terms of the series expansion of this determinant. Roughly speaking, the divergent terms will be given by

$$\textbf{divp}\ \Gamma \sim -\int_0^\infty \frac{d\tau}{\tau^3} \left[1 - \frac{1}{3}(e\tau)^2 \mathcal{F}\right]. \tag{2.98}$$

The τ^{-3} term will give rise to a divergence that can ultimately be associated with the vacuum energy; therefore, it may be reabsorbed by simply redefining said constant value accordingly. The τ^{-1} term, on the other hand, gives rise to a logarithmic divergence that is proportional to $-\mathcal{F}$, which is just the classical Maxwell Lagrangian; it can therefore be compensated by a renormalization of the electric charge (or the field strength). All further terms will behave regularly, meaning that the 1-loop effective action can be regularized as

$$\Gamma = \int dv_x \int_0^\infty d\tau \frac{e^{-\tau m^2}}{16\pi^2 \tau^3} \left[1 - \frac{1}{3}(e\tau)^2 \mathcal{F} - \frac{(e\tau)^2 \mathcal{G}}{\mathfrak{Im}\left(\cosh\left(e\tau\sqrt{2(\mathcal{F} + i\mathcal{G})}\right)\right)}\right]. \tag{2.99}$$

We may see this action as the volume integral of an effective Lagrangian density. Combining this with the classical Lagrangian density, which only consists of the pure electromagnetic sector, the result up to first order corrections becomes

$$\mathcal{L} = -\mathcal{F} + \int_0^\infty d\tau \frac{e^{-\tau m^2}}{16\pi^2\tau^3}\left[1 - \frac{1}{3}(e\tau)^2\mathcal{F} - \frac{(e\tau)^2\mathcal{G}}{\mathfrak{Im}\left(\cosh\left(e\tau\sqrt{2(\mathcal{F}+i\mathcal{G})}\right)\right)}\right]. \quad (2.100)$$

This is the so-called Euler-Heisenberg effective Lagrangian. If the electromagnetic field is weak, it reduces to its more familiar expression

$$\mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{e^4}{1440\pi^2 m^4}\left[(\mathbf{E}\cdot\mathbf{B})^2 + \frac{7}{4}(\mathbf{E}^2 - \mathbf{B}^2)^2\right], \quad (2.101)$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field vectors, respectively [33].

To conclude the discussion on this Lagrangian, we may study the cases where either $\mathbf{E}$ or $\mathbf{B}$ vanish. In the former case, we see that (2.100) reduces to

$$\mathcal{L}(B) = -\frac{1}{2}B^2 - \frac{1}{16\pi^2}\int_0^\infty \frac{d\tau}{\tau^3}e^{-\tau m^2}\left[\frac{e\tau B}{\sinh(e\tau B)} - 1 + \frac{1}{6}(e\tau B)^2\right], \quad (2.102)$$

while in the latter, we obtain

$$\mathcal{L}(E) = \frac{1}{2}E^2 - \frac{1}{16\pi^2}\int_0^\infty \frac{d\tau}{\tau^3}e^{-\tau m^2}\left[\frac{e\tau E}{\sin(e\tau E)} - 1 - \frac{1}{6}(e\tau E)^2\right]. \quad (2.103)$$

We may see that, while the term inside the integral in $\mathcal{L}(B)$ behaves regularly over all its integration domain, the integral in $\mathcal{L}(E)$ runs through infinitely many poles given by $\tau_n = \frac{\pi n}{eE}$, $n = 1, 2, ...$ If we rotate back into Minkowski signature, we may view the integral as taken over a contour that surrounds these poles. This deformation of the integral will result in the Lagrangian gaining an imaginary part

$$\mathfrak{Im}\,\mathcal{L}(E) = \frac{e^2E^2}{16\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp\left(-\frac{n\pi m^2}{eE}\right), \quad (2.104)$$

and so does the effective action after integration over spacetime. In other words, by the results obtained in Section 2.1.5, we may see that a constant electric field will be able to create particles over time; the stronger its magnitude E, the higher the probabilities of particle creation and therefore, the final production of them. On the other hand, a purely magnetic background field will not be able to produce any particles over time. This effect is widely known as the Schwinger effect [21]; while the required energies needed to make this effect clearly noticeable are currently outside of the scope of standard Quantum Electrodynamics experiments, in recent years there have been several proposals of analogue systems in which similar effects can be observed [34, 35]. Similar results are obtained for spinor Quantum Electrodynamics, once the considerations of Section 2.1.4 are taken into account.

2.2.3 The Gilkey-Seeley-DeWitt expansion

Going back to more general discussions, it is hardly ever the case that we can find a closed-form solution to (2.82) in terms of elementary functions for arbitrary operators $\mathcal{Q} = -\nabla^2 + M(x)$ (where $M(x)$ is a potential-like term). However, if we are only interested in the small proper time behaviour of the effective action, and using (2.86) as a base, it is possible to construct a perturbative expansion [36]

$$K(\tau; x, x') = \frac{1}{(4\pi\tau)^{d/2}} \Delta^{1/2}_{VVM}(x, x') \exp\left(-\frac{\sigma(x, x')}{2\tau}\right) \sum_{n=0}^{\infty} \tau^n c_n(x, x') \tag{2.105}$$

that works in this regime (it is important to notice that the expansion does not extend all the way to infinite proper times). The so-called Gilkey-Seeley-DeWitt (GSDW) coefficients $c_n(x, x')$ are smooth functions that depend solely on the space-time geometry and the field interactions as encoded in the operator $\mathcal{Q}$. Plugging this ansatz back into (2.80), we see that the general behaviour of the integrand will be given by $\tau^{n-1-d/2}$, meaning the only terms that would give rise to divergencies are those for which $n \leq \frac{d}{2}$. In $d = 4$, that means only the c_0, c_1 and c_2 terms will need to be addressed.

The explicit calculation of these coefficients can be performed by simply introducing (2.105) into (2.82) and imposing it to be satisfied at all orders in τ. Doing so yields a set of relations between the GSDW coefficients that can be solved recursively starting from $c_0(x, x')$. At the coincidence limit, which is the relevant result on account of (2.80) and will be defined for any function $F(x, x')$ by

$$[F] := \lim_{x' \to x} F(x, x'), \tag{2.106}$$

the first three GSDW coefficients for the general operator described above are given by the formulas

$$[c_0] = \mathbb{I}, \tag{2.107}$$

$$[c_1] = \frac{1}{6} R \, \mathbb{I} - M, \tag{2.108}$$

$$\begin{aligned}[c_2] = & \left(\frac{1}{72} R^2 - \frac{1}{180} R^{\mu\nu} R_{\mu\nu} + \frac{1}{180} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - \frac{1}{30} \nabla^2 R \right) \mathbb{I} \\ & + \frac{1}{2} M^2 - \frac{1}{6} R M + \frac{1}{6} \nabla^2 M + \frac{1}{12} \Omega_{\mu\nu} \Omega^{\mu\nu}, \end{aligned} \tag{2.109}$$

where $R_{\mu\nu\rho\sigma}, R_{\mu\nu}, R$ correspond to the Riemann curvature tensor, Ricci tensor and Riemann curvature scalar of the spacetime under study, respectively, and we define $\Omega_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$. To understand this notation, let us first look at a simple case of a single quantum field on a classical background A_μ, where roughly

$$\nabla_\mu = \partial_\mu + A_\mu \, ; \tag{2.110}$$

in this case we may find

$$\Omega_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{2.111}$$

which is the expression for the field strength associated with A_μ; for an Abelian field, it reduces to the standard formulation of $F_{\mu\nu}$ used in electromagnetic models. For a system with multiple fields (or a single field with internal and/or spacetime indices), we may see $\Omega_{\mu\nu}$ as a collection of matricial objects "indexed" by the (μ, ν) pair; for instance, in a more general Yang-Mills theory the field A^μ acquires a matrix structure in the Lie algebra space, which we denote by the indices (a, b), implying $(\Omega_{\mu\nu})^a{}_b = (F_{\mu\nu})^a{}_b$. In the case of a vector field propagating through a curved spacetime, we may see that $(\Omega_{\mu\nu})^\rho{}_\sigma = R^\rho{}_{\sigma\mu\nu}$. In both cases, the product $\Omega_{\mu\nu}\Omega^{\mu\nu}$ should be understood as the result of taking the trace with respect to the matrix indices.

With all the divergencies fully characterized, the only step left in order to regularize the theory and arrive at a nonsingular expression for the 1-loop effective action would be to modify the original Lagrangian with the corresponding counterterms needed to compensate the terms generated by (2.107)-(2.109). For higher-dimensional theories or for more precise calculations of the heat kernel, which would then translate into more specific derivations of the effective action, we would need to continue this process term by term until the desired level of accuracy is reached. However, this process quickly becomes computationally demanding, with the number and complexity of contributions growing larger and larger at every step. In the next Section we will review an alternative scheme for expanding the heat kernel that will allow us to facilitate some of the calculations.

2.2.4 Heat kernel resummation formulas

The result obtained in the previous Section provides a universal and straightforward method for expanding the heat kernel into a power series in τ, which in turn allows for a method to identify and characterize all its divergencies needed in order to regularize the effective action. However, as already pointed out, it is an increasingly costly method from the computational point of view. There is also an additional consideration to take into account: the original purpose for defining the effective action was to find a way to avoid the diagrammatic expansion procedure most commonly used in Particle Physics in order to understand scattering amplitudes. Our main objective in this regard would be to avoid performing any perturbative expansion of Γ at all. While that is not possible outside of some trivial cases, the question of whether it is feasible to find a different procedure that allows us to preserve some more nonperturbative details of the effective action still remains.

One of the ways to tackle this question is through the proposal of resummation schemes. The basic idea behind a resummation scheme is to present a different perturbative expansion for the heat kernel to the one given in (2.105) that allows for some specific features of the system to be extracted as global prefactors to the infinite series. Over the past few decades, several resummation formulas have been proposed for different systems [37–44] where the properties of the operator $\mathcal{Q}$ allow for a more specific resolution of (2.82). In this Section and the following ones, we will present one such resummation formula for a small handful of different systems.

We begin our discussion by presenting the following differential operator

$$\mathcal{Q}_{\text{quad}} := -\partial^2 + \alpha + \beta_\mu (x-x')^\mu + \frac{1}{4}(\gamma^2)_{\mu\nu}(x-x')^\mu (x-x')^\nu, \tag{2.112}$$

where α, β, γ are coefficients independent of the quantum field and x'^μ is chosen in the same way as in the previous Sections; we will also from now on take the abbreviated notation $\bar{x}^\mu := (x-x')^\mu$ for better readability. This operator appears in any system consisting of a quantum scalar field interacting with an at most quadratic potential classical background. The heat kernel equation (2.82) associated with $\mathcal{Q}_{\text{quad}}$ can be solved exactly [45], provided some sensible assumptions hold.

The heat kernel operator can be proven to be closely related to the Green function, or propagator, of the theory described by $\mathcal{Q}_{\text{quad}}$. Indeed, let us recover the integral expression for the logarithm of a finite matrix A that was presented above:

$$\log A = -\int_0^\infty \frac{d\tau}{\tau} e^{-\tau A}. \tag{2.113}$$

Taking derivatives on both sides with respect to the matrix A, we find the equivalent relation

$$A^{-1} = \int_0^\infty d\tau e^{-\tau A}. \tag{2.114}$$

Once again, it is possible to extend both (2.113) and (2.114) to be valid on infinite-dimensional operators. The left-hand side, $(\mathcal{Q}_{\text{quad}})^{-1}$, can then be identified with the propagator G_{quad} associated with $\mathcal{Q}_{\text{quad}}$. In terms of their matrix elements, we can therefore conclude that

$$G(x, x') := \int_0^\infty d\tau\ K(\tau; x, x') \tag{2.115}$$

as a general result for any operator $\mathcal{Q}$ with associated heat kernel $K(\tau) = e^{-\tau\mathcal{Q}}$.

For the operator in (2.112), we wish to find a solution to the equation

$$\left(-\partial^2 + \alpha + \beta_\mu \bar{x}^\mu + \frac{1}{4}(\gamma^2)_{\mu\nu}\bar{x}^\mu\bar{x}^\nu\right) G_{\text{quad}}(x, x') = \delta(x, x'). \tag{2.116}$$

By performing a Fourier transform from spacetime coordinates into momentum coordinates, this becomes

$$\left(p^2 + \alpha + i\beta_\mu \partial_{p_\mu} - \frac{1}{4}(\gamma^2)_{\mu\nu}\partial_{p_\mu}\partial_{p_\nu}\right) G_{\text{quad}}(p) = 1. \tag{2.117}$$

If we were to ignore the derivative terms, the solution would be a straightforward

$$G_{\text{quad}}(p) = (p^2 + \alpha)^{-1} = \int_0^\infty d\tau\ e^{-\alpha\tau - p^2\tau}, \tag{2.118}$$

where in the last equality we have once again used (2.114). For the complete equation, we look for a more general ansatz

$$G_{\text{quad}}(p) = \int_0^\infty d\tau\ \exp\left(-\alpha\tau + p_\mu A^{\mu\nu}(\tau)p_\nu + p_\mu B^\mu(\tau) + C(\tau)\right). \tag{2.119}$$

By comparing with (2.115), it becomes obvious that the exponential inside the integral in (2.119) is precisely $K_{\text{quad}}(\tau; p)$. Introducing (2.119) into (2.117) and following the calculation (with some small nuances described in [45]), we can obtain an explicit expression for $G_{\text{quad}}(p)$ and in turn $K_{\text{quad}}(\tau; p)$. Performing once again a Fourier transform back into position space, the resulting heat kernel is

$$K_{\text{quad}}(\tau; x, x') := \frac{e^{-\tau\alpha}}{(4\pi\tau)^{d/2}} \frac{\exp\left(-\frac{1}{4}\tilde{\sigma}^{\mu}(x,x') A^{-1}_{\mu\nu}(\tau)\tilde{\sigma}^{\nu}(x,x') - C(\tau)\right)}{\det{}^{1/2}(\tau^{-1}A(\tau))}, \tag{2.120}$$

where we have defined

$$\tilde{\sigma}_{\mu}(x,x') := \bar{x}_{\mu} + B_{\mu}(\tau) \tag{2.121}$$

and the coefficients become

$$A_{\mu\nu}(\tau) := \left[\gamma^{-1}\tanh(\gamma\tau)\right]_{\mu\nu}, \tag{2.122}$$

$$B_{\mu}(\tau) := 2\beta^{\nu}\left[\gamma^{-2}(1-\operatorname{sech}(\gamma\tau))\right]_{\nu\mu}, \tag{2.123}$$

$$C(\tau) := \beta^{\mu}\left[-\tau\gamma^{-2} + \gamma^{-3}\tanh(\gamma\tau)\right]_{\mu\nu}\beta^{\nu} + \frac{1}{2}\left[\log\cosh(\gamma\tau)\right]^{\mu}{}_{\mu}. \tag{2.124}$$

The hyperbolic functions included in these identities should be understood as defined by their respective power series expansions around $\gamma = 0$. Doing so ensures that only even powers of γ are present in the heat kernel, as required from its implementation in (2.112).

Let us now look at a more general system, consisting of a real scalar quantum field interacting via a Yukawa coupling to a scalar background field. This kind of system is characterized by its differential operator

$$\mathcal{Q}_Y := -\partial^2 + V(x), \tag{2.125}$$

where $V(x)$ is an arbitrary potential term that does not depend on the quantum field. If $V(x)$ was an at most quadratic potential, it would be straightforward to show that (2.125) is an operator of the form (2.112) under the simple identification

$$\begin{cases} \alpha & \mapsto V(x') \\ \beta_{\mu} & \mapsto \left(\partial_{\mu}V\right)\big|_{x'} \\ (\gamma^2)_{\mu\nu} & \mapsto 2\left(\partial_{\mu}\partial_{\nu}V\right)\big|_{x'} \end{cases}, \tag{2.126}$$

meaning the heat kernel in (2.120) would be equally valid under such a translation. The main claim for this Section is that, even for a more general potential $V(x)$, it is possible to take (2.120) as the leading order in a heat kernel expansion formula

$$K_Y(\tau; x, x') = K_{\text{quad}}(\tau; x, x') \sum_{j=0}^{\infty} \tau^j a_j(x, x'), \tag{2.127}$$

in a manner analogous to how (2.86) is expanded to (2.105) [46]. Additionally, as we shall elaborate on shortly, performing the expansion in this alternative way will allow us to effectively resum all the contributions of the first two derivatives of V out of the perturbative series, once we take the coincidence limit that is needed in the 1-loop effective action.

Introducing the heat kernel (2.127) into (2.82) and forcing it to be satisfied at every order in τ leads to a recursive relation between the coefficients

$$\begin{aligned} -(j+1+\bar{x}^\alpha \partial_\alpha) a_{j+1}(x, x') &= (-\partial^2 + \mathfrak{S}) a_j(x, x') \\ &+ \sum_{n=1}^{\lfloor j/2 \rfloor} \frac{B_{2n}}{(2n)!} \left(4(2^{2n}-1) \beta^\alpha \left(\gamma^{2(n-1)}\right)_{\alpha\beta} + 2^{2n} \bar{x}^\alpha \left(\gamma^{2n}\right)_{\alpha\beta} \right) \partial^\beta a_{j+1-2n}(x, x'), \end{aligned} \tag{2.128}$$

where B_k is the k-th Bernoulli number, $\lfloor \cdot \rfloor$ denotes the floor function, β and γ^2 are to be replaced as indicated in (2.126), and we employ the shorthand notation for $(\gamma^{2k})^\alpha{}_\beta := (\gamma^2)^\alpha{}_{\mu_1} (\gamma^2)^{\mu_1}{}_{\mu_2} \ldots (\gamma^2)^{\mu_{k-1}}{}_\beta$, with $(\gamma^0)^\alpha{}_\beta \equiv \delta^\alpha{}_\beta$ (reminder that we are working on a Euclidean metric, and as such indices are raised and lowered with the Kronecker δ). We also define the "effective potential"

$$\mathfrak{S}(x, x') := V(x) - \alpha - \bar{x}^\alpha \beta_\alpha - \frac{1}{4} \bar{x}^\alpha (\gamma^2)_{\alpha\beta}\, \bar{x}^\beta. \tag{2.129}$$

We shall now prove the statement presented above. Specifically, the resummation scheme here proposed has the special property that none of the coefficients $a_j(x, x')$, nor their derivatives, will depend on any of the invariants that can be built by using only the potential V and its first two derivatives, that is to say, none of those contained in the set

$$\{V, \quad \partial^\mu V(\gamma^{2k})_{\mu\nu}\partial^\nu V, \quad \delta^{\mu\nu}(\gamma^{2k})_{\mu\nu}\}_{k\geq 0}, \tag{2.130}$$

when taken to their coincidence limit $x' \to x$. For the sake of brevity we will be referring to these particular invariants as *chains* from this point on.

