Nonlinear Dyn
https://doi.org/10.1007/s11071-024-10536-8

RESEARCH

Free vibrations of shallow inextensible cables: a perturbation approach

Marco Lepidi

Received: 24 June 2024 / Accepted: 18 October 2024

Abstract The statics and dynamics of catenary cables are classic matters of theoretical and applied mechanics. The paper systematizes a methodological strategy to achieve analytical solutions for the nonlinear static problem and linearized free dynamic problem of inextensible shallow cables. First, the one-dimensional continuum model of elastically extensible, perfectly flexible cables is revisited to state the nonlinear differential equations governing the static and dynamic equilibria as parametric expressions of the cable shallowness and extensibility. Second, an original hierarchical generalization of the Force Method is presented as methodological solution strategy. The key is the systematic application of perturbation schemes to equilibrium equations, indeformability constraints and geometric compatibility conditions. As a principal point of strength, the proposed strategy allows the unified and consistent treatment of the static and dynamic problems, while requiring the sole assumption of cable shallowness as postulate a priori. As major achievements, fully analytical high-order solutions are obtained for the asymptotic approximation of (i) the catenary configuration in the static field, and (ii) the natural frequencies and classical modal forms in the linearized dynamic field. Parametric analyses of the results highlight that high-order terms determine significant qualitative and quantitative effects on the modal solutions, including competing softening or hardening effects in the natural frequencies.

Keywords Cable dynamics · Perturbation methods · Asymptotic techniques · Force method · Analytical solutions · Catenary curves · Modal analysis

M. Lepidi (✉)
DICCA - Università di Genova,
Via Montallegro 1, 16154 Genoa, Italy
e-mail: marco.lepidi@unige.it

1 Introduction

Cables are extensively employed in a variety of mechanical applications owing to their unrivaled performance in efficiently sustaining and transferring loads, synergically collaborating with other structural elements, and realizing challenging and appealing architectures. On the other side, their natural properties of extreme flexibility and low damping make cables strongly vulnerable to the development of important dynamic phenomena. The scientific and technical importance of this research topic is continuously fueling formulations of advanced analytical and computational models, investigations of static and dynamic responses, analyses of rich scenarios of regular and nonregular behaviors [11,35,36].

From the historical viewpoint, the geometrical differential problem of determining the equilibrium configuration assumed by a perfectly flexible cable suspended between fixed supports under self-weight was first presented as a mathematical contest at the end of the seventeenth century by Jacob Bernoulli. The topic received significant contributions by several outstand-

Published online: 11 November 2024

ing mathematicians, including Leibniz, Euler, Huygens, Johann Bernoulli, among the others [2,15,44].

From the mechanical viewpoint, searching the static equilibrium configuration of suspended cables can be regarded as a statically indeterminate shape-finding problem. Under the ideal hypothesis of axial indeformability, the static configuration assumed by the cable is mathematically described by a transcendental function, mapped in the vertical (gravitational) plane by the *catenary curve*. Relaxing the assumption of axial indeformability determines a slight change in the static configuration, mathematically described by a pair of transcendental cartesian functions that parametrically map the *elastic catenary curve* [13]. Although analytically expressible, both configurational functions parametrically depend on the horizontal reaction at the supports—playing the role of hyperstatic unknown—which has to be numerically assessed a posteriori by solving nonlinear compatibility equations, which impose the indeformability constraint (for the catenary) or a pair of coupled boundary conditions (for the elastic catenary). Consequently, the systematic need to complete the solution process by recurring to numerical solvers suggests considering catenaries and elastic catenaries as *quasi-analytical* solutions [41]. Recently, perturbation methods based on the assumption of cable shallowness have been proposed to achieve *fully analytical*—although asymptotically approximate—solutions for the static problem of elastically extensible cables under general loads [30,31], as well as for the catenary configuration functions and equivalent stiffness matrix of inclined inextensible cables under self-weight [22].

Determining fully analytical expressions of the static cable configuration with the highest possible accuracy is not a mere speculative challenge, considering that static equilibria are typically adopted as a reference for dynamic analyses. Indeed, static pretensions and curvatures can strongly influence the hardening or softening effects in the linear and nonlinear dynamics of prestressed curved continua [16,29]. Furthermore, numerous studies have shown that accurate analytical solutions of the static problem are the fundamental background for addressing direct and inverse problems in cable dynamics, including—for instance—the assessment of the linearized modal properties [17,32,46], the description of internal superharmonic and subharmonic resonances, occurrence of nonlinear interactions, onset of bifurcation instabilities and transition to nonregular oscillation regimes [9,18,37,42], the investigation of aerodynamic phenomena [28,47], the evaluation of damage and temperature effects [24,25], the performance of active control strategies [1,8] and the reliability of identification procedures [26,38]. Focusing in particular on the free oscillation problem, most of the scientific literature has been devoted to linear and nonlinear dynamics of extensible cables, starting from the seminal monograph by Irvine [13]. In contrast, fewer research contributions have focused on inextensible cables, in which the property of perfect inextensibility can be (ideally) attributed on the base of energetic considerations [29]. Enforcing the inextensibility as internal constraint, while simplifying kinematics, poses challenging tasks in the formulation and solution of the dynamic equations. The major conceptual issue concerns the consistent treatment of the internal forces (and geometric stiffness), since the axial tension has to be considered a reactive force unrelated to kinematic quantities by constitutive laws.

Within this stimulating framework, this paper presents a systematic guideline to achieve fully analytical solutions for inextensible shallow cables. First, the traditional formulation of the mechanical model is revisited to state the nonlinear equations governing the static and dynamic problems as parametric expressions of the cable shallowness and extensibility (Sect. 2). Second, an original hierarchical variant of the Force Method, based on the systematic application of asymptotic perturbation theories, is presented as methodological solution strategy. With respect to other approaches, the proposed strategy allows a unified and fully consistent treatment of the static and dynamic problems, starting from the sole assumption of cable shallowness (Sect. 3). As major achievement, high-order fully analytical solutions are provided for the unsolved eigenproblem (modal problem) governing the linearized dynamics of horizontal cables in the small-amplitude oscillation regime (Sect. 4). The satisfying agreement of high-order asymptotic solutions with numerical results is verified. Concluding remarks are finally pointed out.

2 Mechanical model

The mechanical model describing the static and dynamic behavior of a suspended cable is a one-

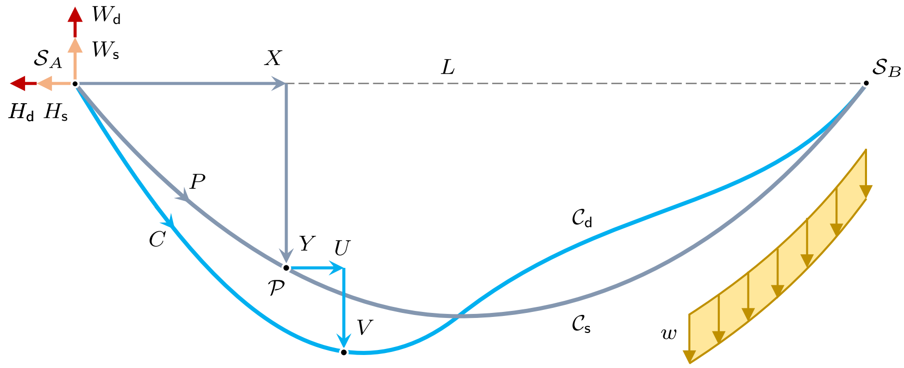

Fig. 1 Inextensible cable suspended between fixed supports

dimensional continuum. The continuum is assumed homogeneous, elastically extensible in the axial direction and perfectly flexible in the transversal directions. Accordingly, the deformed configuration is fully characterized by strictly positive axial strain $\varepsilon(S)$ and axial tension $N(S)$, that depend on the curvilinear abscissa S spanning the total arc-length L_o of the natural configuration $\mathcal{C}_o$. Mechanically, the natural configuration $\mathcal{C}_o$ is strainless and tensionless, by definition. Mathematically, its definition a priori is unnecessary (except for the length L_o), without insufficiency of information for the problem statement. In the absence of transversal (flexural and shear) rigidities, the axial tension is required to be locally collinear to the equilibrium configuration (*collinearity condition*). Consequently, searching the static response of the cable under gravity loads consists in solving a geometric shape-finding problem in the two-dimensional space, pursuing the unknown curvilinear configuration $\mathcal{C}_s$ satisfying the static equilibrium in the vertical plane (Fig. 1). Therefore, studying the free dynamic response of the cable consists in determining the unknown curvilinear configuration $\mathcal{C}_d$ assumed under inertial forces, taking the static configuration $\mathcal{C}_s$ as reference.

In this Section, the partial differential equations governing the static and dynamic problems for extensible cables are formulated. As a minor character of originality, the classic physical–mathematical formulation is specially adapted to express the variable coefficients of all equations as explicit parametric functions of the static and dynamic axial strains. This peculiarity facilitates the transition to the mechanical model of (ideally) inextensible cables, characterized by an infinite axial rigidity that allows null axial strains to coexist with finite and not-null axial tensions.

2.1 Static problem

The cable is supposed hanging between fixed supports $\mathcal{S}_A$ and $\mathcal{S}_B$, under the effect of uniform (S-independent) self-weight density w per unit natural length. Horizontal cables, characterized by supports placed at the same level, are considered. The cable *span* L indicates the length of the rectilinear horizontal *chord* between the supports. The curvilinear abscissa $S \in [0, L_o]$, that identifies the generic particle point $\mathcal{P}(S)$, is chosen as the only independent variable. Solving the static problem requires to determine the cartesian configurational functions $X(S)$ and $Y(S)$ that parametrically locate the static equilibrium position of each particle point $\mathcal{P}(S)$.[1] The positions of all points define the static configuration $\mathcal{C}_s$ in the vertical plane. The full set of static unknowns includes also the horizontal and vertical reactions H_s and W_s at support $\mathcal{S}_A$, the axial strain $\varepsilon_s(S)$, the axial tension $N_s(S)$, and the curvilinear abscissa $P(S)$ spanning the strained configuration $\mathcal{C}_s$.[2]

According to differential geometry, the configurational functions $X(S)$ and $Y(S)$ are required to satisfy the nonlinear differential relation

$$\left(\frac{\partial X}{\partial P}\right)^2 + \left(\frac{\partial Y}{\partial P}\right)^2 = 1 \tag{1}$$

[1] Here the nomenclature and positive signs of all the variables preserve the classic conventions adopted in the pivotal monograph by Irvine [13].

[2] From the mathematical viewpoint, it is important to note that, according to physical reasons, the unknowns $X(S)$, $Y(S)$, $\varepsilon_s(S)$, $N_s(S)$ and $P(S)$ can be assumed to be sufficiently continuous and differentiable functions. Function $P(S)$ can also be considered monotonically non-decreasing.

which is an inherent property of curves in a two-dimensional space and will be referred to as *static geometric constraint* in the following.

Following the classic formulation and methodology providing quasi-analytical solutions [13], the static equilibrium of the finite-length element of cable between the support $\mathcal{S}_A$ and the generic point $\mathcal{P}(S)$ is governed by a pair of partial differential equations

$$N_\mathsf{s}\frac{\partial X}{\partial P} = H_\mathsf{s} \tag{2}$$

$$N_\mathsf{s}\frac{\partial Y}{\partial P} = W_\mathsf{s} - wS \tag{3}$$

expressing the balance of forces acting in the horizontal and vertical directions, respectively. Physically, the former equation states that the horizontal component $T_\mathsf{s} := N_\mathsf{s}(\partial X/\partial P)$ of the axial tension is constant and equal to the support reaction H_s. The latter equation states that the vertical component $Q_\mathsf{s} := N_\mathsf{s}(\partial Y/\partial P)$ of the axial tension differs from the support reaction W_s, except for $S = 0$. The two tension components are coupled by the differential relation $Q_\mathsf{s} = T_\mathsf{s}(\partial Y/\partial X)$, which expresses the *static collinearity condition*.

Furthermore, the linear elastic law ruling the relation between the axial tension and the axial strain, intended as the *unit extension* $\varepsilon_\mathsf{s} := \partial P/\partial S - 1$, reads

$$N_\mathsf{s} = EA\,\varepsilon_\mathsf{s} \tag{4}$$

where EA is the uniform axial rigidity. The variables $X(S)$ and $Y(S)$ obey to boundary conditions $X(0) = 0$, $X(L_\mathsf{o}) = L$ and $Y(0) = 0$, $Y(L_\mathsf{o}) = 0$.

The problem governed by Eqs. (1)–(4) can be solved quasi-analytically. The map of the quasi-analytical solution in the vertical plane describes (in parametric form with S playing the role of parameter) the so-called *elastic catenary* curve of the configuration $\mathcal{C}_\mathsf{s}$.

In alternative to the classic approach, the static equilibrium Eqs. (2)–(3) can be mathematically treated by (i) replacing the axial tension N_s with its components T_s and Q_s at the left hands, (ii) differentiating left and right hands with respect to variable X, and finally (iii) applying a change of the differentiation variable (from S to $X(S)$) with the recursive chain rule $(\partial/\partial X) = (\partial/\partial S)(\partial S/\partial P)(\partial P/\partial X)$ to the right hands. Finally, the equilibrium equations for the generic infinitesimal-length element of cable read

$$\frac{\partial T_\mathsf{s}}{\partial X} = 0 \tag{5}$$

$$\frac{\partial Q_\mathsf{s}}{\partial X} = -\frac{w}{1+\varepsilon_\mathsf{s}}\frac{\partial P}{\partial X} \tag{6}$$

within the domain $X \in [0, L]$. The derivative at the right hand can be expressed in the form

$$\frac{\partial P}{\partial X} = \left[1 + \left(\frac{\partial Y}{\partial X}\right)^2\right]^{1/2} \tag{7}$$

by manipulating the geometric constraint (1). By recalling the static collinearity condition and substituting Eq. (5) into Eq. (6), the nonlinear differential equation governing the static problem becomes

$$H_\mathsf{s}\frac{\partial^2 Y}{\partial X^2} = -\frac{w}{(1+\varepsilon_\mathsf{s})}\left[1 + \left(\frac{\partial Y}{\partial X}\right)^2\right]^{1/2} \tag{8}$$

without approximations. The boundary conditions are $Y(0) = 0$ and $Y(L) = 0$. The equation is characterized by the unknown coefficient H_s at the left hand and the variable coefficient affecting the right-hand term, depending on the axial strain ε_s, which can be expressed as explicit function of the derivative $\partial Y/\partial X$.

In summary, the static problem of the extensible cable is governed by the nonlinear ordinary differential Eq. (8) with a variable coefficient, to be solved in the configurational function $Y(X)$ and the horizontal component of static tension H_s. A fully analytical solution of the Eq. (8) is not available.

2.2 Dynamic problem

The extensible cable is supposed to freely oscillate in the gravitational plane under the action of the inertial forces generated by the mass density m per unit natural length. By taking the static equilibrium configuration $\mathcal{C}_\mathsf{s}$ as reference, the dynamic equilibrium configuration $\mathcal{C}_\mathsf{d}$ is determined by the time-dependent functions $U(S,t)$ and $V(S,t)$, describing the horizontal and vertical displacements of the generic particle point $\mathcal{P}(S)$. The full set of dynamic unknowns includes also the horizontal and vertical reactions $H(t)$ and $W(t)$ at the support $\mathcal{S}_A$, the axial strain $\varepsilon(S,t)$ and axial tension

$N(S,t)$, and the curvilinear abscissa $C(S,t)$ spanning the strained configuration $\mathcal{C}_{\mathsf{d}}$.[3]

According to differential geometry, the displacement functions $U(S,t)$ and $V(S,t)$ are required to satisfy the nonlinear differential relation

$$\left(\frac{\partial X}{\partial C}+\frac{\partial U}{\partial C}\right)^2+\left(\frac{\partial Y}{\partial C}+\frac{\partial V}{\partial C}\right)^2=1 \tag{9}$$

which is an inherent property of curves in a two-dimensional space and will be referred to as *dynamic geometric constraint* in the following.

Following a formulation and methodology similar to those adopted for the static problem, the dynamic equilibrium of the finite-length element of cable between the support $\mathcal{S}_A$ and the generic point $\mathcal{P}(S)$ is governed by a pair of partial differential equations

$$N\left[\frac{\partial X}{\partial C}+\frac{\partial U}{\partial C}\right]=H+\int_0^S m\frac{\partial^2 U(S,t)}{\partial t^2}\,dS \tag{10}$$

$$N\left[\frac{\partial Y}{\partial C}+\frac{\partial V}{\partial C}\right]=W-\int_0^S w\,dS+\int_0^S m\frac{\partial^2 V(S,t)}{\partial t^2}\,dS \tag{11}$$

where the left-hand terms can be recognized as the horizontal component $T := N(\partial X/\partial C + \partial U/\partial C)$ and vertical component $Q := N(\partial Y/\partial C + \partial Y/\partial C)$ of the axial tension. It is straightforward to demonstrate that the two components are internally coupled by the differential relation $Q = T(\mathfrak{D}_V/\mathfrak{D}_U)$, where the auxiliary functions $\mathfrak{D}_U = 1 + \partial U/\partial X$ and $\mathfrak{D}_V = \partial Y/\partial X + \partial V/\partial X$ have been introduced. This differential relation expresses the *dynamic collinearity condition*.

The dynamic equilibrium Eqs. (10)–(11) can be mathematically treated by (i) replacing the axial tension N with its components T and Q and applying the Gauss-Green formulas at the left hands, (ii) applying a change of integration variable (from S to $X(S)$) with the recursive chain rule $dS = (\partial S/\partial P)(\partial P/\partial X)\,dX$ to the right hands, and finally (iii) satisfying integral equations by equating integrand functions and boundary terms of the right and left hands. Finally, the equilibrium equations for the generic infinitesimal-length element of cable—without approximations—read

$$\frac{\partial T}{\partial X}=\frac{m}{1+\varepsilon_{\mathsf{s}}}\frac{\partial P}{\partial X}\frac{\partial^2 U}{\partial t^2} \tag{12}$$

$$\frac{\partial Q}{\partial X}=-\frac{w}{1+\varepsilon_{\mathsf{s}}}\frac{\partial P}{\partial X}+\frac{m}{1+\varepsilon_{\mathsf{s}}}\frac{\partial P}{\partial X}\frac{\partial^2 V}{\partial t^2} \tag{13}$$

within the domain $X \in [0, L]$. The geometric boundary conditions read

$$U(0)=0,\quad U(L)=0,\quad V(0)=0,\quad V(L)=0 \tag{14}$$

while the mechanical boundary conditions

$$T(0)=H,\qquad Q(0)=W \tag{15}$$

can be employed to assess the support reactions, once the tension components have been determined.

By properly separating the static and dynamic parts of the tension components $T = T_{\mathsf{s}} + T_{\mathsf{d}}$ and $Q = Q_{\mathsf{s}} + Q_{\mathsf{d}}$, the equilibrium Eqs. (12)–(13) become

$$\frac{\partial T_{\mathsf{d}}}{\partial X}=\frac{m}{1+\varepsilon_{\mathsf{s}}}\frac{\partial P}{\partial X}\frac{\partial^2 U}{\partial t^2} \tag{16}$$

$$\frac{\partial Q_{\mathsf{d}}}{\partial X}=\frac{m}{1+\varepsilon_{\mathsf{s}}}\frac{\partial P}{\partial X}\frac{\partial^2 V}{\partial t^2} \tag{17}$$

where Eqs. (5)–(6) have been employed to eliminate statically equilibrated terms. The mechanical boundary conditions become $T_{\mathsf{d}}(0) = H_{\mathsf{d}}$, $Q_{\mathsf{d}}(0) = W_{\mathsf{d}}$.

