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Calderón-Zygmund theory on trees and sparse domination for the radial Bergman projection

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Contents

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Introduction

The purpose of this thesis is to provide new developments of some aspects of harmonic analysis on trees. The classical Calderón-Zygmund theory is adapted to this discrete setting in the doubling case, and a new general Calderón-Zygmund theory for non-locally doubling trees is introduced. Moreover, sparse domination techniques are exploited to investigate the Bergman projection, an operator of great relevance in the study of trees. In general terms, the main themes on which this work is based can be grouped into three aspects: trees and the Bergman projection, Calderón-Zygmund theory, and sparse domination.

Trees and Bergman projection

In mathematics, trees are a fundamental class of discrete structures that is extensively studied from various perspectives, for example in graph theory or set theory. It is also possible to develop different aspects of analysis on trees, both for the intrinsic interest of considering a tree as the underlying space and because they can serve as a natural discrete analogue of continuous spaces. Their discrete nature provides a framework for applying classical harmonic analysis, leading to new results and a deeper understanding of the relationship between geometry and analysis on discrete structures.

A tree is a connected graph without cycles that can be viewed as a discrete model of a continuous manifold. It is possible to associate to every tree a function q defined on the tree such that $q(x) + 1$ is the number of neighbors of a vertex x. The particular family of homogeneous trees, where q is constant, i.e. all vertices have the same number of neighbors, is often studied as the discrete counterpart of symmetric spaces of rank one. Among other reasons, this is also motivated by the existence of a natural embedding of homogeneous trees into the hyperbolic disk, as shown in detail in [12]. This analogy has been exploited in both directions in order to spread concepts and problems from one setting to the other and vice versa. In this perspective, general trees can be thought of as discrete models of more general noncompact symmetric spaces. Indeed, the function q can be viewed as the discrete analogue of the sectional curvature in a Riemannian manifold. We will see that the assumption that q is bounded, both from below and above, plays a crucial role in the analysis on trees. This condition actually reflects the requirement on manifolds that the curvature remains between two negative constants. In this thesis, the function q will always be assumed to be bounded from below. We will say that the tree has bounded geometry if it is also bounded from above, while it has unbounded geometry otherwise.

A key operator that appears in complex and harmonic analysis on both continuous and discrete settings is the Bergman projection. In the classical case, the Bergman projection is an operator that projects the L^2 space of functions on a domain (typically a domain in the complex plane) into the subspace of holomorphic functions, which is usually referred to as the holomorphic Bergman space. The underlying measure belongs to the family of the so-called Bergman measures, that are designed to capture the geometry of the underlying domain. When moving to the discrete setting of trees, the Bergman projection still serves a similar purpose, but in this context it is common to consider Bergman spaces of harmonic functions instead of holomorphic ones. Indeed, a definition of holomorphic function on discrete structures is not clearly stated in the literature, while a natural definition of laplacian, and then of harmonic function, can be given here by means of a local mean property. A function defined on a tree is said to be harmonic at a vertex if its value at the aforementioned vertex is given by the mean of the values of the function in its neighbors. Just as in the continuous holomorphic case, the harmonic Bergman space on trees is a reproducing kernel Hilbert space. Its kernel is called the Bergman kernel and the Bergman projection turns out to be an integral operator acting by integration against the Bergman kernel.

The boundedness of the extension of the Bergman projection to L^p spaces is broadly studied and well-known in a variety of continuous contexts. The discrete harmonic Bergman projection and its boundedness properties primarily motivated the work of this thesis, representing in some sense both the beginning and the end. In fact, the starting point was the horocyclic harmonic Bergman projection on homogeneous trees. Initially, we studied it by finding an explicit formula for the Bergman kernel, and then we obtained the classical bounds by analyzing the kernel. Then, the idea was to study the validity of our result in more general trees. The following theorem states the boundedness properties of $\mathcal{P}$ on radial trees and can be obtained as a corollary of the results presented in the next two paragraphs (with the exception of the H^1 and BMO results, that are independent). One is a general boundedness result for operators defined on trees, while the other is a sparse domination result specific to the radial Bergman projection.

Theorem A (Corollary 3.2.12, Thereom 3.2.13, Theorem 3.2.14)**.** *Let $\mathfrak{X}$ be a radial tree. The associated Bergman projection $\mathcal{P}$ maps boundedly:*

1. $L^1(\mathfrak{X})$ *into* $L^{1,\infty}(\mathfrak{X})$;
2. $L^p(\mathfrak{X})$ *into* $L^p(\mathfrak{X})$ *for all* $p \in (1, \infty)$;
3. $H^1(\mathfrak{X})$ *into* $H^1(\mathfrak{X})$;
4. $BMO(\mathfrak{X})$ *into* $BMO(\mathfrak{X})$.

Calderón-Zygmund theory

The Calderón-Zygmund theory is a core part of modern harmonic analysis and is primarily concerned with the study of singular integral operators. The foundational work of Calderón and Zygmund in the 1950s, particularly in their papers [7] and [8], laid the ground for this theory. Their study, along with that of many other mathematicians such as Stein and Weiss, has had impactful consequences in modern analysis.

Singular integral operators, which often appear as convolutions with kernels possessing specific smoothness and decay properties, play a fundamental role in various areas of mathematical analysis, specifically in the study of function spaces such as the Lebesgue spaces, Hardy spaces, and bounded mean oscillation spaces, among others, and have broad applications in partial differential equations, for example. This theory provides the tools to analyze the boundedness and behavior of such operators. A key aspect is the development of the so-called Calderón-Zygmund decomposition, which enables one to break down the functions, and then the operators acting on them, into simpler, more manageable parts, making the analysis of their behavior more tractable.

An integral operator T is an operator that acts on a function f by integrating it against a kernel K, namely

$$Tf(x) = \int_X K(x,y)f(y)\mathrm{d}\sigma(y), \tag{1}$$

where (X,σ) is a general measure space. The space can be more or less general, it may be asked to satisfy different properties, or be endowed with a richer structure. Of course, depending on the properties of (X,σ), the results that can be obtained on integral operators will vary. A singular integral is an integral operator with a kernel that is not absolutely integrable and may not be defined everywhere on X. The most famous example is the Hilbert transform, which is a singular integral operator acting on functions on $\mathbb{R}$ and given by

$$\mathcal{H}f(x) = \int_{\mathbb{R}} \frac{f(y)}{x-y}\mathrm{d}y$$

whenever $x \notin \operatorname{supp}(f)$. The Hilbert transform and its boundedness on Lebesgue spaces $L^p(\mathbb{R})$ were broadly studied at the beginning of the 20th century and they strongly motivated the investigation of Calderón and Zygmund and the development of their theory. Indeed, the study of the Hilbert transform first led to considering a more general class of operators whose kernels are singular on the diagonal

$$\Delta = \big\{(x,y) \in X \times X : x = y\big\},$$

so that (1) is meaningful only away from the support of the function f. A wide class of these operators is that of Calderón-Zygmund operators, defined on functions acting on $\mathbb{R}^d$, bounded on $L^2(\mathbb{R}^d)$ and whose kernel off-diagonal is assumed to satisfy the size condition

$$|K(x,y)| \leq \frac{1}{|x-y|^d} \text{ for } x \neq y,$$

and a Lipschitz condition, i.e. there exists $\delta > 0$ such that

$$|K(x,z) - K(y,z)| + |K(z,x) - K(z,y)| \leq \Big(\frac{|x-y|}{|x-z|^d}\Big)^\delta$$

for all $|x-y| < |x-z|/2$. Sometimes the Lipschitz condition is replaced by the weaker Hörmander condition

$$\sup_{v\in\mathbb{R}^d, r>0} \sup_{x,y\in B(v,r)} \int_{\mathbb{R}^d\setminus B(v,2r)} |K(z,x) - K(z,y)| \mathrm{d}z < +\infty.$$

The integral (1) may generally fail to converge absolutely for f in many natural function spaces, so that a precise definition requires it to be understood as the limit of suitable converging integrals. The theory mainly focuses on proving the boundedness of these operators in various function spaces, usually showing the weak $(1,1)$ boundedness and exploiting interpolation methods. The most important result in Calderón-Zygmund theory asserts that such operators are bounded on $L^p(\mathbb{R}^d)$ for $p \in (1,\infty)$, meaning that there exists a constant $C_p > 0$ such that

$$\|Tf\|_p \leq C_p \|f\|_p.$$

This theorem is a cornerstone of harmonic analysis, as it provides a rigorous understanding of how singular integrals act on various function spaces, and can be easily adapted to more general doubling contexts.