Looking at (2.128), there are only three possible places where these chains might appear when taking the coincidence limit:

i) Explicitly on the recurrence relation or its derivatives; after a quick inspection it is evident that all the potentially "problematic" terms will ultimately be contracted with derivatives of the coefficients $a_j(x, x')$ instead, not fully closed in and of themselves.

ii) In the effective potential $\mathfrak{S}$; however, looking at its structure carefully, it becomes apparent that its coincidence limit, as well as the coincidence limit of its first two derivatives, vanish identically. In fact, it becomes apparent that

$$[\partial_{\mu_1}...\partial_{\mu_k}\mathfrak{S}] = (\partial_{\mu_1}...\partial_{\mu_k}V)\big|_{x'} \qquad \forall k \geq 3. \tag{2.131}$$

Thus, its third and higher order derivatives do not vanish in the coincidence limit, but they will always depend on at least third derivatives of the potential, and thus cannot give rise to any of the chains considered.

iii) Implicitly from the expressions of the coefficients $a_j(x, x')$ or their derivatives. This will shortly be proven not to be possible as well.

Indeed, we can self-evidently conclude that there is a well-defined ordering given by $[a_0], [a_1], [\partial_\mu a_1], [\partial_\mu\partial_\nu a_1], [a_2], ...,$ where every element on the list requires the knowledge of all previous ones in order to be determined via (2.128) or derivatives of it. The initial condition in (2.82) can be shown to imply that

$$a_0(x, x') = 1 \tag{2.132}$$

identically [47] (we may set $j = -1$ in the recurrence relation to see that $\partial_\mu a_0(x, x') = 0$; therefore, a_0 must be a constant, and our choice of normalization ensures it is precisely 1). Its coincidence limit, $[a_0]$, definitely does not depend on any of the chains in (2.130). On the other hand, by analysing the structure of the recursive relation and having already ruled out the other possibilities, we see the only way in which chains might appear in the calculation of any given (derivatives of) $a_j(x, x')$ is if any of the previous elements in the ordering does, or at least if it depends on some "half-chain" (say, if some derivative $\partial_\mu \partial_\nu a_k(x, x')$ depends on $(\gamma^2)_{\mu\nu}$ for some $k < j$) that could be contracted with other pieces in the recursive equation to give rise to a full chain. Since we know for a fact that the first element in the sequence, $a_0(x, x')$, does not present such properties, we can conclude by induction that none of them will.

In other words, with the resummation scheme here presented, we can prove that all the information about the system that can be derived using only the potential and its first two derivatives is fully contained within the global prefactor K_{quad}. The trace of the heat kernel needed to regularize and eventually derive the 1-loop effective action will be

$$K_Y(\tau; x, x) = \frac{e^{-\tau V} \exp\left(\partial^\mu V [\gamma^{-3}(\gamma\tau - 2\tanh(\frac{1}{2}\gamma\tau))]_{\mu\nu} \partial^\nu V\right)}{(4\pi)^{d/2} \det{}^{1/2}\left((\gamma\tau)^{-1}\sinh(\gamma\tau)\right)} \sum_{j=0}^{\infty} \tau^j [a_j], \tag{2.133}$$

where the first few coefficients read

$$[a_0] = a_0(x,x) = 1, \tag{2.134}$$

$$[a_1] = a_1(x,x) = 0, \tag{2.135}$$

$$[a_2] = a_2(x,x) = 0, \tag{2.136}$$

$$[a_3] = a_3(x,x) = -\frac{1}{60}\partial^2\partial^2 V. \tag{2.137}$$

Just from this small sample, it is self-evident that the coefficients take a much simpler form than their GSDW counterparts, while agreeing with previous results [48, 49]. The effect is more noticeable as more and more terms are introduced in the calculation.

The expression in (2.133) is completely regular in τ; it is also a fully real expression, provided $(\gamma^2)_{\mu\nu} \sim \partial_\mu\partial_\nu V$ is positive definite. If at least one of its eigenvalues $(\lambda_-)^2$ is negative, then the corresponding eigenvalue λ_- of γ will be purely imaginary, in exchange turning

$$\sinh(\lambda_-\tau) \mapsto i\,\sin(|\lambda_-|\tau). \tag{2.138}$$

The heat kernel will thus develop an infinite series of poles on the τ-axis, wherever the sine vanishes. To be able to perform the integral (2.78), we would need to follow a similar procedure to the one in Section 2.2.2 and modify the integration contour to circumvent the poles on the τ axis. In doing so, the effective action acquires an imaginary part, leading to a process of particle production, which seems to point at a "Schwinger-like" process for scalar fields under specific conditions (namely, a strong enough potential whose Hessian is not positive definite).

2.2.5 Electromagnetic backgrounds

The results discussed in the previous Section can actually be generalized to other systems of interest. A simple step forward in this direction can be taken by studying the interaction of a massive charged complex quantum scalar field with a purely electromagnetic classical background. The Lagrangian for such model in Euclidean signature takes the form

$$\mathcal{L}_E := (D_\mu \phi)^\dagger (D^\mu \phi) + m^2 \phi^\dagger \phi + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.139}$$

where m is the mass of the scalar field and $F_{\mu\nu}$ the electromagnetic field strength tensor, while the covariant derivative is defined as

$$D_\mu = \partial_\mu + ieA_\mu, \tag{2.140}$$

where $A^\mu(x)$ is the background electromagnetic field. Expanding the covariant derivative and dropping the pure electromagnetic term in the Lagrangian, since it is fully classical and independent of the scalar field, the operator associated with such a system will be

$$\mathcal{Q}_{SQED} := -\partial^2 - 2ieA^\mu \partial_\mu - ie\ \partial_\mu A^\mu + m^2 + e^2 A^2. \tag{2.141}$$

Ignoring temporarily the term linear in derivatives, we can compare (2.141) and (2.125) and see that both operators are of the same form after the definition

$$V_{SQED}(x) := m^2 + e^2 A^2 - ie\ \partial_\mu A^\mu. \tag{2.142}$$

It would be ideal, then, to be able to use the exact same results obtained for the Yukawa interaction in this new system, the only change being the definition of the potential term.

It remains to be seen if the term linear in derivatives in (2.141) poses any complication. Let us first consider the case of a constant electromagnetic field strength $F_{\mu\nu}$. In this particular case, we can rewrite

$$A_\mu(x) = \frac{1}{2} \bar{x}^\nu F_{\nu\mu} \tag{2.143}$$

in order to still satisfy the defining relation $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Substituting this definition into the operator, we see that V_{SQED} is composed of a constant term, akin to the coefficient α in (2.112), and a term quadratic in $\bar{x}$, akin to γ^2, but no term that could be compared to β. The structure of (2.120) with these coefficients ensures that only even powers of $\bar{x}$ may appear in the heat kernel. Therefore, acting on it with the term that is linear in derivatives, which is proportional to $\bar{x}^\mu F_{\mu\nu}\partial^\nu$, will always yield terms of the form $\bar{x}^\mu F_{\mu\nu}\bar{x}^\nu$, vanishing identically on account of the antisymmetry of $F_{\mu\nu}$. In short, thanks to this property of the electromagnetic tensor, the same heat kernel that satisfies the equation without a term linear in derivatives also satisfies the equation with it.

In general, for any system characterized by an operator

$$\mathcal{Q}_T = \mathcal{Q}_{\text{quad}} + \bar{x}_\mu T^{\mu\nu}\partial_\nu, \tag{2.144}$$

where $T^{\mu\nu}$ is an antisymmetric tensor, it is possible to show that the same heat kernel that solves the equation for $\mathcal{Q}_{\text{quad}}$ also solves it for $\mathcal{Q}_T$. For a general electromagnetic field we can ensure this is the case by temporarily choosing the Fock-Schwinger gauge [50], defined by

$$\bar{x}^\mu A_\mu(x) = 0, \tag{2.145}$$

in order to write the potential A^μ as an infinite series in terms of $F_{\mu\nu}$ and its derivatives [51]

$$A_\mu(x) = \sum_{k=0}^{\infty} \frac{1}{k!(k+2)} \bar{x}^{\mu_1}...\bar{x}^{\mu_k}\bar{x}^\rho \left(\partial_{\mu_1}...\partial_{\mu_k}F_{\rho\mu}\right)\big|_{x'} . \tag{2.146}$$

Introducing this definition into (2.141) and keeping only the terms at most quadratic in $\bar{x}$ to draw the comparison to (2.144), we can ensure that, for this truncation, the heat kernel will once again be given by (2.120) with the appropriate redefinitions of the coefficients α, β and γ^2.

In the same way that was discussed for the Yukawa interaction, we propose that the heat kernel for a fully general electromagnetic background can then be seen as a series expansion of the form (2.127), with

$$\begin{cases} \alpha = m^2 \\ \beta_\mu = \frac{i}{3} e \; \partial^\lambda F_{\lambda\mu} \\ (\gamma^2)_{\mu\nu} = -e^2 (F^2)_{\mu\nu} - ie \; \partial^\lambda \partial_{(\mu} F_{\nu)\lambda} \end{cases} , \tag{2.147}$$

where $F_{\mu\nu}$ and its derivatives are evaluated at x', and the idempotent symmetrization of indices has been denoted by the parenthesis; $F_{\mu\nu}$ is treated as a matrix in its spacetime indices, meaning $(F^2)_{\mu\nu} = F_{\mu\rho} F^\rho{}_\nu$. Furthermore, in the same way that this resummation scheme allowed for several contributions of the potential and its derivatives to be taken out of the perturbative expansion, the heat kernel here described presents a similar property. It is possible to demonstrate that, after taking the coincidence limit, none of the coefficients $a_j(x, x')$ will depend on any chains constructed purely from products of $F_{\mu\nu}$, i.e.

$$\{(F^k)^\mu{}_\mu\}_{k \geq 0} \tag{2.148}$$

(in addition to a trivial resummation of the mass term). The recursive relation (2.128) gets expanded with some new terms proportional to A^μ (which naturally come from the extra term linear in derivatives that was not present before), but none of them introduce any new complications that would prevent us from constructing a proof similar to the one presented in Section 2.2.4. A short discussion regarding these new terms will be carried on shortly in order to tackle both scalar and spinor QED at the same time.

Let us, then, move on to the case of a single spinor field in an electromagnetic background. As discussed in the previous chapter, the relevant operator in a spinor theory Lagrangian will be of first order in derivatives, namely

$$\mathcal{Q}^{1/2} = -(\gamma^\mu D_\mu + m), \tag{2.149}$$

where the covariant derivative is defined as

$$D_\mu := \partial_\mu + ie\, A_\mu. \tag{2.150}$$

However, following the argumentation in Section 2.1.4, we can look at the "squared" operator instead,

$$\mathcal{Q}_{QED} := \mathcal{Q}^{1/2}(\mathcal{Q}^{1/2})' = -\partial^2 - 2ieA^\mu\partial_\mu + m^2 - 2ie\,\partial_\mu A^\mu - \frac{i}{2}e\sigma^{\mu\nu}F_{\mu\nu} + e^2A^2, \tag{2.151}$$

where $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. The only new feature that distinguishes (2.151) from its scalar counterpart (2.141) is given by the $\sigma^{\mu\nu}F_{\mu\nu}$ term, which modifies the potential to be matrix-valued. In this situation, time-ordering becomes a concern when trying to find the heat kernel matrix elements, specially if we were to derive a general result that preserved gauge invariance. However, due to the specific construction procedure that we have followed in the previous Section (considering the case where $F_{\mu\nu}$ is a constant and expanding upon it, choosing a particular gauge, and only being interested in the series coefficients that explicitly do not depend on the proper time), our only concern regarding this will be to meticulously work with the terms that depend on the potential and its derivatives in the recurrence relation, since they will no longer commute with each other. This does not meaningfully modify any of the relevant arguments from the previous Section, meaning the heat kernel can once again be given in the form (2.120), this time with the coefficients

$$\begin{cases} \alpha = m^2 - \frac{i}{2}e\sigma^{\mu\nu}F_{\mu\nu}, \\ \beta_\mu = -ie\left(\frac{1}{3}\partial^\rho F_{\mu\rho} + \frac{1}{2}\sigma^{\rho\lambda}\partial_\mu F_{\rho\lambda}\right), \\ (\gamma^2)_{\mu\nu} = -e^2(F^2)_{\mu\nu} - ie\left(\partial^\rho\partial_{(\mu}F_{\nu)\rho} + \sigma^{\rho\lambda}\partial_\mu\partial_\nu F_{\rho\lambda}\right) \end{cases}, \tag{2.152}$$

with all tensor fields evaluated at x'. Once again, we state the claim that such a heat kernel is able to resum all contributions of the form

$$\{(F^k)^\mu{}_\mu, \quad (\sigma^{\mu\nu}F_{\mu\nu})^k\}_{k\geq 0} \tag{2.153}$$

(as well as the mass term) into the global prefactor and out of the perturbative expansion. After introducing the heat kernel into (2.79), expanding all the hyperbolic functions in power series and imposing that the heat kernel equation is satisfied at all orders of τ, we obtain a recurrence relation

$$\begin{aligned}
&-(j+1+\bar{x}^\alpha\partial_\alpha)a_{j+1}(x,x') = (-\partial^2+\mathfrak{S}+2A^\mu\partial_\mu)a_j(x,x')\\
&\quad+\sum_{n=1}^{\lfloor\frac{j}{2}\rfloor}\frac{B_{2n}}{(2n)!}\left(4(2^{2n}-1)\beta^\alpha\left(\gamma^{2(n-1)}\right)_{\alpha\beta}+2^{2n}\bar{x}^\alpha\left(\gamma^{2n}\right)_{\alpha\beta}\right)\partial^\beta a_{j+1-2n}(x,x')\\
&-\sum_{n=1}^{\lfloor\frac{j+1}{2}\rfloor}\frac{B_{2n}}{(2n)!}\left(4(2^{2n}-1)\beta^\alpha\left(\gamma^{2(n-1)}\right)_{\alpha\beta}+2^{2n}\bar{x}^\alpha\left(\gamma^{2n}\right)_{\alpha\beta}\right)A^\beta a_{j+1-2n}(x,x').
\end{aligned} \tag{2.154}$$

Most of the terms in (2.154) are exactly the same as previously discussed for Yukawa interactions. The main difference comes from the terms that explicitly include A^μ in the first and third lines. In order to prove that these new terms do not introduce any complications that might invalidate the proof as it was delineated in Section 2.2.4, it suffices to know that only the terms contracted with the leading order of A^μ (in the expression derived from taking the Fock-Schwinger gauge) and the first term of $(\gamma^2)_{\mu\nu}$ have the potential for generating any relevant chains, since β_μ and all other terms of A^μ and $(\gamma^2)_{\mu\nu}$ will depend on at least one derivative of $F_{\mu\nu}$. However, the term $\bar{x}^\alpha(\gamma^{2n})_{\alpha\beta}A^\beta$ on the third line vanishes identically in that case due to the antisymmetry of $F_{\mu\nu}$, and the term $A^\mu\partial_\mu a_j(x,x')$ on the first line can only generate a chain for $a_{j+1}(x,x')$ if any of the previous coefficients themselves do at least generate a half-chain, which we know from the induction argument that it is not the case.

Even though the Fock-Schwinger gauge has been used in order to justify many of the steps in this procedure, both the global prefactor and the perturbative coefficients will depend, at least in the coincidence limit, on manifestly gauge invariant objects (since they will all be geometric quantities built from $F_{\mu\nu}$ and its derivatives), meaning the results are ultimately gauge independent. In $d = 4$ in particular, it can be shown [52] that all of the chains that we have discussed may be written in terms of the electromagnetic invariants $\mathcal{F}$ and $\mathcal{G}$, meaning the heat kernel here presented effectively resums all their contributions, leaving only terms with derivatives of them in the series expansion.

Once again, the heat kernel will present an infinite series of poles if at least one of the eigenvalues of γ^2 becomes negative; since in this setup we know that $\gamma^2 \sim e^2F^2 +$ derivatives of F, then this will happen whenever the eigenvalues of F^2 become negative. A simple study of these, which is particularly convenient in $d = 4$ because they will also depend only on the electromagnetic invariants, shows that this is the case when the electric field dominates over the magnetic field. The resulting divergence affects all the terms in the heat kernel expansion, leading to an expanded, local version of the effective Euler-Heisenberg Lagrangian derived in Section 2.2.2.

2.2.6 Inhomogeneous fields

In Section 2.2.5 it was argued that, even though the operators for scalar and spinor QED presented a linear term in spatial derivatives that distinguished them from the general formula (2.112), it was still possible to utilize the results from Section 2.2.4 because said term ultimately became inconsequential on account of its specific structure (2.144). Nevertheless, it is sensible to ask the question of whether it is still possible to find a resummation scheme for systems where that is not the case.