Separating the static and dynamic parts of the tension components allows also to express the dynamic collinearity condition in the alternative forms $Q_{\mathsf{d}} = (T_{\mathsf{s}}+T_{\mathsf{d}})(\mathfrak{D}_V/\mathfrak{D}_U)-Q_{\mathsf{s}}$ or $T_{\mathsf{d}} = (Q_{\mathsf{s}}+Q_{\mathsf{d}})(\mathfrak{D}_U/\mathfrak{D}_V)-T_{\mathsf{s}}$. By partial differentiation, the relations

$$\frac{\partial T_{\mathsf{d}}}{\partial X}=\left(\frac{\partial Q_{\mathsf{s}}}{\partial X}+\frac{\partial Q_{\mathsf{d}}}{\partial X}\right)\frac{\mathfrak{D}_U}{\mathfrak{D}_V}+ +\frac{(Q_{\mathsf{s}}+Q_{\mathsf{d}})}{\mathfrak{D}_V^2}\frac{\partial(\mathfrak{D}_U\mathfrak{D}_V)}{\partial X}-\frac{\partial T_{\mathsf{s}}}{\partial X} \tag{18}$$

$$\frac{\partial Q_{\mathsf{d}}}{\partial X}=\left(\frac{\partial T_{\mathsf{s}}}{\partial X}+\frac{\partial T_{\mathsf{d}}}{\partial X}\right)\frac{\mathfrak{D}_V}{\mathfrak{D}_U}+ +\frac{(T_{\mathsf{s}}+T_{\mathsf{d}})}{\mathfrak{D}_U^2}\frac{\partial(\mathfrak{D}_U\mathfrak{D}_V)}{\partial X}-\frac{\partial Q_{\mathsf{s}}}{\partial X} \tag{19}$$

[3] From the mathematical viewpoint, it is be important to note that, according to physical reasons, the unknowns $U(S)$, $V(S)$, $\varepsilon(S)$, $N(S)$ and $C(S)$ can be assumed to be sufficiently continuous and differentiable functions. Function $C(S)$ can also be considered monotonically non-decreasing.

are obtained and can be used to eliminate one of the two components from the equilibrium Eqs. (16)–(17).

In summary, the free dynamic problem of the extensible cable is governed by a coupled pair of nonlinear partial differential equations with variable coefficients (16)–(17), to be solved in the displacement variables $U(X,t)$, $V(X,t)$ and the dynamic part of the tension components $T_d(X,t)$, $Q_d(X,t)$. Since the tension components are related to each other by the dynamic collinearity condition, the problem can be reformulated in one *or* the other component (*principal unknown force*).

3 Inextensible cables

The assumption of cable inextensibility in the static response is introduced by zeroing the static axial strain ($\varepsilon_s = 0$), which is equivalent to impose the relation

$$\frac{\partial P}{\partial S} = 1 \tag{20}$$

and—naturally—entails also that the arc-length of the static configuration curve

$$L_s = \int_0^L \left[1 + \left(\frac{\partial Y}{\partial X}\right)^2\right]^{1/2} dX \tag{21}$$

is equal to the natural length L_o. The identity $L_s = L_o$ is crucial in solving the static problem and will be referred to as *static inextensibility condition* in the following.

The assumption of cable inextensibility in the dynamic response is introduced by zeroing the axial strain ($\varepsilon = \varepsilon_s + \varepsilon_d = 0$). By virtue of the static inextensibility, the dynamic inextensibility reduces to zeroing the dynamic axial strain ($\varepsilon_d = 0$), which is equivalent to impose the relation

$$\frac{\partial C}{\partial P} = 1 \tag{22}$$

and—remarkably—implies that the displacement functions are coupled by the nonlinear differential relation

$$2\frac{\partial U}{\partial X} + \left(\frac{\partial U}{\partial X}\right)^2 + 2\frac{\partial Y}{\partial X}\frac{\partial V}{\partial X} + \left(\frac{\partial V}{\partial X}\right)^2 = 0 \tag{23}$$

which also entails that the arc-length L_d of the dynamic configuration curve is equal to the natural length L_o (demonstration can be found in the Appendix). Relation (23) has a crucial role in solving the dynamic problem and will be referred to as *dynamic inextensibility condition* in the following.

In this Section, the static and dynamic inextensibility conditions are enforced into the nondimensional equations governing the cable response. Therefore, the linearized equations of motion governing the regime of small-amplitude oscillations in the neighborhood of the reference static configuration $\mathcal{C}_s$ are formulated.

3.1 Nondimensional equations

The static and dynamic problem can be conveniently formulated in nondimensional form by defining dimensionless variables and introducing a minimal set of independent dimensionless parameters, sufficient to fully characterize the mechanical system. To this end, the cable span L, the static horizontal reaction H_s and the circular (square) frequency $\Omega_r^2 = wL_0/(2mL^2)$ can be selected as reference length, force and (inverse of) time, respectively. Therefore, the dimensionless variables

$$x = \frac{X}{L}, \quad y = \frac{Y}{L}, \quad p = \frac{P}{L}, \quad u = \frac{U}{L}, \quad v = \frac{V}{L}, \tag{24}$$

the dimensionless forces

$$h_s = \frac{2H_s}{wL_o}, \quad h_d = \frac{2H_d}{wL_o}, \quad \nu_s = \frac{2W_s}{wL_o}, \quad \nu_d = \frac{2W_d}{wL_o}, \quad t_s = \frac{2T_s}{wL_o}, \quad t_d = \frac{2T_d}{wL_o}, \quad q_s = \frac{2Q_s}{wL_o}, \quad q_d = \frac{2Q_d}{wL_o} \tag{25}$$

and the dimensionless quantities

$$\delta = \frac{wL}{8H_s}, \qquad \Lambda = \frac{L_o}{L} \tag{26}$$

are introduced. The dimensionless time $\tau = \Omega_r t$ is also defined. The relation $4\Lambda\,\delta\,h_s = 1$ holds by construction. The fundamental mechanical parameter δ is strictly positive by definition, and represents (one eighth of) the ratio between the approximate cable weight wL and the static horizontal reaction H_s. The geometric parameter Λ is known as *aspect ratio* [14], and expresses the ratio between the natural length of the cable and the length of the chord. Clearly, the aspect ratio of inextensible cables must satisfy the *admissibility inequality* $\Lambda > 1$.

By imposing the static inextensibility conditions and introducing dimensionless variables and parameters, Eq. (8) governing the static problem of inextensible cables assumes the classic form known as *funicular equation*, reading

$$y'' = -8\delta \left[1 + \left(y'\right)^2\right]^{1/2} \tag{27}$$

where the apex indicates differentiation with respect to variable $x \in [0, 1]$. The equation is complemented by the boundary conditions $y(0) = 0$ and $y(1) = 0$.

By imposing the static inextensibility conditions and introducing dimensionless variables and parameters, Eqs. (16)–(17) governing the dynamic problem of inextensible cables have nondimensional form

$$t_\mathsf{d}' = \ddot{u} \left[1 + \left(y'\right)^2\right]^{1/2} \tag{28}$$

$$q_\mathsf{d}' = \ddot{v} \left[1 + \left(y'\right)^2\right]^{1/2} \tag{29}$$

where the overdot indicates differentiation with respect to dimensionless time τ. The dynamic indeformability condition reads in nondimensional form

$$2u' + \left(u'\right)^2 + 2y'v' + \left(v'\right)^2 = 0 \tag{30}$$

and provides a nonlinear internal coupling between the displacement variables. The equations are complemented by geometric boundary conditions $u(0) = 0$, $u(1) = 0$ and $v(0) = 0$, $v(1) = 0$. The mechanical boundary conditions $t_\mathsf{d}(0) = h_\mathsf{d}$, $t_\mathsf{d}(0) = \nu_\mathsf{d}$ can be employed to relate the reactions to the tension components at the left support.

One of the tension component t_d *or* q_d can be eliminated by employing the dynamic collinearity conditions (18)–(19), expressed in nondimensional variables, to replace the derivatives at the left hand of Eqs. (28)–(29) with the expressions

$$\begin{aligned} t_\mathsf{d}' = {} & \left(q_\mathsf{s}' + q_\mathsf{d}'\right) \frac{1+u'}{y'+v'} + \\ & + \left(q_\mathsf{s} + q_\mathsf{d}\right) \frac{\left[\left(1+u'\right)\left(y'+v'\right)\right]'}{\left(y'+v'\right)^2} \end{aligned} \tag{31}$$

$$\begin{aligned} q_\mathsf{d}' = {} & t_\mathsf{d}' \frac{y'+v'}{1+u'} + \\ & + \left(t_\mathsf{s} + t_\mathsf{d}\right) \frac{\left[\left(1+u'\right)\left(y'+v'\right)\right]'}{\left(1+u'\right)^2} - q_\mathsf{s}' \end{aligned} \tag{32}$$

where the static equilibrium Eq. (5), reading $t_\mathsf{s}' = 0$ in nondimensional form, has been taken into account.

3.2 Linearized equations

Within the limits of the small-amplitude oscillation regime, the nonlinear equations governing the cable dynamics can be linearized in the neighborhood of the reference static configuration $\mathcal{C}_\mathsf{s}$. If the horizontal component t_d of the axial tension is selected as principal unknown force, Eq. (32) must be used to eliminate the vertical component q_d of the axial tension (secondary unknown force). After linearization, the governing equations of motion read

$$t_\mathsf{d}' = \ddot{u} \left[1 + \left(y'\right)^2\right]^{\frac{1}{2}} \tag{33}$$

$$h_\mathsf{s} \left[v' \left(1 + (y')^2\right)\right]' + (y' t_\mathsf{d})' = \ddot{v} \left[1 + \left(y'\right)^2\right]^{\frac{1}{2}} \tag{34}$$

where the displacement variables are internally coupled by the linearized form of the dynamic indeformability condition (30), reading

$$u' + y'v' = 0 \tag{35}$$

while all the static forces in Eq. (34) have been expressed as function of the horizontal reaction h_s only, by employing the static solution $t_\mathsf{s} = h_\mathsf{s}$ and the static collinearity condition $q_\mathsf{s}' = (t_\mathsf{s} y')' = h_\mathsf{s} y''$. Alternately, if the vertical component q_d of the axial tension is selected as principal unknown force, Eq. (31) must be used to eliminate the horizontal component t_d of the axial tension (secondary unknown force). After linearization, the dynamic equilibrium is governed by Eq.

(29) and a second Eq. similar to (34), here not reported for the sake of synthesis.

4 Asymptotic solutions

In the absence of fully-analytical solutions and in alternative to quasi-analytical (catenary) and numerical solution, the nonlinear static problem and the linearized undamped free dynamic problem (*modal problem*) can be addressed by recurring to perturbation methods. Perturbation techniques are adaptable and powerful methodological tools that have wide applications throughout a variety of scientific research fields, ranging from direct problems concerning linear and nonlinear dynamics, stability and bifurcation [27,33,34] to inverse problems focused on modal identification, damping and damage detection, optimal spectral design [19,21,23]. Perturbation methods are also well-established strategies to study classical and emerging problems in cable mechanics, including static behaviors [3,22,31], linear and nonlinear dynamic phenomena [20,35,43], aerodynamic instabilities [4,47], responses to traveling forces and masses [5,6], active vibration control [7,45].

In this Section, as primary character of novelty, two original perturbation schemes are formulated to provide a general, consistent and unitary strategy to solve the static problem and modal problem for inextensible cables. The leading idea is to adapt classical perturbation schemes to work in the framework of indeformable (inextensible) but statically and dynamically indeterminate (hyperstatic) systems, by following an ordered solution scheme that resembles or generalizes—at each perturbation order—the Force Method for deformable structures. Therefore, the solution algorithm is referred as *generalized Force Method* in the following. The primary objective is to pursue fully analytical static and modal solutions, expressed as explicit parametric functions of the assigned mechanical data (aspect ratio Λ).

4.1 Quasi-analytical solutions

The inextensible cable can be classified as a statically indeterminate system, with unitary degree of hyperstaticity. The hyperstatic unknown is the horizontal reaction H_{s}. Accordingly, the funicular Eq. (27) is a nonlinear static equilibrium equation in the configuration variable $y(x)$, where the role of hyperstatic unknown is played by the dimensionless parameter δ.

The funicular equation can be solved analytically by exploiting the properties of hyperbolic functions. The solution is the well-known catenary function

$$y = \frac{\sinh(4\delta x)\sinh(4\delta(1-x))}{4\delta} \tag{36}$$

where the hyperstatic unknown δ has to be assessed a posteriori. The catenary function is symmetric over the x-domain and attains its maximum at the cable midspan ($x = 1/2$), as expected. Having assumed the span as reference length, the maximum value $s = y(1/2)$ can be referred to as cable *sag-to-span* ratio.

The role of compatibility condition for the assessment of the hyperstatic unknown is played by the *static inextensibility condition* $L_{\mathsf{s}} = L_{\mathsf{o}}$, which reads $\Lambda_{\mathsf{s}} = \Lambda$ in nondimensional form, where

$$\Lambda_{\mathsf{s}} = \int_0^1 \left[1 + \left(y'\right)^2\right]^{1/2} dx = \frac{\sinh(4\delta)}{4\delta} \tag{37}$$

is the *catenary arc-length*, which is a monotonically increasing function for positive δ-values.

In summary, the catenary function (36) must be considered—as a whole—a quasi-analytical parametric function, since the parameter δ cannot be determined from the compatibility equation $\sinh(4\delta) = 4\Lambda\delta$, except numerically. The solution is unique, because the meaningless negative solution has to be discarded.

By assuming the quasi-analytical catenary configuration as known static reference, significant contributions to the modal solutions for inextensible cables are attributed to Rohrs [39], Saxon and Cahn [40] and Goodey [10], under different simplifying assumptions.

4.2 Asymptotic static solutions

Founding the perturbation strategy on a solid physical base requires some preliminary mechanical considerations. The leading idea is that the static equilibrium configuration of structural cables tends to be characterized by spatial closeness to the rectilinear chord between the supports. This geometrical property, typically exalted by the cable lightness (small w) and high levels of static tension (large H_{s}) is classically referred to as cable *shallowness* in the literature.

According to the assumption of cable shallowness, the solution $y(x)$ of the funicular Eq. (27) can be postulated a priori to be significantly smaller than the inter-support distance. This postulate is mathematically equivalent to assume that the solution $y(x)$ can be expressed as an asymptotically convergent series of n terms $y_i(x)$ with growing orders of smallness. The ordering of successive terms is provided by increasing integer powers of a small dimensionless ordering parameter $\epsilon \ll 1$, acting as addend multipliers. The series reads

$$y_{[n]}(x) = \sum_{i=1}^{n} \epsilon^i\, y_i(x) = \\ = \epsilon\, y_1(x) + \epsilon^2 y_2(x) + \cdots + \\ + \epsilon^i y_i(x) + \cdots + \epsilon^n y_n(x) \tag{38}$$

where functions $y_i(x)$ serve as independent unknown configurational variables. Individually, the variable $y_i(x)$ can be denoted as i-th *configurational sensitivity*.

Coherently with the key assumption of cable shallowness, the setting of the perturbation scheme must be completed by imposing the smallness of the hyperstatic unknown $\delta = wL/(8H_\mathsf{S})$. Therefore, the ordering

$$\delta_{[n]} = \epsilon\delta_1 + \epsilon^2\delta_2 + \cdots + \epsilon^i\delta_i + \cdots + \epsilon^n\delta_n \tag{39}$$

can be introduced. The scaled quantities δ_i serve as independent unknowns. Individually, the quantity δ_i can be denoted as i-th *hyperstatic sensitivity*.

Within the perturbation framework, solving the static problem consists in assessing all the hyperstatic sensitivities δ_i required to determine the configurational sensitivities $y_i(x)$ for $i = 1, \ldots, n$. Subsequently, after complete reabsorption of the ordering parameter ϵ, the solution $y_{[n]}(x)$, reconstructed up to n-th order, is expected to asymptotically tend to the exact catenary solution $y(x)$ for growing orders n. From the operational viewpoint, the series $y_{[n]}(x)$ must be necessarily truncated at a non-infinite number n of terms. Although truncation is inevitable, the achievable accuracy is controllable, since a certain *approximation order* n implies residual difference of the order $\mathcal{O}(\epsilon^{n+1})$. As character of originality with respect to similar perturbation schemes applied to static problems of shallow cables, it might be remarked that only *unknown* physical quantities require to be ordered a priori, whereas the correct ordering of the mechanical data (aspect ratio Λ) is discovered only a posteriori, for consistency reasons.

The variable expression (38) and the parameter ordering (39) can be introduced in Eq. (27). Then, expansion and collection of same ϵ-power terms gives an ordered hierarchical system of nonlinear *Perturbation Equilibrium Equations* (PPE). By retaining orders up to the sixth ($n = 6$), the system reads

$$\epsilon^1 : y_1'' = -8\delta_1 \tag{40}$$

$$\epsilon^2 : y_2'' = -8\delta_2 \tag{41}$$

$$\epsilon^3 : y_3'' = -8\delta_3 - 4\delta_1 (y_1')^2 \tag{42}$$

$$\epsilon^4 : y_4'' = -8\delta_4 - 8\delta_1 y_1' y_2' - 4\delta_2 (y_1')^2 \tag{43}$$

$$\epsilon^5 : y_5'' = -8\delta_5 - 8\delta_1 y_1' y_3' - 8\delta_2 y_1' y_2' + \\ - 4\delta_1 (y_2')^2 - 4\delta_3 (y_1')^2 + \delta_1 (y_1')^4 \tag{44}$$

$$\epsilon^6 : y_6'' = -8\delta_6 - 8\delta_1 y_2' y_3' - 8\delta_2 y_1' y_3' - 8\delta_3 y_1' y_2' + \\ - 8\delta_1 y_1' y_4' - 4\delta_2 (y_2')^2 - 4\delta_4 (y_1')^2 + \\ + \delta_2 (y_1')^4 + 4\delta_1 y_2' (y_1')^3 \tag{45}$$

while homogeneous boundary conditions $y_i(0) = 0$ and $y_i(1) = 0$ must be imposed at each order. Naturally, higher orders ($n > 6$) can be taken into account to improve the solution accuracy, if necessary.

The coupled system of PPE can be attacked by analytically solving the equations in cascade of substitutions. The uncoupled linear equation at the lowest order provides the *generating solution* $y_1(x)$. Thereafter (for $i > 1$), the i-th perturbation order ϵ^i systematically establishes a linear non-homogeneous equation stating that the second derivative of the i-th unknown variable $y_i''(x)$ equates a known term at right hand, consisting in a polynomial function of the first derivatives of all the lower-order variables. Since the generating solution is a second x-degree (quadratic) function, the higher ϵ-order solutions are polynomials of increasing x-degrees (cubic, quartic, quintic,…). Specifically, after integration and imposition of boundary conditions, the solution of the i-th order (power of the perturbation parameter ϵ) is a polynomial function including terms up to the $(i + 1)$-th degree (power of the variable x)

$$\epsilon^i : y_i(x) = \sum_{j=0}^{i+1} c_{ij} x^j = \\ = c_{i1} x + c_{i2} x^2 + \cdots + c_{i(i+1)} x^{i+1} \tag{46}$$

where coefficients c_{ij}—here not reported for the sake of synthesis—generally depend on all the unknown hyperstatic sensitivities δ_k (with $k \leq i$).