The classical Calderón-Zygmund theory was initially developed on $\mathbb{R}^d$. However, the study of integral operators extends beyond the case of Euclidean spaces. The underlying space can be a general measure space, and many examples of integral operators arise in a variety of contexts. Significant developments of the theory have taken place, for instance, in spaces of homogeneous type or in nondoubling settings. The goal of this thesis is to explore integral operators on trees equipped with the Gromov metric (parallel to the Euclidean metric), which can give rise to both doubling and nondoubling situations, depending on the measure. In order to guarantee the desired bounds for an integral operator defined on a tree $\mathfrak{X}$, the associated kernel $K : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ will be asked to satisfy a Hörmander-type condition

$$\sup_{Q\in\mathcal{D}} \sup_{x,y\in Q} \sum_{z\in\mathfrak{X}\setminus Q} |K(z,x) - K(z,y)|\mu(z) < \infty \tag{2}$$

and, in the nondoubling case, the following analog of the size condition

$$\sup_{Q\in\mathcal{D}} \sup_{x\in Q} \sum_{z\in Q^{(1)}\setminus Q} |K(x,z)|\mu(z) < \infty, \tag{3}$$

where $\mathcal{D}$ is a suitable family of subsets of the tree and $Q^{(1)}$ is a specific superset of Q.

Theorem B (Theorem 3.1.3, Theorem 3.1.8, Theorem 3.1.9)**.** *Let* $(\mathfrak{X}, \mu)$ *be a general tree equipped with any measure and let* T *be an integral operator with symmetric kernel* $K\colon \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$*. If* T *is bounded on* $L^2(\mathfrak{X})$ *and* K *satisfies* (2) *and* (3)*, then* T *maps boundedly:*

1. $L^1(\mathfrak{X})$ *into* $L^{1,\infty}(\mathfrak{X})$*;*
2. $L^p(\mathfrak{X})$ *into* $L^p(\mathfrak{X})$ *for all* $p \in (1, \infty)$*;*
3. $H^1(\mathfrak{X})$ *into* $L^1(\mathfrak{X})$*;*
4. $L^\infty(\mathfrak{X})$ *into* $BMO(\mathfrak{X})$*.*

Furthermore, if the space is doubling, the same results hold, requiring only that K *satisfies* (2)*, since in this context* (2) *implies* (3)*.*

Sparse domination

Sparse domination is a powerful and elegant technique in harmonic analysis that has gained significant attention over the past two decades, offering a novel approach to understanding the boundedness and behavior of operators. It is mainly employed in the study of singular operators and maximal functions, and it has further applications in harmonic analysis, such as in weighted inequalities, in the study of the $T(1)$ theorem and of the A_2 conjecture. The sparse domination approach simplifies the analysis of operators, especially in settings where traditional methods, such as those based on kernel estimates or direct computation, may be cumbersome or difficult to apply. This approach has proved to be a key tool in analyzing the behavior of operators in both classical and more abstract settings, including spaces with more elaborate geometric structures.

The key concept is that of a sparse collection of sets. A family of sets is called sparse if for each set there exists a subset that constitutes its *core*, in the sense that it contains a fairly large portion of its measure, and the family of all cores is actually disjoint. Roughly speaking, a collection is sparse if it does not overlap except in a set of very small measure. A sparse collection $\mathcal{S}$ can be used to dominate the action of a given operator T, providing bounds on its behavior in terms of a sparse operator, which is defined as a sum of means over sparse sets, namely

$$\mathcal{A}_{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \mathbb{1}_Q(x).$$

More precisely, the goal is to prove a pointwise bound of the form

$$Tf(x) \lesssim \mathcal{A}_{\mathcal{S}}(x)$$

or a weaker bound in a bilinear form sense

$$\langle Tf_1, f_2 \rangle \lesssim \langle \mathcal{A}_{\mathcal{S}} f_1, f_2 \rangle,$$

in order to transfer to T the good properties that $\mathcal{A}_{\mathcal{S}}$ enjoys.

The sparse estimates find their historical origin in the seminal work of Andrei K. Lerner (see [33]), motivated by the search for sharp weighted norm inequalities. Indeed, sparse domination techniques have proven to be successful also in deriving weighted estimates, that is in understanding how operators behave when dealing with weight functions and their boundedness properties on the corresponding weighted spaces. In this context, the goal is to determine how the presence of the weight influences the operator's norm and then to find a condition that a weight w must satisfy to guarantee that

$$T : L^p(X, w) \to L^p(X, w)$$

boundedly. Sparse domination provides a powerful method for handling these estimates by implying them in a clear and straightforward way. This allows for precise bounds on operators such as maximal functions, singular integrals, or Riesz transforms in weighted spaces.

The classical theory of sparse domination was originally developed for doubling measures, but recent advancements have aimed to extend these techniques to different contexts, including higher-dimensional spaces, more general types of integral operators and in particular to nondoubling settings. In [18] and [48] the authors provide sparse domination for Calderón-Zygmund operators for some nondoubling measures that are still assumed to satisfy certain growth properties. On the other hand, in [19], they deal with the family of *balanced* measures, measures satisfying a weaker condition than doubling, by introducing a significant alteration to the notion of sparse form. In particular, an additional term is needed to address the absence of the doubling property. In light of this result, in this work we present a modified sparse domination result for the Bergman projection on trees, with respect to a particular class of measures which are not only nondoubling, but also non-locally doubling. The sparse-like domination theorem makes use of an extra form, denoted by $\mathcal{E}_{\mathcal{S}}$, and tailored to $\mathcal{P}$.

Theorem C (Theorem 4.1.2, Corollary 4.1.4)**.** *Let $\mathfrak{X}$ be any radial tree and let $f_1, f_2 \in L^1(\mathfrak{X})$. There exists a sparse collection $\mathcal{S}$ such that*

$$|\langle \mathcal{P} f_1, f_2 \rangle| \lesssim \langle \mathcal{A}_{\mathcal{S}} f_1, f_2 \rangle + \mathcal{E}_{\mathcal{S}}(f_1, f_2).$$

If the tree has bounded geometry, namely the space is doubling, then $\mathcal{E}_{\mathcal{S}} = 0$ suffices.

The sparse domination stated in the previous theorem allows for deriving weighted inequalities for the Bergman projection, and in particular, for finding an appropriate condition on weights that ensures the boundedness of $\mathcal{P}$ on the corresponding weighted space. Concerning the doubling situation, the right class of weights is given by the Bekollé-Bonami class B_p, while in the nondoubling case the presence of $\mathcal{E}_{\mathcal{S}}$ forces a stronger condition on the weight. It turns out that the appropriate class is a Bekollé-Bonami-type class $\tilde{B}_p$.

Theorem D (Theorem 4.2.5). *Let $\mathfrak{X}$ be any radial tree and let $p \in (1, \infty)$. If $w \in \tilde{B}_p$, then $\mathcal{P} : L^p(\mathfrak{X}, x) \to L^p(\mathfrak{X}, w)$ boundedly.*

The thesis is structured into several chapters, which we outline as follows.