Consider the following unspecified toy model of a quantum field system for which the differential operator becomes

$$\mathcal{Q}_N = -\partial^2 + N^\mu \partial_\mu + \alpha, \tag{2.155}$$

where α is a constant mass-like term and N^μ is an arbitrary vector field, with no particular gauge properties that would allow us to rewrite it analogously to (2.144). This model can serve as a simplified version of some inhomogeneous field theories, which have garnered some interest in recent years [53]. We can start by considering the case where $N^\mu = N_0^\mu$ is just a constant (in spacetime, meaning $\partial_\mu N_0^\nu = 0$) vector. In this situation, the exact heat kernel can be obtained once again from the Green function equation [54] Fourier-transformed into momentum coordinates

$$\left(p^2 + iN_0^\mu p_\mu + \alpha\right) G_N(p) = 1 \Longrightarrow G_N(p) = \left(p^2 + iN_0^\mu p_\mu + \alpha\right)^{-1}, \tag{2.156}$$

which can be rewritten in an integral form as

$$G_N(p) = \int d\tau e^{-\tau(p^2 + iN^\mu p_\mu + \alpha)} = \int d\tau K_N(p), \tag{2.157}$$

and finally Fourier-transformed back into

$$K_0(\tau; x, x') = \frac{1}{(4\pi\tau)^{d/2}} \exp\left(-\tau\alpha - \frac{1}{4\tau}(\tau N_0 - \bar{x})^2\right). \tag{2.158}$$

For a general vector N^μ, it is still possible to perform an expansion around the reference point x', the leading order of which will just be $N^\mu(x')$; we may therefore propose that (2.158) acts as a global prefactor in a perturbative expansion

$$K_N(\tau; x, x') = \frac{1}{(4\pi\tau)^{d/2}} \exp\left(-\tau\alpha - \frac{1}{4\tau}(\tau N(x') - \bar{x})^2\right) \sum_{j=0}^{\infty} \tau^j a_j(x, x'). \tag{2.159}$$

The recurrence relation that can be derived this time from introducing (2.159) into the heat kernel equation is

$$\begin{aligned}\left(j + \bar{x}^\mu \left(\partial_\mu - \frac{1}{2}(N - N(x'))_\mu\right)\right) a_j(x, x') = \\ \left(\partial^2 - (N - N(x'))^\mu \left(\partial + \frac{1}{2}N(x')\right)\right) a_{j-1}(x, x').\end{aligned} \tag{2.160}$$

Given that the proposed system is extremely simple, the only sensible chains that we may look for and resum are

$$\{(N_\mu N^\mu)^k\}_{k\geq 0}. \tag{2.161}$$

We may once again see that no (half-)chains will appear on the coincidence limit of the coefficients unless the previous coefficients do which, since $[a_0] = 1$ by construction, completes the proof by induction. It is important to notice, though, that the coefficient $a_0(x, x')$ is not constant for this system; for instance, we may take derivatives on (2.160) and find

$$[\partial_\mu a_0] = 0, \qquad [\partial_\mu \partial_\nu a_0] = \frac{1}{2}\partial_{(\mu} N_{\nu)}. \tag{2.162}$$

This means that the derivatives of a_0 must be taken into account for the calculations. The first few coefficients, when taken to the coincidence limit, are given by

$$a_0(x, x) = 1, \tag{2.163}$$

$$a_1(x, x) = \frac{1}{2}\partial_\mu N^\mu, \tag{2.164}$$

$$a_2(x, x) = \frac{1}{24}\left(3(\partial_\alpha N^\alpha)^2 - 2\partial_{(\alpha} N_{\beta)}\partial^\alpha N^\beta + 2(\partial_\alpha - N_\alpha)\partial^2 N^\alpha\right), \tag{2.165}$$

$$\begin{aligned} a_3(x, x) = {} & \frac{1}{120}(\partial_\alpha - N_\alpha)\partial^4 N^\alpha + \frac{1}{24}\partial_\alpha N^\alpha \partial_\beta \partial^2 N^\beta - \frac{1}{30}\partial^\beta N^\alpha \partial^2 \partial_{(\alpha} N_{\beta)} \\ & - \frac{1}{144}\partial^2 N^\alpha \partial^2 N_\alpha - \frac{1}{360}\partial_\alpha \partial^\beta N^\alpha \partial^2 N_\beta - \frac{1}{180}\partial_\beta \partial_\gamma N_\alpha \partial^\beta \partial^\gamma N^\alpha \\ & - \frac{1}{90}\partial_\alpha \partial_\gamma N_\beta \partial^\beta \partial^\gamma N^\alpha + \left(\frac{1}{45}\partial^\alpha - \frac{1}{24}N^\alpha\right)\partial^\beta N_\alpha \partial_\beta \partial_\gamma N^\gamma \\ & + \frac{1}{48}(\partial_\alpha N^\alpha)^3 - \frac{1}{24}\partial_\alpha N^\alpha \partial_{(\beta} N_{\gamma)}\partial^\gamma N^\beta - \frac{1}{24}N^\alpha \partial_\beta N^\beta \partial^2 N_\alpha \\ & + \frac{1}{48}N^\alpha N^\beta \partial_\gamma N_\beta \partial^\gamma N_\alpha. \end{aligned} \tag{2.166}$$

2.3 Conclusions and future directions

Over the last few Sections, we have covered the basic formulation of the effective action for a quantum field system. On its own, the effective action presents an alternative, but potentially powerful, procedure for studying scattering amplitudes and any related properties of the fields, while also connecting to the standard diagrammatic procedure if necessary. Particularly, for those systems where the classical action is quadratic in the fields, it allows for an exact, nonperturbative derivation of the transition amplitude $\langle 2|1\rangle [J]$. For more general systems, the 1-loop effective action serves as a first-order approximation that helps study the quantum corrections to a clasical field theory.

In order to actually derive an expression for the 1-loop effective action, we made use of the heat kernel definition to write Γ as an integral, over a single parameter τ, of an operator whose matrix elements satisfy a well-defined differential equation (at least in Euclidean signature). While the heat kernel equation is solvable in some specific cases, for a more general system there is a need to perform some kind of series expansion in τ and recursively determine its coefficients.

While the GSDW expansion is completely general and applicable to any model, there is no denying that the computational cost of deriving the coefficients becomes very high, very fast. Additionally, in no way does it help to the original objective of finding some nonperturbative information about the system. Thus, resummation schemes have been proposed over the years to rewrite the GSDW expansion in a more powerful way. One such resummation formula has been fully discussed and shown to be applicable to a handful of very different systems. The results agree with previous observations, while simplifying the calculations and giving a much more detailed leading order term.

Nevertheless, it remains to be seen if this procedure can be further generalized. There are several conceptual and computational complications that may arise when dealing with heat kernels for more complex systems. As a way of illustrating this, consider a system consisting of a spinor coupled to an axial vector field S^μ on a flat spacetime [55, 56], which may also be seen as the interaction between a spinor field and the spacetime torsion tensor [57]. The relevant operator

$$\mathcal{Q}_S^{1/2} = -(\gamma^\mu(\partial_\mu + i\eta\gamma_5 S_\mu) + m) \tag{2.167}$$

can be squared into

$$\mathcal{Q}_S = -\partial^2 + 2i\eta\gamma_5\sigma^{\mu\nu}S_\mu\partial_\nu + m^2 - \eta^2 S^2 - i\eta\gamma_5\partial_\mu S^\mu - \frac{i}{2}\eta\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}, \tag{2.168}$$

where $\mathcal{S}_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu$ and γ_5 is the chiral element associated with the γ matrices [58]. Setting S^μ to be a constant vector (which may be of interest in some models of Schwinger effect in the pressence of Lorentz-violating background fields [59]) makes this operator appear to be of the form (2.155) with the appropriate definitions of α and N^μ, and therefore should be solvable in terms of the heat kernel (2.159). However, in this situation the vector

$$N^\mu = 2i\,\eta\gamma_5\,S_\mu\sigma^{\nu\mu} \tag{2.169}$$

is a matrix-valued vector with a nontrivial matrix structure given by the spinor indices of the $\sigma^{\mu\nu}$ matrices. There is some structure to be found in N^μ thanks to the Clifford algebra constraints, namely that its anticommutators satisfy

$$\{N^\mu, N^\mu\} = 8\eta^2(S^2\eta^{\mu\nu} - S^\mu S^\nu)\mathbb{I}, \tag{2.170}$$

allowing for the Green equation to be solved exactly in momentum space

$$G_S(p) = (p^2 + iN^\mu p_\mu + \alpha)^{-1} = \frac{p^2 - iN^\mu p_\mu + \alpha}{(p^2+\alpha)^2 + 4\eta^2[p^2S^2 - (p_\mu S^\mu)^2]}. \tag{2.171}$$

However, trying to extract a related heat kernel operator out of this propagator has proven to be highly nontrivial. If we were to try and get the result directly from the integral expression (2.157) and perform the Fourier transform, we likewise find the complication that the term $\exp(-i\tau\, p_\mu N^\mu)$ is the exponential of a matrix that is not proportional to the identity, meaning it is not possible to just perform the integral as if it were a Gaussian; a series expansion is needed instead. While it is possible to prove that $N^\mu N_\mu \propto \mathbb{I}$, allowing us to separate the exponential into a cosh(...) and a sinh(...) term, the arguments inside both of them make it so that the resulting integrals do not seem to accept elementary solutions. While in the Abelian case we were able to circumnavigate the complication of a matrix-valued term, that does no longer seem to be the case. Applying the same workflow as in the Yukawa and electromagnetic cases becomes highly nontrivial due to the commutation relations present all throughout the calculations. It is possible that employing techniques from the worldline formalism for heat kernel calculations may help alleviate the issue; additional comparative studies may be of great interest in this regard. Alternative approaches have been developed for this problem by considering specific representations of the γ-matrices, thus pointing to a possible resolution of this problem [60].

Beyond that, there is no shortage of possible avenues to extend these results. Obvious candidates are finding a generalization of the resummation scheme for non-Abelian gauge theories like that of the strong interaction [61, 62], and of course curved spacetimes (both of which would once again require developing a satisfactory way of dealing with matrix-valued elements in the operator). From an experimental perspective, the resummed version of the proper-time expansion for the electromagnetic Lagrangian can lead to an enhanced analysis of some accessible phenomena, serving as a better approximation than the locally-constant approximation frequently used [63]; in particular, it may be associated with testable nonlinear QED processes such as light-by-light scattering [64] and vacuum birefringence [65].

Chapter 3

Black holes semiclassical effects

Black holes are some of the most intriguing objects in all of Theoretical and Mathematical Physics. In the context of this work, they are one of the very few instances where the strength of gravitational interactions becomes comparable to that of the other fundamental forces. As such, understanding their behaviour becomes of paramount importance to the future of Quantum Gravity as a field of research. The first part of this chapter will be focused on general aspects of black holes, as well as some of their specific quirks that become fundamental in this line of thinking. In order to simplify the computational complications and focus only on the (already abundant) conceptual challenges, we will be following the recent surge of interest in two-dimensional gravity models that admit black hole solutions and capture, to some extent, the features of those arising in Einstein's theory of gravity. One such model, originally proposed by Callan, Giddings, Harvey and Strominger and later expanded upon over the years, will be the main focus of the second part of this Chapter.

3.1 Black holes

3.1.1 General Relativity basics

The following paragraphs will serve as a quick introduction of the main elements and notations used in the study of General Relativity, as a whole, and black hole models, in particular. A more in-depth treatment may be found in [66], while [67] provides a full deep dive into the mathematical formalism here sketched.

As briefly posed in the Introduction of this document, in the context of General Relativity, gravity is not understood as an interaction between objects that live in a spacetime, but rather as an intrinsic property of the spacetime itself. Mathematically, this manifests in the spacetime being defined as a d-dimensional manifold $\mathcal{M}$ (for the most part, we will be interested in the cases of $d = 4$ and $d = 2$), the curvature of which is what we would then experience as a gravitational pull. The manifold $\mathcal{M}$ is additionally endowed with a metric structure that allows us to formally define distances and angles (or more accurately, spacetime intervals) between the points in spacetime. The traditional Euclidean space definitions of scalars (including the coordinate functions themselves), vectors and tensors used in Newtonian mechanics are then adapted to this new formalism.

Let us solidify the concept of a manifold's curvature, which for our purposes should be done intrinsically, as our current understanding of spacetime does not generally contemplate it as being embedded into a higher-dimensional space. A key challenge in that regard is understanding how tensor fields change between different points around the manifold; unlike flat Euclidean space, it is not possible to directly compare the value of these fields at two different points, even infinitesimally close ones, given how the metric structure is generally nontrivial. The rate of change of a vector field is instead given by the covariant derivative, whose coordinate-dependent representation is given by

$$\nabla_\mu v^\nu := \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\rho} v^\rho, \tag{3.1}$$

with analogous definitions for other types of tensors, assuming that the spacetime is torsion-free. Here, the ordinary partial derivative $\partial_\mu v^\nu$ merely measures the change in the vector's components along the different coordinate curves, which would be sufficient in Euclidean space. The second coefficient, known as the connection (given in General Relativity by the Christoffel symbols $\Gamma^\nu{}_{\mu\rho}$), quantifies how the local coordinate basis vectors themselves change from point to point. For the class of manifolds that will be of interest in this document, the Christoffel symbols can be found to depend only on the metric tensor.

A vector v is said to be parallel-transported along a curve in $\mathcal{M}$ whose tangent vector is t if

$$t^\mu \nabla_\mu v^\nu = 0. \tag{3.2}$$

A curve $\gamma(\tau)$ in $\mathcal{M}$, then, is said to be a geodesic if its own tangent vector v can be parallel-transported along itself, that is, if

$$v^\mu \nabla_\mu v^\nu = 0. \tag{3.3}$$

Expanding the expression for the vector components and Christoffel symbols, we arrive at

$$\frac{d^2\gamma^\mu}{d\tau^2} + \Gamma^\mu{}_{\rho\lambda}\frac{d\gamma^\rho}{d\tau}\frac{d\gamma^\lambda}{d\tau} = 0, \tag{3.4}$$

where $\{\gamma^\mu(\tau)\}$ are the coordinate functions of the point $\gamma(\tau)$; this last equation will be of great relevance in following Sections. For any point P in $\mathcal{M}$ and vector v defined in its tangent space, there exists a unique geodesic that passes through P with tangent vector v. Geodesics are usually presented as the definition of the "straightest possible lines" in curved spacetime.

Moving on, we may take a moment to analyze the definition in (3.2) and realize that parallel transport is a path-dependent object; given the same start and end points, the vectors given by parallel transporting the same original vector along two different curves may be very different when examined at the endpoint. For infinitesimally close points in $\mathcal{M}$, the difference between the action of two different paths will be given by the commutator $[\nabla_\mu, \nabla_\nu]$ which, even though neither of the covariant derivatives is a tensor in itself, acts as a tensor of type (1,3), called the Riemann curvature tensor,

$$[\nabla_\mu, \nabla_\nu]v^\lambda = R^\lambda{}_{\mu\nu\rho}v^\rho, \tag{3.5}$$

with similar relations when acting on other types of tensors. The necessary and sufficient condition for a spacetime to be flat will then be that this mismatch vanishes identically at every point, $R^\lambda{}_{\mu\nu\rho} = 0$. Given (3.5), it is self-evident that the components of the Riemann tensor can be written in terms of the Christoffel symbols and, in turn, of the metric tensor.

The Riemann tensor can be further mathematically decomposed into three main contributions

$$\begin{aligned} R_{\mu\nu\rho\lambda} = C_{\mu\nu\rho\lambda} + \frac{1}{d-2}(R_{\mu\rho}g_{\nu\lambda} + R_{\nu\lambda}g_{\mu\rho} - R_{\nu\rho}g_{\mu\lambda} - R_{\mu\lambda}g_{\nu\rho}) \\ - \frac{1}{(d-1)(d-2)}R(g_{\mu\rho}g_{\nu\lambda} - g_{\nu\rho}g_{\mu\lambda}), \end{aligned} \tag{3.6}$$

where $C_{\mu\nu\rho\lambda}$ is a totally traceless object known as the Weyl tensor, $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ is the Ricci tensor, and $R = R^\mu{}_\mu$ the Ricci scalar of curvature. This decomposition is valid in general for any $d \geq 3$, and it accounts for different kinds of deformations that a set of points in $\mathcal{M}$ may undergo due to the curvature of it as they are transported through the manifold. For $d = 3$, however, the Weyl tensor vanishes identically, since the Riemann and Ricci tensors have the same number of degrees of freedom. As we shall briefly see, the fact that the only contributions to the curvature come from the Ricci tensor implies that gravity can only exist where matter is located; in other words, gravity cannot propagate through spacetime if $d < 4$ [68, 69]. As for $d = 2$, it is immediate to see that not only will $C_{\mu\nu\rho\lambda}$ vanish again, but the entire expression as a whole breaks down on account of the $(d-2)^{-1}$ factors. This mostly comes down to the entire Riemann tensor being reduced to a single independent component, proportional to the Gaussian curvature of the manifold, which is closely related to its (topologically invariant) Euler characteristic. We will discuss what this means from a dynamical point of view in Section 3.2.1.

Let us again come back to the general solution with d left unspecified. The simplest possible purely gravitational action that can be built from nontrivial invariants using the metric tensor is the so-called Einstein-Hilbert action

$$S_{\text{Einstein-Hilbert}}[g] = \int d^d x \sqrt{-g}\, R, \tag{3.7}$$

up to additional terms that may be needed when considering spacetimes with a boundary [70]. In order to consider theories of matter interacting with gravity, we would need to add its corresponding action S_m. Applying the tools of variational calculus, the principle of stationary action leads to the field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{3.8}$$

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$ is the stress-energy or energy-momentum tensor associated with the matter in the system, and κ is a coupling constant (by comparing to Newtonian gravity we can determine $\kappa \propto G c^{-4}$, where G and c are Newton's constant and the vacuum speed of light, respectively). Here we see the close relation between the Ricci components of the curvature tensor and the matter content of the theory; note however, that these equations do not impose any constraint on the Weyl tensor at all. In some applications, specifically those related to cosmological evolution, an additional term proportional to Λ is added to the Einstein-Hilbert Lagrangian, serving as a measure of the energy of spacetime itself [71]. Imposing that the action is invariant under diffeomorphic transformations leads to the generalized conservation of energy equation

$$\nabla_\mu T^{\mu\nu} = 0. \tag{3.9}$$

3.1.2 Black hole solutions

The expression (3.8) presents a system of nonlinear coupled second order partial differential equations on the metric tensor. Solving these for a completely generic metric and matter distribution is highly nontrivial; for more complex systems it is common to resort to numerical methods [72]. However, it is possible (albeit not simple) to find closed-form solutions for specific relevant cases.