Remarkably, the lowest order approximation provided by the generating solution $y_1 = 4\delta_1 x(1-x)$ describes the so-called *parabolic cable*, having sag-to-span ratio $s_1 = y_1(1/2)$ equal to δ_1. Physically, this means that δ can be interpreted—to a literally first approximation—as the sag-to-span ratio of the parabolic inextensible cable characterized by the aspect ratio Λ.

Within the perturbation framework, the generalized Force Method requires to univocally assessed all the hyperstatic sensitivities δ_i a posteriori, with the aim of identifying the unique geometrically compatible solution among all the statically determinable polynomial functions $y_{[n]}(x)$ satisfying the Perturbation Equilibrium Equations. The static inextensibility condition serving as compatibility equation is imposed in the form $\Lambda_{\mathsf{p}} = \Lambda$. The quantity Λ_{p} is the *polynomial arc-length* or—rigorously—the length of the polynomial curve mapped by the series function $y_{[n]}$, which is determinable as

$$\Lambda_{\mathsf{p}} = \int_0^1 \left[1 + \left(y'_{[n]}\right)^2\right]^{1/2} dx \tag{47}$$

where the derivative

$$\begin{aligned} y'_{[n]} &= \sum_{i=1}^{n} \epsilon^i y'_i(x) = \sum_{i=1}^{n} \epsilon^i \sum_{j=0}^{i+1} (c_{ij} x^j)' = \\ &= \sum_{i=1}^{n} \epsilon^i (c_{i1} + 2c_{i2}x + \cdots + (i+1)c_{i(i+1)}x^i) \end{aligned} \tag{48}$$

is understood. By operating coherently with the perturbation scheme, the integrand term in Eq. (47) can be expressed as ϵ-power series, and the linearity of integration can be invoked. Consequently, the arc-length Λ_{p} is asymptotically expressible in form of the ϵ-power series $\Lambda_{\mathsf{p}[n]} = \Lambda_{\mathsf{p}0} + \epsilon\,\Lambda_{\mathsf{p}1} + \epsilon^2 \Lambda_{\mathsf{p}2} + \cdots + \epsilon^i \Lambda_{\mathsf{p}i} + \cdots + \epsilon^n \Lambda_{\mathsf{p}n}$, which tends to the catenary length Λ_{s} for growing approximation orders n. Solving analytically the integrals returns that the lowest-order terms of the series are $\Lambda_{\mathsf{p}0} = 1$ and $\Lambda_{\mathsf{p}1} = 0$, while the high-order terms read

$$\Lambda_{\mathsf{p}2} = \tfrac{8}{3}\delta_1^2 \tag{49}$$

$$\Lambda_{\mathsf{p}3} = \tfrac{16}{3}\delta_1\delta_2 \tag{50}$$

$$\Lambda_{\mathsf{p}4} = \tfrac{8}{3}\delta_2^2 + \tfrac{16}{3}\delta_1\delta_3 + \tfrac{32}{15}\delta_1^4 \tag{51}$$

$$\Lambda_{\mathsf{p}5} = \tfrac{16}{3}\delta_1\delta_4 + \tfrac{16}{3}\delta_2\delta_3 + \tfrac{128}{15}\delta_1^3\delta_2 \tag{52}$$

$$\begin{aligned} \Lambda_{\mathsf{p}6} = {} & \tfrac{8}{3}\delta_3^2 + \tfrac{16}{3}\delta_2\delta_4 + \tfrac{16}{3}\delta_1\delta_5 + \tfrac{64}{5}\delta_1^2\delta_2^2 + \\ & + \tfrac{128}{15}\delta_1^3\delta_3 + \tfrac{256}{315}\delta_1^6 \end{aligned} \tag{53}$$

where, from the viewpoint of the perturbation strategy, it is worth noting that the arc-length $\Lambda_{\mathsf{p}[n]}$ depends on all the hyperstatic sensitivities δ_i up to $i = n-1$.

Before finalizing the solution of the static problem, the partial results provided by Eqs. (49)–(53) deserve a brief discussion. In synthesis, the generalized Force Method for inextensible cables requires (i) assigning the aspect ratio Λ as the only data, and (ii) enforcing the compatibility condition to assess the hyperstatic unknown δ. Outside the perturbation framework, satisfying the *admissibility inequality* $\Lambda > 1$ is sufficient to obtain an admissible solution of the compatibility condition $\Lambda_{\mathsf{s}} = \Lambda$. Differently, within the framework of the perturbation strategy, the data must satisfy an extra requirement. Indeed, Eqs. (49)–(53) disclose that the compatibility condition $\Lambda_{\mathsf{p}} = \Lambda$ can be satisfied without violating the assumption of cable shallowness (mathematically $\delta \in \mathcal{O}(\epsilon)$) if and only if the assigned aspect ratio respects the requirement $\Lambda = 1 + \mathcal{O}(\epsilon^2)$, for the sake of consistency with the arc-length $\Lambda_{\mathsf{p}} = 1 + \mathcal{O}(\epsilon^2)$. Consequently, the requirement $\Lambda = 1 + \mathcal{O}(\epsilon^2)$ will be denoted as *consistency requirement* in the following. This discussion demonstrates that the aspect ratio Λ has not to be postulated, but is mathematically *required* to be quasi-unitary. Practically, the consistency is automatically verified by introducing the new mechanical parameter Λ_2 that describes the small natural extra-length exceeding the inter-support distance, so that the aspect ratio can be expressed as $\Lambda = 1 + \epsilon^2\Lambda_2$. Methodologically, it can be remarked that the ordering of the natural cable length (aspect ratio $\Lambda = 1+\epsilon^2\Lambda_2$) is a necessary consequence of the cable shallowness ($\delta = \epsilon\delta_1 + \epsilon^2\delta_2 + \cdots + \epsilon^n\delta_n$), so that it does not requires to be assumed or imposed a priori. Physically, the mathematical findings formally confirm the intuitive idea that inextensible but shallow cables must have natural length greater (inextensibility condition), but not much greater (shallowness postulate) than the distance between the supports.

Once the aspect ratio is conveniently ordered in the form $\Lambda = 1+\epsilon^2\Lambda_2$ to automatically respect the consis-

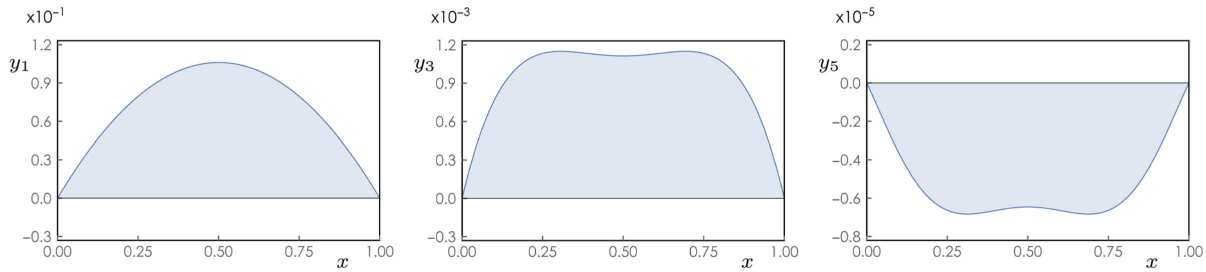

Fig. 2 Configurational sensitivities $y_i(x)$ of increasing orders ($i = 1, 3, 5$) for an inextensible cable with aspect ratio $\Lambda = 103/100$

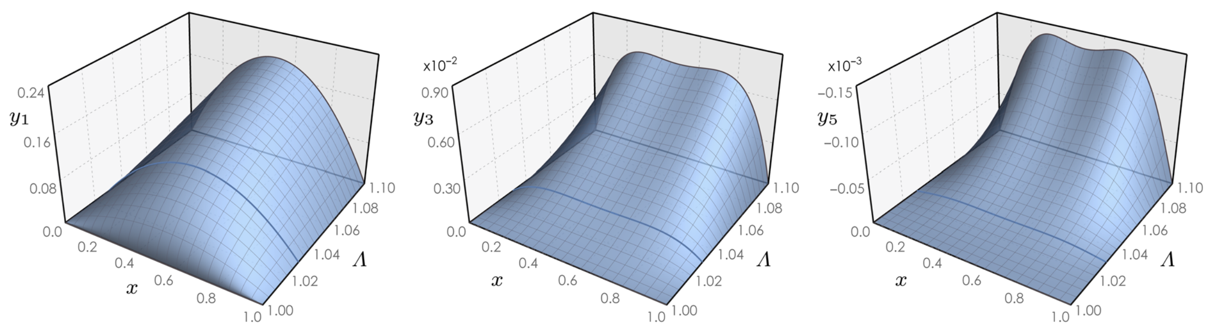

Fig. 3 Configurational sensitivities $y_i(x)$ of increasing orders ($i = 1, 3, 5$) for inextensible cables with different aspect ratios Λ

tency requirement, the inextensibility condition can be imposed in the asymptotic form $\Lambda_{p[n]} = 1 + \epsilon^2 \Lambda_2$. By employing solutions (49)–(53) at the left hand and collecting terms of the same ϵ-order, an ordered hierarchy of algebraic linear *Perturbation Compatibility Equations* (PCE) in the unknown hyperstatic sensitivities δ_i is stated. Low-order PCE ($n = 0, 1$) are identically satisfied. High-order PCE ($n > 2$) are homogeneous equations. By solving in cascade, the solutions read

$$\delta_1 = \tfrac{1}{4} \Lambda_1 \tag{54}$$

$$\delta_3 = -\tfrac{3}{80} \Lambda_1 \Lambda_2 \tag{55}$$

$$\delta_5 = \tfrac{321}{22400} \Lambda_1 \Lambda_2^2 \tag{56}$$

where the auxiliary quantity $\Lambda_1 = (6\Lambda_2)^{1/2}$ has been introduced. All the even coefficients $\delta_2, \delta_4, \ldots$ are systematically null (meaning that the alternative ordering $\delta_{[n]} = \epsilon\delta_1 + \epsilon^3\delta_3 + \cdots + \epsilon^{2n+1}\delta_{2n+1}$ could be adopted for the hyperstatic unknown without loss of generality).

Once all the hyperstatic sensitivities δ_i are assessed, the unique compatible configurational function $y_{[n]}(x)$ is determined by substituting solutions (54)–(56) into the coefficients c_{ij} of the solutions (46). After substitutions, the odd i-th configurational sensitivities read

$$y_1(x) = \Lambda_1 x(1 - x) \tag{57}$$

$$y_3(x) = \tfrac{1}{20} \Lambda_1 \Lambda_2 x(1 - x)(17 - 40x(1 - x)) \tag{58}$$

$$y_5(x) = \tfrac{1}{5600} \Lambda_1 \Lambda_2^2 x(1 - x)(560\, x\, c(x) - 519) \tag{59}$$

where the cubic function $c(x) = (x-1)(4x-3)(4x-1)$. All the even sensitivities $y_2(x), y_4(x), \ldots$ are systematically null (meaning that the alternative ordering of the configurational function $y_{[n]} = \epsilon y_1 + \epsilon^3 y_3 + \cdots + \epsilon^{2n+1} y_{2n+1}$ could be adopted without loss of generality). The configurational function $y_{[n]}(x)$ is symmetric, as expected, because all the sensitivities $y_i(x)$ are symmetric.

To summarize, the primary physical–mathematical achievement is that a fully analytical—although asymptotically approximate—solution, depending only on the aspect ratio Λ, has been determined for the configurational cable function $y_{[n]}(x)$. From the mechanical viewpoint, it is interesting to recognize that the lowest-order asymptotic approximation of the configurational function $y_{[1]}(x)$ describes the well-known and widely used parabolic configuration. The full set ($i = 1, 3, 5$) of functions $y_i(x)$ is illustrated in Fig. 2 for a partic-

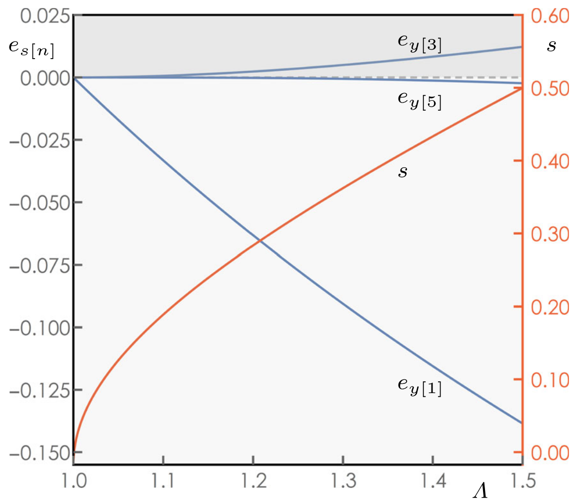

Fig. 4 Approximation accuracy of the asymptotic configuration function $y_{[n]}$ at midspan. Differences $e_{s[n]}$ at increasing orders ($n = 1, 3, 5$) for inextensible cables with growing aspect ratios Λ, corresponding to increasing sag s

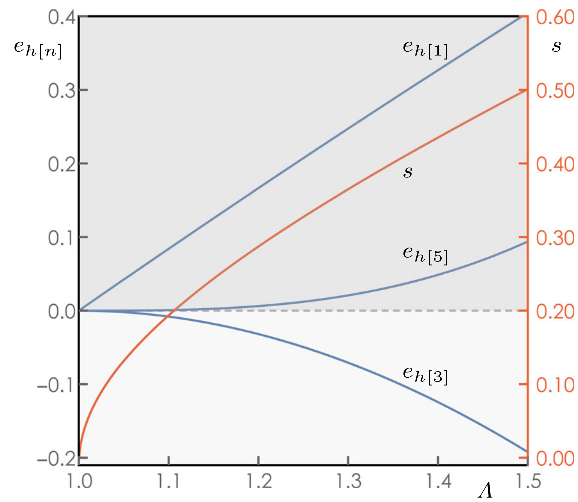

Fig. 5 Approximation accuracy of the asymptotic horizontal reaction $h_{[n]}$. Differences $e_{h[n]}$ at increasing orders ($n = 1, 3, 5$) for inextensible cables with growing aspect ratios Λ, corresponding to increasing sag s

ular shallow cable and in Fig. 3 for cables with varying aspect ratios Λ in the large range $(1, 11/10]$, that approximately corresponds to catenary sag s in the range $(0, 1/5]$. Qualitatively, it can be observed that high-order contributions provide either amplification (positive sensitivity $y_3(x)$) or deamplification (negative sensitivity $y_5(x)$) of the parabolic configuration. Quantitatively, the growing smallness of the function amplitudes for increasing orders can be appreciated.

Parametric analyses focused on the sag of the exact catenary function s compared with the sag of the asymptotic approximations $s_{[n]} = y_{[n]}(1/2)$ for increasing aspect ratios Λ are illustrated in Fig. 4. The approximation accuracy is measured by the relative difference $e_{s[n]} = (s_{[n]} - s)/s$. Qualitatively, the comparison shows that the lowest order approximation underestimates the exact sag (negative $e_{s[1]}$), whereas the higher order approximations provide either small overestimations (positive $e_{s[3]}$) or very small underestimations (negative $e_{s[5]}$). Quantitatively, the approximation accuracy achievable at the highest order is excellent, with relative differences $|e_{s[5]}| < 0.0025$ for large aspect ratio $\Lambda = 15/10$, corresponding to *not-so-shallow* cables with sag $s \simeq 1/2$ or even larger.

Once the hyperstatic unknown δ and the configurational function $y(x)$ have been (asymptotically) determined, the horizontal reaction h_s can be determined according to the (exact) relation $h_s = 1/(4\Lambda\delta)$. By consistently employing the series expansion of the variables δ and the ordering of the parameter Λ, the asymptotic expression the horizontal reaction can be determined in the series form $h_{[n]} = \epsilon^{-1}h_{-1} + \epsilon^0 h_0 + \epsilon^1 h_1 + \epsilon^2 h_2 + \cdots + \epsilon^i h_i + \cdots + \epsilon^n h_n$. It is particularly important to remark that—consistently with the perturbation scheme—the lowest-order approximation $h_{[-1]} \in \mathcal{O}(\epsilon^{-1})$ and, consequently, the j-th term of the series belong to the i-th $= (j-2)$-th order $\mathcal{O}(\epsilon^i)$. Methodologically, this finding is a necessary consequence of the assumption of cable shallowness (39), and cannot be introduced as independent hypothesis. Physically, this mathematical result formalizes the mechanical concept that shallow cables have large horizontal reactions, which—as literally first approximation—are inversely proportional to the small midspan sag. Recalling the consistency condition $\Lambda = 1 + \epsilon^2\Lambda_2$ and the asymptotic expression $\delta_{[n]}$, after new series expansion and collection of the same ϵ-power terms, the odd coefficients of the series read

$$h_{-1} = \frac{1}{\Lambda_1} \tag{60}$$

$$h_1 = -\frac{17\,\Lambda_1}{20} \tag{61}$$

$$h_3 = \frac{913\,\Lambda_1\Lambda_2}{6720} \tag{62}$$

while even coefficients $h_2, h_4, \ldots$ are systematically null. Starting from the definitions of the vertical reaction $\nu_{\mathsf{s}} = h_{\mathsf{s}} y'(0)$ and axial tension $n_{\mathsf{s}}(x) = h_{\mathsf{s}}[1 + (y'(x))^2]^{1/2}$, analogous procedures can be applied to achieve their asymptotic expressions $\nu_{[n]} = \epsilon^{-1} h_{-1} + \epsilon^0 \nu_0 + \epsilon^1 \nu_1 + \epsilon^2 \nu_2 + \cdots + \epsilon^i \nu_i + \cdots + \epsilon^n \nu_n$ and $n_{[n]}(x) = \epsilon^{-1} n_{-1}(x) + \epsilon^0 n_0(x) + \epsilon^1 n_1(x) + \epsilon^2 n_2(x) + \cdots + \epsilon^i n_i(x) + \cdots + \epsilon^n n_n(x)$, not reported here for the sake of conciseness.

Parametric analyses focused on the exact horizontal reactions h_{s} compared with the asymptotic approximations $h_{[n]}$ for increasing aspect ratios Λ are illustrated in Fig. 5. The approximation accuracy is measured by the relative difference $e_{h[n]} = (h_{\mathsf{s}} - h_{[n]})/h_{\mathsf{s}}$. Qualitatively, the comparison shows that the lowest order approximation overestimates the exact reaction (positive $e_{h[1]}$), whereas the higher order approximations provide either small underestimations (negative $e_{h[3]}$) or very small overestimations (positive $e_{h[5]}$). Quantitatively, the approximation accuracy achievable at the highest order is satisfying, with relative differences $|e_h| < 0.1$ for large aspect ratio $\Lambda = 15/10$, corresponding to *not-so-shallow* cables with sag-to-span ratios larger than 1/2.