In Chapter 1, we provide the necessary preliminaries for the subsequent chapters, with a particular focus on the underlying space of trees and the key mathematical tools we will use. We discuss important concepts linked with dyadic analysis and sparse domination, techniques that are particularly useful for analyzing operators in non-standard settings.

Chapter 2 is dedicated to the classical Calderón-Zygmund theory in the context of homogeneous trees endowed with the natural Bergman measures, which turn out to be doubling. In this section, we present an in-depth and quite technical study of the Bergman projection, a specific example that allows us to explore in detail the shape of the associated kernel.

In Chapter 3, we address the case of general trees with Bergman measures, that lead to possibly nondoubling settings. Specifically, we focus on the Bergman projection in the particular case of radial trees, but with a different approach from that used in the previous chapter. Here, instead of analyzing the kernel directly, we construct an orthonormal basis for the Bergman space and treat the projection as a *Bergman shift*, in analogy with the well known Haar shifts. Nevertheless, we do not investigate every single shift that composes the kernel as a single product, but we gather together some tied products in order to exploit some joint cancellations enjoyed by the orthonormal basis. This approach provides an alternative and innovative perspective, that proves to be more efficient.

Finally, in Chapter 4, we present a sparse domination result for the Bergman projection, which is pure for the doubling case, while it requires an additional term for the nondoubling case. In this part, we provide a proof based on a scheme that is adaptable to other examples, but that uses specific estimates and features of this particular operator already found in Chapter 3. As a corollary, we obtain boundedness results on weighted spaces, allowing us to characterize an appropriate class of Bekollé-Bonami-type weights.

The work of this thesis is contained in the following papers:

1. Filippo De Mari, Matteo Monti and Elena Rizzo. Horocyclic harmonic Bergman spaces on homogeneous trees. To appear on *Analysis and Applications*, arXiv:2309.15047, 2023.

2. José M. Conde Alonso, Filippo De Mari, Matteo Monti, Elena Rizzo and Maria Vallarino. Endpoint estimates and sparse domination in nonhomogeneous trees. Preprint, arXiv:2410.23047, 2024.

Chapter 1

Preliminaries

1.1 Trees

Analysis on trees has emerged as a powerful tool in the study of operators and functions, particularly in settings where discrete models of continuous objects provide new insights. In fact, homogeneous trees can be embedded in a natural way into the hyperbolic disk and are usually interpreted as discrete counterparts of symmetric spaces of rank one. Just as the hyperbolic disk offers a rich structure for the study of functions in complex analysis, discrete structures that mimic the properties of continuous spaces offer a similar setting that frequently proves to be more manageable. Many problems and techniques from classical harmonic analysis can be transferred to the framework of trees, whose discrete nature often provides a clearer path for generalizing results to more abstract contexts.

This section gives an introduction to trees, with a specific focus on the underlying geometric structure that we will need across this work. For more details we refer to [10, 20, 21, 26].

1.1.1 Basic notions

A *graph* is a pair $(X, \mathcal{E})$ where X is the set of *vertices* and $\mathcal{E}$ is the family of *edges*, that are two-element subsets of X. The graph is said to be *oriented* if the edges are ordered pairs and it is said to be *unoriented* otherwise. Two vertices x and y are *neighbors* if $\{x, y\} \in \mathcal{E}$ and in this case we write $x \sim y$. A *path* $[x_0, x_n]$ in the graph $(X, \mathcal{E})$ is a finite sequence $x_0, x_1, \ldots, x_n$ such that $\{x_i, x_{i+1}\} \in \mathcal{E}$ for every $i = 0, \ldots, n-1$ and the graph is said to be *connected* if for every $x, y \in X$ there exists a path joining them, i.e. such that $x_0 = x$ and $x_n = y$. A *chain* is a path where $x_i \neq x_{i+2}$ for every $i = 0, \ldots, n-2$ and a *loop* is a chain such that $x_0 = x_n$.

Throughout this thesis we will focus on operators acting on spaces of functions defined on trees, which are a special family of graphs.

Definition 1.1.1. A *tree* is an unoriented, connected and loop free graph.

By abuse of notation we will refer to a tree just by the set of its vertices $\mathfrak{X}$. Notice that given any two vertices $x, y \in \mathfrak{X}$ there always exists a unique chain $[x, y]$ linking them. This fact induces a natural metric on the tree, the *edge counting distance* $d\colon \mathfrak{X}\times\mathfrak{X} \to \mathbb{N}$, such that $d(x, y)$ is given by the number of edges belonging to the chain $[x, y]$. Furthermore, we will only be interested in *locally finite* trees, namely trees in which every vertex belongs to a finite number of edges. This allows us to associate to every tree (and more in general to every locally finite connected graph) the function $q\colon \mathfrak{X} \to \mathbb{N}$ such that $q(x) + 1$ is the number of neighbors of x.

Definition 1.1.2. A tree is said to be *q-homogeneous* if the function q is constant. In this case, q is called the *degree* of $\mathfrak{X}$.

The q-homogeneous tree will be denoted by $\mathfrak{X}_q$. The set of vertices of a homogeneous tree is always infinite if $q \neq 0$ and it is just $\mathbb{Z}$ when $q = 1$. We will consider q-homogeneous trees with $q \geq 2$.

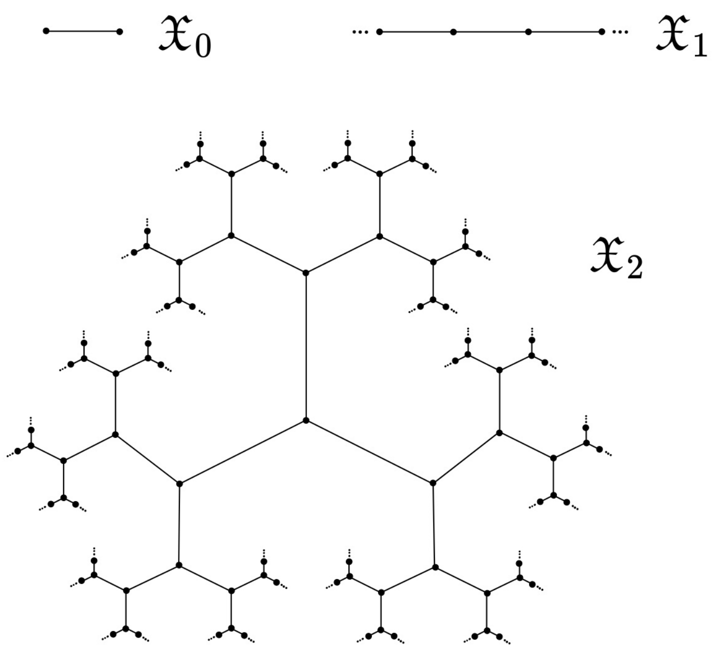

Figure 1.1: The 0-homogeneous tree and portions of the 1-homogeneous tree and the 2-homogeneous tree.

Every tree holds a natural structure of boundary. An *infinite chain* is an infinite sequence $(x_i)_{i\in\mathbb{N}}$ of vertices of $\mathfrak{X}$ such that $x_i \neq x_{i+2}$ for every $i \in \mathbb{N}$. Denote by $I(\mathfrak{X})$ the set of infinite chains in $\mathfrak{X}$ and consider the following equivalence relation on it. Given $(x_i)_{i\in\mathbb{N}}, (y_j)_{j\in\mathbb{N}}$ in $I(\mathfrak{X})$ we say that they are equivalent if there exists $m \in \mathbb{Z}$ such that $x_i = y_{i+m}$ for every i sufficiently large. In such case we write $(x_i)_{i\in\mathbb{N}} \sim (y_j)_{j\in\mathbb{N}}$.