One such solution is the one obtained for the metric tensor outside of a static, spherically symmetric distribution of mass (which we may for simplicity take as point-like). Since there is no matter in the whole studied region, we may set $T_{\mu\nu} = 0$ everywhere. By looking into (3.8), we may see that this implies

$$R_{\mu\nu} = 0. \tag{3.10}$$

The most general metric that satisfies these conditions can be written in spherical coordinates as

$$ds^2 = -A(r)\ dt^2 + B(r)\ dr^2 + r^2\ d\Omega^2, \tag{3.11}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\ d\varphi^2$ is the angular differential. We can find $R_{\mu\nu}$ in terms of $A(r)$ and $B(r)$ and impose the vacuum solution condition. By additionally imposing that the spacetime becomes asymptotically flat as we move to $r \to \infty$ and that the gravitational interaction for any nonrelativistic test particle that is sufficiently far away from the mass distribution reduces to the well-known Newtonian limit, we find that the solution is given by the Schwarzschild metric [73]

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2\ d\Omega^2. \tag{3.12}$$

For simplicity's sake, from now on we will work in units where $c = 1$. The metric apparently has two locations at which it diverges, namely $r = 0$ and $r = 2GM$; however, we can verify that only one of them is truly a spacetime singularity.

While $R_{\mu\nu}$ and R vanish identically, it is important to notice that the Weyl tensor part of the Riemann decomposition does not. In fact, we may calculate the Kretschmann curvature invariant $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} = \dfrac{48G^2M^2}{r^6}$, which indeed shows that the curvature is nonvanishing everywhere in spacetime, and in fact diverges as we approach $r = 0$. The point $r = 2GM$, on the other hand, has a well-defined curvature, so the apparent divergence we see in the metric must be just a mathematical artifact associated with a poor coordinate choice. To see this, we may rewrite the metric using the equivalent Kruskal coordinates, giving

$$ds^2 = \frac{32G^3M^3}{r} \exp\left(-\frac{r}{2GM}\right)(-dx_+^2 + dx_-^2) + r^2 d\Omega^2. \tag{3.13}$$

These coordinates relate to the original ones via

$$x_-^2 - x_+^2 = \left(\frac{r}{2GM} - 1\right) \exp\left(\frac{r}{2M}\right). \tag{3.14}$$

The metric in (3.13) is clearly regular at $r = 2GM$, showing that the Schwarzschild spacetime can be defined for all $r > 0$ or, equivalently, $x_-^2 - x_+^2 > -1$. Given the spherical symmetry of the problem, we may drop the angular part of the metric from here on, and focus solely on radially incoming or outgoing matter; in Kruskal coordinates, null incoming and outgoing geodesics will be given by the 45°-angle lines $x_+ \pm x_- =$ constant. It is also common practice to further define $x_+ \pm x_- = \tan\sigma_\pm$; in doing so, the infinite ranges of $x_\pm$ get compressed to finite $(-\frac{\pi}{2}, \frac{\pi}{2})$ ranges in $\sigma_\pm$. The $r = 0$ singularity is represented by two straight lines at $\sigma_+ + \sigma_- = \pm\frac{\pi}{2}$. While this procedure produces a metric of the form $\Omega^2 g_{\mu\nu}$, with Ω^2 potentially reaching infinite values, it is always possible to redefine an unphysical conformal metric $\hat{g} = \Omega^{-2} g$ and use it instead to absorb the infinities [74]. The resulting Penrose diagram is represented in Figure 3.1.

For future black hole solutions covered in this document we will be presenting their Penrose diagrams directly; as such, it will prove convenient to get used to the terminology surrounding them. Roughly speaking, a Penrose diagram warps

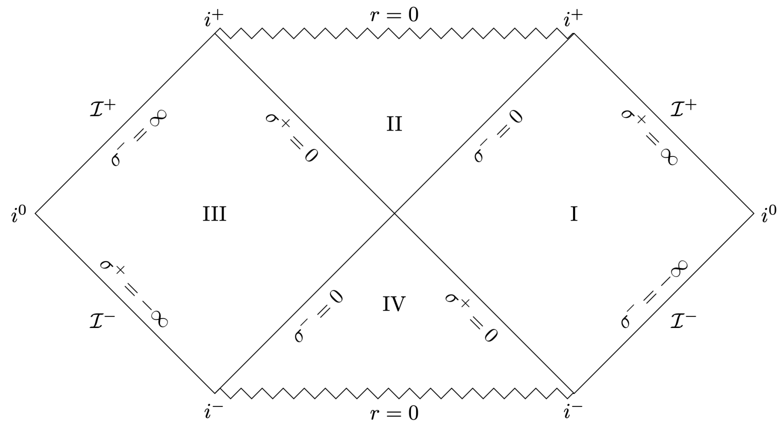

Figure 3.1: Penrose diagram corresponding to the Schwarzschild black hole spacetime. The spacetime is divided into four sectors, with I and II being the exterior and interior region of the black hole, respectively. Sectors III and IV correspond to a "parallel universe" and a "white hole" region, respectively; these appear as the mathematical consequence of maximally extending the coordinates, but are usually excluded by physically motivated arguments.

the representation of spacetime to "bring the infinities to a finite position". As such, it is not a very useful representation of distances between points. Instead, a Penrose diagram is specially useful in studying relationships of causality between events (since null lines are still represented by 45°-angle straight lines), as well as the asymptotic behaviour of fields, the metric included. This is mainly because all the possible directions in which a field can go into infinity get collapsed into the points and straight lines that conform the borders of the diagram, namely:

i) $\mathcal{I}^-$, the past null infinity, from which all incoming null geodesics originate.

ii) $\mathcal{I}^+$, the future null infinity, to which all outgoing null geodesics progress.

iii) $i^\pm$, similarly to $\mathcal{I}^\pm$, represent the future and past infinities that mark the "endpoints" of timelike geodesics.

iv) i^0 is the "space infinity" for all spacelike hypersurfaces, like those characterized by $t =$ constant, as $r \to \infty$.

v) The singularity at $r = 0$. The upper line (future singularity) becomes the end point of all incoming null and timelike geodesics that cross the event horizon. Conversely, the lower line (past singularity) would be the origin of all timelike and null geodesics in the white hole region; as stated in Figure 3.1, it is usually taken as a mathematical construct rather than a physical object.

The event horizon, given by the line $\sigma^- = 0$ (and $\sigma^+ = 0$ if Sectors III and IV are considered), is defined as the boundary of the region obtained by tracing the entire infinity $\mathcal{I}^+$ back to its past. It is clearly a null object that separates the outside region I (and therefore the future infinities) from the interior region II. It is impossible for an object inside region II to escape to infinity, since that would require traveling faster than light in order to cross from II to I. Instead, its inevitable fate is to head towards the singularity, at which point it is currently not understood what happens; as the curvature of the theory becomes infinitely large, purely classical theories lose their predictive power, marking the necessity for a higher energy (possibly semiclassical or fully quantum) model.

Another notion that will be of special interest is that of the apparent horizon. Roughly speaking, this horizon delimits the boundary of a region of trapped surfaces, that is, the region for which all non-spacelike geodesic paths converge. For the Schwarzschild solution it is trivial to see that all ingoing null geodesics start at large values of r and eventually reach $r = 0$. On the other hand, the behaviour of outgoing geodesics changes drastically depending on the region: while those in Sector III will reach higher and higher values of r (which, if we take two different geodesics and consider the angular part of the metric, would lead to them spreading apart), those in Sector II will inevitably collapse towards the singularity. Therefore, the trapped region coincides with Sector II, and the apparent horizon is precisely the boundary of it. In general, for static black hole solutions the apparent horizon will coincide

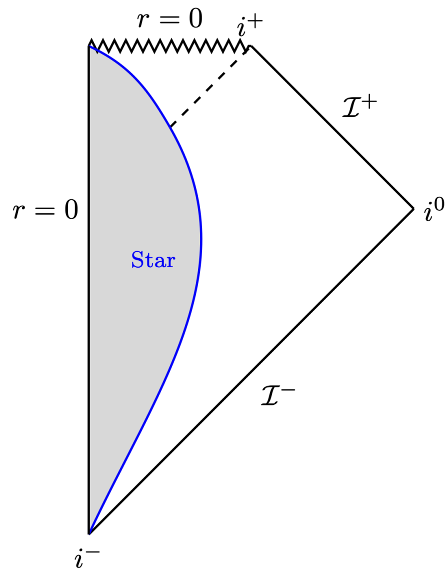

Figure 3.2: Penrose diagram for the gravitational collapse of a static star. The spacetime outside the star will be given by the Schwarzschild metric, while the inside (grey area) metric is left undefined. If the star is massive enough, its collapse will reach a point where an event horizon and singularity form. The apparent horizon is not shown in this image, as it would require a precise description of the metric inside the star.

with the event horizon. The differences between them become more apparent when considering dynamic black hole models like those in Sections 3.2.2 to 3.2.4.

The Schwarzschild solution presents us with a static and ultimately eternal black hole. A more realistic model for black holes is that they originate at some point from the collapse of a massive object [75] into a volume small enough that the horizon at $r = 2GM$ lies outside of it. The object needs to be massive enough that the repulsive interactions between the electrons or even neutrons in its atoms are not enough to counteract the gravitational collapse [76]. The Penrose diagram for a black hole formed via gravitational collapse is given in Figure 3.2.

From the strictly classical point of view, once formed black holes are completely determined by three simple independent parameters: their mass, their electric charge, and their angular momentum [77–79]. Any other information about the object(s) that formed the black hole in the first place is lost and irrecoverable. This so-called "no-hair theorem" poses a big problem when put in the context of the second law of Thermodynamics, since one could theoretically decrease the entropy of the universe by simply dropping matter into a black hole. In order to solve this, and inspired by a previous assertion by Hawking that the area of a black hole event horizon can never decrease over time [80], Bekenstein proposed that the entropy of a black hole is proportional to its surface area [81].

The fact that black holes have an associated entropy immediately implies via thermodynamical arguments that they must also have a temperature, which should be measurable in terms of radiation emitted from it. Indeed, in another work by Hawking [82], this process of radiation was eventually characterized; black holes behave like a black body system, emitting radiation at a temperature

$$T_H = \frac{\hbar c^3}{8\pi k_B G M}, \tag{3.15}$$

where k_B is Boltzmann constant. This result is not possible from a completely classical point of view; it constitutes the first inherently quantum property of a black hole. This effect can be visualized in several ways, including the creation of particle pairs near the horizon where, instead of immediately annihilating each other, one falls inside the black hole and the other escapes to infinity. Alternatively, we can define vacuum states in the infinite past and future regions $\mathcal{I}^{\pm}$ and check that, since the spacetime is not stationary due to the gravitational collapse of the star, they are not equivalent quantizations; the vacuum state in the infinite past is seen as a thermal state from the point of view of an observer at the infinite future [83]. Yet another interpretation of this phenomenon can be derived from the trace anomaly of the energy-momentum tensor [84, 85]; we shall follow this point of view in 2-dimensional theories.

In any case, the fact that black holes radiate seems to indicate that, as time goes on, they will start losing mass and eventually evaporate. How this process ultimately comes to be is still an object of debate, mainly because it is unclear how the singularity behaves throughout all of it. A major concern is that, even though the matter that falls in a black hole could be of any kind, the black hole itself seemingly only radiates in the way a black body does. This is, in essence, the conflict of the information loss paradox; some relevant references on the matter can be found in [86–89]. All in all, there is still a lot to be understood with regards to the evolution of black holes, and the problem only becomes even more obscured by the great amount of computational complexity that is involved in General Relativity calculations due to its immense amount of degrees of freedom. That is why, over the last few decades, there has been an increasing interest in building and studying appropriate 2D models for black hole solutions that help alleviate the mathematical ordeal of these calculations, while still keeping most of the relevant information intact. We shall follow the same path in the following Section.

3.2 Black hole evolution in 2D

3.2.1 Gravity models in $d = 2$

For the following Sections, we will be considering gravity models in 2-dimensional spacetimes. The main reason behind this choice is the relative simplification that all the equations undergo in comparison to the usual 4D formulation. An example of this is that, when examining the Riemann tensor symmetry properties, it becomes clear that there is only one independent component $R_{\mu\nu\rho\lambda}$. The general decomposition in (3.6) is no longer valid; instead, the entire Riemann tensor can be uniquely determined from

$$R_{\mu\nu\rho\lambda} = \frac{1}{2}R(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\rho}). \tag{3.16}$$

This has immediate consequences on the Einstein-Hilbert action (3.7); in 2 dimensions, the left hand side of (3.8) vanishes identically. The Einstein-Hilbert action will not yield a dynamical local field theory, becoming a topological invariant instead.

Thus, in order to define gravity in a 2D spacetime we must search for alternatives that still satisfy all the relevant conditions (namely, diffeomorphism invariance and dependence on at most second order derivatives of $g_{\mu\nu}$ so that the equations of motion are at most of second order in derivatives). One such model is the so-called Jackiw-Teitelboim gravity [90]

$$S_{JT}[g,\phi] = \int d^2x\sqrt{-g}\phi(R-\Lambda), \tag{3.17}$$

where Λ is a cosmological constant term and ϕ is scalar field that couples to gravity, called a dilaton. Dilaton models have been studied in depth over the past years, eventually spreading out into a big family of models [91]. The CGHS model, which is the main topic of the remaining part of the text, is one of them.

Additionally, it can be proven [92] that any metric in 2D can be written in a locally conformally flat form

$$ds^2 = -e^{2\rho(x^+,x^-)}dx^+dx^-, \tag{3.18}$$

where $x^{\pm}$ are null coordinates like the ones defined in Section 3.1.2 The entire dynamics of the metric can then be read from the field $\rho(x^+,x^-)$. On the subject of conformal transformations, let us briefly consider a simple system of a massless scalar coupled to d-dimensional gravity through

$$S_m[\varphi] = -\frac{1}{2}\int d^dx\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{d-2}{4(d-1)}R\varphi^2\right) \tag{3.19}$$

(in the special case $d=2$ it is immediate to see that the second term vanishes and the theory reduces to a minimally coupled model). The classical action S_m is not only invariant under diffeomorphisms, but also under conformal (or Weyl) transformations

$$g_{\mu\nu} \mapsto \Omega^2(x)g_{\mu\nu}\,, \qquad \varphi \mapsto \Omega^{-\frac{d-2}{2}}(x)\varphi. \tag{3.20}$$

In the same way that diffeomorphism invariance constrains the energy-momentum tensor to satisfy $\nabla_\mu T^{\mu\nu} = 0$, Weyl invariance introduces yet another constraint

$$T^{\mu}_{\ \mu} = 0. \tag{3.21}$$

However, as discussed in Section 2.2.3, once we consider the quantum theory there will be some divergent contributions that need to be regularized. Following the standard GSDW approach, we see that these contributions are associated with the coincidence limit of the coefficients $c_j(x, x')$ for which $j \leq \frac{d}{2}$. While the divergences associated with $j < \frac{d}{2}$ simply get rescaled under a Weyl transformation, the logarithmic divergence that comes with $c_{\frac{d}{2}}$ gets shifted by a finite amount. This modifies the trace of the energy-momentum tensor, which will now be proportional to said coefficient. This is known as the trace anomaly of the energy-momentum tensor; it is an inevitable conclusion of trying to quantize the theory given by S_m, as there is no alternative regularization scheme that preserves both diffeomorphism and Weyl invariance simultaneously [93, 94].

Given that the coefficients c_j are written in terms of the invariants of the theory via (2.107 - 2.109), for our massless scalar 2D theory the trace anomaly will be proportional to $c_1 \propto R$ (incidentally, for a 4D theory the relevant coefficient would be c_2). Additionally, we can further utilize the mathematical formalism of Conformal Field Theory and, in particular, operator product expansions (OPEs) to closely relate the trace anomaly,

$$\langle T^{\mu}_{\ \mu} \rangle = \frac{c}{24} R, \tag{3.22}$$

to the central charge of the system c [95], which counts the degrees of freedom of the system; for a single scalar field, $c = 1$. For this and the following Section, we will leave the central charge unspecified but assume $c > 0$; systems with $c < 0$ will come into play in Section 3.2.4. The effective action term that generates the trace anomaly can be shown to be

$$\Gamma_{\text{Polyakov}}[g] = -\frac{c}{96\pi} \int d^2x \sqrt{-g} R \,\Box^{-1} R, \tag{3.23}$$

which will be of great relevance when constructing a full model of semiclassical gravitational interactions.

In the case of a 2-dimensional gravity theory, the trace anomaly (3.22) can be used in conjunction with energy-momentum conservation (which we must still satisfy even at the quantum level, $\nabla_\mu \langle T^{\mu\nu} \rangle = 0$) to completely determine $\langle T_{\mu\nu} \rangle$ up to some integration functions which characterize the quantum states. This will be of major importance when studying Hawking radiation, since the flux of outgoing radiation will be given in conformally flat coordinates by $\langle T_{--} \rangle$. All in all, we have a powerful connection between the geometric properties of the theory (represented by R) and the Hawking radiation emitted by its black hole solutions, through the quantum corrections obtained from the effective action. We will explore these ideas for the CGHS model, starting in the following Section.