4.3 Asymptotic modal solutions

The linearized dynamic problem governed by partial differential Eqs. (33)–(34) can be conveniently transferred in the frequency domain by applying the Fourier Transform, which applies to the generic dynamic function $f(x,\tau) : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ as the integral $\mathcal{F}[f(x,\tau)] = \phi_f(x,\omega) = \int_{-\infty}^{\infty} f(x,\tau) \exp(-\imath\omega\tau)\, d\tau$, where $\omega \in \mathbb{R}$ is the Fourier variable and $\phi_f(x,\omega) : \mathcal{D} \times \mathbb{R} \to \mathbb{R}$ is the transformed function. After Fourier transformation, the equations of motion become

$$\phi_t' + \omega^2 \phi_u \left[1 + (y')^2\right]^{1/2} = 0 \tag{63}$$

$$h_{\mathsf{s}}\left[\phi_v' \left(1 + (y')^2\right)\right]' + (y'\phi_t)' + \\ + \omega^2 \phi_v \left[1 + (y')^2\right]^{1/2} = 0 \tag{64}$$

where $\phi_u(x,\omega) = \mathcal{F}[u(x,\tau)]$, $\phi_v(x,\omega) = \mathcal{F}[v(x,\tau)]$ and $\phi_t(x,\omega) = \mathcal{F}[t_{\mathsf{d}}(x,\tau)]$ are the transformed variables. The corresponding boundary conditions read $\phi_u(0,\omega) = 0, \phi_u(1,\omega) = 0, \phi_v(0,\omega) = 0, \phi_v(1,\omega) = 0$. If the transformed inertial forces are conveniently normalized by setting $\omega^2 = h_r \beta^2$, Eqs. (63)–(64) become

$$\phi_t' + h_r \beta^2 \phi_u \left[1 + (y')^2\right]^{1/2} = 0 \tag{65}$$

$$h_s\left[\phi_v' \left(1 + (y')^2\right)\right]' + (y'\phi_t)' + \\ + h_r \beta^2 \phi_v \left[1 + (y')^2\right]^{1/2} = 0 \tag{66}$$

where $\beta \in \mathbb{R}$ is the normalized dimensionless Fourier variable and $h_r = 1/\Lambda_1$ plays the role of normalization factor for the inertial forces. Physically, the quantity h_r corresponds to the lowest order asymptotic approximation h_{-1} of the static horizontal reaction. Moreover, by applying the Fourier Transform to the dynamic indeformability condition (35), the internal coupling

$$\phi_u' = -y'\phi_v' \tag{67}$$

is established between the transformed functions $\phi_v(x,\omega)$ and $\phi_u(x,\omega)$.

In synthesis, the Fourier transformation determines a new coupled linear problem governed by a pair of homogeneous ordinary differential equations with (known and unknown) variable coefficients. From the mathematical viewpoint, the problem can be classified as an internally constrained Sturm-Liouville multi-variable eigenproblem, in which the quantity β plays the role of unknown eigenvalue, while the β-dependent variables $\phi_u(x,\beta)$, $\phi_v(x,\beta)$ and $\phi_t(x,\beta)$ play the role of unknown eigenfunctions (where the relation $\omega^2 = h_r\beta^2$ is used for the change of Fourier variable). Infinite eigensolutions, or *eigenquartets* $(\beta, \phi_u(x,\beta), \phi_v(x,\beta), \phi_t(x,\beta))$ exist. From the physical viewpoint, the quantities β and ω can be interpreted as wavenumber and circular frequency of free harmonic cable oscillations, spatially shaped by the mode $\boldsymbol{\phi}(x,\omega) = (\phi_u(x,\omega), \phi_v(x,\omega), \phi_t(x,\omega))$, where the eigenfunctions $\phi_u(x,\omega)$, $\phi_v(x,\omega)$ and $\phi_t(x,\omega)$ can be regarded as modal components related to the displacements $u(x,\tau)$ and $v(x,\tau)$ and the horizontal force $t_{\mathsf{d}}(x,\tau)$, respectively. Therefore, solving the eigenproblem is equivalent to perform a classical modal analysis for the inextensible cable.

Coherently with the methodological approach to the static problem and in the absence of analytically exact eigensolutions, the eigenproblem governed by Eqs. (65)–(66) and the internal constraint (67) can be attacked by employing a perturbation method. For the sake of asymptotic rigor, the static horizontal force h_s and the catenary function $y(x)$, playing the role of known coefficients in the problem equations, are replaced by their asymptotic expressions

$$h_{[n]} = \epsilon^{-1} h_{-1} + \epsilon^{1} h_1 + \epsilon^{3} h_3 + \cdots + \\ + \epsilon^{i} h_i + \cdots + \epsilon^{n} h_n \tag{68}$$

$$y_{[n]}(x) = \epsilon^{1} y_1(x) + \epsilon^{3} y_3(x) + \cdots + \\ + \epsilon^{i} y_i(x) + \cdots + \epsilon^{n} y_n(x) \tag{69}$$

where only the not-null odd ϵ-power terms have been considered. The lowest horizontal reaction sensitivities h_i (for $i = -1, 1, 3$) and configurational sensitivities $y_i(x)$ (for $i = 1, 3, 5$) have been determined analytically in Sect. 4.2 (see Eqs. (57)–(59) and (60)–(62), respectively). The last known coefficient is the normalization factor h_r, which is $h_r = h_{[-1]}$ by definition and can be ordered accordingly $h_r = \epsilon^{-1} h_{-1}$. Therefore, following a standard perturbation scheme, the unknown coefficient β—playing also the role of eigenvalue—is postulated to be asymptotically expressible in the form

$$\beta_{[n]} = \epsilon^{0} \beta_0 + \epsilon^{1} \beta_1 + \epsilon^{2} \beta_2 + \cdots + \\ + \epsilon^{i} \beta_i + \cdots + \epsilon^{n} \beta_n \tag{70}$$

where the quantities β_i must be regarded as independent unknowns. The unknown β-dependent variables $\phi_u(x)$, $\phi_v(x)$ and $\phi_t(x)$—playing also the role of eigenfunctions—are postulated to be asymptotically expressible in the form

$$\phi_{u[n]}(x) = \sum_{i=2}^{n} \epsilon^{i} \phi_{ui}(x) = \\ = \epsilon^{2} \phi_{u2}(x) + \epsilon^{3} \phi_{u3}(x) + \cdots + \\ + \epsilon^{i} \phi_{ui}(x) + \cdots + \epsilon^{n} \phi_{un}(x) \tag{71}$$

$$\phi_{v[n]}(x) = \sum_{i=1}^{n} \epsilon^{i} \phi_{vi}(x) = \\ = \epsilon^{1} \phi_{v1}(x) + \epsilon^{2} \phi_{v2}(x) + \cdots + \\ + \epsilon^{i} \phi_{vi}(x) + \cdots + \epsilon^{n} \phi_{vn}(x) \tag{72}$$

$$\phi_{t[n]}(x) = \sum_{i=0}^{n} \epsilon^{i} \phi_{ti}(x) = \\ = \epsilon^{0} \phi_{t0}(x) + \epsilon^{1} \phi_{t1}(x) + \cdots + \\ + \epsilon^{i} \phi_{ti}(x) + \cdots + \epsilon^{n} \phi_{tn}(x) \tag{73}$$

where the functions $\phi_{ui}(x)$, $\phi_{vi}(x)$ and $\phi_{ti}(x)$ play the role of independent unknown β-dependent variables. The i-th quantity β_i is referred to as i-th *eigenvalue sensitivity*, or i-th *β-eigensensitivity*, while the i-th quantities $\phi_{ui}(x)$, $\phi_{vi}(x)$ and $\phi_{ti}(x)$ are referred to as i-th *eigenfunction sensitivities*, or i-th *ϕ-eigensensitivity*.

Once all the eigenvalue sensitivities β_i and eigenfunction sensitivities $\phi_{ui}(x)$, $\phi_{vi}(x)$ and $\phi_{ti}(x)$ are determined (for $i = 1, \ldots, n$), the series $\beta_{[n]}$ and $\phi_{u[n]}(x)$, $\phi_{v[n]}(x)$, $\phi_{t[n]}(x)$ are expected to asymptotically tend to the exact eigensolution for growing n-values (increasing *approximation orders*). Leaving aside other mathematical considerations, the different ordering of smallness postulated for the eigenfunctions may deserve some mechanical justifications. First, the order $\mathcal{O}(\epsilon)$ postulated for the eigenfunction $\phi_v(x)$ complies with the physical requirement that the small amplitudes of the vertical oscillations in linearized free dynamics cannot be larger than the catenary sag of the horizontal inextensible cable (recall that $\delta \in \mathcal{O}(\epsilon)$). Second, the order $\mathcal{O}(\epsilon^2)$ attributed to the eigenfunction $\phi_u(x)$ reflects the physical evidence that the linearized dynamics of horizontal cables tends to be dominated by vertical oscillations (recall that $\phi_v(x) \in \mathcal{O}(\epsilon)$), whose small amplitudes prevail over the very small amplitudes of horizontal oscillations. Third, the order $\mathcal{O}(\epsilon^0)$ assigned to the eigenfunction $\phi_t(x)$ sustains the physical condition that the dynamic (oscillating) contribution to the horizontal tension cannot be larger than the static pretension (recall that $h_s \in \mathcal{O}(\epsilon^{-1})$), in order to preserve the geometric stiffness of the cable in the linearized dynamic regime.

The assumed ordering of the constant and variable coefficients (68)–(69) and the postulated asymptotic expressions for the eigenvalue (70) and eigenfunctions (71)–(73) can be introduced into the equations of motions (63)–(64). After expansion and collection of same ϵ-power terms, an ordered hierarchy of *Modal Perturbation Equations* (MPE) is obtained is obtained. By considering orders up to and including the fourth ($n = 4$), the system reads

ϵ^0:

$$\phi'_{t0} = 0 \tag{74}$$

$$h_{-1} \mathfrak{L}_0 \phi_{v1} = 0 \tag{75}$$

ϵ^1:

$$\phi'_{t1} = \mathfrak{d}_{t1}(\beta_0, \phi_{u2}) \tag{76}$$

$$h_{-1}\mathfrak{L}_0\phi_{v2} = \mathfrak{d}_{v1}(\phi_{t0}) + + \beta_1\mathfrak{a}_0(\beta_0, \phi_{v1}) \tag{77}$$

ϵ^2:

$$\phi'_{t2} = \mathfrak{d}_{t2}(\beta_0, \beta_1, \phi_{u2}, \phi_{u3}) \tag{78}$$

$$h_{-1}\mathfrak{L}_0\phi_{v3} = \mathfrak{d}_{v2}(\beta_0, \beta_1, \phi_{t1}, \phi_{v1}, \phi_{v2}) + + \beta_2\mathfrak{a}_0(\beta_0, \phi_{v1}) \tag{79}$$

ϵ^3:

$$\phi'_{t3} = \mathfrak{d}_{t3}(\beta_0, \ldots, \beta_2, \phi_{u2}, \ldots, \phi_{u4}) \tag{80}$$

$$h_{-1}\mathfrak{L}_0\phi_{v4} = \mathfrak{d}_{v3}(\beta_0, \ldots, \beta_2, \phi_{t0}, \phi_{t2}, \phi_{v1}, \ldots, \phi_{v3}) + + \beta_3\mathfrak{a}_0(\beta_0, \phi_{v1}) \tag{81}$$

ϵ^4:

$$\phi'_{t4} = \mathfrak{d}_{t4}(\beta_0, \ldots, \beta_3, \phi_{u2}, \ldots, \phi_{u5}) \tag{82}$$

$$h_{-1}\mathfrak{L}_0\phi_{v5} = \mathfrak{d}_{v4}(\beta_0, \ldots, \beta_3, \phi_{t1}, \phi_{t3}, \phi_{v1}, \ldots, \phi_{v4}) + + \beta_4\mathfrak{a}_0(\beta_0, \phi_{v1}) \tag{83}$$

where the pair of equations at each order ($MPE_{1,2}$) is equipped by four geometric boundary conditions $\phi_{ui}(0) = 0$, $\phi_{ui}(1) = 0$, $\phi_{vi}(0) = 0$, $\phi_{vi}(1) = 0$. The equilibrium relations $\phi_{ti}(0) = \phi_{hi}$ at the left support cannot be imposed directly as mechanical boundary conditions (because the modal horizontal reactions ϕ_{hi} are unknown), but can be invoked to employ the horizontal reactions ϕ_{hi} as hyperstatic unknowns at each order.

From the methodological viewpoint, it is important to recognize that the second equation of each ϵ^i-order is identically characterized by

- the linear differential operator $\mathfrak{L}_0 = \partial^2/\partial x^2 + \beta_0$, acting on the eigensensitivity $\phi_{v(i+1)}$ at the left hand
- the zeroth order coefficient, $\mathfrak{a}_0 = -2h_{-1}\beta_0\phi_{v1}$, multiplying the eigensensitivity β_i at the left hand,

while the *defects of homogeneity* $\mathfrak{d}_{ti}$ and $\mathfrak{d}_{vi}$ at the i-th order are reported in the Appendix. It is remarkable that the i-th pair of defects $\mathfrak{d}_{ti}$ and $\mathfrak{d}_{vi}$ depend on the static solution sensitivities up to $y_{(i-1)}$ and $h_{(i-1)}$.

Similarly, the asymptotic static solutions (68)–(69) and the postulated asymptotic expressions for the eigensolutions (70)–(73) can be introduced into the dynamic indeformability condition (67). After expansion and collection of same ϵ-power terms, an ordered hierarchy of *Perturbation Indeformability Conditions* (PIC) is obtained. The equations not identically satisfied read

$$\epsilon^2 : \phi'_{u2} = -y'_1\phi'_{v1} \tag{84}$$

$$\epsilon^3 : \phi'_{u3} = -y'_1\phi'_{v2} \tag{85}$$

$$\epsilon^4 : \phi'_{u4} = -y'_3\phi'_{v1} - y'_1\phi'_{v3} \tag{86}$$

$$\epsilon^5 : \phi'_{u5} = -y'_3\phi'_{v2} - y'_1\phi'_{v4} \tag{87}$$

$$\epsilon^6 : \phi'_{u6} = -y'_5\phi'_{v1} - y'_3\phi'_{v3} - y'_1\phi'_{v5} \tag{88}$$

where it is worth noting that the Perturbation Indeformability Condition governing the order ϵ^i is necessary to assess the unknown ϕ_{ui}, contributing to the Modal Perturbation Equation governing the order $\epsilon^{(i-1)}$.

The two hierarchies of MPE and PIC can be solved in cascade, by formulating a problem-specific strategy of attack. Methodologically, the solution strategy can be regarded as a sort of recursive, but hierarchically delayed, application of the generalized Force Method at each order, technically similar to the strategy adopted for the static problem. According to this general idea, the lowest zeroth order is solved to determine the *generating eigenquartet* $(\beta_0, \phi_{t0}, \phi_{v1}, \phi_{u2})$, while at the generic higher order ϵ^i all the eigenquartet sensitivities $(\beta_i, \phi_{ti}, \phi_{v(i+1)}, \phi_{u(i+2)})$ that are

- dynamically balanced, that is, satisfy the MPE (equilibrium equations in the frequency domain)
- geometrically compatible, that is, satisfy the PIC (indeformability conditions) and all—except one—the geometric boundary conditions of the MPE,

are determined as parametric functions of the support reaction ϕ_{hi}, playing the role of hyperstatic unknown. Finally, the hyperstatic unknown is determined a posteriori, with delay of one perturbation order, by imposing the last unsatisfied geometric boundary condition—serving as Compatibility Condition CC—at the order $\epsilon^{(i-1)}$. The perturbation strategy based on the generalized Force Method is sketched in Fig. 6, while the principal steps of the algorithm work as follows.

4.3.1 Zeroth order

The lowest order (power ϵ^0) of the MPE is governed by an uncoupled pair of homogeneous Eqs. (74)–(75), having fixed coefficients. The problem unknowns are the *generating eigenvalue* β_0 and the *generating eigenfunctions* ϕ_{t0} (modal force) and ϕ_{v1} (modal vertical

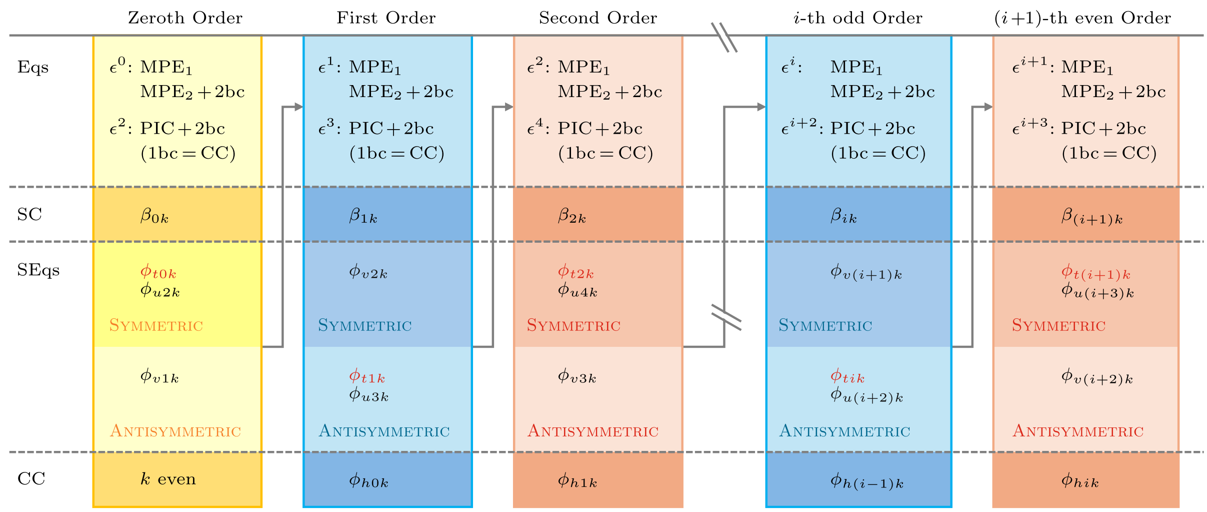

Fig. 6 Perturbation strategy based on the generalized Force Method for the linear eigenproblem. For each order, the scheme illustrates the governing Equations (Eqs) with boundary condition (bc), the eigenvalue sensitivity resulting from the Solvability Conditions (SC), the symmetric and antisymmetric eigenfunction sensitivities resulting from the Solvable Equations (SEqs), and the hyperstatic unknown resulting from the Compatibility Conditions (CC). Arrows indicate solutions used to specify equations of the next order. Solutions in red are dependent on the hyperstatic unknown to be determined at the next order. Solutions in black are dependent on the hyperstatic unknown to be determined at the same order (except for the zeroth order)

displacement). In parallel, the lowest order (power ϵ^2) of the PIC states a non-homogeneous Eq. (84) that relates the MPE solutions with the *generating eigenfunction* ϕ_{u2} (modal horizontal displacement), which can be counted as extra unknown.

The first MPE (74) is trivially solvable in the generating eigenfunction ϕ_{t0}, which results to be constant. By recalling the equilibrium relations $\phi_{t0}(0) = \phi_{h0}$ at the left support, the generating horizontal eigenreaction ϕ_{h0} can be assigned the role of lowest order hyperstatic unknown, to be determined at the next order.