Definition 1.1.3. The *boundary* of $\mathfrak{X}$ is the set $\Omega := I(\mathfrak{X})/\sim$.

Notice that if we fix a reference point o (as it will be done in Section 1.1.2) this definition is equivalent to defining the boundary just as the set of all infinite chains starting in o. The main idea is that a boundary point can be seen as the limit point of an infinite chain in $\mathfrak{X}$. In fact it was shown in [12] that a homogeneous tree of odd degree q (then such that every vertex has an even number $q+1$ of neighbors) can be isometrically embedded in the unit disk $\mathbb{D}$ endowed with the hyperbolic metric. Through this embedding the boundary of the tree corresponds almost everywhere with the boundary of the disk, the unit circle.

1.1.2 Rooted vs hanged trees

A tree $\mathfrak{X}$ is said to be *rooted* if we fix a reference vertex o, called the *origin* of the tree, and define the *modulus* of a vertex x as its distance from the origin, namely $|x|_o := d(o,x)$.

For $x \neq o$ we call the *sector* of x with respect to o the subset of $\mathfrak{X}$ given by

$$T_x^o := \{y \in \mathfrak{X} : [x,o] \subseteq [y,o]\}$$

and we say that x is the *generator* of T_x^o, while we simply define the sector generated by o as the whole tree, that is $T_o^o := \mathfrak{X}$. Roughly speaking, T_x^o is the set containing x and all the vertices lying below x, namely all its descendants with respect to o.

For every $x \in \mathfrak{X}$ the set of its *successors* is given by

$$S_o(x) := \{y \in \mathfrak{X} : y \sim x,\ |y|_o = |x|_o + 1\},$$

while its *predecessor* $x_o^{(1)}$ is the only neighbor of x such that $|x_o^{(1)}|_o = |x|_o - 1$. Observe that the predecessor is actually a function from $\mathfrak{X} \setminus \{o\}$ to $\mathfrak{X}$, and for every $n \in \mathbb{N}$ we can compose it n times, obtaining a function from $\mathfrak{X} \setminus B_d(o, n-1)$ to $\mathfrak{X}$, which returns the n-th predecessor $x_o^{(n)}$ of a vertex with respect to the origin, with the convention that $x_o^{(0)}$ is just x, i.e. the 0-th composition is the identity on $\mathfrak{X}$.

Given two vertices $x, y \in \mathfrak{X}$, their *confluent* with respect to o, denoted by $x \wedge_o y$, is the only vertex such that $[x,o] \cap [y,o] = [x \wedge_o y, o]$. Notice that $x \wedge_o y$ is just the last vertex in common between the paths connecting the origin to x and y, respectively.

All these notions are represented in Figure 1.2.

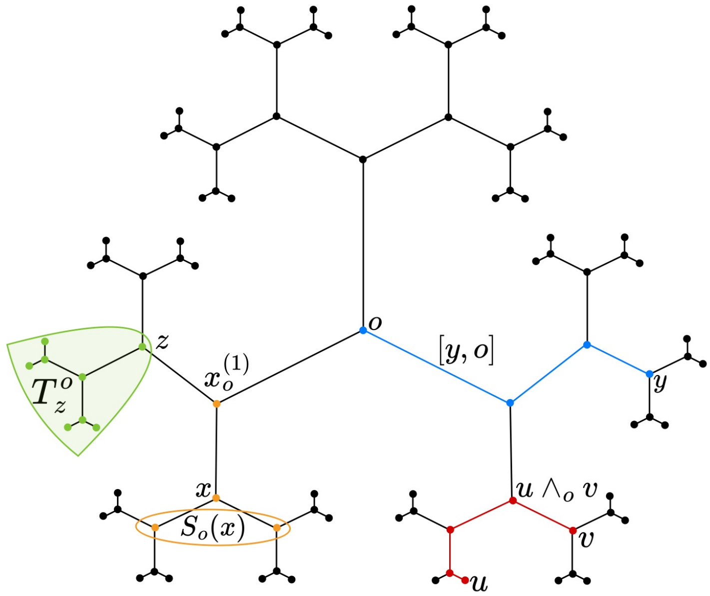

Figure 1.2: Rooted 2-homogeneous tree. On the left, the green area is the first portion of the sector T_z^o generated by z. Then, the neighbors of x are split in the predecessor and the successors. Next, the confluent of two vertices u and v. Finally, the blue path is $[y, o]$.

In this work a special class of rooted trees will be considered, the so-called radial trees.

Definition 1.1.4. A rooted tree $\mathfrak{X}$ is said to be *radial* if the function q associated to $\mathfrak{X}$ is radial, namely if $q(x) := q(|x|_o)$.

As in the homogeneous case, we will ask q to be greater or equal than 2, so that in this case $q\colon \mathbb{N} \to \mathbb{N} \setminus \{0, 1\}$.

Remark 1.1.5. Once the origin o is fixed, we will always omit the index o from the notation, namely we will write $|x|$, T_x, $S(x)$, $p(x)$, $x \wedge y$.

The hanged version of a tree provides a different visualization, requiring passage through the boundary. Let us consider a reference boundary point $\omega \in \Omega$, which will play a similar role as that of the origin o in the rooted version. It will be possible to give notions analogous to the ones already presented for the rooted tree, here with respect to ω instead of o. Before doing that, we introduce the definitions of horocycles, horoballs and horocyclic index.

Given two vertices $x, y \in \mathfrak{X}$, their *confluent* with respect to ω, denoted by $x \wedge_\omega y$, is the only vertex such that $[x, \omega) \cap [y, \omega) = [x \wedge_\omega y, \omega)$. Now let $x, y \in \mathfrak{X}$, they are said to be equivalent if $d(x, x \wedge_\omega y) = d(y, x \wedge_\omega y)$. Every equivalence class induced on the tree by this equivalence relation is a *horocycle* with respect to ω. It is straightforward to see that a horocycle is an infinite subset of $\mathfrak{X}$. The set of horocycles with respect to ω is indexed on $\mathbb{Z}$ and forms a partition of $\mathfrak{X}$. If we fix a reference horocycle H_0^ω, it is easy to see that for all $x \in \mathfrak{X}$ the map $y \mapsto d(x, x \wedge_\omega y) - d(y, x \wedge_\omega y)$ is constant on H_0^ω. Thus, for every $k \in \mathbb{Z}$ the k-th horocycle H_k^ω given by

$$H_k^\omega := \left\{ x \in \mathfrak{X} : d(x, x \wedge_\omega y) = d(y, x \wedge_\omega y) + k \text{ for some } y \in H_0^\omega \right\}$$

is well defined. Roughly speaking, we enumerate the horocycles with negative integers in the direction of ω and with positive integers in the opposite direction (see Figure 1.3). The *horoball* with respect to ω at level $n \in \mathbb{N}$ is the set of the form

$$HB_n^\omega := \bigcup_{j \leq n} H_j^\omega.$$

Finally, for all $x \in \mathfrak{X}$, the *horocyclic index* $|x|_\omega$ is the unique integer such that $x \in H^\omega_{|x|_\omega}$.

As already said, the boundary point ω is in this context the substitute of the origin o of the rooted version. In this perspective, the last definitions have a similar natural interpretation as well. Indeed, the horocyclic index takes the role of the modulus of a vertex, while horocycles and horoballs are the counterparts of spheres and balls (with respect to d), respectively.