3.2.2 The CGHS model

As pointed above, the Callan-Giddings-Harvey-Strominger model [96] is a particular construction of dilatonic gravity; it was originally conceived as a low energy limit of a higher dimensional string theory, but for the purposes of this document its main point of interest is that it provides a solvable and renormalizable theory of 2D gravity [97]. The classical action for this system is given by

$$S_{\mathrm{CGHS}}[g, \phi, f] = \frac{1}{2\pi} \int d^2x \sqrt{-g} \left[e^{-2\phi} (R + 4(\nabla\phi)^2 + 4\lambda^2) - \frac{1}{2} \nabla_\mu f \, \nabla^\mu f \right], \quad (3.24)$$

where ϕ is the dilaton field, λ^2 is a cosmological constant and f is a minimally coupled massless scalar field, which as stated above is automatically conformally coupled (in general we may take a set of N such fields f_i, $i = 1, ..., N$; we will leave the sum over i implicit in our expressions). From this action we can obtain the equations of motion:

i) From the variation with respect to $g_{\mu\nu}$:

$$2e^{-2\phi}[\nabla_\mu \nabla_\nu \phi + g_{\mu\nu}((\nabla\phi)^2 - \nabla^2\phi - \lambda^2)] - (T^f)_{\mu\nu} = 0, \tag{3.25}$$

where T^f is the energy-momentum tensor of the matter fields

$$(T^f)_{\mu\nu} = \frac{1}{2}\left[\partial_\mu f_i \, \partial_\nu f_i - \frac{1}{2} g_{\mu\nu}(\nabla f_i)^2\right]. \tag{3.26}$$

As previously discussed, the purely geometric part of the Einstein equations that would normally arise from this variation vanishes identically in $d = 2$; thus, we may identify the entire left-hand side of (3.25) with the full energy-momentum tensor of the CGHS model.

ii) From the variation with respect to ϕ:

$$e^{-2\phi}[R + 4\lambda^2 + 4\nabla^2\phi - 4(\nabla\phi)^2] = 0, \tag{3.27}$$

which will be referred to as the dilaton equation of motion.

iii) From the variation with respect to f_i:

$$\Box f_i = 0, \tag{3.28}$$

which was to be expected since a minimally coupled scalar should satisfy the Klein-Gordon equation.

In the conformal gauge (3.18), classically we have ${(T^f)^\mu}_\mu \propto (T^f)_{+-} = 0$ due to conformal symmetry, and these equations get transformed into

$$e^{-2(\rho+\phi)}[-4\partial_+\partial_-\phi + 4\partial_+\phi\,\partial_-\phi + 2\partial_+\partial_-\rho + \lambda^2 e^{2\rho}] = 0, \tag{3.29}$$

$$e^{-2\phi}[2\partial_+\partial_-\phi - 4\partial_+\phi\,\partial_-\phi - \lambda^2 e^{2\rho}] = 0, \tag{3.30}$$

$$\partial_+\partial_- f_i = 0, \tag{3.31}$$

together with the constraints that come from fixing $g_{\pm\pm} = 0$:

$$e^{-2\phi}[2\partial_\pm^2\phi - 4\partial_\pm\rho\,\partial_\pm\phi] = (T^f)_{\pm\pm}. \tag{3.32}$$

The first two of these can be combined to give a free field equation

$$\partial_+\partial_-(\rho - \phi) = 0, \tag{3.33}$$

which means the fields ρ and ϕ differ from each other by a sum $\omega_+(x^+) + \omega_-(x^-)$; however, by diffeomorphism invariance we can rescale and shift the coordinates to make these $\omega_\pm$ vanish and set $\rho = \phi$; we will refer to this particular choice as the Kruskal gauge. In this setup, the equations of motion simplify to

$$\partial_+\partial_- e^{-2\phi} = -\lambda^2, \tag{3.34}$$

$$\partial_\pm^2(e^{-2\phi}) = -(T^f)_{\pm\pm}. \tag{3.35}$$

If we momentarily consider that there are no scalar fields f_i, then the equations above have a general solution

$$e^{-2\phi} = -\lambda^2 x^+ x^- + \frac{M}{\lambda}, \tag{3.36}$$

where M is an (at this point) arbitrary constant of integration. From this we can read the components of $g_{\mu\nu}$ and find the Ricci curvature

$$R = 8e^{-2\phi}\partial_+\partial_-\phi = 4\frac{\partial_+(e^{-2\phi})\partial_-(e^{-2\phi})}{e^{-2\phi}} - 4\partial_+\partial_-(e^{-2\phi}) = \frac{4M\lambda}{\frac{M}{\lambda} - \lambda^2 x^+ x^-}. \tag{3.37}$$

This curvature is clearly divergent along the curve

$$x^+x^- = \frac{M}{\lambda^3}, \tag{3.38}$$

and in general we can perform similar tricks to those in Section 3.1.2 to see that this solution is yet another static black hole solution, with mass M and event horizon located at $x^\pm = 0$. Its Penrose diagram is qualitatively the same as Figure 3.1, the main difference being that, in the same way the Schwarzschild metric becomes asymptotically flat (i.e. Minkowski) in the infinities of region I, the CGHS solution approaches the equivalent spacetime in this setup, which is called the linear dilaton vacuum solution and is characterized by

$$R = \nabla^2\phi = 0, \qquad (\nabla\phi)^2 = \lambda^2. \tag{3.39}$$

This can also be seen to be the global solution in the particular case when $M = 0$. In the coordinates $x^\pm$, the flat spacetime metric is naturally defined in the region $x^+ > 0$ and $x^- < 0$. Upon introducing the coordinates $\sigma^\pm$ via $\lambda x^\pm = \pm e^{\pm\lambda\sigma^\pm}$, the solution assumes the simple form

$$ds^2 = -\,d\sigma^+ d\sigma^-, \tag{3.40}$$

where now $\sigma^\pm$ cover the entire spacetime. As for the dilaton field, if we were to define the coordinates $\sigma^\pm = t \pm \sigma$ so that the metric takes the well-known form $-dt^2 + d\sigma^2$, then we would see

$$\phi = -\frac{\lambda}{2}\left(\sigma^+ - \sigma^-\right) = -\lambda\sigma. \tag{3.41}$$

This linear dependence of the dilaton on the spacetime coordinates motivates the designation of the spacetime as the “linear dilaton vacuum”.

Let us now take a look at the case of an infalling null shell of matter; this can be represented by the energy-momentum tensor

$$(T^f)_{++} = \frac{m}{\lambda x_0^+}\delta(x^+ - x_0^+), \tag{3.42}$$

where m is the energy of the shell. The solution to the constraints and equations of motion becomes

$$e^{-2\phi} = -\lambda^2 x^+ x^- - \frac{m}{\lambda x_0^+}(x^+ - x_0^+)\,\Theta(x^+ - x_0^+), \tag{3.43}$$

where we have introduced the Heaviside Θ function. For a more general setup, we can imagine a continuous drip of incoming null mass throughout some interval $[x_i^+, x_f^+]$. The solution in this case becomes

$$e^{-2\phi} = -\lambda^2 x^+ \left(x^- + \frac{P(x^+)}{\lambda^2}\right) + \frac{m(x^+)}{\lambda}, \tag{3.44}$$

where we have defined

$$P(x^+) = \int_{x_i^+}^{x_f^+} dy^+ \; T_{++}(y^+), \tag{3.45}$$

$$m(x^+) = \lambda \int_{x_i^+}^{x_f^+} dy^+ \; y^+ \, T_{++}(y^+). \tag{3.46}$$

In this model we will have an event horizon at $x^- = -\lambda^{-1} P(x_f^+)$ (requiring a global knowledge of the spacetime), which denotes the entire region of points that will be unable to reach the infinity at $\mathcal{I}_R^+$. On the other hand, we have a separate apparent horizon that follows the curve given by

$$x^- = -\frac{P(x^+)}{\lambda^2}. \tag{3.47}$$

This describes the frontier of all trapped surfaces, i.e. the boundary of the region for which, locally, both ingoing and outgoing geodesics converge. We can visualize this by momentarily imagining that our 2D spacetime comes from dropping the angular

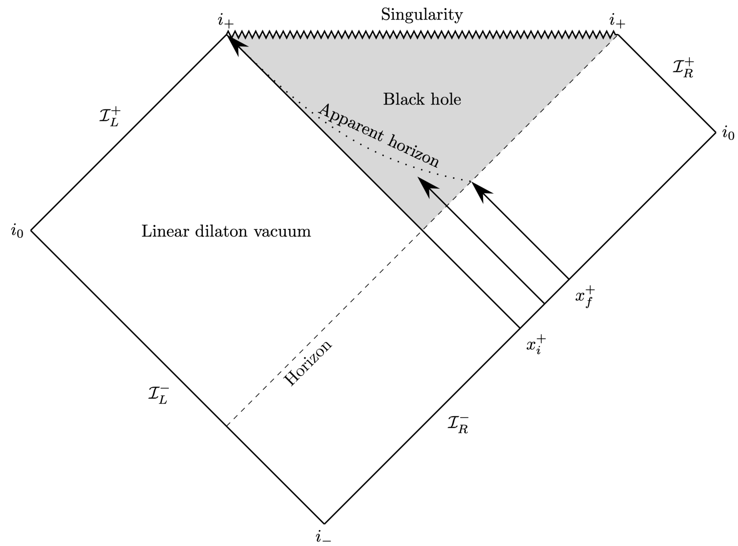

Figure 3.3: Penrose diagram for the formation of a black hole by a continuous flow of infalling null energy. The apparent horizon does no longer coincide with the event horizon, since the mass of the black hole grows over time.

contribution of a spherically symmetric 4D one; the area of the 2-spheres that would be located at each point in our 2D spacetime will be proportional to $e^{-2\phi}$, so the trapped region corresponds to the set of points for which the area always decreases, meaning $\partial_\pm(e^{-2\phi}) < 0$ or, equivalently, $\partial_\pm\phi > 0$. The apparent horizon will lie on the boundary of this region where either $\partial_+\phi$ or $\partial_-\phi$ changes sign; looking at our construction, $\partial_-\phi = \frac{\lambda^2}{2}e^{2\phi}x^+ > 0$ everywhere, but $\partial_+\phi$ does change signs when the line described by $\partial_+\phi = 0$ is crossed, which is the one described above. While the one in this model increases in size until meeting with the event horizon, it is possible to have mechanisms that generate shrinking apparent horizons, making previously trapped surfaces become free again; we shall see one such case in the following Section. The full process for the black hole formation is shown in Figure 3.3.

From this information we could go ahead and find the Hawking radiation emitted by this system. Given (3.22), we know that in conformal gauge and for a set of N

scalars f_i ($c = N$) we have

$$\langle (T^f)_{+-} \rangle = -\frac{N}{12}\partial_+\partial_-\rho. \tag{3.48}$$

Using the conservation of energy-momentum and with our knowledge of the metric, we have

$$\langle (T^f)_{\pm\pm} \rangle = -\frac{N}{12}\left(\partial_\pm\rho\,\partial_\pm\rho - \partial_\pm^2\rho + t_\pm\right), \tag{3.49}$$

where $t_\pm$ are some integration functions. We can fix these by imposing that the energy-momentum tensor vanishes in the infinite past of the vacuum region. In order to define this region, as well as the infinite future we need to study Hawking radiation, it is best to work in asymptotically flat coordinates, i.e. those for which the metric locally becomes that of flat spacetime in the regions studied. Before the gravitational collapse it is trivial to see that the solution becomes the linear dilaton vacuum once again. The conformal factor in terms of $x^\pm$ is

$$e^{2\rho} = -\frac{1}{\lambda^2 x^+ x^-}, \tag{3.50}$$

which can be plugged back into the conformal gauge definition to get

$$ds^2 = -\frac{dx^+}{\lambda x^+}\frac{dx^-}{-\lambda x^-} = -d\sigma^+ d\sigma^-, \tag{3.51}$$

where the sign is attached to the defined-negative x^-. We can integrate each asymptotically flat coordinate $\sigma^\pm$ to arrive at the simple relation

$$e^{\pm\lambda\sigma^\pm} = \pm\lambda x^\pm. \tag{3.52}$$

As for the region after the collapse, in the limit $x^+ \to \infty$ the term that goes with $m(x^+)$ can be dropped from (3.44). Following the same procedure as above, we may define

$$\begin{cases} e^{\lambda\hat{\sigma}^+} = \lambda x^+ \\ e^{-\lambda\hat{\sigma}^-} = -\lambda\left(x^- + \frac{P(x_f^+)}{\lambda}\right) \end{cases}. \tag{3.53}$$

In the infinite past, imposing that (3.49) vanishes leads to the identification $t^{\pm}(\sigma^{\pm}) = 0$. We may then make use of the transformation law of the energy-momentum tensor to bring $t_{\pm}$ to the $x^{\pm}$ coordinates before the f-shell collapses, resulting in

$$t_{\pm}(x^{\pm}) = \frac{1}{(2x^{\pm})^2}. \tag{3.54}$$

The same coordinates extend to the region above the shell, meaning that as we approach $x^{+} \to \infty$ we may approximate the outgoing flux by

$$\langle (T^f)_{--} \rangle_{\mathcal{I}_R^+} = \frac{N}{48} \left[\left(x^{-} + \frac{P(x_f^+)}{\lambda} \right)^{-2} - \frac{1}{(2x^{-})^2} \right]. \tag{3.55}$$

Finally, if we perform a change of coordinates into $\hat{\sigma}^{\pm}$, we arrive at the desired expression

$$\langle (T^f)_{--} \rangle_{\mathcal{I}_R^+} = \frac{N\lambda^2}{48} \left[1 - \frac{1}{\left(1 + \frac{P(x_f^+)}{\lambda} e^{\lambda \hat{\sigma}^{-}} \right)^2} \right]. \tag{3.56}$$

We can easily see that this flux vanishes as $\hat{\sigma}^{-} \to -\infty$ and tends to a constant as $\hat{\sigma}^{-} \to \infty$. The Hawking temperature associated with this radiation flow can be obtained by comparing the expression above with the general law for the energy flux emitted by a black body in $d = 2$, which satisfies $\Phi \propto NT^2$. The resulting temperature $T = \frac{\lambda}{2\pi}$ is independent of the mass of the black hole; unlike black hole solutions in 4D, where the temperature diverges as the mass of the black hole decreases, the temperature at which the CGHS solution radiates tends to a constant at late times, after a transition period. Additionally, we may notice that the only information that survives from the continuous flow of mass that formed the black hole, no matter how simple or complicated it was, is the final momentum $P(x_f^+)$.

3.2.3 The RST correction

If we were to integrate (3.56) over all $\mathcal{I}_R^+$, we would find that the total Hawking radiation emitted is infinite. However, this should come as no surprise, since the model in the previous Section essentially describes a black hole that emits indefinitely without ever losing its own energy, which is of course not a physically reasonable object. We need to consider the backreaction that the emission of radiation exerts on the black hole itself; while at early times this will not make a big difference, as the black hole evolves its contribution will become more and more notable.

The CGHS action (3.24) describes a fully classical spacetime, while Hawking radiation comes as a first order quantum effect due to the trace anomaly; as previously discussed, its contribution can be derived from an effective action, whose explicit form is given by the Polyakov term (3.23). If we considered an action of the form $S_{\text{CGHS}} + \Gamma_{\text{Polyakov}}$, we would obtain a 1-loop corrected system that could be used to study the backreaction; however, the resulting equations of motion are not solvable in terms of simple functions, having to rely on numerical methods [98, 99]. The RST (Russo-Susskind-Thorlacius) model provides an alternative redefinition of the action that solves this problem [100].

Before that, though, let us take a deeper look at the Polyakov term. As it is currently written, the operator $\Box^{-1}$ makes it so the action becomes nonlocal, since it involves finding the Green's function associated with the d'Alembertian, which requires knowledge of the entire spacetime. This nonlocality may lead to several complications down the line, but we may recover a local formulation by rewriting

$$\Gamma_{\text{Polyakov}}[g, \psi] = -\frac{c}{48\pi} \int d^2x \sqrt{-g} \left(\frac{1}{2} (\nabla\psi)^2 + \psi R \right), \tag{3.57}$$

where ψ is an auxiliary, massless scalar field; on the solution to its own equation of motion, we can recover the original formulation (3.23). The nonlocality of the Polyakov term will effectively be hidden behind the field ψ, since its value can only be fully determined by the boundary conditions of the model, which will be nonlocal in nature. Therefore, while not in the original formulation of the RST model, it will be of use to keep this auxiliary field.

The RST model proposes the addition of yet another extra term to the action, which takes the form

$$S_{\mathrm{RST}}[g,\phi] = -\frac{c}{48\pi}\int d^2x\sqrt{-g}\;\phi R. \tag{3.58}$$

The main reasoning behind this term is to preserve our ability to impose the Kruskal gauge on the (ρ, ϕ) fields, which will allow us to simplify and solve the equations of motion. The equations of motion for the full system will differ slightly from those obtained in the classical CGHS model. There will be a new equation for the auxiliary field ψ (which will just be $\Box\psi = R$), while the dilaton equation (3.27) receives a correction $R \to R\left(1 + \frac{c\,e^{2\phi}}{48}\right)$, and the equation derived from the metric (3.25) is modified to include the energy-momentum tensors of the Polyakov and RST terms; the matter sector equation will obviously see no changes. Combining the dilaton equation and the tracelessness condition for $T_{\mu\nu}$, we arrive at an equivalent relation

$$R + 2\,\Box\phi = 0, \tag{3.59}$$

which can then be plugged into the equation of motion for ψ to obtain

$$\Box(\psi + 2\phi) = 0. \tag{3.60}$$

This allows us to write $\psi = -2\phi + w$, where w is an arbitrary free field, that is, one that satisfies $\Box w = 0$.

In the same way that we did for the classical CGHS model, we may start the analysis of RST by finding its static solutions first. We could expect to once again find a configuration that reminds us of the Schwarzschild solution; in terms of the dilaton field, we may sketch this generic metric as [101]

$$ds^2 = -g(\phi)\, dt^2 + \frac{h^2(\phi)}{g(\phi)}\, d\phi^2, \tag{3.61}$$

which can in turn be written in the more familiar form

$$ds^2 = -g(x)dt^2 + \frac{dx^2}{g(x)}, \tag{3.62}$$

provided that x is defined such that $\frac{dx}{d\phi} = h(\phi)$. The condition (3.59) can be seen to imply

$$\left(\frac{g'}{h}\right)' = 2\left(\frac{g}{h}\right)' \Longrightarrow g' = 2g - \chi h, \tag{3.63}$$

where the $'$ denotes differentiation with respect to ϕ, and χ is an integration constant that can be fixed to $\chi = -2\lambda$ by imposing that the metric becomes asymptotically flat at infinity (in particular, that it reduces to the linear dilaton vacuum) and using the dilaton equation of motion. We may integrate the expression above to get

$$g(\phi) = G(\phi)e^{2\phi}, \tag{3.64}$$

where the function $G(\phi)$ satisfies $G'(\phi) = 2\lambda h(\phi)e^{-2\phi}$. The dilaton equation can then be solved in terms of this $G(\phi)$ function to give

$$\frac{c}{24} - 2e^{-2\phi} = \left(1 + \frac{A}{G}\right) G' \tag{3.65}$$

and, integrating once more,

$$\Omega(\phi) := e^{-2\phi} + \frac{c}{24}\phi = G + A \log G + \frac{M}{\lambda}, \tag{3.66}$$

where A and M are integration constants. Similarly to the case of the CGHS model, M will be identified as the mass of the black hole solution.