The second MPE (75) states the linear eigenproblem $\phi''_{v1} + \beta_0^2 \phi_{v1} = 0$, which is analytically solvable. By imposing the boundary conditions $\phi_{v1}(0) = 0$ and $\phi_{v1}(1) = 0$, infinite eigenpairs are determined. The k-th eigenpair is composed by the generating eigenvalues β_{0k} and generating eigenfunctions ϕ_{v1k}, reading

$$\beta_{0k} = k\pi, \tag{89}$$

$$\phi_{v1k} = c_1 \sin(k\pi x) \tag{90}$$

with c_1 playing the role of undetermined amplitude. Each k-value corresponds to a different eigenquartet (with $k \in \mathbb{N}$). Remarkably, the generating eigensolutions can be distinguished into antisymmetric ($k = 2, 4, 6, \ldots$) and symmetric ($k = 1, 3, 5, \ldots$) functions.

Once the MPE solutions have been determined, the right hand of the PIC (84) is known. Therefore, the generating eigenfunction ϕ_{u2} can be determined by integrating and imposing the boundary conditions $\phi_{u2}(0) = 0$ and $\phi_{u2}(1) = 0$. Two cases must be distinguished

- *Even modes* ($k = 2, 4, 6, \ldots$)—The generating eigen functions ϕ_{v1k} are antisymmetric and the unknowns ϕ_{u2} always admit compatible and not trivial solutions. The k-th solution ϕ_{u2k} is

$$\phi_{u2k} = \Phi_{u2k}(x) \sin(k\pi x) + \\ + \Psi_{u2k}(x) \cos(k\pi x) + \Gamma_{u2k}(x) \tag{91}$$

  where the auxiliary x-dependent polynomial functions Φ_{u2k}, Ψ_{u2k} and Γ_{u2k} are reported in the Appendix. Generating eigenfunctions ϕ_{u2k} can be proved to be symmetric functions.
- *Odd modes* ($k = 1, 3, 5, \ldots$)—The generating functions ϕ_{v1k} are symmetric, but the unknowns ϕ_{u2k} do not admit any compatible solution, since the boundary conditions cannot be satisfied (for $c_1 \neq 0$).

The impossibility to satisfy the PIC for symmetric generating eigenfunction ϕ_{v1k} imposes to consider only even modes generated by values $k = 2p$ (with $p \in \mathbb{N}$). Consequently, all the generating eigensolutions are characterized by equispaced generating eigenvalues β_{0k}, antisymmetric generating eigenfunctions ϕ_{v1k} and symmetric generating eigenfunctions ϕ_{u2k} in all the higher orders.

From the mechanical viewpoint, the existence of only antisymmetric modes in the vertical oscillations of horizontal inextensible cables, based on the lowest order approximation of the catenary configuration (quadratic function y_1 in Eq. (84)), is coherent with the null elongation that characterizes antisymmetric modes in the transversal dynamics of parabolic extensible cables. As minor remark, it may be worth noting that the PIC are *strong* conditions of inextensibility, imposed locally on each infinitesimal element of the cable. Therefore, the existence of odd modes (corresponding to symmetric eigenfunctions ϕ_{v1k} and antisymmetric eigenfunctions ϕ_{u2k}) cannot be excluded if the strong inextensibility condition is relaxed and replaced by the *weak* condition imposing globally the identity between the dynamic and natural lengths ($\Lambda_{\mathsf{d}} = \Lambda_{\mathsf{o}}$) of the cable.

4.3.2 First order

The first order (power ϵ^1) of the MPE is governed by a coupled pair of non-homogeneous Eqs. (76)–(77), having variable coefficients. The problem unknowns are the eigenvalue sensitivity β_1 and the eigenfunction sensitivities ϕ_{t1} (modal force) and ϕ_{v2} (modal vertical displacement). In parallel, the first order (power ϵ^3) of the PIC states a non-homogeneous Eq. (85) that relates the MPE solutions (antisymmetric function ϕ_{v2k}) with the eigenfunction sensitivity ϕ_{u3} (modal horizontal displacement), countable as extra unknown.

The first MPE (76) must be solved in the first eigensensitivity ϕ_{t1} by integrating the defect of homogeneity $\mathfrak{d}_{t1}(\beta_0, \phi_{u2})$. By specifying the integrand function for the k-th solutions β_{0k} and ϕ_{u2k}, known from the lower order, the solution ϕ_{t1} for the k-th eigenquartet is the antisymmetric (minus ϕ_{h1k}) function

$$\phi_{t1k}(x) = \Phi_{t1k}(x)\sin(k\pi x) + \\ + \Psi_{t1k}(x)\cos(k\pi x) + \Gamma_{t1k}(x) + \phi_{h1k} \quad (92)$$

where the auxiliary x-dependent polynomial functions Φ_{t1k}, Ψ_{t1k} and Γ_{t1k} are reported in the Appendix. By recalling the equilibrium relations $\phi_{t1}(0) = \phi_{h1}$ at the left support, the first sensitivity ϕ_{h1k} of the horizontal eigenreaction has been employed as first order hyperstatic unknown, to be determined at the next order.

The second MPE (77) is not immediately solvable, because the differential operator $\mathfrak{L}_0$ is singular for β_{0k}, by construction. By invoking the Fredholm Alternative theorem, a non trivial solution ϕ_{v2} exists only if the defect of homogeneity $\mathfrak{d}_{v1}(\phi_{t0}) + \beta_1\mathfrak{a}_0(\beta_0, \phi_{v1})$ is orthogonal to the solution φ of the adjoint differential problem. Since the second MPE (77) can be demonstrated to state a self-adjoint problem (with self-adjoint boundary conditions), the solvability condition actually requires orthogonality between the defect of homogeneity and the self-solution $\varphi = \phi_{v1}$. By specifying the defect of homogeneity for the lower order solutions of the k-th eigenquartet, the solvability condition requires

$$\int_0^1 \phi_{v1k}\left[\mathfrak{d}_{v1}(\phi_{h0}) + \beta_1\mathfrak{a}_0(\beta_{0k}, \phi_{v1k})\right]dx = 0 \quad (93)$$

which is a linear algebraic equation for the first eigenvalue sensitivity β_1. Formally, the k-th solution reads

$$\beta_{1k} = -\frac{\int_0^1 \phi_{v1k}\ \mathfrak{d}_{v1}(\phi_{h0k})dx}{\int_0^1 \phi_{v1k}\ \mathfrak{a}_0(\beta_{0k}, \phi_{v1k})dx} \quad (94)$$

whose numerator can be demonstrated to identically vanish (because the integrand functions is antisymmetric), while denominator certainly differs from zero. Consequently the first eigenvalue sensitivity is identically null $\beta_{1k} = 0$ for all the eigenquartets. Enforcing this result into the defect of homogeneity makes Eq. (77) solvable. Therefore, by integrating and imposing boundary conditions $\phi_{v2}(0) = 0$ and $\phi_{v2}(1) = 0$, the solution ϕ_{v2k} for the k-th eigenquartet is the symmetric function

$$\phi_{v2k}(x) = c_2\sin(k\pi x) + \Psi_{v2k}\cos(k\pi x) + \Gamma_{v2k} \quad (95)$$

where the auxiliary ϕ_{h0}-dependent constants Ψ_{v2k} and Γ_{v2k} are reported in the Appendix. The boundary conditions state an algebraic problem of two equations in the two integration constants. Since the two equations are linearly dependent, only one unknown can be assessed, while the other (constant c_2) remains indeterminate. Nonetheless, the constant c_2 can be considered null ($c_2 = 0$) without loss of generality, since it multiplies an asymptotically redundant contribution that is parallel to the generating solution ϕ_{v1k}.

Once the MPE solutions have been determined, the right hand of the PIC (85) is known and the equation can be solved for the unknown ϕ_{u3}. By integrating and imposing one of the two remaining boundary condition $\phi_{u3}(0) = 0$, the solution ϕ_{u3} for the k-th eigenquartet is the antisymmetric (minus Γ_{u3k}) function

$$\phi_{u3k}(x) = \Phi_{u3k} \sin(k\pi x) + \Psi_{u3k}(x) \cos(k\pi x) + \Gamma_{u3k} \tag{96}$$

where the auxiliary ϕ_{h0}-dependent constants Φ_{u3k} and Γ_{u3k} and polynomial function $\Psi_{u3k}(x)$ are reported in the Appendix. To complete the solution strategy according to the generalized Force Method, the last boundary condition $\phi_{u3}(1) = 0$ is used as a sort of compatibility condition to determine the hyperstatic unknown ϕ_{h0}, remained undetermined from the lower order (delay of one perturbation order). The compatibility condition is satisfied only if the hyperstatic unknown is identically null $\phi_{h0} = 0$ for all k-values (all eigenquartets). Once the hyperstatic unknown has been assessed, the ϕ_{h0}-dependent solutions ϕ_{t0} (constant), ϕ_{v2k} (symmetric) and ϕ_{u3k} (antisymmetric) are fully determined a posteriori and, specifically, discovered to be null.

4.3.3 Second order

The second order (power ϵ^2) of the MPE is governed by a coupled pair of non-homogeneous Eqs. (78)–(79), with variable coefficients. The problem unknowns are the eigenvalue sensitivity β_2 and the eigenfunction sensitivities ϕ_{t2} (modal force) and ϕ_{v3} (modal vertical displacement). In parallel, the second order (power ϵ^4) of the PIC states a non-homogeneous Eq. (86) that relates the MPE solutions with the eigenfunction sensitivity ϕ_{u4} (modal horizontal displacement), which can be counted as extra unknown.

The first MPE (78) must be solved in the second eigensensitivity ϕ_{t2} by integrating the defect of homogeneity $\mathfrak{d}_{t2}(\beta_0, \beta_1, \phi_{u2}, \phi_{u3})$. By specifying the integrand function for the k-th solutions β_{0k}, β_{1k}, ϕ_{u2k}, ϕ_{u3k}, known from the lower orders, the defect of homogeneity results to be null. Accordingly, the second eigenfunction sensitivity ϕ_{t2} is constant. By recalling the equilibrium relations $\phi_{t2}(0) = \phi_{h2}$ at the left support, the second sensitivity ϕ_{h2k} of the horizontal eigenreaction has been employed as first order hyperstatic unknown, to be determined at the next order.

The second MPE (79) is not immediately solvable, because the differential operator $\mathfrak{L}_0$ is again singular for β_{0k}, by construction. Similarly to the previous order, the Fredholm Alternative theorem can be invoked to find a non trivial solution ϕ_{v3} by satisfying the solvability condition on the defect of homogeneity $\mathfrak{d}_{v2}(\beta_0, \beta_1, \phi_{t1}, \phi_{v1}, \phi_{v2}) + \beta_2 \mathfrak{a}_0(\beta_0, \phi_{v1})$. By specifying the defect of homogeneity for the lower order solutions of the k-th eigenquartet, the solvability condition states a linear algebraic equation for the second eigenvalue sensitivity β_2. Formally, the k-th solution reads

$$\beta_{2k} = -\frac{\int_0^1 \phi_{v1k}\, \mathfrak{d}_{v2}(\beta_{0k}, \beta_{1k}, \phi_{t1k}, \phi_{v1k}, \phi_{v2k}) dx}{\int_0^1 \phi_{v1k}\, \mathfrak{a}_0(\beta_{0k}, \phi_{v1k}) dx} \tag{97}$$

and, after substitution and integration (note that the integration eliminates the dependence on the hyperstatic unknown ϕ_{h1k}), the solution reads

$$\beta_{2k} = -\frac{\Lambda_1^2 \left(114 + k^2\pi^2\right)}{12\, k\pi} + \frac{k\pi h_1}{2h_{-1}} \tag{98}$$

which is the lowest not null eigenvalue sensitivity. Enforcing this result into the defect of homogeneity makes Eq. (79) solvable. Therefore, by integrating and imposing boundary conditions $\phi_{v3}(0) = 0$ and $\phi_{v3}(1) = 0$, the solution ϕ_{v3k} for the k-th eigenquartet is the antisymmetric (minus ϕ_{h1k}) function

$$\begin{aligned} \phi_{v3k}(x) = c_3 \sin(k\pi x) + \Phi_{v3k}(x) \sin(k\pi x) + \\ + \Psi_{v3k}(x) \cos(k\pi x) + \Gamma_{v3k}(x) \end{aligned} \tag{99}$$

where the auxiliary ϕ_{h1k}-dependent polynomial functions $\Phi_{v3k}(x)$, $\Psi_{v3k}(x)$ and $\Gamma_{v3k}(x)$ are reported in the Appendix. Similarly to the previous order, the two algebraic equations provided by the boundary conditions are linearly dependent and allow the assessment of one integration constant, leaving the other (constant c_3) indeterminate. Nonetheless, the constant c_3 can be considered null ($c_3 = 0$) without loss of generality.

Once the MPE solutions have been determined, the right hand of the PIC (86) is known and the equation can be solved for the unknown ϕ_{u4}. By integrating and imposing one of the two remaining boundary condition $\phi_{u4}(0) = 0$, the solution ϕ_{u4} for the k-th eigenquartet is the symmetric (minus ϕ_{h1k}) function

$$\phi_{u4k}(x) = \Phi_{u4k}(x)\sin(k\pi x) + \Psi_{u4k}(x)\cos(k\pi x) + \Gamma_{u4k}(x) \quad (100)$$

where the auxiliary ϕ_{h1k}-dependent polynomial functions $\Phi_{u4k}(x)$, $\Psi_{u4k}(x)$ and $\Gamma_{u4k}(x)$ are reported in the Appendix. To complete the solution strategy, the last boundary condition $\phi_{u4}(1) = 0$ is used as compatibility condition to determine the hyperstatic unknown ϕ_{h1k}, remained undetermined from the lower order. Specifically, the compatibility condition establishes a linear algebraic equation giving, for the k-th eigenquartet, the first sensitivity of the horizontal reaction

$$\phi_{h1k} = -2c_1 k\pi h_{-1}\Lambda_1 \quad (101)$$

which can be remarked to be directly proportional to the mode number k. Once the hyperstatic unknown has been assessed, the ϕ_{h1k}-dependent solutions ϕ_{t1} (antisymmetric), ϕ_{v2k} (antisymmetric) and ϕ_{u3k} (symmetric) are fully determined a posteriori.

4.3.4 Third order and higher odd orders

The third order (power ϵ^3) of the MPE is governed by a coupled system of two non-homogeneous Eqs. (80)–(81), with variable coefficients. The problem unknowns are the eigenvalue sensitivity β_3 and the eigenfunction sensitivities ϕ_{t3} (modal force) and ϕ_{v4} (modal vertical displacement). In parallel, the third order (power ϵ^5) of the PIC states a non-homogeneous Eq. (85) that relates the MPE solutions with the eigenfunction sensitivity ϕ_{u5} (modal horizontal displacement), which can be counted as extra unknown. Third order is the highest odd order whose solution deserves to be briefly discussed, due to some distinctive mathematical and algorithmic peculiarities with respect to lower odd orders. Higher odd orders (power $\epsilon^5, \epsilon^7, \ldots$) can be systematically attacked with the same methodological scheme and return formally similar solutions.

The first MPE (80) must be solved in the third eigensensitivity ϕ_{t3} by integrating the defect of homogeneity $\mathfrak{d}_{t3}(\beta_0, \ldots, \beta_2, \phi_{u2}, \ldots, \phi_{u4})$. By specifying the integrand function for the k-th solutions $\beta_{0k}, \ldots, \beta_{2k}$, $\phi_{u2k}, \ldots, \phi_{u4k}$, known from the lower orders, the solution ϕ_{t3} for the k-th eigenquartet is the antisymmetric (minus ϕ_{h3k}) function

$$\phi_{t3k}(x) = \Phi_{t3k}(x)\sin(k\pi x) + \Psi_{t3k}(x)\cos(k\pi x) + \Gamma_{t3k}(x) + \phi_{h3k} \quad (102)$$

where the auxiliary x-dependent polynomial functions Φ_{t3k}, Ψ_{t3k} and Γ_{t3k} are reported in the Appendix. By recalling the equilibrium relations $\phi_{t3}(0) = \phi_{h3}$ at the left support, the first sensitivity ϕ_{h3k} of the horizontal eigenreaction has been employed as first order hyperstatic unknown, to be determined at the next order.

The second MPE (81) is not immediately solvable, due to the singularity of the differential operator $\mathfrak{L}_0$ for β_{0k}. The solvability conditions for the existence of a non trivial solution ϕ_{v4} is imposed on the defect of homogeneity $\mathfrak{d}_{v3}(\beta_0, \ldots, \beta_2, \phi_{t0}, \phi_{t2}, \phi_{v1}, \ldots, \phi_{v3}) + \beta_3\mathfrak{a}_0(\beta_0, \phi_{v1})$, giving rise to in a linear algebraic equation in the third eigenvalue sensitivity β_3. By specifying the defect of homogeneity for the lower order solutions of the k-th eigenquartet, the solution indicates that the third eigenvalue sensitivity is identically null $\beta_{3k} = 0$. Enforcing this result into the defect of homogeneity makes Eq. (81) solvable. Therefore, by integrating and imposing the boundary conditions $\phi_{v4}(0) = 0$ and $\phi_{v4}(1) = 0$, the solution ϕ_{v4} for the k-th eigenquartet is the symmetric function

$$\phi_{v4k}(x) = c_4\sin(k\pi x) + \Psi_{v4k}\cos(k\pi x) + \Gamma_{v4k} \quad (103)$$

where the auxiliary ϕ_{h2k}-dependent constants Ψ_{v4k} and Γ_{v4k} are reported in the Appendix. The constant c_4, remained undetermined because the boundary conditions establish two linearly dependent equations if the two integration constants, can be considered null ($c_4 = 0$) without loss of generality.

Once the MPE solutions have been determined, the right hand of the PIC (87) is known and the equation can be solved for the unknown ϕ_{u5}. By integrating and imposing one of the two remaining boundary condition $\phi_{u5}(0) = 0$, the solution ϕ_{u5} for the k-th eigenquartet is the antisymmetric function

$$\phi_{u5k}(x) = \Phi_{u5k}(x)\sin(k\pi x) + \Psi_{u5k}(x)\cos(k\pi x) + \Gamma_{u5k}(x) \quad (104)$$

where the auxiliary ϕ_{h2k}-dependent polynomial functions $\Phi_{u5k}(x)$, $\Gamma_{u5k}(x)$ and $\Psi_{u5k}(x)$ are reported in the Appendix. The last boundary condition $\phi_{u5}(1) = 0$ is used as compatibility condition to determine the hyperstatic unknown ϕ_{h2k}, remained undetermined from the lower order. The compatibility condition is satisfied only if the hyperstatic unknown is identically null $\phi_{h2k} = 0$ for all k-values (all eigenquartets). Once the hyperstatic unknown has been assessed, the ϕ_{h2k}-dependent solutions ϕ_{t2k} (constant), ϕ_{v4k} (symmetric) and ϕ_{u5k} (antisymmetric) are discovered to be null.

The eigensolution sensitivities provided by higher odd orders present mathematical properties that are formally similar to solutions of the third order. By applying the same methodological scheme to attack the governing equations, the generic higher order (power ϵ^i, with i odd) provides, as result of the first MPE, the eigenfunction sensitivity ϕ_{ti} for the k-th eigenquartet

$$\phi_{tik}(x) = \Phi_{tik}(x)\sin(k\pi x) + \\ + \Psi_{tik}(x)\cos(k\pi x) + \Gamma_{tik}(x) + \phi_{hik} \quad (105)$$

which is an antisymmetric function, where the i-th sensitivity ϕ_{hik} of the horizontal eigenreaction plays the role of hyperstatic unknown, to be determined at the following $(i+1)$-th (even) order. As solvability condition for the second MPE, the generic higher odd order requires identical nullity of the i-th eigenvalue sensitivity β_{ik}. Finally, the compatibility condition requires the hyperstatic unknown $\phi_{h(i-1)k}$, remained undetermined from the previous $(i-1)$-th (even) order, to be null. After substitution, the eigenfunction sensitivities $\phi_{t(i-1)k}$ (constant), $\phi_{v(i+1)k}$ (symmetric) and $\phi_{u(i+2)k}$ (antisymmetric) turn out to be systematically null.