Now we can state all the analogous definitions to those given in the previous setting. Let $x \in \mathfrak{X}$, the *sector* of x is the subset of $\mathfrak{X}$ given by

$$T_x^\omega := \{ y \in \mathfrak{X} : [x, \omega) \subseteq [y, \omega) \}$$

and we say that x is the *generator* of T_x^ω. Just as in the rooted case, T_x^ω is the set containing x and all the vertices lying below x with respect to ω. For every $x \in \mathfrak{X}$ the set of its *successors* with respect to ω is given by

$$S_\omega(x) := \{ y \in \mathfrak{X} : y \sim x,\ |y| = |x|_\omega + 1 \},$$

while its *predecessor* $x_\omega^{(1)}$ is the only neighbor of x such that $|x_\omega^{(1)}|_\omega = |x|_\omega - 1$. Notice that in this case the predecessor is a function defined on the whole of $\mathfrak{X}$, as well as all its powers for every $n \in \mathbb{N}$.

Again we represent these notions in Figure 1.3.

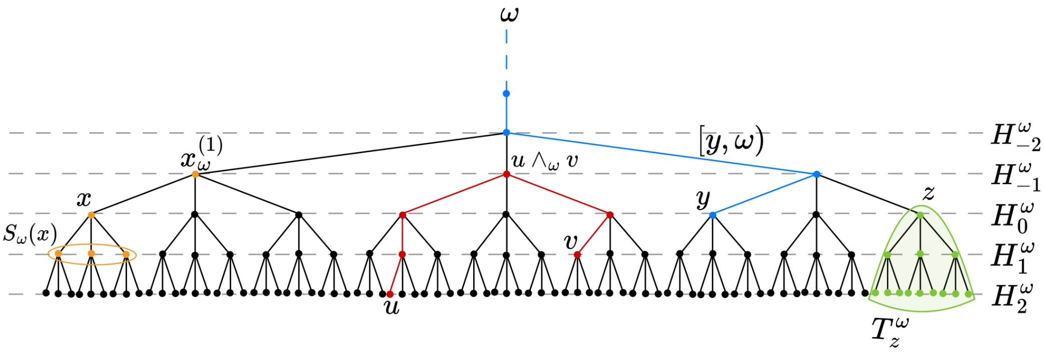

Figure 1.3: Hanged 3-homogeneous tree. On the left, the neighbors of x are split in the predecessor and the successors. Next, the confluent of two vertices u and v. Then, the blue path is $[y, \omega)$. Finally, the green area is the first portion of the sector T_z^ω generated by z.

The horocyclic trees are a special family of hanged trees on which we will focus here.

Definition 1.1.6. A hanged tree $\mathfrak{X}$ is said to be *horocyclic* if the function q associated to $\mathfrak{X}$ is horocyclic, namely constant on horocycles, that is if $q(x) := q(|x|_\omega)$.

Assuming again $q \geq 2$, one obtains here $q : \mathbb{Z} \to \mathbb{N} \setminus \{0, 1\}$.

Remark 1.1.7. In this context as well, once we fix a point $\omega \in \Omega$, we omit the index ω in the notation. This means that there will be no difference in notation between the rooted and hanging versions of the tree, so it will be crucial to clarify the context we are dealing with.

Remark 1.1.8. In [12] the authors provide an embedding of rooted homogeneous trees of odd degree (where every vertex has an even number of neighbors) into the hyperbolic disk. This embedding is such that the automorphisms generated by rotations and translations may be represented by automorphisms of the disk. Moreover, it allows to spread the notion of harmonic function from one context to the other. In this setting the structure of the tree mimics the hyperbolic geometry's growth from the origin. On the other hand, the tree hanging from a point on the boundary has a parallel with the upper half plane $\mathbb{U}$. Just as $\mathbb{U}$ is a model of a non compact and boundary defined space, this visualization of the tree reflects this same concept, with its structure resembling the discrete analogue of the continuous upper half plane. The reference point of the boundary to which the tree is hanged plays the role of the point at infinity in $\mathbb{U}$, while the rest of the boundary represents the real axis.

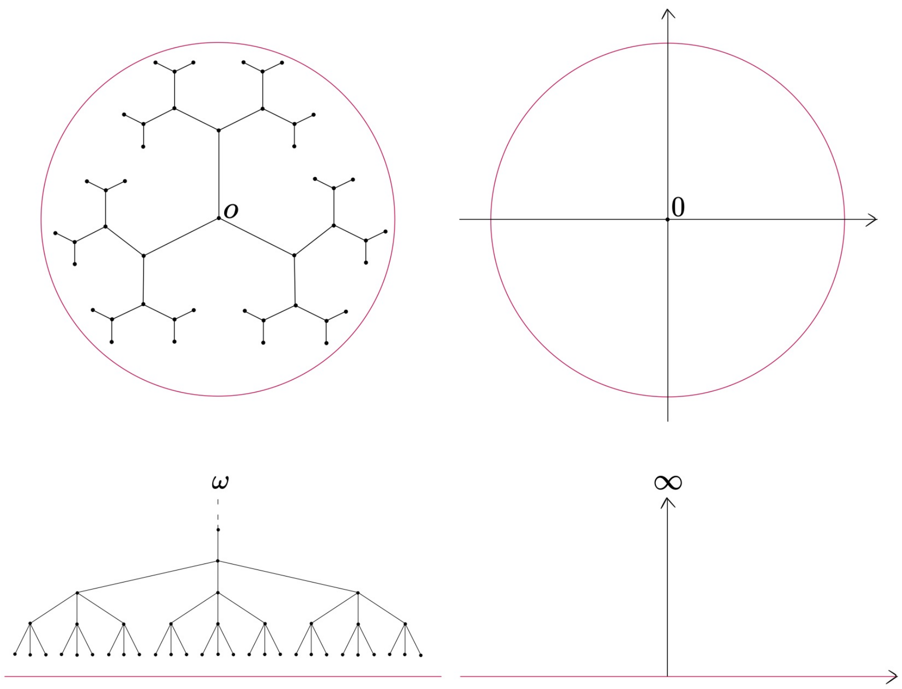

Figure 1.4: Similarities between the rooted homogeneous tree and the disk, and between the hanged homogeneous tree and the upper half plane, respectively.

1.1.3 Gromov metric and Bergman measures

Even if the most natural metric that one can think of on a tree is perhaps the edge counting distance, from now on the tree will always be endowed with the Gromov metric. In fact in the case of homogeneous trees, the counting distance is the discrete version of the hyperbolic metric on the disk, while the Gromov metric corresponds to the Euclidean one (see [5]). As it will be clear later on, in particular starting from Section 1.3.2, the Gromov metric plays a key role in the development of this work, especially thanks to the peculiar shape of its balls. See [30] and [1] for more details about Gromov metric in general and in discrete settings, respectively. We present here the rooted and the hanged versions of this metric.

Definition 1.1.9. The *rooted Gromov metric* $\rho_o\colon \mathfrak{X}\times\mathfrak{X}\to\mathbb{R}$ is defined by

$$\rho_o(x,y)=\begin{cases}0 & \text{if } x=y,\\ e^{-|x\wedge_o y|_o} & \text{if } x\neq y.\end{cases}$$

The *hanged Gromov metric* $\rho_\omega\colon \mathfrak{X}\times\mathfrak{X}\to\mathbb{R}$ is defined by

$$\rho_\omega(x,y)=\begin{cases}0 & \text{if } x=y,\\ e^{-|x\wedge_\omega y|_\omega} & \text{if } x\neq y.\end{cases}$$

In both versions, when o or ω will be fixed, the Gromov metric will be just denoted by ρ. We denote by $\lfloor\cdot\rfloor$ the floor function.