The free field equation for w can be expressed in terms of g and h as $w' = C\dfrac{h}{g}$. After introducing all the pieces into the energy-momentum tensor equation and performing some algebraic manipulations, we see that the integration constants A and C must be related by

$$A = \frac{c}{192\lambda^2} C(C + 4\lambda). \tag{3.67}$$

Therefore, the most general static solution to the complete RST model equations of motion is given by the "Schwarzschild-like" metric defined by $g(\phi)$ and $h(\phi)$, where both of them depend on the function $G(\phi)$, in turn implicitly fixed by (3.66) up to a constant C (or equivalently A). In order to study what this constant C entails, let us first write the explicit expressions for the energy-momentum tensor associated with the quantum corrections introduced by the Polyakov effective action

$$(T_{\mu\nu})_{\text{Polyakov}} = \frac{c}{24}\left[\frac{1}{2}\nabla_\mu\psi\nabla_\nu\psi - \nabla_\mu\nabla_\nu\psi + g_{\mu\nu}\left(\Box\psi - \frac{1}{4}\nabla_\mu\psi\nabla^\mu\psi\right)\right], \tag{3.68}$$

and the RST term

$$(T_{\mu\nu})_{\text{RST}} = \frac{c}{24}(g_{\mu\nu}\Box\phi - \nabla_\mu\nabla_\nu\phi), \tag{3.69}$$

since those are the new pieces whose asymptotic behaviour we need to determine. Using (t, ϕ) coordinates, we may see that in the limit $t \to \infty$ (and therefore $g(\phi) \to 1$ due to the metric being asymptotically flat) we get

$$(T_{tt})_{\text{Polyakov}} + (T_{tt})_{\text{RST}} \to \frac{c}{96}(C + 2\lambda)^2. \tag{3.70}$$

Therefore, by setting $C = -2\lambda$ (or equivalently $A = -\frac{c}{48}$), we arrive at a solution whose energy density at infinity vanishes. This corresponds to a so-called Boulware vacuum state. By imposing instead that the energy density is that of a thermal state given by a black body which, as we saw in the previous Section, for a CGHS background has an associated temperature $T = \frac{\lambda}{2\pi}$, we obtain a Hartle-Hawking type of solution. This state is found when $A = 0$, which corresponds to two possible values of C, namely $C = 0$ and $C = -4\lambda$. These two choices give rise to different definitions of w and ψ that may lead to different conclusions regarding the entropy of the black hole solution [102]. Nevertheless, these differences between the two possible Hartle-Hawking states lay outside the scope of this work, and so will not be discussed.

All in all, we may regard the parameter C (or A) as a way to "select" the quantum state of the system from which we observe the solutions of the equations of motion. From our static solution we can reach the Boulware and Hartle-Hawking states, as well as infinitely many others given by the values these constants can take. As we shall promptly see, it is not possible to reach the other most famous quantum state, the Unruh state, by merely changing the value of A. This is to be expected since the Unruh state cannot correspond to a fully static solution of the model.

Let us now write the metric in a conformally flat way. This can be done if we define the coordinates

$$x^{\pm} = \pm\frac{1}{\lambda}\exp\left(\pm\lambda\left[t \pm \frac{1}{2\lambda}\log G(\phi)\right]\right), \tag{3.71}$$

in terms of which the metric becomes (3.18) with $\rho = \phi$, that is, in the Kruskal gauge. In terms of these coordinates, the solution (3.66) is given by

$$\Omega(\phi) := e^{-2\phi} + \frac{c}{24}\phi = -\lambda^2 x^+ x^- + A\log(-\lambda^2 x^+ x^-) + \frac{M}{\lambda}, \tag{3.72}$$

which differs from (3.36) by the new A-dependent term. The coordinates $x^{\pm}$ have the same domain as before, that being $-\infty < x^- < 0 < x^+ < \infty$.

In the Kruskal gauge, the free field equation for w may be solved as

$$w(x^+, x^-) = w_+(x^+) + w_-(x^-), \tag{3.73}$$

where the functions $w_{\pm}$ will be fixed by the boundary conditions (i.e. by the choice of quantum state). Using this, we can write the equation of motion associated with

$S_{\text{CGHS}} + \Gamma_{\text{Polyakov}} + S_{\text{RST}}$ (now with the matter term that depends on f_i), as well as the conformally flat metric constraints, as

$$\begin{cases} T_{\pm\pm} = 0 = 2e^{-2\phi}[2(\partial_\pm\phi)^2 - \partial_\pm^2\phi] + \dfrac{c}{24}\left(\partial_\pm^2\phi - 2t_\pm\right) + \dfrac{1}{2}(\partial_\pm f_i)^2 \\ \\ T_{+-} = 0 = \left(2e^{-2\phi} - \dfrac{c}{24}\right)\partial_+\partial_-\phi - 4e^{-2\phi}\partial_+\phi\partial_-\phi - \lambda^2 \end{cases}, \tag{3.74}$$

where we have introduced the functions (each prime now representing a derivative of the function with respect to its respective argument)

$$t_\pm = \frac{1}{2}w_\pm'' - \frac{1}{4}(w_\pm')^2, \tag{3.75}$$

in analogy with the $t_\pm$ functions of the previous Section.

Inspired by the static case, we may look for a solution of these equations in terms of $\Omega(\phi) := e^{-2\phi} + \dfrac{c}{24}\phi$. Plugging this definition back into (3.74) we obtain the equations

$$\begin{cases} T_{\pm\pm} = 0 = \partial_\pm^2\Omega - \dfrac{c}{12}t_\pm + \dfrac{1}{2}(\partial_\pm f_i)^2 \\ T_{+-} = 0 = -\partial_+\partial_-\Omega - \lambda^2, \end{cases} \tag{3.76}$$

the second of which can be solved as

$$\Omega = -\lambda^2 x^+ x^- + u_+(x^+) + u_-(x^-) \tag{3.77}$$

and introduced into the first one to obtain

$$u_\pm'' = \frac{c}{12}t_\pm - \frac{1}{2}(\partial_\pm f_i)^2. \tag{3.78}$$

A quick comparison with (3.72) lets us see that, for a static solution,

$$u_+(x^+) + u_-(x^-) = A\log(-\lambda^2 x^+ x^-) + \frac{M}{\lambda}, \tag{3.79}$$

which immediately implies that the functions $t_\pm$ are given by

$$t_\pm = -\frac{A}{12c(x^\pm)^2}. \tag{3.80}$$

The fact that they both take the same form is a property exclusive to static solutions; it will not be the case in general, when we have a dynamic solution. To see this, let us once again consider the value of the energy-momentum tensor as we approach the asymptotically flat regions. The asymptotically flat coordinates are once again given by (3.52); using these coordinates, we may find that in $\mathcal{I}_R^\pm$ the energy-momentum tensor components become

$$\langle T_{\pm\pm}\rangle_{\mathcal{I}_R^\pm} = \frac{\lambda^2 c}{48}\left(1 + \frac{48A}{c}\right), \tag{3.81}$$

while the $(+-)$ component vanishes identically. We can immediately see that, as expected, the expression above vanishes for the Boulware state $A = -\frac{c}{48}$, while for the Hartle-Hawking states $A = 0$ we obtain the energy density expected of a thermal gas at temperature $T = \frac{\lambda}{2\pi}$ (specifically, we get only half of it, since each component accounts for left- or right-moving particles, which contribute in the same way to the total radiation). However, the Unruh state corresponds to a state for which there is no incoming radiation at $\mathcal{I}_R^-$ and an outgoing thermal flux at $\mathcal{I}_R^+$. In other words, we would need A to take on each of these values at each infinity, which leads to the result

$$t_+(x^+) = \frac{1}{(2x^+)^2}, \qquad t_-(x^-) = 0, \tag{3.82}$$

and the (no longer static) solution

$$\Omega(\phi) = -\lambda^2 x^+ x^- - \frac{c}{48}\log(\lambda x^+) + \frac{M}{\lambda}. \tag{3.83}$$

To conclude this Section, we shall now briefly study how the solutions behave after we once again introduce a shell of incoming null matter,

$$\Omega(\phi) = e^{-2\phi} + \frac{c}{24}\phi = \Omega_0(\phi) - \frac{m}{\lambda x_0^+}(x^+ - x_0^+)\Theta(x^+ - x_0^+), \tag{3.84}$$

where Ω_0 will refer to either (3.72) or (3.83) for different states. The curvature of spacetime associated with these solutions will be

$$R = 8e^{-2\phi}\partial_+\partial_-\phi = \frac{8e^{-2\phi}}{\Omega'}\left(\partial_+\partial_-\Omega - \frac{\Omega''}{(\Omega')^2}\partial_+\Omega\,\partial_-\Omega\right), \tag{3.85}$$

meaning the singularity will be found wherever $\Omega' = 0$. As for the apparent horizon, it will still lie on the curve where $\partial_+\phi = 0$.

The Hartle-Hawking solution becomes relatively similar to that of the classical CGHS model

$$\Omega(\phi) = e^{-2\phi} + \frac{c}{24}\phi = -\lambda^2 x^+ x^- + \frac{M}{\lambda} - \frac{m}{\lambda x_0^+}(x^+ - x_0^+)\Theta(x^+ - x_0^+), \tag{3.86}$$

the only difference being the term linear in ϕ. Before the f-shell comes into play, there is a singularity along the curve

$$x^+x^- = \frac{M}{\lambda^3} - \frac{c}{48\pi\lambda^2}\left(1 - \log\frac{c}{48\pi}\right), \tag{3.87}$$

which is displaced from its original position in the CGHS model by the corrections introduced through the term that is dependent on c. Since the event horizon (which for a static black hole coincides with the apparent horizon) is still located at the null lines $x^+x^- = 0$, the singularity becomes naked if the mass M is smaller than the critical value

$$M_{\text{crit}} = \frac{\lambda c}{48}\left(1 - \log\frac{c}{48\pi}\right). \tag{3.88}$$

The existence of a naked singularity goes against the cosmic censorship hypothesis in its original form, but might be allowed in more modern interpretations under some conditions [103]. Note that if c grows sufficiently large then $M_{\text{crit}} < 0$ and any value of $M > 0$ will give a singularity that is hidden behind the horizon. For a set of N massless scalar fields this would happen at $c = N \geq 131$; this is one of the reasons why, in its original formulation, the RST formalism is studied in the large N limit. We will discuss another reason behind this choice in the next Section.

In any case, after the shell of null energy collapses into the black hole, we may notice that (3.86) can be rewritten as

$$\Omega(\phi) = e^{-2\phi} + \frac{c}{24}\phi = -\lambda^2 x^+ \left(x^- + \frac{m}{\lambda^3 x_0^+} \right) + \frac{M+m}{\lambda}. \tag{3.89}$$

This is the exact same black hole solution, with the mass increased to $\hat{M} = M + m$ and the value of x^- shifted by a constant. The singularity and apparent horizon get shifted accordingly, but no other relevant new behaviours appear; even the Hawking radiation at the infinite future remains unchanged, since we know it is independent of the mass of the black hole in this formalism.

Moving on to the Boulware state solution, we have

$$\Omega(\phi) = -\lambda^2 x^+ x^- - \frac{c}{48}\log(-\lambda^2 x^+ x^-) + \frac{M}{\lambda} - \frac{m}{\lambda x_0^+}(x^+ - x_0^+)\Theta(x^+ - x_0^+). \tag{3.90}$$

Once again, let us see the spacetime before the null shell collapses first. Briefly commenting on the $M = 0$ case, the solution to this particular case is once again a simple dilaton vacuum (or, as discussed in Section 3.2.2, the quadrant of it that the coordinates $x^\pm$ cover, which may be brought into the entire spacetime by defining the coordinates $\sigma^\pm$). For $M > 0$, the singularity of this solution will be found along the curve

$$\frac{48}{c}(-\lambda^2 x^+ x^-) - \log\left(\frac{48}{c}(-\lambda^2 x^+ x^-)\right) = 1 - \frac{48M}{c\lambda}. \tag{3.91}$$

This curve actually lies outside of the range of $x^\pm$ if the central charge is positive; incidentally, for fields that satisfy $c < 0$, such as the ones we will discuss in the next Section, we may express this curve in terms of the Lambert W function, which satisfies $W(z)e^{W(z)} = z$. The resulting expression is

$$x^+x^- = \frac{c}{48\lambda^2}\, W_k\left(-\exp\left(\frac{48M}{c\lambda} - 1\right)\right), \qquad k = 0, -1, \tag{3.92}$$

with $W_0(z), W_{-1}(z)$ being the two real branches of the Lambert function when $z < 0$. We will see the significance of these branches momentarily, but for the time being let us come back to the usual $c > 0$ case.

In the limit $x^+x^- \to 0$, which corresponds to $\phi \to -\infty$, we may approximate

$$\Omega(\phi) \approx e^{-2\phi} \approx -\log(-\frac{c}{48}\lambda^2 x^+x^-) \tag{3.93}$$

and find that the curvature diverges to $R \to -\infty$ as we approach the limit. Thus, we conclude that there is a null singularity that follows the axes $x^\pm = 0$. Trying to find the apparent horizon by setting $\partial_\pm\Omega = 0$ gives a curve

$$\lambda^2 x^+x^- = -\frac{c}{48}, \tag{3.94}$$

which does lie in the spacetime region covered by $x^\pm$. However, upon closer inspection we may see that the sign of $\partial_\pm\Omega$ does not actually change when we move away from this curve in each direction, meaning we cannot classify this curve as an apparent horizon. Instead, the line describes the throat of a wormhole; going back to the idea of the CGHS model as a 2D section of a spherically symmetric 4D spacetime, we may see that the throat would represent a local minimum for the radius of the 2-spheres located at each point in the CGHS spacetime. As the fields are defined to be in the Boulware state, there is no Hawking radiation to be found in the asymptotic region of spacetime. From outside the throat of the wormhole, there is no fundamental difference between the metric described by this solution and that of a static star of the same size.

Let us now look at the region after the massless matter shell collapses. For simplicity sake, we will focus on the case where $M = 0$; the addition of M will not change the dynamics of the spacetime after the collapse in any way, beyond the singularity at $x^{\pm} = 0$ and the redefinition of the mass and shift of the coordinates by a constant factor that was already seen in the Hartle-Hawking case. As discussed above, the singularity will be found along the curve given by

$$\frac{48}{c}(-\lambda^2 x^+ x^-) - \log\left(\frac{48}{c}(-\lambda^2 x^+ x^-)\right) = 1 + \frac{48m}{c\lambda}\left(\frac{x^+}{x_0^+} - 1\right), \tag{3.95}$$

where the only change when compared to (3.91) is the mass-related term. This time, the solution exists entirely within the range of $x^{\pm}$. The two branches of the curve will once again be given by the Lambert function

$$\lambda^2 x^+ x^- = \frac{c}{48} W_k\left(-\exp\left(\frac{48m}{c\lambda}\left[1 - \frac{x^+}{x_0^+}\right] - 1\right)\right), \qquad k = 0, -1. \tag{3.96}$$

If we consider this as the definition of two different curves $x^-(x^+)$, we may in turn see those as two solutions that branch out to each side of the curve $\lambda^2 x^+ x^- = -\frac{c}{48}$, which is the only solution of (3.96) at the point $x^+ = x_0^+$ and also coincides with the "innermost" of the two apparent horizons that can also be found. This horizon would be the continuation of the wormhole throat in the $M > 0$ case, showing that the wormhole destabilizes after the matter shell falls in. The upper branch ($k = 0$) of the singularity is always timelike and it rapidly approaches $x^- = 0$, while the lower branch ($k = -1$) starts as a spacelike object as it expands outwards, until it eventually crosses the outermost horizon given by $\lambda^2 x^+ \left(x^- + \frac{m}{\lambda x_0^+}\right) = -\frac{c}{48}$ in a finite proper time [104]. It is at this point that the singularity becomes timelike and naked, implying that our semiclassical prescription will most likely break down in some way. Figure 3.4 shows the Penrose diagram for the spacetime accessible to an outside observer, as well as the region causally connected to the lower branch of the singularity; for the remainder of this document we will focus on this region of spacetime.

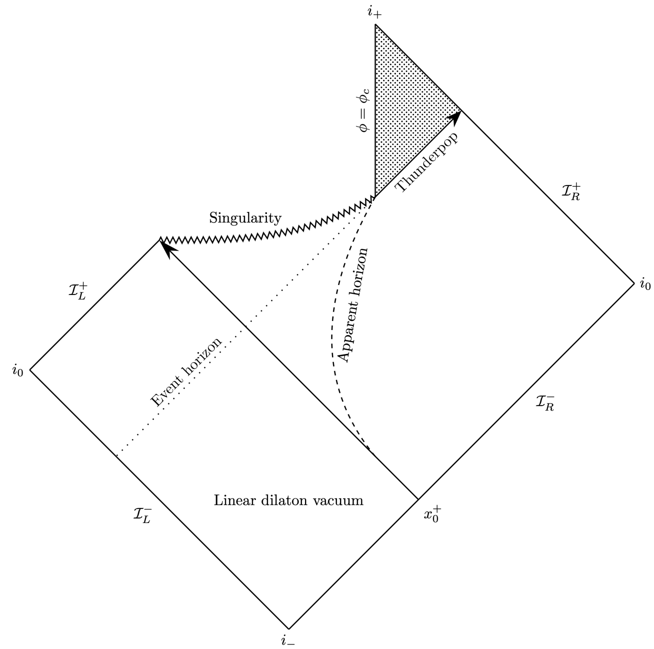

Figure 3.4: Evolution of the RST black hole, from its original formation up to the creation of a naked singularity. The greyed out area at the top is determined by the boundary conditions after the collision; some possible constructions force the emission of a "thunderpop" of energy from the point of merging to the null future infinity. In the original formulation, after the thunderpop is emitted the apparent horizon curve is extended as the (timelike) reflective boundary line $\phi = \phi_c$.