4.3.5 Fourth order and higher even orders

The fourth order (power ϵ^4) of the MPE is governed by a coupled system of two non-homogeneous Eqs. (82)–(83), with variable coefficients. The problem unknowns are the eigenvalue sensitivity β_4 and the eigenfunction sensitivities ϕ_{t4} (modal force) and ϕ_{v5} (modal vertical displacement). In parallel, the fourth order (power ϵ^6) of the PIC states a non-homogeneous Eq. (85) that relates the MPE solutions with the eigenfunction sensitivity ϕ_{u6} (modal horizontal displacement), which can be counted as extra unknown. Fourth order is the highest even order whose solution deserves to be briefly discussed, due to some distinctive mathematical and algorithmic peculiarities with respect to lower even orders. Higher even orders (power $\epsilon^6, \epsilon^8, \ldots$) can be systematically attacked with the same methodological scheme and return formally similar solutions.

The first MPE (82) must be solved in the second eigensensitivity ϕ_{t4} by integrating the defect of homogeneity $\mathfrak{d}_{t4}(\beta_0, \ldots, \beta_3, \phi_{u2}, \ldots, \phi_{u5})$. By specifying the integrand function for the k-th solutions $\beta_{0k}, \ldots, \beta_{1k}$ and $\phi_{u2k}, \ldots, \phi_{u5k}$, known from the lower orders, the defect of homogeneity results to be null. Accordingly, the fourth eigenfunction sensitivity ϕ_{t4} is constant and can be expressed as $\phi_{t4}(0) = \phi_{h4}$, where the fourth sensitivity ϕ_{h4k} of the horizontal eigenreaction is the hyperstatic unknown to be determined at the next order.

The second MPE (83) is not immediately solvable, because the differential operator $\mathfrak{L}_0$ is again singular for β_{0k}, by construction. The solvability conditions for the existence of a non trivial solution ϕ_{v5} is imposed on the defect of homogeneity $\mathfrak{d}_{v4}(\beta_0,\ldots,\beta_3, \phi_{t1}, \phi_{t3}, \phi_{v1},\ldots,\phi_{v4}) + \beta_4\mathfrak{a}_0(\beta_0, \phi_{v1})$, giving rise to in a linear algebraic equation in the fourth eigenvalue sensitivity β_4. By specifying the defect of homogeneity for the lower order solutions of the k-th eigenquartet, the solution reads

$$\beta_{4k} = -\frac{\mathsf{b}_k^{41}\Lambda_1^2 h_1}{24\,k\pi h_{-1}} - \frac{\mathsf{b}_k^{42}\Lambda_1^4 - \mathsf{b}_k^{43}\Lambda_1^2\Lambda_2}{1440\,k^3\pi^3} + \\ -\frac{k\pi h_1^2}{8h_{-1}^2} + \frac{k\pi h_3}{2h_{-1}} \quad (106)$$

where the auxiliary quantities b_k^{41}, b_k^{42} and b_k^{43} are reported in the Appendix. Enforcing this result into the defect of homogeneity makes Eq. (83) solvable. Therefore, by integrating and imposing boundary conditions $\phi_{v5}(0) = 0$ and $\phi_{v5}(1) = 0$, the solution for the k-th eigenquartet is the antisymmetric function

$$\phi_{v5k}(x) = c_5\sin(k\pi x) + \Phi_{v5k}(x)\sin(k\pi x) + \\ + \Psi_{v5k}(x)\cos(k\pi x) + \Gamma_{v5k}(x) \quad (107)$$

where the auxiliary ϕ_{h3k}-dependent polynomial functions $\Phi_{v5k}(x)$, $\Psi_{v5k}(x)$ and $\Gamma_{v5k}(x)$ are reported in the Appendix. The constant c_5, remained undetermined because the boundary conditions establish two linearly dependent equations in the two integration constants, can be considered null ($c_5 = 0$) without loss of generality.

Once the MPE solutions have been determined, the right hand of the PIC (88) is known and the equation can be solved for the unknown ϕ_{u6}. By integrating and imposing one of the two remaining boundary condition $\phi_{u6}(0) = 0$, the solution ϕ_{u6} for the k-th eigenquartet is the symmetric function

$$\phi_{u6k}(x) = \Phi_{u6k}(x)\sin(k\pi x) + \\ + \Psi_{u6k}(x)\cos(k\pi x) + \Gamma_{u6k}(x) \quad (108)$$

where the auxiliary ϕ_{h3k}-dependent polynomial functions $\Phi_{u6k}(x)$, $\Gamma_{u6k}(x)$ and $\Psi_{u6k}(x)$ are reported in the Appendix. The last boundary condition $\phi_{u6}(1) = 0$ is used as compatibility condition to determine the hyperstatic unknown ϕ_{h3k}, remained undetermined from the lower order. Specifically, the compatibility condition establishes a linear algebraic equation giving, for the k-th eigenquartet, the third sensitivity of the eigenreaction

$$\phi_{h3k} = -2c_1 k\pi h_1 \Lambda_1 - \frac{c_1 h_{-1}\left(\mathsf{h}_k^{31}\Lambda_1^3 - \mathsf{h}_k^{32}\Lambda_1\Lambda_2\right)}{10\,k\pi} \quad (109)$$

where the auxiliary quantities h_k^{31} and h_k^{32} are reported in the Appendix. Two contributions, one being directly proportional and the other inversely proportional to the mode number k, can be recognized. Once the hyperstatic unknown has been determined, the ϕ_{h3k}-dependent eigenfunction sensitivities ϕ_{t3k} (antisymmetric), ϕ_{v5k} (antisymmetric) and ϕ_{u6k} (antisymmetric) become fully determined a posteriori.

The eigensolution sensitivities provided by higher even orders present mathematical properties that are formally similar to solutions of the fourth order. By applying the same methodological scheme to attack the governing equations, the generic higher order (power $\epsilon^{(i+1)}$, with $(i+1)$ even) provides, as result of the first MPE, constant eigenfunction sensitivity $\phi_{t(i+1)k} = \phi_{h(i+1)k}$ for the k-th eigenquartet, where the $(i+1)$-th sensitivity $\phi_{h(i+1)k}$ of the horizontal eigenreaction plays the role of hyperstatic unknown, to be determined at the following $(i+2)$-th (odd) order. The solvability condition for the second MPE returns the (not null) $(i+1)$-th eigenvalue sensitivity $\beta_{(i+1)k}$. Finally, the compatibility condition allows the assessment of the (not null) hyperstatic unknown ϕ_{hik}, remained undetermined from the previous i-th (odd) order. Consequently, not null eigenfunction sensitivities ϕ_{tik} (antisymmetric), $\phi_{v(i+2)k}$ (antisymmetric) and $\phi_{u(i+3)k}$ (symmetric) are obtained.

4.3.6 Asymptotic reconstructed solutions

Perturbation solutions of the linear eigenproblem up to the fourth order (corresponding to the highest powers ϵ^4 and ϵ^6 for the governing MPE and PIC, respectively) guarantee fully analytical reconstruction of the asymptotic series approximations

- up the fourth order ($n = 4$) for the eigenvalue

$$\beta_{[4]k} = \epsilon^0\beta_{0k} + \epsilon^2\beta_{2k} + \epsilon^4\beta_{4k} \quad (110)$$

- up to the sixth order ($n = 6$) for the eigenfunctions of the horizontal and vertical displacements

$$\phi_{u[6]k}(x) = \epsilon^2\phi_{u2k}(x) + \epsilon^4\phi_{u4k}(x) + \epsilon^6\phi_{u6k}(x) \quad (111)$$

$$\phi_{v[6]k}(x) = \epsilon^1\phi_{v1k}(x) + \epsilon^3\phi_{v3k}(x) + \epsilon^5\phi_{v5k}(x) \quad (112)$$

- up to the fourth order ($n = 4$) for the axial force and horizontal reaction eigenfunctions

$$\phi_{t[4]k}(x) = \epsilon^1\phi_{t1k}(x) + \epsilon^3\phi_{t3k}(x) \quad (113)$$

$$\phi_{h[4]k} = \epsilon^1\phi_{h1k} + \epsilon^3\phi_{h3k} \quad (114)$$

for all the eigenquartets (even $k = 2, 4, 6, \ldots$). Null contributions have been excluded from the reconstructed series. Reconstruction of the eigensolutions is finalized by complete reabsorption of the ordering parameter ϵ, by inverting the parameter ordering $\Lambda_2 = (\Lambda - 1)/\epsilon^2$. From the mechanical viewpoint, it may be interesting to remark that the catenary curve differs from the parabolic curve for the static configurational sensitivity y_3 (and highers), which affects dynamic equations – and consequently dynamic solutions – starting from the second order (eigensensitivities β_{2k}, ϕ_{v3k}, ϕ_{u4k}, ϕ_{t3k}).

4.3.7 Numerical Ritz–Rayleigh solutions

In the absence of exact analytical solutions, the asymptotic solutions of the modal problem require validation by comparison with numerical results. Among the other methodologies, the Ritz–Rayleigh technique [12] can be adapted to attack the governing equations. Accordingly, the unknown mode $\boldsymbol{\phi}(x) = (\phi_t(x), \phi_u(x), \phi_v(x))$ must be expressed as complete and convergent series

$$\boldsymbol{\phi}(x) = \sum_{i=1}^{m} a_i \boldsymbol{\psi}_i(x) = \sum_{i=1}^{m} a_i (\psi_{ti}(x), \psi_{ui}(x), \psi_{vi}(x)) \tag{115}$$

where the Ritz–Rayleigh modes $\boldsymbol{\psi}_i(x)$ are composed by known functions $\psi_{ti}(x)$, $\psi_{ui}(x)$, $\psi_{vi}(x)$, which are required to satisfy the geometric boundary conditions.

Eigenfunction $\phi_v(x)$ is selected as principal unknown. For the sake of completeness and convergence, the corresponding Ritz–Rayleigh functions are chosen as

$$\psi_{vi}(x) = \sin(2i\pi x) \tag{116}$$

which are automatically (for all $i = 1, \ldots, m$) compatible with the boundary conditions $\psi_{vi}(0) = \psi_{vi}(1) = 0$, and also satisfy the symmetry of the problem.

The Ritz–Rayleigh functions of eigenfunction $\phi_u(x)$ can be conveniently selected to systematically respect the indeformability condition (67). By enforcing indeformability and imposing boundary conditions $\psi_{ui}(0) = \psi_{ui}(1) = 0$, the known Ritz–Rayleigh functions read

$$\psi_{ui}(x) = \frac{\mathsf{c}^r_{ui}(x)\cos(2i\pi x) + \mathsf{s}^r_{ui}(x)\sin(2i\pi x) + \mathsf{n}^r_{ui}}{i^2\pi^2 + 16\,\delta^2} \tag{117}$$

where the x-dependent auxiliary functions $\mathsf{s}^r_{ui}(x)$, $\mathsf{c}^r_{ui}(x)$ and n^r_{ui} are determined quasi-analytically by differentiating the exact catenary function (36). Indeed, their expressions can be obtained analytically (as reported in the Appendix), but depend on the hyperstatic unknown δ, which has be to assessed numerically.

Finally, the Ritz–Rayleigh functions of eigenfunction $\phi_t(x)$ can be conveniently selected to systematically satisfy the first modal Eq. (65). Consequently, the Ritz–Rayleigh functions can be expressed in the form $\psi_{ti}(x) = \beta^2 \chi_{ti}(x)$, where the known sub-functions read

$$\chi_{ti}(x) = \frac{\mathsf{c}^r_{ti}(x)\cos(2i\pi x) + \mathsf{s}^r_{ti}(x)\sin(2i\pi x) + \mathsf{n}^r_{ti}}{(4\,i^2\pi^2 + 64\,\delta^2)(i^2\pi^2 + 64\,\delta^2)} \tag{118}$$

and the x-dependent auxiliary functions $\mathsf{s}^r_{ti}(x)$, $\mathsf{c}^r_{ti}(x)$ and $\mathsf{n}^r_{ti}(x)$ are determined quasi-analytically (as reported in the Appendix) by differentiating the exact catenary function (36).

Since expressions (117), (118) provide automatic satisfaction of the indeformability condition and the first modal equation, the eigenproblem reduces to the second modal Eq. (66). The algorithm requires to: (i) replace all the eigenfunctions with their Ritz–Rayleigh series, (ii) premultiplying the governing equations by the virtual eigenfunction $\delta\phi_v(x) = \sum_{i=1}^{m} \delta a_i\, \psi_{vi}(x)$, (iii) integrating (analytically or numerically) over the entire x-domain, (iv) collect and zeroing all the terms multiplied by the same virtual amplitude δa_i. The procedure transforms the differential Eq. (66) into a coupled linear system of m algebraic equations reading

$$(\mathbf{A} + \beta^2\mathbf{B})\mathbf{a} = \mathbf{0} \tag{119}$$

where the entries of the m-by-m matrices $\mathbf{A}$ and $\mathbf{B}$ are

$$A_{ij} = \int_0^1 h_s \psi_{vj} \left[\psi'_{vi}\left(1 + (y')^2\right)\right]' dx \tag{120}$$

$$B_{ij} = \int_0^1 \psi_{vj}(y'\chi_{ti})' + h_r \psi_{vj}\psi_{vi}\left[1 + (y')^2\right]^{\frac{1}{2}} dx \tag{121}$$

while the m-by-1 vector of Ritz–Rayleigh amplitudes is $\mathbf{a} = (a_1, a_2, \ldots, a_m)$. Therefore, the ℓ-th Ritz–Rayleigh eigenvalue β^R_ℓ can be assessed by searching numerically the ℓ-th root (for $\ell = 1, \ldots, m$) of the characteristic equation $\det(\mathbf{A} + \beta^2\mathbf{B}) = 0$. The ℓ-th Ritz–Rayleigh modal functions can be obtained as $\phi^R_{v\ell}(x) = \boldsymbol{\psi}_v \cdot \mathbf{a}_\ell$, $\phi^R_{u\ell}(x) = \boldsymbol{\psi}_u \cdot \mathbf{a}_\ell$, $\phi^R_{t\ell}(x) = \boldsymbol{\psi}_t \cdot \mathbf{a}_\ell$, where $\mathbf{a}_\ell$ is the ℓ-th eigenvector and $\boldsymbol{\psi}_v = (\psi_{v1}, \psi_{v2}, \ldots, \psi_{vm})$, $\boldsymbol{\psi}_u = (\psi_{u1}, \psi_{u2}, \ldots, \psi_{um})$, $\boldsymbol{\psi}_t = (\psi_{t1}, \psi_{t2}, \ldots, \psi_{tm})$ are the vectors of Ritz–Rayleigh functions. Naturally, growing numbers m of functions increase the accuracy of the Ritz–Rayleigh results in approximating the exact eigensolutions.

4.3.8 Asymptotic results and numerical validation

Asymptotic reconstructed solutions for the eigenvalues β_k of the lowest eigenquartets ($k = 2, 4, 6, 8$) are shown in Fig. 7 versus increasing aspect ratios in the range $\Lambda \in (1, 11/10]$. Independently of the mode number k, the major qualitative remarks concerning the

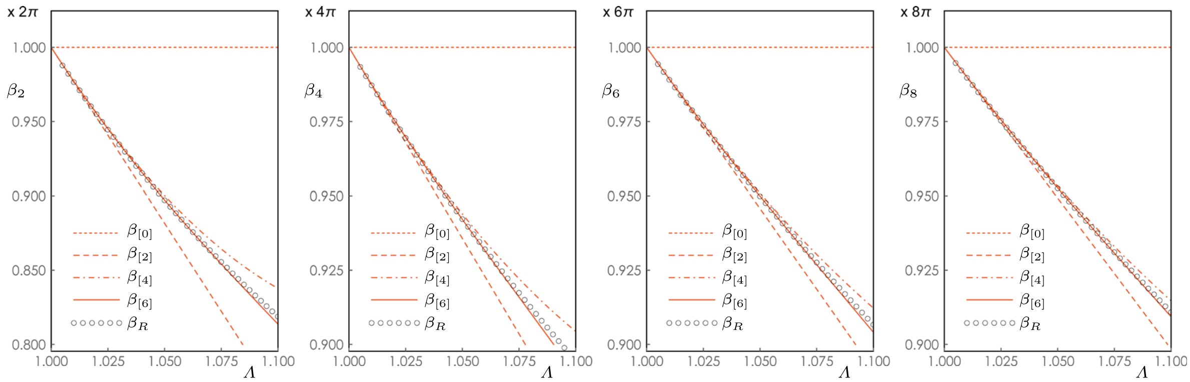

Fig. 7 Asymptotic approximations $\beta_{v[n]}$ of the eigenvalues for the shallow inextensible cables with aspect ratio $\Lambda \in (1, 11/10]$