Proposition 1.1.10. *For every $x \in \mathfrak{X}$ and $r > 0$,*

$$B_{\rho_o}(x,r) = \begin{cases} \{x\} & \text{if } r \leq e^{-|x|_o}, \\ T_{x(|x|_o+\lfloor\log(r)\rfloor)} & \text{if } e^{-|x|_o} < r \leq 1, \\ \mathfrak{X} & \text{if } r > 1. \end{cases}$$

Proof. Let $x, y \in \mathfrak{X}$, $x \neq y$ and $r > 0$. If $y \in B_{\rho_o}(x,r)$ then we have that

$$e^{-|x|_o} \leq \rho_o(x,y) = e^{-|x \wedge_o y|_o} < r$$

since $x \wedge_o y \in [x, o]$, so $|x \wedge_o y|_o \leq |x|_o$. Thus

1. if $r \leq e^{-|x|_o}$, then there is no $y \in \mathfrak{X} \setminus \{x\}$ such that $e^{-|x \wedge_o y|_o} < r$ and $B_{\rho_o}(x,r) = \{x\}$;
2. if $e^{-|x|_o} < r \leq 1$, then $|x \wedge_o y|_o > -\log(r) \geq \lfloor -\log(r)\rfloor$ gives

   $$|x|_o \geq |x \wedge_o y|_o \geq \lfloor -\log(r)\rfloor + 1 = -\lfloor\log(r)\rfloor,$$

   and this means that y must lie in the sector of the predecessor of x which has modulus equal to $-\lfloor\log(r)\rfloor$, that is $B_{\rho_o}(x,r) = T_{x(|x|_o+\lfloor\log(r)\rfloor)}$;
3. if $r > 1$, then every $y \in \mathfrak{X}$ is such that $e^{-|x\wedge_o y|_o} < r$ and $B_{\rho_o}(x,r) = \mathfrak{X}$.

□

Proposition 1.1.11. *Let $x \in \mathfrak{X}$ and $r > 0$,*

$$B_{\rho_\omega}(x,r) = \begin{cases} \{x\} & \text{if } r \leq e^{-|x|_\omega}, \\ T_{x(|x|_\omega+\lfloor\log(r)\rfloor)} & \text{if } r > e^{-|x|_\omega}. \end{cases}$$

Proof. Analogous to Proposition 1.1.10. The only difference being that the horocyclic index can be negative (while the modulus cannot), so that case 3 falls into case 2. □

We introduce now the measures that will be considered all along this thesis while dealing with radial or horocyclic trees. It is worth noticing that we will never endow the tree with the counting measure, which could appear to be the most straightforward choice in a discrete setting.

The construction of our measures is mainly motivated by the study of the *reference measures* introduced in [13] in the context of radial homogeneous trees and by the particular subclass of reference measures studied in [25]. The measures considered in [25], given by $q^{-\alpha|x|_o}$ for $\alpha > 1$, are a natural counterpart of the classical family of measures usually considered on the hyperbolic disk, the Bergman measures $(1-|z|^2)^{\alpha-2}\mathrm{d}z$, $\alpha>1$. In particular, in both cases they are radial and decreasing. Concerning radial trees, the idea is just to generalize this class of measures. On the other hand, for what concerns horocyclic homogeneous trees, we are interested in finding the discrete version of the measures $\mathrm{Im}(z)^{\alpha-2}\mathrm{d}z$, $\alpha>1$, classically investigated on the upper half plane, and then to generalize them to the nonhomogeneous case. In this case, the idea is to find measures that are constant on horocycles and decreasing while moving away from the reference point on the boundary.

Let $\mathfrak{X}$ be a radial tree with fixed origin o, consider for every $\alpha>1$

$$\mu_\alpha^o(x) := \mu_\alpha^o(|x|_o) = \prod_{i=0}^{|x|_o-1} q(i)^{-\alpha}, \quad x\in\mathfrak{X}.$$

Now, let $\mathfrak{X}$ be a horocyclic tree hanged to the boundary point ω, for all $\alpha>1$ we define

$$\mu_\alpha^\omega(x) := \mu_\alpha^\omega(|x|_\omega) = \begin{cases} 1 & \text{if } |x|_\omega = 0,\\ \displaystyle\prod_{i=0}^{|x|_\omega-1} q(i)^{-\alpha} & \text{if } |x|_\omega \geq 1,\\ \displaystyle\prod_{i=|x|_\omega}^{-1} q(i)^{\alpha} & \text{if } |x|_\omega \leq -1.\end{cases}$$

Notice that for a homogeneous horocyclic tree this just reads

$$\mu_\alpha^\omega(x) = q^{-\alpha|x|_\omega}, \quad x\in\mathfrak{X}.$$

Both in the radial and horocyclic cases, we will refer to these families of measures as *Bergman measures.*

Proposition 1.1.12. *The pair $(\mathfrak{X},\mu_\alpha^o)$ is a finite measure space for every $\alpha>1$.*

Proof. It is just an immediate computation,

$$\mu_\alpha^o(\mathfrak{X}) = \sum_{x\in\mathfrak{X}} \mu_\alpha^o(x) = \sum_{n=0}^{\infty}\sum_{x\in S_d(o,n)} \mu_\alpha^o(x) = \sum_{n=0}^{\infty}\prod_{i=0}^{n-1} q(i)\prod_{j=0}^{n-1} q(j)^{-\alpha}$$
$$= \sum_{n=0}^{\infty}\prod_{i=0}^{n-1}\frac{1}{q(i)^{\alpha-1}} \leq \sum_{n=0}^{\infty}\left(\frac{1}{2^{\alpha-1}}\right)^n < \infty.$$

□

Conversely, since every horocycle is infinte, it is straightforward to see that the total measure of $\mathfrak{X}$ with respect to μ_α^ω is not finite. Nevertheless, sectors with respect to ω do have finite μ_α^ω measure, and this feature will play a fundamental role in what follows.

Lemma 1.1.13. *For every $x \in \mathfrak{X}$, $\mu_\alpha^\omega(T_x^\omega) \simeq \mu_\alpha^\omega(x)$ and so it is finite.*

Proof. It is obvious that $\mu_\alpha^\omega(x) \leq \mu_\alpha^\omega(T_x^\omega)$, so we only have to show that $\mu_\alpha^\omega(T_x^\omega) \lesssim \mu_\alpha^\omega(x)$.

Let $x \in \mathfrak{X}$ be such that $|x|_\omega \geq 0$,

$$\begin{aligned}
\mu_\alpha^\omega(T_x^\omega) &= \sum_{y \in T_x^\omega} \mu_\alpha^\omega(y) = \sum_{k=|x|_\omega}^{\infty} \sum_{y \in T_x^\omega \cap H_k} \mu_\alpha^\omega(y) \\
&= \sum_{k=|x|_\omega}^{\infty} \prod_{i=|x|_\omega}^{k-1} q(i) \prod_{j=0}^{k-1} q(j)^{-\alpha} \\
&= \prod_{j=0}^{|x|_\omega-1} \frac{1}{q(j)^\alpha} \sum_{k=|x|_\omega}^{\infty} \prod_{i=|x|_\omega}^{k-1} \frac{1}{q(i)^{\alpha-1}} \\
&= \mu_\alpha^\omega(x) \sum_{k=|x|_\omega}^{\infty} \prod_{i=|x|_\omega}^{k-1} \frac{1}{q(i)^{\alpha-1}} \\
&\leq \mu_\alpha^\omega(x) \sum_{k=|x|_\omega}^{\infty} \left(\frac{1}{2^{\alpha-1}}\right)^{k-|x|_\omega} \\
&\simeq \mu_\alpha^\omega(x) < \infty.
\end{aligned}$$