The exact behaviour of the system after the merging of the singularity and the apparent horizon cannot be derived from the RST system alone. First of all, the energies reached in the proximities of the singularity way surpass those where semi-classical approximations are thought to remain applicable. As such, the RST model can only be taken as-is up until the point where the singularity becomes naked. Beyond that point (and arguably, even before that), until we have a more accurate description of singularities and Quantum Gravity as a whole, the best we can do is resort to some sensible phenomenological boundary conditions in order to patch the rest of spacetime up to the future infinities. One such particularly interesting proposal is considering that, after the singularity merges with the apparent horizon, it immediately disappears, leaving behind only the linear dilaton vacuum again. This is actually compatible with the solution (3.90) because ϕ takes a vacuum value along the x^- line that passes through the merging point. The derivatives of ϕ, however, are not continuous on the boundary between the black hole and vacuum sections. This translates into a breaking of energy-momentum conservation when crossing from a region to another.

We can also see this by finding out the radiated flux of the black hole at the future infinity and integrating (only up to the value of the asymptotically flat coordinate $\hat{\sigma}^-$ that corresponds to the merging point) in order to find the total energy emitted by the black hole until the singularity becomes timelike. In order to match the total initial energy given by m to the final energy after the singularity disappears, the black hole has to emit a finite amount of null negative energy from the point of merging into the future null infinity. This phenomenon is commonly know as a thunderpop, in contrast to the thunderbolt event proposed by Hawking [105], which consists of an infinitely large burst of null negative energy that a naked singularity emits when it forms, breaking the unitarity of the theory altogether.

A similar (qualitatively speaking) result is found when studying the fields in an Unruh state. Given the form of (3.83), we find that the singularity at $x^+ = 0$ and the apparent horizon at $\lambda^2 x^+ x^- = -\dfrac{c}{48}$ from the Boulware solution remain, while the $x^- = 0$ singularity becomes a new apparent horizon (given that the Unruh solution does not have an x^- dependence in the logarithm, there is nothing that prevents us from performing an extension of this coordinate into positive values and studying the behavior of Ω in this region in order to confirm this is the case). The other singularity may be found once again from imposing $\Omega'(\phi) = 0$; the resulting curve crosses both horizons at specific points and, while it remains spacelike in the region $x^- > 0$, after colliding with the "outer" horizon it once again becomes timelike and naked. The results obtained in this Section can be further generalized to include more realistic infalling energy flows like we did in Section 3.2.2.

3.2.4 Negative central charges

The RST model is perhaps the most well-known solvable semiclassical model for black hole evolution. It is, however, not free of concerns; while the thunderpop event can be attributed to the intrinsic limitations of semiclassical theories (since its energy is comparable to the Planck scale in dilaton gravity, it may well be that the thunderpop disappears when a more complex quantum gravity model is applied), there is also the inevitable thunderbolt that ultimately breaks the unitarity of the theory, as well as energy conservation [106]. This phenomenon is intrinsically tied with the existence of naked singularities and is model-independent; therefore, as long as we do not fully understand singularities we will have no way of dealing with the later stages of black hole evolution.

Before that, though, there are a couple other more immediate concerns that need to be addressed. As it stands, the term (3.58) has been introduced as a way to eliminate the nonlinearities of the equations of motion and obtain a solvable model. It does so by preserving the Kruskal gauge that allows us to set $\rho = \phi$ in the conformal gauge for the metric. But this setup feels somewhat *ad hoc* in that there is no specific reason behind the choice beyond mathematical solvability. One could argue that there are other equally valid ways to reformulate the equations; a well-known alternative to the RST formulation is given by the Bose-Parker-Peleg (BPP) construction [107], in which the chosen counterterm is instead

$$S_{\mathrm{BPP}}[g,\phi] = \frac{c}{24\pi}\int d^2x\sqrt{-g}\,((\nabla\phi)^2 - \phi R), \tag{3.97}$$

in order to maintain the explicit conformal symmetry of CGHS. Solving the equations of motion of $S_{\mathrm{CGHS}} + \Gamma_{\mathrm{Polyakov}} + S_{BPP}$, we arrive at solutions that do not develop a permanent naked singularity once the initial thunderpop is emitted, leaving behind a so-called "remnant" instead (which somehow stores all the information of the matter that formed the black hole that was not emitted by Hawking radiation, "solving" the information paradox). Later results show that these two models are part of a larger one-parameter family of possible counterterms [108]

$$S_{\text{local}}[g, \phi] = \frac{c}{24\pi} \int d^2x \sqrt{-g} \, [(1-2a)(\nabla\phi)^2 + (a-1)\phi R]. \tag{3.98}$$

We may quickly check that $a = 0$ corresponds to the BPP model, while $a = \frac{1}{2}$ leads to the RST one. All of these models can be solved without recurring to numerical methods; in the case of a single shell of infalling null matter, the solution reads (with M and m being the masses / energies associated with the original static solution and the shell, respectively)

$$e^{-2\phi} + \frac{ac}{12}\phi = -\lambda^2 x^+ x^- - \frac{c}{48}\log(-\lambda^2 x^+ x^-) + \frac{M}{\lambda} - \frac{m}{\lambda x_0^+}(x^+ - x_0^+)\Theta(x^+ - x_0^+). \tag{3.99}$$

The general construction of a singularity and an apparent horizon that eventually meet due to the backreaction of Hawking radiation is the same. Coupling the "end-point" of this dynamical solution to a new static one, whose mass $\hat{M} \neq M$ in general, gives rise to the thunderpop phenomenon in order to preserve energy conservation. The particular models with $a = 0$ and $a = \frac{1}{2}$ are usually preferred because they allow to set $\hat{M} = M$ while still being compatible with energy conservation.

There is yet another wrinkle in these constructions that needs to be ironed. As already pointed out, the classical CGHS model has two main symmetries that allow us to work out the solutions to the equations of motion (namely, diffeomorphism and conformal invariance). In constructing these models, we should account for these symmetries during the quantization process; for example, the path integral of S_{CGHS} is integrating over many different metrics that should be considered the same due to symmetries. This is usually enforced through the introduction of a Faddev-Popov $b - c$ system of ghost fields [109] that gets rid of the redundancy. However, introducing new fields will of course inevitably modify the central charge of the system; the ghost fields, in particular, have an assigned $c = -26$. Additionally, we may include the ϕ and ρ fields, as they are additional degrees of freedom of the theory; each one of them contributes with $c = 1$.

The most straightforward way of introducing these fields into the RST (or any equivalent) model is through a direct modification $c \to c - 24$. However, doing so would imply that ϕ, ρ and the ghost fields somehow are also included in the Hawking radiation and become observable at the future infinity. In most relevant applications of the RST model, the limit $N \to \infty$ is taken so that this contribution can be safely neglected. We propose instead an alternative process that will allow us to work for arbitrary N, which is to consider that the reparametrization fields are not coupled to the metric $g_{\mu\nu}$, but to the conformally-equivalent, on-shell flat auxiliary metric $\hat{g}_{\mu\nu} = e^{-2\phi} g_{\mu\nu}$ [110]. In other words, the effective action term associated with these fields will once again be given by a Polyakov action, this time of the form

$$\Gamma^{-}_{\text{Polyakov}}[\hat{g}] = -\frac{c^{-}}{96\pi} \int d^2x \sqrt{-g}\; R(\hat{g}) \Box^{-1}_{\hat{g}} R(\hat{g}), \tag{3.100}$$

where $c^{-} < 0$ (we will also rename $\Gamma_{\text{Polyakov}} \to \Gamma^{+}_{\text{Polyakov}}$ and $c \to c^{+} > 0$ for the regular fields from here on for ease of distinction). All in all, we may consider that the full effective action for the CGHS model is given by

$$\Gamma_{\text{total}}[g, \hat{g}, \phi, f] = S_{\text{CGHS}}[g, \phi, f] + \Gamma^{+}_{\text{Polyakov}}[g] + \Gamma^{-}_{\text{Polyakov}}[\hat{g}] + S_{\text{local}}[g, \phi], \tag{3.101}$$

which we can write in terms of $g_{\mu\nu}$ as

$$\begin{aligned} \Gamma_{\text{total}}[g, \phi, f] = S_{\text{CGHS}} &- \frac{c^{+} + c^{-}}{96\pi} \int d^2x \sqrt{-g}\; R \Box^{-1} R \\ &+ \frac{c^{+}}{24\pi} \int d^2x \sqrt{-g} \left[(1 - 2a)(\nabla\phi)^2 + (a - 1)\phi R\right] \\ &+ \frac{c^{-}}{24\pi} \int d^2x \sqrt{-g} \left[(\nabla\phi)^2 - \phi R\right]. \end{aligned} \tag{3.102}$$

We can work out the equations of motion in Kruskal gauge and find solutions in a similar manner as the previous Sections. The static solution obtained considering that the matter fields are in the Boulware state will be

$$\Omega(\phi) = e^{-2\phi} + \frac{a\, c^{+}}{12}\phi = -\lambda^2 x^{+} x^{-} - \frac{c^{+} + c^{-}}{48} \log(-\lambda^2 x^{+} x^{-}) + \frac{M}{\lambda}. \tag{3.103}$$

The parameter a can be fixed with some physically motivated conditions. In particular, if we enforce that the linear dilaton vacuum remains a solution at both the classical and 1-loop corrected level, we need to impose, in addition to $M = 0$, the specific value

$$a = \frac{c^+ + c^-}{2\, c^+}. \tag{3.104}$$

We can quickly check that the case $c^+ \gg c^-$ (or simply $c^- = 0$) reduces to RST as it was formulated above.

There are now two new scenarios to study; first of all and as consistency check, if $c^+ = 0$ (we still may assume that there is a nonvanishing classical stress-energy tensor, but there is no trace anomaly associated with it), then the equations of motion can be written in terms of the more relevant metric $\hat{g}$, which in conformal gauge reads as $\hat{g}_{+-} = -\frac{1}{2}e^{2(\rho-\phi)}$. In Kruskal gauge, this translates into $\hat{g}$ taking once again the form of the linear dilaton vacuum spacetime. Hence, there is no singularity, no apparent horizon and no Hawking radiation process at all. This falls in line with our previous conditions with regards to the ghost fields.

As for the more general case $c^+ + c^- < 0$, we may see that the solution for a single shell of infalling matter will be

$$\begin{aligned}\Omega &= e^{-2\phi} + \kappa\phi \\ &= -\lambda^2 x^+ x^- - \frac{\kappa}{2}\log(-\lambda^2 x^+ x^-) - \frac{m}{\lambda x_0^+}(x^+ - x_0^+)\Theta(x^+ - x_0^+),\end{aligned} \tag{3.105}$$

where we have defined $\kappa = \dfrac{c^+ + c^-}{24} \leq 0$ for future convenience. This is where we present the main result for this model: once again, the singularity, which would lie on the curve $\Omega' = 0$, is absent since the dilaton would need to satisfy $e^{-2\phi} = \dfrac{\kappa}{2} < 0$. However, there is still an apparent horizon on the area past x_0^+ that follows the curve

$$\lambda^2 x^- = -\frac{\kappa}{2x^+} - \frac{m}{\lambda x_0^+}. \tag{3.106}$$

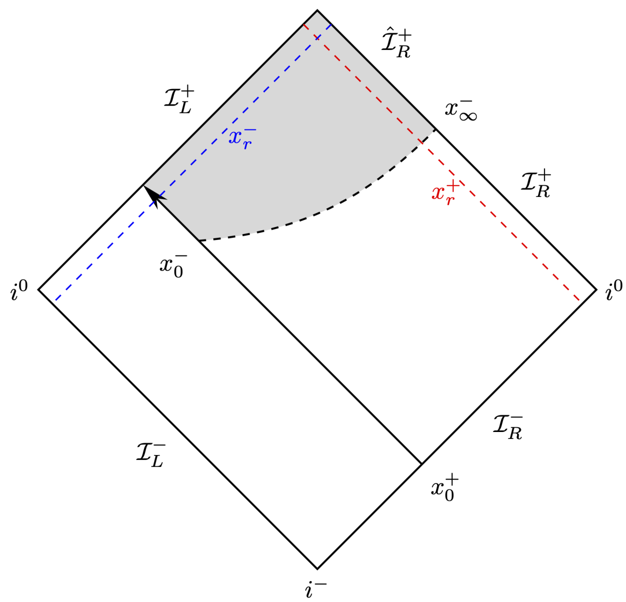

Figure 3.5: Evolution of the system for $c^+ + c^- < 0$; after the null shell of infalling matter passes, there is an apparent horizon that expands up to a constant value. In order to parametrize the flux in the future asymptotic regions, we may work on an arbitrary null reference geodesic $x_r^\pm$ and take the appropriate limit.

Notice that, due to the sign of κ, the horizon originates at

$$x_0^- = -\frac{1}{\lambda^2 x_0^+}\left[\frac{m}{\lambda} + \frac{\kappa}{2}\right] \tag{3.107}$$

and then expands as $x^+ \to \infty$ to $x_\infty^- = -\dfrac{m}{\lambda^3 x_0^+}$. In order for x_0^- to lie inside the range of x^-, it is necessary that $\dfrac{m}{\lambda} > \dfrac{|\kappa|}{2}$; the Penrose diagram for this situation is sketched in Figure 3.5, and for the purposes of this document that will be the only case we will consider. The question of what exactly happens when the opposite is true remains still open. One possibility is that the apparent horizon simply appears whenever the curve defined by (3.106) reaches values of x^- that do lie in the established domain; it might be of interest for future studies to see what this "retarded" horizon formation physically implies.

There are three asymptotic regions of interest in this system, denoted in the Penrose diagram. All of them are located above the matter collapse at x_0^+, since the $\mathcal{I}_L$ region before the collapse will just be the linear dilaton vacuum once again. Therefore, we may take the solution to be given by

$$\Omega = e^{-2\phi} + \kappa\phi = -\lambda^2 x^+ \left(x^- + \frac{m}{\lambda x_0^+} \right) - \frac{\kappa}{2} \log(-\lambda^2 x^+ x^-) + \frac{m}{\lambda} \tag{3.108}$$

for the remaining of this Section. We may try to utilize the same arguments as for the CGHS and RST models in order to describe the energy flux in the outside region $\mathcal{I}_R$ by defining appropriate asympotically flat coordinates, which before the collapse will be $\sigma^\pm$ associated with the linear dilaton vacuum we covered in Section 3.2.2 and after the collapse will be given by

$$ds^2 \sim -\frac{dx^+ dx^-}{-\lambda^2 x^+ (x^- - x_\infty^-)} = -d\sigma^+ \frac{dx^-}{-\lambda(x^- - x_\infty^-)} = -d\sigma^+ d\hat{\sigma}^-, \tag{3.109}$$

and evolving the energy-momentum tensor from before the matter collapse to the future asymptotic region. In doing so, we find that the region is complete (meaning, the asymptotically flat coordinate $\hat{\sigma}^-$ covers the entire $(-\infty, \infty)$ range), unlike the RST model. The resulting energy flux is once again that of a Hawking radiation process of the form (3.56), with N being replaced by c^+ (as a reminder, by construction we should not see c^- in the Hawking radiation process).

The interior region $\hat{\mathcal{I}}_R$ is undoubtedly more complicated to treat. At the time of writing this document, no suitable set of asymptotically flat coordinates has been found. The main complication seems to be that, while it is possible to approximate $\Omega \sim \kappa\rho$ for $x^- \to 0$ and find a set of asymptotically flat coordinates in this way, this approximation does not hold as satisfactorily as we approach $x^- = x_\infty^-$. As such, the results obtained in this way are not to be taken as an accurate representation of the real system. In order to get a more illustrative representation of the behavior of the interior regions, we will follow a slightly different approach. We will first illustrate this for the $\mathcal{I}_L$ region, since it will prove to be more approachable.