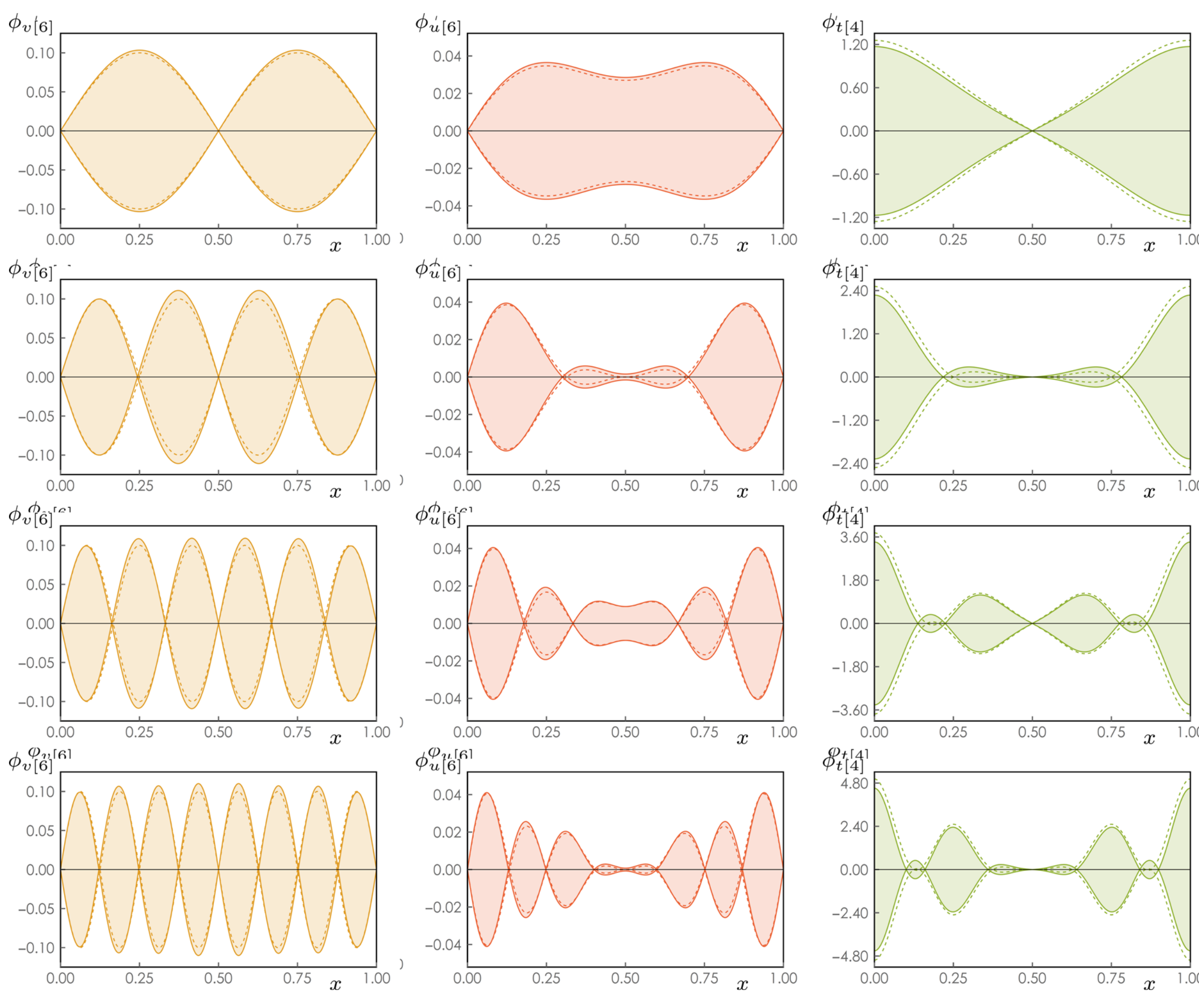

Fig. 8 Asymptotic approximations $\phi_{v[6]}$ (yellow), $\phi_{u[6]}$ (red), $\phi_{t[4]}$ (green) of the eigenfunctions (with amplitude $c_1 = \pm 1/10$) for shallow inextensible cables with aspect ratio $\Lambda = 103/100$: $k = 2$ (first row), $k = 4$ (second row), $k = 6$ (third row), $k = 8$ (fourth row). Comparison with lowest order approximations $\phi_{v[1]}$ (dashed yellow), $\phi_{u[2]}$ (dashed red), $\phi_{t[1]}$ (dashed green)

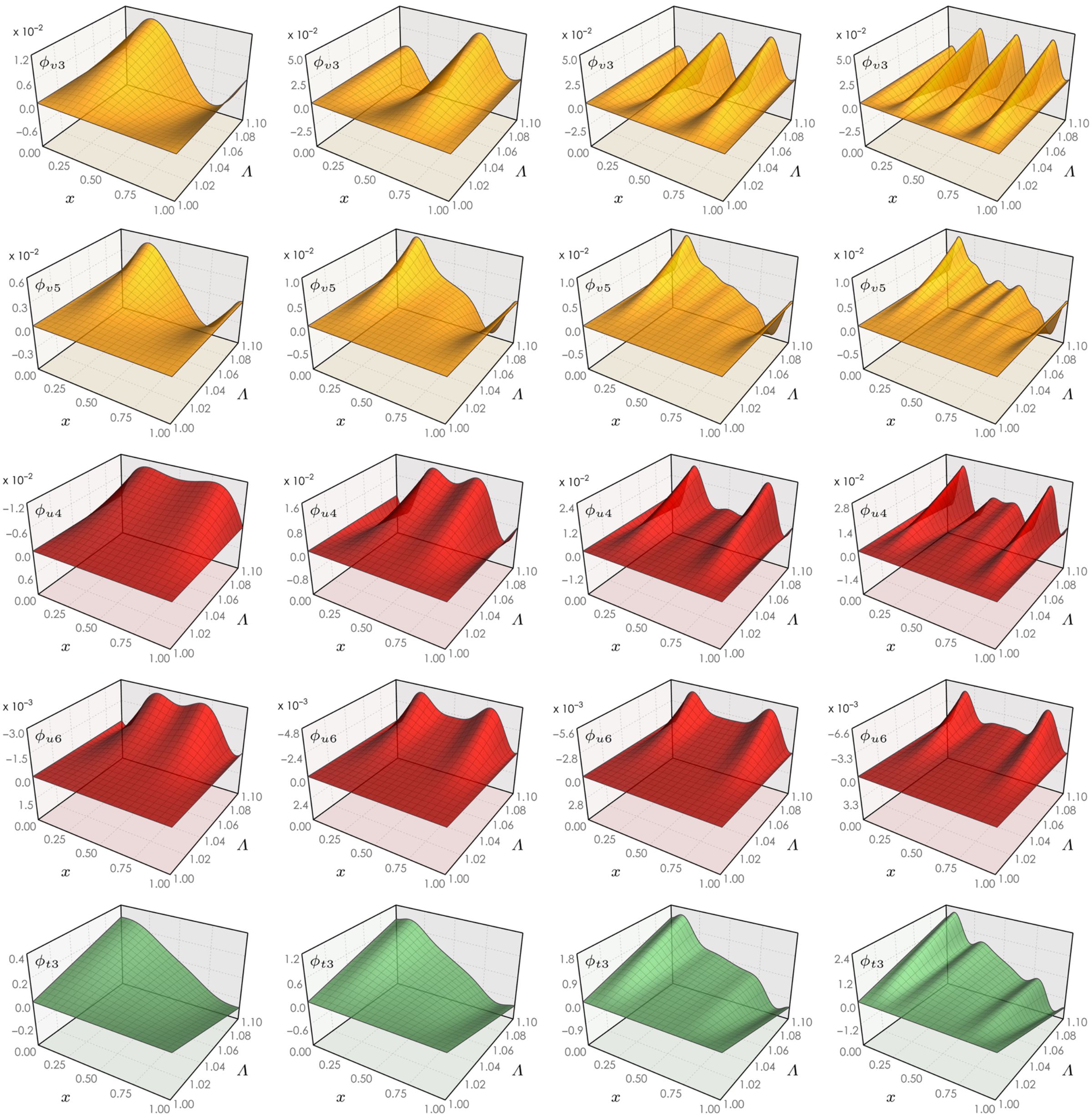

Fig. 9 Eigenfunction sensitivities ϕ_{v3k} ϕ_{v5k} (yellow), ϕ_{u4k}, ϕ_{u6k} (red), ϕ_{t3k} (green) for the shallow inextensible cable with aspect ratios $\Lambda \in (1, 11/10]$: $k = 2$ (first column), $k = 4$ (second column), $k = 6$ (third column), $k = 8$ (fourth column)

different eigensensitivities are that: (i) the second sensitivity β_{2k} determines small *softening* effects (frequency decrement versus increasing aspect ratio), while (ii) the fourth sensitivity β_{4k} causes very small *hardening* (frequency increment versus increasing aspect ratio) effects, and finally (iii) extremely small softening effects are finally caused by the sixth sensitivity β_{6k} (analytically determined after some cumbersome algebra that cannot be reported here for the sake of synthesis). These competing high-order ($n = 2, 3, 4, \ldots$) effects correct the Λ-independent low-order ($n = 0, 1$) approximations of the eigenvalue in a non-negligible manner. Physically, this achievement means that the natural frequencies of catenary cables (described by high orders) can be appreciably different from the natural frequencies of parabolic cables (first order).

The differences grow for increasing aspect ratios, as expected. Quantitatively, the comparison with the corresponding ℓ-th Ritz–Rayleigh eigenvalue β_ℓ^R ($\ell = 1, 2, 3, 4$) allows to appreciate the growing accuracy of the asymptotic approximations for increasing orders n of the reconstructed solutions $\beta_{[n]k}$. The comparison highlights also that the alternate softening or hardening behaviors described by the eigensensitivities β_{ik} contributing to the asymptotic solutions $\beta_{[n]k}$ determine alternate underestimation ($n = 2, 6, \ldots$) or overestimation ($n = 0, 4, \ldots$) of the numerical solutions β_ℓ^R, respectively.

Reconstructed solutions for all the eigenfunctions are shown in Fig. 8 for the lowest eigenquartets ($k = 2, 4, 6, 8$) of a shallow cable with the particular aspect ratio $\Lambda = 103/100$. The geometrical properties of the eigenfunctions $\phi_{u[n]}$ (symmetric) and $\phi_{v[n]}$, $\phi_{t[n]}$ (antisymmetric) can be appreciated, regardless of the approximation order. Comparing the high order (continuous lines) with the low order (dashed lines) solutions highlights that—for the same amplitude c_1—low-order approximations slightly underestimate the eigenfunctions $\phi_{u[n]}$ and $\phi_{v[n]}$ (modal displacements), while slightly overestimate the eigenfunctions $\phi_{t[n]}$ (modal force). Also, high orders determine slight shifts of all the eigenfunction nodes. All the eigenfunction sensitivities are also illustrated in Fig. 9, where their increasing smallness can be appreciated for growing aspect ratios.

Conclusions

Physical–mathematical problems concerning the statics and free dynamics of inextensible shallow cables have been addressed, with the primary objective of achieving fully analytical solutions, expressible as parametric functions of the geometrical and inertial data.

First, the traditional formulation of the mechanical one-dimensional continuum model describing the static and dynamic responses of elastically extensible, perfectly flexible cables has been revisited to state the nonlinear equilibrium equations as parametric expressions of cable shallowness and extensibility. Therefore, the nonlinear conditions for cable inextensibility in the static and dynamic fields have been derived as internal constraints between the configurational variables.

Second, an original hierarchical variant of the Force Method is presented as methodological solution strategy, based on the systematic application of perturbation schemes to equilibrium equations, indeformability constraints and compatibility conditions. With respect to other approaches, the proposed strategy allows a unified and fully consistent treatment of the static and dynamic problems, requiring the sole geometric assumption of cable shallowness as postulate a priori.

Technically, the perturbation scheme for the static problem consists of a two-step algorithm that requires stating and recursively solving in closed form: (i) a linear differential system of Perturbation Equilibrium Equations (PEE) and, (ii) a linear algebraic system of perturbation compatibility equations. Subsequently, taking the asymptotic static solution as a reference, the perturbation scheme for the linearized free dynamic problem consists of a three-step algorithm that requires stating and recursively solving in closed form: (i) a linear differential system of Modal Equilibrium Equations (MPE), then (ii) a linear differential system of Perturbation Indeformability Conditions (PIC), and finally, (iii) the hierarchy of boundary conditions serving as Compatibility Conditions (CC) for the perturbation sensitivities of the hyperstatic unknown. The outlined perturbation strategy can also be regarded as a generalizable methodological tool, whose technical principles can be systematically extended to search for analytical (asymptotically convergent) solutions of nonlinear static equilibrium problems and linearized eigenproblems related to statically indeterminate structures.

As a significant achievement in the attack on the nonlinear static problem, highly accurate fully analytical solutions have been obtained for the asymptotic approximation of the catenary configuration assumed by horizontal inextensible cables hanging between fixed supports under self-weight. As major achievement in the attack on the linearized dynamic problem, fully analytical—although asymptotically approximate—solutions have been obtained for the unsolved eigenproblem (modal problem) governing the free undamped response of horizontal cables in the small-amplitude oscillation regime. Parametric analyses of the results have highlighted that high-order terms determine qualitative and quantitative effects in the eigensolutions, including competing softening or hardening effects in the eigenvalues (natural frequencies). In comparison, low-order approximations (parabolic cable configurations) have been recognized to determine underestimations or overestimations of the eigenfunctions (modal displacements and modal forces). The asymptotic solutions of the modal problem have been successfully validated by a comparison with the numerical results obtained by applying a proper discretization technique.

Acknowledgements The Author is grateful to Professor Angelo Luongo for inspiring the conception and development of the present work and of many other contributions in the fields of linear and nonlinear dynamics, stability and bifurcation. His tireless efforts in transferring to younger colleagues the scientific passion for the beauty and rigor of analytical solutions, as well as for the power and elegance of perturbation techniques, are sincerely acknowledged. The present work can be understood as an attempt to address the challenging open problem concerning the free oscillations of inextensible cables, presented (in linearized form) in Chapter 5 of the book *Mathematical models of beams and cables* [29].

Funding The authors have not disclosed any funding.

Data availibility Enquiries about data availability should be directed to the authors.

Declarations

Conflict of interest Relevant data can be made available upon request. The Author declare no conflict of interest.

A Appendix

A.1 Indeformability conditions

The total length L_{d} of the cable in the dynamic equilibrium configuration can be expressed in the form

$$L_{\mathsf{d}} = \int_0^{L\mathsf{d}} dC = \int_0^L \frac{\partial C}{\partial X}\, dX = \tag{122}$$

$$= \int_0^L \left[\left(1 + \frac{\partial U}{\partial X}\right)^2 + \left(\frac{\partial Y}{\partial X} + \frac{\partial V}{\partial X}\right)^2 \right]^{1/2} dX \tag{123}$$

where the chain rule and the dynamic geometric constraint (9) have been used. By expanding and imposing the dynamic indeformability condition (22), the relation

$$L_{\mathsf{d}} = \int_0^L \left[1 + \left(\frac{\partial Y}{\partial X}\right)^2\right]^{1/2} dX = L_{\mathsf{s}} \tag{124}$$

is demonstrated.

A.2 Configurational sensitivities

The Λ_2-dependent coefficients d_{ij} of the configurational sensitivities (57)–(59) are

$$\begin{aligned}
&d_{11} = \Lambda_1, && d_{12} = -\Lambda_1\\
&d_{31} = \tfrac{17}{20}\Lambda_1\Lambda_2, && d_{32} = -\tfrac{57}{20}\Lambda_1\Lambda_2,\\
&d_{33} = 4\Lambda_1\Lambda_2, && d_{34} = -2\Lambda_1\Lambda_2\\
&d_{51} = -\tfrac{519}{5600}\Lambda_1\Lambda_2^2, && d_{52} = -\tfrac{1161}{5600}\Lambda_1\Lambda_2^2,\\
&d_{53} = \tfrac{11}{5}\Lambda_1\Lambda_2^2, && d_{54} = -\tfrac{51}{10}\Lambda_1\Lambda_2^2\\
&d_{55} = \tfrac{24}{5}\Lambda_1\Lambda_2^2, && d_{56} = -\tfrac{8}{5}\Lambda_1\Lambda_2^2
\end{aligned} \tag{125}$$

where the relation $\Lambda_1 = (6\Lambda_2)^{1/2}$ can be recalled.

A.3 Defects of homogeneity

The i-th defect of homogeneity $\mathfrak{d}_{ti}$ in the first equations of the Modal Perturbation Equations (74)–(83) is

$$\mathfrak{d}_{t1} = -\beta_0^2\phi_{u2}h_{-1} \tag{126}$$

$$\mathfrak{d}_{t2} = -\beta_0^2\phi_{u3}h_{-1} - 2\beta_0\beta_1\phi_{u2}h_{-1} \tag{127}$$

$$\begin{aligned}\mathfrak{d}_{t3} = {}& -\beta_0^2\phi_{u4}h_{-1} - 2\beta_0(\beta_2\phi_{u2} + \beta_1\phi_{u3})h_{-1} +\\ & - \beta_1^2\phi_{u2}h_{-1} - \tfrac{1}{2}\beta_0^2(y_1')^2\phi_{u2}h_{-1}\end{aligned} \tag{128}$$

$$\begin{aligned}\mathfrak{d}_{t4} = {}& -\beta_0^2\phi_{u5}h_{-1} - 2\beta_0(\beta_3\phi_{u2} + \beta_2\phi_{u3})h_{-1} +\\ & - \beta_1^2\phi_{u3}h_{-1} - 2\beta_1\beta_2\phi_{u2}h_{-1} - 2\beta_1\phi_{u4}h_{-1} +\\ & - \tfrac{1}{2}\beta_0^2\phi_{u3}(y_1')^2h_{-1} - \beta_1\beta_0\phi_{u2}(y_1')^2h_{-1}.\end{aligned} \tag{129}$$

The i-th defect of homogeneity $\mathfrak{d}_{vi}$ in the second equations of the Modal Perturbation Equations (74)–(83) is

$$\mathfrak{d}_{v1} = -y_1'\phi_{t0}' - y_1''\phi_{t0} \tag{130}$$

$$\begin{aligned}\mathfrak{d}_{v2} = {}& -y_1'\phi_{t1}' - y_1''\phi_{t1} +\\ & - 2\left[\beta_0\beta_1\phi_{v2} + \tfrac{1}{2}\beta_1^2\phi_{v1} + \tfrac{1}{4}\beta_0^2(y_1')^2\phi_{v1}\right]h_{-1} +\\ & - 2\left[y_1'y_1''\phi_{v1}' + \tfrac{1}{2}(y_1')^2\phi_{v1}''\right]h_{-1} - \phi_{v1}''h_1\end{aligned} \tag{131}$$

$$\begin{aligned}\mathfrak{d}_{v3} = {}& -y_3'\phi_{t0}' - y_3''\phi_{t0} - y_1'\phi_{t2}' - y_1''\phi_{t2} +\\ & - 2\left[\beta_0\beta_2\phi_{v2} + \beta_0\beta_1\phi_{v3} + \beta_1\beta_2\phi_{v1} + \tfrac{1}{2}\beta_1^2\phi_{v2}\right]h_{-1} +\\ & - \left[\beta_0^2 y_1'y_2'\phi_{v1} + \beta_0\beta_1(y_1')^2\phi_{v1} + \tfrac{1}{2}\beta_0^2(y_1')^2\phi_{v2}\right]h_{-1} +\\ & - 2\left[y_1'y_1''\phi_{v2}' + \tfrac{1}{2}(y_1')^2\phi_{v2}''\right]h_{-1} - \phi_{v2}''h_1\end{aligned} \tag{132}$$

$$\begin{aligned}\mathfrak{d}_{v4} = {}& -y_1'\phi_{t3}' - y_3'\phi_{t1}' - y_1''\phi_{t3} - y_3''\phi_{t1} +\\ & - 2\left[\beta_1\beta_3\phi_{v1} + \beta_1\beta_2\phi_{v2} + \beta_0\beta_3\phi_{v2}\right]h_{-1} +\end{aligned}$$

$$
\begin{aligned}
&-2\left[\beta_0\beta_2\phi_{v3}+\beta_0\beta_1\phi_{v4}+\tfrac{1}{2}\beta_2^2\phi_{v1}+\tfrac{1}{2}\beta_1^2\phi_{v3}\right]h_{-1}+\\
&-\left[\beta_0\beta_2(y_1')^2\phi_{v1}+\beta_0\beta_1(y_1')^2\phi_{v2}+\beta_0^2 y_1' y_3'\phi_{v1}\right]h_{-1}+\\
&-\tfrac{1}{2}\left[\beta_0^2(y_1')^2\phi_{v3}+\beta_1^2(y_1')^2\phi_{v1}-{}^{1}\beta_0^2(y_1')^4\phi_{v1}\right]h_{-1}+\\
&-2\left[y_1'y_1''\phi_{v3}'+\tfrac{1}{2}(y_1')^2\phi_{v3}''+(y_1'y_3''\phi_{v1}')'\right]h_{-1}+\\
&-2\left[y_1'y_1''\phi_{v1}'+\tfrac{1}{2}(y_1')^2\phi_{v1}''\right]h_1-\phi_{v3}''h_1-\phi_{v1}''h_3.
\end{aligned}
\tag{133}
$$