Now let $x \in \mathfrak{X}$ be such that $|x|_\omega < 0$,

$$\begin{aligned}
\mu_\alpha^\omega(T_x^\omega) &= \sum_{y \in T_x^\omega} \mu_\alpha^\omega(y) = \sum_{k=|x|_\omega}^{\infty} \sum_{y \in T_x^\omega \cap H_k} \mu_\alpha^\omega(y) \\
&= \sum_{k=0}^{\infty} \prod_{i=|x|_\omega}^{k-1} q(i) \prod_{j=0}^{k-1} q(j)^{-\alpha} + \sum_{k=|x|_\omega}^{-1} \prod_{i=|x|_\omega}^{k-1} q(i) \prod_{j=k}^{-1} q(j)^{\alpha} \\
&= \prod_{i=0}^{|x|_\omega-1} \frac{1}{q(i)^\alpha} \sum_{k=0}^{\infty} \prod_{j=|x|_\omega}^{k-1} \frac{1}{q(j)^{\alpha-1}} + \prod_{i=|x|_\omega}^{-1} q(i)^\alpha \sum_{k=|x|_\omega}^{-1} \prod_{j=|x|_\omega}^{k-1} \frac{1}{q(j)^{\alpha-1}} \\
&\leq \prod_{i=0}^{|x|_\omega-1} \frac{1}{q(i)^\alpha} \sum_{k=0}^{\infty} \left(\frac{1}{2^{\alpha-1}}\right)^{k-|x|_\omega} + \mu_\alpha^\omega(x) \sum_{k=|x|_\omega}^{-1} \left(\frac{1}{2^{\alpha-1}}\right)^{k-|x|_\omega} \\
&\lesssim 1 + \mu_\alpha^\omega(x) \lesssim \mu_\alpha^\omega(x) < \infty.
\end{aligned}$$

□

Remark 1.1.14. Arguing as in the first case of the previous proof it is seen that also $\mu_\alpha^o(T_x^o) \simeq \mu_\alpha^o(x)$ for every $x \in \mathfrak{X}$.

1.1.4 Doubling and nondoubling settings

In the study of metric measure spaces, it is often important to distinguish between doubling and nondoubling spaces. Broadly speaking, the doubling condition limits how quickly the measure of sets can grow when they are dilated. Indeed, the balls in spaces satisfying the doubling property have a more regular and controlled growth, which makes many analytic tools easier to handle. In contrast, nondoubling spaces do not satisfy this condition, meaning that the measure of a set can grow more rapidly when it is scaled. This lack of control over the growth of measures introduces new complications, as the arguments and techniques that rely on the doubling condition no longer apply. As a result, analysis in nondoubling spaces requires novel approaches and methods to overcome the complexities introduced by the unbounded growth of the measure. The remainder of this work will focus on exploring these two settings in the context of trees.

Definition 1.1.15. A metric measure space (X, δ, σ) is *doubling* if there exists $C > 0$ such that

$$\sigma\Big(B_\delta(x, 2r)\Big) \leq C\sigma\Big(B_\delta(x, r)\Big)$$

for every $x \in X$ and $r > 0$.

Let $\mathfrak{X}$ be a radial or horocyclic tree endowed with the corresponding Gromov metric ρ and a measure μ_α for some $\alpha > 1$. The following result describes under which condition $(\mathfrak{X}, \rho, \mu_\alpha)$ is a doubling space.

Proposition 1.1.16. *The metric measure space $(\mathfrak{X}, \rho, \mu_\alpha)$ is doubling if and only if the function q associated to $\mathfrak{X}$ is bounded.*

Proof. Let $x \in X$ and $r > 0$. By Propositions 1.1.10 and 1.1.11 it is immediate to see that if $B_\rho(x, r) = \{x\}$ then either $B_\rho(x, 2r) = \{x\}$ or $B_\rho(x, 2r) = T_x$, while if $B_\rho(x, r) = T_x$ then either $B_\rho(x, 2r) = T_x$ or $B_\rho(x, 2r) = T_{p(x)}$. By Lemma 1.1.13 and Remark 1.1.14 we know that $\mu_\alpha(T_x) \simeq \mu_\alpha(x)$, meaning that

$$\frac{\mu_\alpha(T_x)}{\mu_\alpha(x)} \simeq 1,$$

while

$$\frac{\mu_\alpha(T_{p(x)})}{\mu_\alpha(T_x)} \simeq \frac{\mu_\alpha(p(x))}{\mu_\alpha(x)} = q(|x| - 1)^\alpha.$$

Hence, $(\mathfrak{X}, \rho, \mu_\alpha)$ is doubling if and only if q is bounded from above. □

1.2 Maximal functions and weights

This section is devoted to introducing the theory of weights, for which we need to briefly recall some concepts related to maximal functions. Roughly speaking, a weight is a nonnegative function that complies with certain properties. The theory of weights is mainly inspired by the study of weighted inequalities in very diverse subjects within harmonic analysis. In the context of singular integrals a better understanding of this topic was obtained in the 1970s, especially thanks to the work of Muckenhoupt. His characterization of the weights for which the Hardy-Littlewood maximal function is bounded on the weighted L^p space motivates the introduction of the A_p classes. We present the theory in the setting of $\mathbb{R}^d$ with the Lebesgue measure in analogy with Chapters 2 and 7 of [29], to which we refer for the proofs.

1.2.1 Maximal functions

From now on integral averages over subsets $E \subseteq \mathbb{R}^d$ will be denoted in the following way

$$\langle f \rangle_E = \frac{1}{|E|} \int_E f.$$

Definition 1.2.1. Let f be a locally integrable function on $\mathbb{R}^d$. The *centered Hardy-Littlewood maximal function* of f is defined as

$$\mathcal{M}f(x) = \sup_{\substack{B \text{ ball} \\ c(B)=x}} \langle |f| \rangle_B,$$

where $c(B)$ denotes the center of the ball B.

It is clear that $\mathcal{M}f = \mathcal{M}(|f|)$ and that $\mathcal{M}$ is a positive sublinear operator. Furthermore, $\|\mathcal{M}f\|_\infty \leq \|f\|_\infty$ so that $\mathcal{M}$ is bounded on $L^\infty(\mathbb{R}^d)$, while it is easy to see that it is not bounded on L^1.

Note that in the definition of $\mathcal{M}$ the supremum is taken over centered balls. Different maximal functions can be defined by taking the supremum over uncentered balls, centered cubes or uncentered cubes. It is easy to see that all these definitions are pointwise equivalent, up to a constant. Consequently, every boundedness result that holds for $\mathcal{M}$, hold for the three other maximal operators as well.

Theorem 1.2.2 (Theorem 2.1.6, [29])**.** *There exists $C > 0$ such that for every $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$*

$$\left|\left\{x \in \mathbb{R}^d : \mathcal{M}f(x) > \lambda\right\}\right| \leq \frac{C}{\lambda}\|f\|_1,$$

in other words the Hardy-Littlewood maximal operator is of weak-type $(1,1)$. By interpolation it is bounded on $L^p(\mathbb{R}^d)$ for every $p \in (1,\infty)$.

This theorem is a classical result and we omit the proof, that relies on Vitali's covering lemma.

Remark 1.2.3. The importance of maximal operators is essential. For instance the weak-type $(1,1)$ boundedness of $\mathcal{M}$ is crucial to prove Lebesgue's differentiation Theorem.

1.2.2 Weights and A_p classes

Definition 1.2.4. A *weight* is a function $w : \mathbb{R}^d \to [0, \infty]$ which is locally integrable and takes values in $(0, \infty)$ almost everywhere.

It follows that if $1/w$ is also locally integrable, then it is a weight. Given a weight w and a subset $E \subseteq \mathbb{R}^d$ measurable, we denote by

$$w(E) = \int_E w(x)\mathrm{d}x$$

the w-measure of E. If E is bounded then $w(E) < \infty$ since $w \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. The weighted L^p spaces are the spaces of functions such that

$$\int_{\mathbb{R}^d} |f(x)|^p w(x)\mathrm{d}x < \infty$$

equipped with the obvious norm. They are denoted by $L^p(\mathbb{R}^d, w)$ or just $L^p(w)$ (also, in more general contexts, the underlying space is omitted while it is fixed once and for all).