The main idea behind this process is to find the (ingoing) flux $\langle T_{++}\rangle$ that reaches an arbitrary reference null geodesic x_r^-. This can be done comfortably in terms of the (x^+, x^-) coordinates by using the equations of motion of the system as done in previous Sections; the result of said calculation is given by

$$\langle T_{x^+x^+}\rangle(x^+, x_r^-) = -\frac{c^+}{12}\left[\frac{\lambda^4}{\kappa^2}(x_r^- - x_\infty^-)^2 + \frac{\lambda^2}{\kappa x^+}(x_r^- - x_\infty^-)\right]. \tag{3.110}$$

Then, in order to study the asymptotic region, we will be taking the limit where $x_r^- \to 0$. However, in order to determine the actual physical flux we need to connect with the asymptotically flat coordinates of $\mathcal{I}_L$ when performing this limit; this can most straightforwardly be done by reevaluating the above flux in terms of the appropriate affine parameter, along the x_r^- geodesic:

$$\langle T^f_{\hat{\sigma}^+\hat{\sigma}^+}\rangle \equiv \langle T^f_{x^+x^+}\rangle(x^+, x_r^-)\left(\frac{dx^+}{d\hat{\sigma}^+}\right)^2_{x^-=x_r^-}. \tag{3.111}$$

In the above expression we have kept the $\hat{\sigma}$ notation to avoid further confusion. The value of $\left(\frac{dx^+}{d\hat{\sigma}^+}\right)_{x^-=x_r^-}$ can be explicitly derived from the geodesic equation (3.4), which in this context takes the simple form

$$\frac{d^2x^+}{d\hat{\sigma}^{+2}} + \Gamma^+_{++}\frac{dx^+}{d\hat{\sigma}^+}\frac{dx^+}{d\hat{\sigma}^+} = 0 \ . \tag{3.112}$$

Making contact with the expected solution before the null shell of matter collapses, i.e. imposing that

$$\left.\frac{dx^+}{d\hat{\sigma}^+}\right|_{x^+=x_0^+} = \left.\frac{dx^+}{d\sigma^+}\right|_{x^+=x_0^+} = \lambda x_0^+, \tag{3.113}$$

it is possible to find a unique solution for $\frac{dx^+}{d\hat{\sigma}}$, which allows us to finally derive

$$\langle T^f_{\hat{\sigma}^+\hat{\sigma}^+}\rangle = \frac{c^+\lambda m}{12\kappa^2}\frac{x^+}{x_0^+}\left(\kappa + \frac{m}{\lambda}\frac{x^+}{x_0^+}\right)\exp\left[-\frac{4m}{\kappa\lambda}\left(1 - \frac{x^+}{x_0^+}\right)\right], \tag{3.114}$$

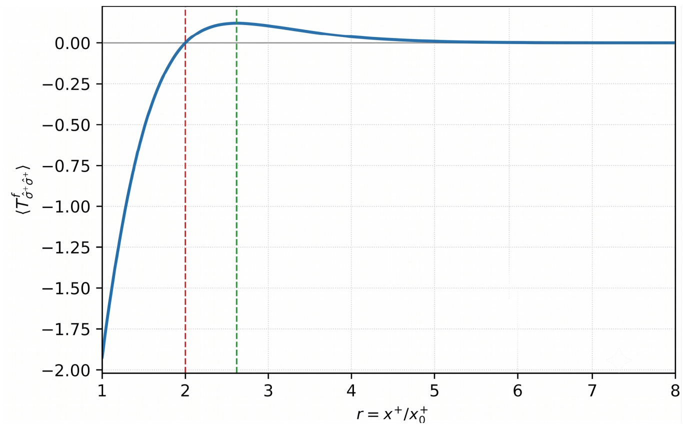

Figure 3.6: Energy-momentum flux that reaches $\mathcal{I}_L$, as a function of $\dfrac{x^+}{x_0^+}$. For $\kappa < \dfrac{m}{\lambda}$, there is both a negative as well as a negative contribution. When integrating over the entire asymptotic region, these contributions may cancel each other in the specific case $\kappa = -2\dfrac{m}{\lambda}$.

where x^+ should be understood as $x^+(\hat{\sigma}^+)$, for the region above the collapsing shell. The sign of the flux depends on the combination $\kappa + \dfrac{m}{\lambda}\dfrac{x^+}{x_0^+}$; Figure 3.6 shows the behavior of $\langle T^f_{\hat{\sigma}^+\hat{\sigma}^+}\rangle$ when $\kappa < -\dfrac{m}{\lambda}$. It gains a negative value at $x^+ = x_0^+$, which then turns positive and exponentially decays as $x^+ \to \infty$. For a system with $\kappa \geq -\dfrac{m}{\lambda}$, the flux remains nonnegative for all values of x^+. We may furthermore calculate the total energy radiated across $\mathcal{I}_L$ to obtain

$$E_L = \frac{c^+}{48}\left(\lambda + \frac{2m}{\kappa}\right). \tag{3.115}$$

Once again, the sign will be governed by the specific values of κ, m and λ; it will for the case where the apparent horizon is formed directly over the null infalling shell. Incidentally, the case where $E_L = 0$ corresponds to that where the apparent horizon forms exactly at $x_0^- = 0$.

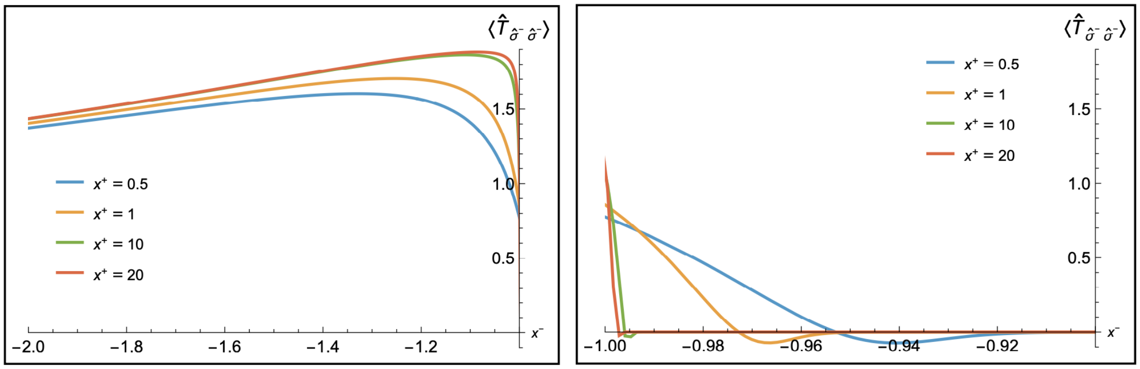

Figure 3.7: Energy-momentum flux that reaches $\mathcal{I}_R$ (on the left) and $\hat{\mathcal{I}}_R$ (on the right) as a function of x^-, parametrized so that $x^-_\infty = -1$, for several values of x^+_r.

As for the $\hat{\mathcal{I}}_R$ region, it is not possible to find a satisfactory solution in terms of elementary functions. As stated above, there is no simple approximation that lets us simplify the expression of Ω in a meaningful way. The exact solution for ρ needed in order to derive the Christoffel symbols and the energy-momentum tensor is given by

$$\rho = \frac{\Omega}{\kappa} + \frac{1}{2} W\left(-\frac{2}{\kappa}\exp\left(\frac{2\Omega}{\kappa}\right)\right), \tag{3.116}$$

where once again $W(z)$ stands for the Lambert W function; since this time $z > 0$ on account of $\kappa < 0$, there is a unique solution for every value of Ω, which will be given by (3.108). We may use this exact expression for the entire R region, including both the interior $\hat{\mathcal{I}}_R$ and the exterior $\mathcal{I}_R$. We can perform the same kinds of computations in order to find the flux on a reference null geodesic x^+_r

$$\langle T^f_{x^\pm x^\pm} \rangle = -\frac{c^+}{12}\left[\frac{1-W}{\kappa^2(1+W)^3}(\partial_\pm \Omega)^2 - \frac{\partial^2_\pm \Omega}{\kappa(1+W)} + \frac{1}{4(x^\pm)^2}\right], \tag{3.117}$$

where we have omitted the argument of $W(z)$ for ease of notation. The change into affine parameters can once again be performed by deriving the geodesic equation and matching the solutions to previously known information (in this case, the properties of the asymptotically flat coordinates in the exterior region and near $x^- = 0$).

Figure 3.7 shows the resulting flux across the entire R region of the Penrose diagram. On the outside region, the flux develops a peak slightly outside the horizon, dropping to a strictly positive value when $x^- = x^-_\infty$. Inside the horizon, a region of negative flux develops before exponentially decaying towards $x^- = 0$. As $x_r^+ \to \infty$, both the negative and positive peak conflate towards the horizon; eventually, the profile collapses onto a step-like configuration. The flux in the exterior region reproduces the thermal plateau that was obtained previously, while vanishing completely exactly at the horizon. This behavior is compatible with the results obtained separately for the exterior region and the interior (strictly near $x^- = 0$), while additionally providing some needed clarification for the region near the apparent horizon.

3.3 Conclusions and future directions

General Relativity is the most advanced description of classical gravity that we have at the current time. Its main characteristic is that it intimately ties the nature of gravitational interactions to the geometry of spacetime itself. The principle of equivalence then allows us to see objects "interacting with gravity" merely as following the geodesics of a curved background. This curvature becomes intrinsically linked to the energy content of the system (represented by its energy-momentum tensor $T_{\mu\nu}$) via the famous Einstein field equations, which can be derived from a classical action in the same way other field theories do. The symmetry of General Relativity, which stems from the simple coordinate map redefinition freedom on a manifold, imposes the conservation of energy-momentum.

Black holes are then a particular set of solutions of the Einstein equations with some specific properties. They correspond to solutions where a region of spacetime becomes trapped by an event horizon, from which no object can escape to infinity. Black holes formed by the collapse of a massive object develop a singularity in their interior, where the curvature of spacetime diverges. These singularities constitute one of the greater puzzles in Theoretical and Mathematical Physics, as they very explicitly point towards a breakdown of the classical formulation of gravity at the high energy limit. The existence of a completely trapped region that does not connect causally to any outside future observer presents itself with yet another set of questions, specifically with regards to entropy and the conservation of information. Once matter falls inside a black hole, it seems that all its structural information disappears except for its mass, charge and angular momentum, which are simply added to the black hole's.

When we introduce semiclassical effects in the mix we get some answers (and even more questions) about the actual evolution of a black hole. The process of Hawking radiation seemingly points to black holes slowly losing their energy via thermal radiation, reducing its size until their eventual complete evaporation. However, the actual final stages of the process will most likely require some greater theory to be fully understood, since the currently running assumption is that black hole evaporation eventually leads to a catastrophic thunderbolt event that breaks all unitarity of our theories.

Over the past few decades, two-dimensional black hole models have become an invaluable asset in order to gain a better understanding of black hole evolution without the additional mathematical complexity of higher-dimensional models. Even though the Einstein-Hilbert action becomes a topological invariant and therefore does not give rise to any equations of motion, losing any predictive power, it is possible to construct alternative gravity models that do not suffer from the same problem. One such family of models is given by dilaton gravity theories, of which the CGHS model is one of them. Its popularity comes originally from its appearance as a low energy limit of some string theories, but also from its closeness to the result of ignoring the angular parts in a spherically symmetric 4D gravity theory. It is also a fully analytically solvable model for gravity in 2D.

Taking advantage of conformal symmetry, we are able to solve the CGHS equation of motion with ease, reproducing results similar to those of a Schwarzschild black hole. When one considers semiclassical corrections, though, it is important to notice that the tracelessness of $T_{\mu\nu}$ imposed by this symmetry is no longer satisfied. This can be seen using effective action methods like the ones described in the previous Chapter. This anomaly is ultimately the source of a quantum correction to the action in the form of the Polyakov term. Additional counterterms may be imposed if we want to maintain the ability to solve this model without the need of numerical methods at the semiclassical level (like the RST or BPP models).

In constructing these models, it becomes apparent that the multitude symmetries of the theory give rise to a great amount of redundancy in our definition of the path integral / effective action. Ghost fields need to be introduced to account for this, modifying yet again our models with terms that have a negative central charge. Since we do not expect these fields to radiate at infinity like the physical ones, we impose that these fields stay in the Boulware state by coupling them to the auxiliary metric $\hat{g}$, which is strictly flat on the Kruskal gauge. Surprisingly, by considering negative charge fields we seem to be able to get rid of the classical singularity altogether thanks to the quantum effects. Instead, after the gravitational collapse of a null shell of matter, this model develops an apparent horizon that envelops another asymptotically flat region, for a total of three different future null infinities. The radiated flux that reaches these regions has been computed, taking a limit approximation approach for those regions for which the asymptotically flat coordinates cannot be found straightforwardly. Hawking radiation can be observed on the region outside the horizon, while more complex flux profiles can be seen in the interior regions.

One of the main concerns when developing a semiclassical model for gravity is the preservation of unitarity. As discussed in Section 3.2.3, the RST model cannot fully predict the entire evolution of spacetime on account of the existence of a naked singularity that breaks unitarity past the point of merging between the singularity and the apparent horizon. While the geometry of spacetime is still “patched up” by introducing physically motivated boundary conditions, the fact remains that correlations between points at the future infinity can be lost when traced back to the past. For our model with negative central charges, the question of unitarity becomes extremely subtle, specially on the R regions since the exterior region is complete, meaning it becomes unclear if it is even possible to study correlations between a point inside the horizon and another outside. All in all, it becomes clear that a better understanding of the entire R side of the Penrose diagram is needed.

Another concern that may arise with regards to this model is that of energy conservation. At this point in time, a full computation of the total energy radiated has not been completed, on account of the complexities regarding the exact flux profile of the R regions. If energy is to be conserved, then the total energy radiated through $\mathcal{I}_L$, $\mathcal{I}_R$ and $\hat{\mathcal{I}}_R$ should equal the initial mass m of the collapsing shell that formed the black hole. Given that both flux profiles acquire positive and negative values, it is indeed possible that their contributions cancel each other identically. The case where $\kappa = -2\dfrac{m}{\lambda}$ becomes particularly interesting since the energy radiated through $\mathcal{I}_L$ vanishes identically, leaving only the two remaining regions to consider. It appears as though, at least in this case, it is not possible to ensure energy conservation, on account of the flux profile becoming akin to a step distribution when $x^+ \to \infty$; additional considerations may need to be taken into account. One such possibility is the introduction of reflective boundary conditions at a specific value of ϕ, in the same way that previous models have considered before in regards to the CGHS model semiclassical corrections.

In any case, the model here presented shows a very interesting pattern in which, by introducing a sufficiently large negative central charge in the theory, it is possible to make the singularity of the RST black hole (or equivalent models) disappear altogether. This of course trivially solves the issue of the naked singularity and the thunderbolt, but also presents a possibility with regards to black hole evolution being determined by the precise nature of the matter content inside it, contrary to the standard treatment of black holes.

Chapter 4

Closing remarks

The search for Quantum Gravity is one of the most exciting, but also nebulous, endeavors in current Theoretical and Mathematical Physics. Without going into the most ambitious models for fully-realized theories of gravity as a quantum system, semiclassical approaches alone already give rise to new ideas and results that can help us get a better view of the subject matter.

Effective actions serve as a gateway into 1-loop quantum corrections to our classical field theories. They are one of the more powerful tools we currently have in understanding how interaction processes work at high energies. More specifically, they serve as an alternative approach to the usual diagrammatic expansion: where the Feynman diagram approach presents an infinite series of increasingly more complex integrals that define each loop level contribution, the effective action can theoretically take care of the entire calculation in a nonperturbative way. Even if that ends up not being possible for many applications, we may still be able to define a first-order approximation given by the 1-loop effective action.

Heat kernel methods are then used in order to study and regularize the divergencies of the effective action. These divergencies dictate the relevant counterterms that need to be introduced in the classical action in order to tackle the semiclassical corrections. While some of them will amount to simple redefinitions of the constants of the theory, like the energy of the vacuum or the interaction constants, others will give rise to completely new effects that result in the breaking of classical symmetries at the quantum level. Examples of the latter include the chiral anomaly for an electromagnetic theory, or the trace anomaly of the energy-momentum tensor associated with the matter content in a curved spacetime.

While the direct calculation of the divergencies of the effective action may not be possible except for the simpler systems, the Gilkey-Seeley-DeWitt prescription allows us to define a short proper time expansion of the heat kernel, whose coefficients can be found from the invariants of the theory studied. The divergencies are then directly tied to the GSDW coefficients, which can be explicitly calculated recursively by imposing the heat kernel equation at every order in τ. Nevertheless, the GSDW expansion suffers from computational limitations as the number of terms grows ever larger and larger. In Sections 2.2.4-2.2.6 of this document, we presented an alternative resummation scheme that can build a slightly longer bridge into the perturbative expansion, extracting as much nonperturbative information as possible. This formalism was applied to several systems of interest, and we speculated over the possibility of more involved generalizations.

One such generalization that would be of great interest is in the study of quantum fields on classical gravitational curved backgrounds. Black holes, in particular, present themselves as one of the more enticing "laboratories" in which to apply and test the tools provided by the effective action. Their most prominent feature is the existence of a trapped region from which no information can reach an outside observer. Most black hole models contain a special region at which the curvature of spacetime becomes infinite, called the singularity, where our classical theory loses its predictive value, pointing at the need of a quantum or semiclassical model.

One of the first results one can derive from a semiclassical approach to gravity is the so-called trace anomaly, which is linked to the breaking of conformal symmetry. In $d = 2$ specifically, we can make use of the GSDW expansion to find the relevant counterterm that needs to be added to the classical theory, which is given by the Polyakov action; additionally, the trace anomaly directly links the values of the energy-momentum tensor to the curvature scalar R. More broadly speaking, the mathematical simplicity of 2-dimensional black hole models gives us a relatively accessible framework in which to study semiclassical models of gravity, with most of the relevant features (such as the existence of Hawking radiation) still intact. The Callan-Giddings-Harvey-Strominger model stands out as an analytically solvable system for which a black hole solution can be found. Introducing the Polyakov term we obtain a fully functional semiclassical black hole model, but one that can only be solved via numerical methods. It is convenient to reinstate exact solvability by introducing extra terms such as the Russo-Susskind-Thorlacius term. The resulting model provides a systematic way of categorizing and studying the different states (among them the famous Boulware, Hartle-Hawking and Unruh states) in which the matter fields may be defined, presenting a rich variety of solutions with varying degrees of complexity.

One such solution leads to a model in which a naked singularity emerges after a finite amount of proper time. The existence of this singularity leads to the emission of an infinitely large (negative) energy burst known as the thunderbolt, as well as a finite one that serves as a way to maintain energy-momentum conservation (up to this infinite emission that may be treatable in some other way) after its appearance, called a thunderpop. While the latter may be an artifact of the semiclassical RST model still not being powerful enough to understand the behaviour of spacetime near singularities, in our current formulation the thunderbolt is an inherent property of naked singularities that breaks the unitarity and predictability of our theories.

In Section 3.2.4 we present an alternative construction that generalizes the RST and other equivalent frameworks to include matter fields with negative central charge, such as the ghost fields associated with the symmetries of the CGHS model. Imposing that they couple not to the metric of the system, but to a conformally equivalent, on-shell flat one, we ensure that no Hawking radiation is emitted and observed coming from these fields. The resulting system develops no singularity whatsoever, trivially solving the issue of the thunderbolt, but it does present an apparent horizon. Ongoing work is being done in understanding the full nature of this system; so far, the outside region of this spacetime behaves in the same way as the CGHS black hole solution, while the interior regions seem to include both positive and negative flux contributions. It remains to be seen whether or not the total energy balance is preserved without the need of considering additional elements in the model, like the imposition of specific boundary conditions.

There are many different directions in which we may continue expanding the frameworks here discussed. Effective actions, in particular, constitute a "multi-purpose" tool with which to study the properties of any high energy system. More powerful resummation schemes may allow us to cover broader ground into non-perturbative quantum effects and first-order corrections to classical field theories. Applying this knowledge to the study of black holes and, more generally, to any gravitational interaction could potentially point us in the future to the development of a quantum theory of gravity. While direct experimental confirmation of any of the results presented in this document may be far away as of now, recent advances in fields like Condensed Matter Physics could lead to the construction of analogue models which could be used to get a better insight into the phenomena that we have covered. In summary, while our current insights into a possible "Theory of Everything" may be limited at best, there is no shortage of avenues through which we may be able to face the challenge.

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