A.4 Auxiliary functions

The auxiliary quantities defining the eigenfunction sensitivity ϕ_{u2k} in Eq. (91) read

$$
\begin{aligned}
\Phi_{u2k}(x)&=c_1\Lambda_1(2x-1)\\
\Psi_{u2k}&=c_1\frac{2\Lambda_1}{k\pi}\\
\Gamma_{u2k}&=c_1\frac{2\Lambda_1}{k\pi}.
\end{aligned}
\tag{134}
$$

The auxiliary quantities defining the eigenfunction sensitivity ϕ_{t1k} in Eq. (92) read

$$
\begin{aligned}
\Phi_{t1k}&=-4c_1\Lambda_1 h_{-1}\\
\Psi_{t1k}(x)&=c_1 k\pi\Lambda_1(2x-1)h_{-1}\\
\Gamma_{t1k}(x)&=c_1 k\pi\Lambda_1(2x+1)h_{-1}.
\end{aligned}
\tag{135}
$$

The auxiliary quantities defining the eigenfunction sensitivity ϕ_{v2k} in Eq. (95) read

$$
\begin{aligned}
\Psi_{v2k}&=-c_1\frac{2\Lambda_1\phi_{h0}}{k^2\pi^2 h_{-1}}\\
\Gamma_{v2k}&=c_1\frac{2\Lambda_1\phi_{h0}}{k^2\pi^2 h_{-1}}.
\end{aligned}
\tag{136}
$$

The auxiliary quantities defining the eigenfunction sensitivity ϕ_{u3k} in Eq. (96) read

$$
\begin{aligned}
\Phi_{u3k}&=\frac{4\Lambda_1^2\phi_{h0}}{k^3\pi^3 h_{-1}}\\
\Psi_{u3k}(x)&=\frac{2\Lambda_1^2\phi_{h0}}{k^2\pi^2 h_{-1}}(1-2x)\\
\Gamma_{u3k}&=-\frac{2\Lambda_1^2\phi_{h0}}{k^2\pi^2 h_{-1}}.
\end{aligned}
\tag{137}
$$

The auxiliary quantities defining the eigenfunction sensitivity ϕ_{v3k} in Eq. (99) read

$$
\begin{aligned}
\Phi_{v3k}(x)&=\frac{c_1\mathsf{s}_{vk}^{31}\Lambda_1^2}{12k^2\pi^2}\\
\Psi_{v3k}(x)&=\frac{c_1\mathsf{c}_{vk}^{31}\Lambda_1^2}{6\pi k}-\frac{2\Lambda_1\phi_{h1k}}{k^2\pi^2 h_{-1}}\\
\Gamma_{v3k}(x)&=\frac{c_1\mathsf{n}_{vk}^{31}\Lambda_1^2}{k\pi}+\frac{2\Lambda_1\phi_{h1k}}{k^2\pi^2 h_{-1}}
\end{aligned}
\tag{138}
$$

where the variable coefficients can be expressed

$$
\begin{aligned}
\mathsf{s}_{vk}^{31}&=\mathsf{s}_{vk}^{310}+\mathsf{s}_{vk}^{312}k^2\pi^2\\
\mathsf{c}_{vk}^{31}&=\mathsf{c}_{vk}^{310}+\mathsf{c}_{vk}^{312}k^2\pi^2
\end{aligned}
\tag{139}
$$

and quantities $\mathsf{s}_{vk}^{31\ell}$, $\mathsf{c}_{vk}^{31\ell}$, n_{uk}^{31} are polynomial functions of the variable x.

The auxiliary quantities defining the eigenfunction sensitivity ϕ_{u4k} in Eq. (100) read

$$
\begin{aligned}
\Phi_{u4k}(x)&=\frac{c_1\left(\mathsf{s}_{uk}^{41}\Lambda_1^3+\mathsf{s}_{uk}^{42}\Lambda_1\Lambda_2\right)}{60k^2\pi^2}+\frac{4\Lambda_1^2\phi_{h1k}}{k^3\pi^3 h_{-1}}\\
\Psi_{u4k}(x)&=\frac{c_1\left(\mathsf{c}_{uk}^{41}\Lambda_1^3+\mathsf{c}_{uk}^{42}\Lambda_1\Lambda_2\right)}{30k^3\pi^3}+\frac{2\Lambda_1^2\phi_{h1k}}{k^2\pi^2 h_{-1}}(1-2x)\\
\Gamma_{u4k}(x)&=\frac{c_1\left(\mathsf{n}_{uk}^{41}\Lambda_1^3+\mathsf{n}_{uk}^{42}\Lambda_1\Lambda_2\right)}{10k^3\pi^3}-\frac{2\Lambda_1^2\phi_{h1k}}{k^2\pi^2 h_{-1}}
\end{aligned}
\tag{140}
$$

where the variable coefficients can be expressed

$$
\begin{aligned}
\mathsf{s}_{uk}^{41}&=\mathsf{s}_{uk}^{410}+\mathsf{s}_{uk}^{412}k^2\pi^2\\
\mathsf{s}_{uk}^{42}&=\mathsf{s}_{uk}^{420}+\mathsf{s}_{uk}^{422}k^2\pi^2\\
\mathsf{c}_{uk}^{41}&=\mathsf{c}_{uk}^{410}+\mathsf{c}_{uk}^{412}k^2\pi^2+\mathsf{c}_{uk}^{414}k^4\pi^4\\
\mathsf{c}_{uk}^{42}&=\mathsf{c}_{uk}^{420}+\mathsf{c}_{uk}^{422}k^2\pi^2\\
\mathsf{n}_{uk}^{41}&=\mathsf{n}_{uk}^{410}+\mathsf{n}_{uk}^{412}k^2\pi^2\\
\mathsf{n}_{uk}^{42}&=\mathsf{n}_{uk}^{420}+\mathsf{n}_{uk}^{422}k^2\pi^2
\end{aligned}
\tag{141}
$$

and quantities $\mathsf{s}_{uk}^{41\ell}$, $\mathsf{s}_{uk}^{42\ell}$, $\mathsf{c}_{uk}^{41\ell}$ $\mathsf{c}_{uk}^{42\ell}$, $\mathsf{n}_{uk}^{41\ell}$, $\mathsf{n}_{uk}^{42\ell}$ are polynomial functions of the variable x.

The auxiliary quantities defining the eigenfunction sensitivity ϕ_{t3k} in Eq. (102) read

$$
\Phi_{t3k}(x)=\frac{c_1\left(\mathsf{s}_{tk}^{31}\Lambda_1^3+\mathsf{s}_{tk}^{32}\Lambda_1\Lambda_2\right)h_{-1}}{30k^2\pi^2}-4c_1\Lambda_1 h_1
$$

$$\Psi_{t3k}(x) = \frac{c_1\left(\mathsf{s}_{tk}^{31}\Lambda_1^3 + \mathsf{s}_{tk}^{32}\Lambda_1\Lambda_2\right)h_{-1}}{60\,k\pi} + \\ + c_1 k\pi(2x-1)\Lambda_1 h_1$$

$$\Gamma_{t3k}(x) = \frac{c_1\left(\mathsf{n}_{tk}^{31}\Lambda_1^3 + \mathsf{n}_{tk}^{32}\Lambda_1\Lambda_2\right)h_{-1}}{60\,k\pi} + \\ + c_1 k\pi(2x+1)\Lambda_1 h_1 \qquad (142)$$

where the variable coefficients can be expressed

$$\begin{aligned}
\mathsf{s}_{tk}^{31} &= \mathsf{s}_{tk}^{310} + \mathsf{s}_{tk}^{312}k^2\pi^2 + \mathsf{s}_{tk}^{314}k^4\pi^4 \\
\mathsf{s}_{tk}^{32} &= \mathsf{s}_{tk}^{320} + \mathsf{s}_{tk}^{322}k^2\pi^2 \\
\mathsf{c}_{tk}^{31} &= \mathsf{c}_{tk}^{310} + \mathsf{c}_{tk}^{312}k^2\pi^2 \\
\mathsf{c}_{tk}^{32} &= \mathsf{c}_{tk}^{320} + \mathsf{c}_{tk}^{322}k^2\pi^2 \\
\mathsf{n}_{tk}^{31} &= \mathsf{n}_{tk}^{310} + \mathsf{n}_{tk}^{312}k^2\pi^2 \\
\mathsf{n}_{tk}^{32} &= \mathsf{n}_{tk}^{320} + \mathsf{n}_{tk}^{322}k^2\pi^2 \qquad (143)
\end{aligned}$$

and quantities $\mathsf{s}_{tk}^{31\ell}, \mathsf{s}_{tk}^{32\ell}, \mathsf{c}_{tk}^{31\ell}\ \mathsf{c}_{tk}^{32\ell}, \mathsf{n}_{tk}^{31\ell}, \mathsf{n}_{tk}^{32\ell}$ are polynomial functions of the variable x.

The auxiliary quantities defining the eigenfunction sensitivity ϕ_{v4k} in Eq. (103) read

$$\begin{aligned}
\Psi_{v4k} &= -\frac{2\Lambda_1\phi_{h2k}}{k^2\pi^2 h_{-1}} \\
\Gamma_{v4k} &= \frac{2\Lambda_1\phi_{h2k}}{k^2\pi^2 h_{-1}}. \qquad (144)
\end{aligned}$$

The auxiliary quantities defining the eigenfunction sensitivity ϕ_{u5k} in Eq. (104) read

$$\begin{aligned}
\Phi_{u5k} &= \frac{4\Lambda_1^2\phi_{h2k}}{k^3\pi^3 h_{-1}} \\
\Psi_{u5k}(x) &= \frac{2\Lambda_1^2\phi_{h2k}}{k^2\pi^2 h_{-1}}(1-2x) \\
\Gamma_{u5k} &= -\frac{2\Lambda_1^2\phi_{h2k}}{k^2\pi^2 h_{-1}}. \qquad (145)
\end{aligned}$$

The auxiliary quantities defining the eigenvalue sensitivity β_{4k} in Eq. (106) read

$$\begin{aligned}
\mathsf{b}_k^{41} &= 114 + k^2\pi^2 \\
\mathsf{b}_k^{42} &= 413820 - 22440k^2\pi^2 - 37k^4\pi^4 \\
\mathsf{b}_k^{43} &= 36\left(31200 - 1246k^2\pi^2 - 3k^4\pi^4\right). \qquad (146)
\end{aligned}$$

The auxiliary quantities defining the eigenfunction sensitivity ϕ_{v5k} in Eq. (107) read

$$\Phi_{v5k}(x) = \frac{c_1(\mathsf{s}_{vk}^{51}\Lambda_1^4 + \mathsf{s}_{vk}^{52}\Lambda_1^2\Lambda_2)}{144\,k^4\pi^4}$$

$$\Psi_{v5k}(x) = \frac{c_1(\mathsf{c}_{vk}^{51}\Lambda_1^4 + \mathsf{c}_{vk}^{52}\Lambda_1^2\Lambda_2)}{360\,k^3\pi^3} - \frac{4c_1\Lambda_1^2 h_1}{k\pi h_{-1}} + \\ - \frac{2\Lambda_1\phi_{h3k}}{k^2\pi^2 h_{-1}}$$

$$\Gamma_{v5k}(x) = \frac{c_1(\mathsf{n}_{vk}^{51}\Lambda_1^4 + \mathsf{n}_{vk}^{52}\Lambda_1^2\Lambda_2)}{15\,k^3\pi^3} + \frac{4c_1\Lambda_1^2 h_1}{k\pi h_{-1}} + \\ + \frac{2\Lambda_1\phi_{h3k}}{k^2\pi^2 h_{-1}} \qquad (147)$$

where the variable coefficients can be expressed

$$\begin{aligned}
\mathsf{s}_{vk}^{51} &= \mathsf{s}_{vk}^{510} + \mathsf{s}_{vk}^{512}k^2\pi^2 + \mathsf{s}_{vk}^{514}k^4\pi^4 + \mathsf{s}_{vk}^{516}k^6\pi^6 \\
\mathsf{s}_{vk}^{52} &= \mathsf{s}_{vk}^{520} + \mathsf{s}_{vk}^{522}k^2\pi^2 + \mathsf{s}_{vk}^{524}k^4\pi^4 \\
\mathsf{c}_{vk}^{51} &= \mathsf{c}_{vk}^{510} + \mathsf{c}_{vk}^{512}k^2\pi^2 + \mathsf{c}_{vk}^{514}k^4\pi^4 \\
\mathsf{c}_{vk}^{52} &= \mathsf{c}_{vk}^{520} + \mathsf{c}_{vk}^{522}k^2\pi^2 + \mathsf{c}_{vk}^{524}k^4\pi^4 \\
\mathsf{n}_{vk}^{51} &= \mathsf{n}_{vk}^{510} + \mathsf{n}_{vk}^{512}k^2\pi^2 \\
\mathsf{n}_{vk}^{52} &= \mathsf{n}_{vk}^{510} + \mathsf{n}_{vk}^{512}k^2\pi^2 \qquad (148)
\end{aligned}$$

and quantities $\mathsf{s}_{vk}^{51\ell}, \mathsf{s}_{vk}^{52\ell}, \mathsf{c}_{vk}^{51\ell}\ \mathsf{c}_{vk}^{52\ell}, \mathsf{n}_{vk}^{51\ell}, \mathsf{n}_{vk}^{52\ell}$ are polynomial functions of the variable x.

The auxiliary quantities defining the eigenfunction sensitivity ϕ_{u6k} in Eq. (108) read

$$\Phi_{u6k}(x) = \frac{c_1\left(\mathsf{s}_{uk}^{61}\Lambda_1^5 + \mathsf{s}_{uk}^{62}\Lambda_1^3\Lambda_2 + \mathsf{s}_{uk}^{63}\Lambda_1\Lambda_2^2\right)}{50400\,k^4\pi^4} + \\ + \frac{4\Lambda_1^2\phi_{h3k}}{k^3\pi^3 h_{-1}} + \frac{8c_1\Lambda_1^3 h_1}{k^3\pi^3 h_{-1}}$$

$$\Psi_{u6k}(x) = \frac{c_1\left(\mathsf{c}_{uk}^{61}\Lambda_1^5 + \mathsf{c}_{uk}^{62}\Lambda_1^3\Lambda_2 + \mathsf{c}_{uk}^{63}\Lambda_1\Lambda_2^2\right)}{25200\,k^5\pi^5} + \\ + \frac{2\Lambda_1^2\phi_{h3k}}{k^2\pi^2 h_{-1}}(1-2x) + \frac{4c_1\Lambda_1^3 h_1}{k\pi h_{-1}}(1-2x)$$

$$\Gamma_{u6k}(x) = \frac{c_1\left(\mathsf{n}_{uk}^{61}\Lambda_1^5 + \mathsf{n}_{uk}^{62}\Lambda_1^3\Lambda_2 + \mathsf{n}_{uk}^{63}\Lambda_1\Lambda_2^2\right)}{8400\,k^5\pi^5} + \\ - \frac{2\Lambda_1^2\phi_{h3k}}{k^2\pi^2 h_{-1}} - \frac{4c_1\Lambda_1^3 h_1}{k^2\pi^2 h_{-1}} \qquad (149)$$

where the variable coefficients can be expressed

$$\mathsf{s}_{uk}^{61} = \mathsf{s}_{uk}^{610} + \mathsf{s}_{uk}^{612}k^2\pi^2 + \mathsf{s}_{uk}^{614}k^4\pi^4 + \mathsf{s}_{uk}^{616}k^6\pi^6$$

$$\mathsf{s}_{uk}^{62} = \mathsf{s}_{uk}^{620} + \mathsf{s}_{uk}^{622} k^2\pi^2 + \mathsf{s}_{uk}^{624} k^4\pi^4$$
$$\mathsf{s}_{uk}^{63} = \mathsf{s}_{uk}^{630} + \mathsf{s}_{uk}^{632} k^2\pi^2 + \mathsf{s}_{uk}^{634} k^4\pi^4$$
$$\mathsf{c}_{uk}^{61} = \mathsf{c}_{uk}^{610} + \mathsf{c}_{uk}^{612} k^2\pi^2 + \mathsf{c}_{uk}^{614} k^4\pi^4 + \mathsf{c}_{uk}^{616} k^6\pi^6$$
$$\mathsf{c}_{uk}^{62} = \mathsf{c}_{uk}^{620} + \mathsf{c}_{uk}^{622} k^2\pi^2 + \mathsf{c}_{uk}^{624} k^4\pi^4 + \mathsf{c}_{uk}^{626} k^6\pi^6$$
$$\mathsf{c}_{uk}^{63} = \mathsf{c}_{uk}^{630} + \mathsf{c}_{uk}^{632} k^2\pi^2 + \mathsf{c}_{uk}^{634} k^4\pi^4$$
$$\mathsf{n}_{uk}^{61} = \mathsf{n}_{uk}^{610} + \mathsf{n}_{uk}^{612} k^2\pi^2 + \mathsf{n}_{uk}^{614} k^4\pi^4$$
$$\mathsf{n}_{uk}^{62} = \mathsf{n}_{uk}^{620} + \mathsf{n}_{uk}^{622} k^2\pi^2 + \mathsf{n}_{uk}^{624} k^4\pi^4$$
$$\mathsf{n}_{uk}^{63} = \mathsf{n}_{uk}^{630} + \mathsf{n}_{uk}^{632} k^2\pi^2 + \mathsf{n}_{uk}^{634} k^4\pi^4 \tag{150}$$

and quantities $\mathsf{s}_{uk}^{61\ell}$, $\mathsf{s}_{uk}^{62\ell}$, $\mathsf{s}_{uk}^{63\ell}$, $\mathsf{c}_{uk}^{61\ell}$, $\mathsf{c}_{uk}^{62\ell}$, $\mathsf{c}_{uk}^{63\ell}$, $\mathsf{n}_{uk}^{61\ell}$, $\mathsf{n}_{uk}^{62\ell}$, $\mathsf{n}_{uk}^{63\ell}$ are polynomial functions of the variable x.

The auxiliary quantities defining the eigenfunction sensitivity ϕ_{h3k} in Eq. (109) read

$$\mathsf{h}_k^{31} = 5(64 - 3k^2\pi^2)$$
$$\mathsf{h}_k^{32} = (960 - 37k^2\pi^2). \tag{151}$$

The auxiliary quantities defining the Ritz–Rayleigh functions $\psi_{ui}(x)$ and $\chi_{ti}(x)$ in Eqs. (117), (118) read

$$\mathsf{c}_u^r(x) = 4i\pi\,\delta\cosh(4\delta(1-2x))$$
$$\mathsf{s}_u^r(x) = -i^2\pi^2\sinh(4\delta(1-2x))$$
$$\mathsf{n}_u^r = -4i\pi\,\delta\cosh(4\delta) \tag{152}$$
$$\mathsf{c}_t^r(x) = -i\pi h_r\left(i^2\pi^2 - 32\delta^2\right)\sinh(8\delta(1-2x))$$
$$\mathsf{s}_t^r(x) = -4\delta h_r\Big(i^2\pi^2 + 64\delta^2 + +3\pi^2 i^2\cosh(8\delta(1-2x))\Big)$$
$$\mathsf{n}_t^r(x) = -2i\pi h_r\left(i^2\pi^2 + 64\delta^2\right)\cosh(4\delta)\times \times\sinh(4\delta(1-2x)). \tag{153}$$

where δ has to assessed by solving numerically the compatibility equation $\sinh(4\delta) = 4\Lambda\delta$.

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