By Theorem 1.2.2, the maximal function $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^d)$ for every $p \in (1, \infty)$. The issue we address is whether there exists a characterization of all weights such that $\mathcal{M}$ is bounded on $L^p(w)$ for every $p \in (1, \infty)$.

Definition 1.2.5. Let $p \in (1, \infty)$, a weight w belongs to the A_p class if

$$\mathcal{M} : L^p(w) \to L^p(w).$$

Now, let $p \in (1, \infty)$ and suppose that

$$\int_{\mathbb{R}^d} \mathcal{M}f(x)^p w(x)\mathrm{d}x \le C_p \int_{\mathbb{R}^d} |f(x)|^p w(x)\mathrm{d}x \tag{1.1}$$

holds for all $f \in L^p(w)$. Then if we fix a ball B, using that

$$\frac{1}{|B|}\int_B |f(y)|\mathrm{d}y \le \mathcal{M}(f\mathbb{1}_B)(x)$$

for every $x \in B$ and applying (1.1) to $f\mathbb{1}_B$, we obtain

$$w(B)\Big(\frac{1}{|B|}\int_B |f(y)|\mathrm{d}y\Big)^p \le \int_B \mathcal{M}(f\mathbb{1}_B)(x)^p w(x)\mathrm{d}x \le C_p \int_B |f(x)|^p w(x)\mathrm{d}x.$$

Therefore,

$$w(B)\Big(\frac{1}{|B|}\int_B |f(y)|\mathrm{d}y\Big)^p \le C_p \int_B |f(x)|^p w(x)\mathrm{d}x$$

holds for every ball B and function $f \in L^p(w)$. By choosing $f = w^{-p'/p}$ we have $f^p w = w^{-p'/p} = f$ and so the two integrands are equal. It follows that

$$\frac{w(B)}{|B|}\Big(\frac{1}{|B|}\int_B |f(y)|\mathrm{d}y\Big)^{p-1} \le C_p.$$

Given the additional assumption that $\inf_{y\in B} w(y) > 0$ for any ball B, one would obtain that

$$\sup_{B \text{ ball}} \Big(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\Big)\Big(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\mathrm{d}x\Big)^{p-1} \leq C_p. \qquad (1.2)$$

By taking $f = (w+\epsilon)^{-p'/p}$, replacing $w(x)$ with $w(x)+\epsilon$ and applying the Lebesgue monotone convergence theorem for $\epsilon \to 0$ it can be seen that (1.2) also holds when $\inf_{y\in B} w(y) = 0$. The previous computation shows that this condition on w is necessary to belong to the A_p class. It can be shown that (1.2) is actually a sufficient condition.

Theorem 1.2.6 (Theorem 7.1.9, [29]). *Let $p \in (1, \infty)$. A weight w belongs to the A_p class if and only if its A_p characteristic constant*

$$[w]_{A_p} := \sup_{B \text{ ball}} \Big(\frac{1}{|B|}\int_B w(x)\mathrm{d}x\Big)\Big(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\Big)^{p-1}$$

is finite. Furthermore, if $w \in A_p$, then

$$\|\mathcal{M}\|_{L^p(w)\to L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p-1}}.$$

In the following proposition, we outline some fundamental properties of A_p weights.

Proposition 1.2.7 (Proposition 7.1.5, [29]). *Let $p \in (1, \infty)$ and $w \in A_p$. The following properties hold:*

1) $[\delta^\lambda(w)]_{A_p} = [w]_{A_p}$, *where* $\delta^\lambda(w)(x) = w(\lambda x)$, $\lambda > 0$;

2) $[\tau^z(w)]_{A_p} = [w]_{A_p}$, *where* $\tau^z(w)(x) = w(x - z)$, $z \in \mathbb{R}^d$;

3) $[\lambda w]_{A_p} = [w]_{A_p}$, $\lambda > 0$;

4) *the function* $w^{-\frac{1}{p-1}}$ *is in the* $A_{p'}$ *class and* $[w^{-\frac{1}{p-1}}]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}$. *Therefore,* $w \in A_2$ *if and only if* $w^{-1} \in A_2$ *and both weights have the same characteristic constant.*

5) $[w]_{A_p} \geq 1$ *and equality holds if and only if* w *is a constant;*

6) *for every* $1 < p < q < \infty$, $[w]_{A_q} \leq [w]_{A_p}$ *and consequently* $A_p \subseteq A_q$.

1.3 Dyadic analysis

In harmonic analysis, dyadic analysis and martingales provide powerful frameworks for studying functions at various scales, allowing for their decomposition and understanding of their structure. Dyadic analysis focuses on the use of dyadic systems to study the local behavior of functions, while martingales arise naturally in the study of functions via their connection to filtrations of σ-algebras. In this context, a martingale often reflects the dyadic structure of the function space being analyzed. In this section, we introduce the basics of dyadic harmonic analysis in the classical setting of $\mathbb{R}^d$ equipped with the Lebesgue measure and then we show how it is possible to equip the tree with a natural dyadic system which carries a martingale structure. Some useful references for this part are [34, 41, 50, 40].

1.3.1 Dyadic systems and martingales

A *cube* in $\mathbb{R}^d$ is a half open set of the form

$$Q = [x_1, x_1 + \ell(Q)) \times \cdots \times [x_d, x_d + \ell(Q)),$$

where $\ell(Q)$ is the side lenght of Q and $c(Q) = \big(x_1 + \frac{1}{2}\ell(Q), \ldots, x_d + \frac{1}{2}\ell(Q)\big)$ is the center of Q. Given a cube Q, $\mathscr{D}(Q)$ is the dyadic sublattice of subcubes of Q, i.e. the family of subcubes of Q obtained by successive dyadic subdivisions. The family $\mathscr{D}(Q)$ can be written as the following disjoint union

$$\mathscr{D}(Q) = \bigcup_{k=0}^{\infty} \mathscr{D}_k(Q),$$

where $\mathscr{D}_k(Q)$ is the k-th generation of dyadic subcubes of Q. The cubes belonging to $\mathscr{D}_1(Q)$ are the *children* of Q, while the *parent* of a cube $R \in \mathscr{D}_k(Q)$ is the only cube $R^{(1)} \in \mathscr{D}_{k-1}(Q)$ such that $R \subseteq R^{(1)}$. The cubes belonging to the dyadic sublattice $\mathscr{D}(Q)$ satisfy some really good properties:

1. they are nested, meaning that two different cubes are either disjoint or one is contained into the other;
2. each generation $\mathscr{D}_k(Q)$ is a partition of Q;
3. the dyadic parent of $R \in \mathscr{D}(Q)$ is unique and $\ell(R^{(1)}) = 2\ell(R)$.

Instead of considering just a cube, it is also possible to define dyadic lattices over all of $\mathbb{R}^d$ by partitioning it with cubes of the same length.

Definition 1.3.1. The *standard dyadic system* in $\mathbb{R}^d$ is defined as

$$\mathscr{D} = \bigcup_{k\in\mathbb{Z}} \mathscr{D}_k,$$

$$\mathscr{D}_k = \Big\{2^k[x_1, x_1 + 1)\times \cdots \times [x_d, x_d + 1) : x_1, \ldots, x_d \in \mathbb{Z}\Big\}.$$

In the framework of dyadic analysis another version of the Hardy-Littlewood maximal function is broadly studied.

Definition 1.3.2. The *dyadic maximal function* associated to the standard dyadic system $\mathscr{D}$ is

$$\mathcal{M}_{\mathscr{D}} f(x) = \sup_{\substack{x \in Q \\ Q \in \mathscr{D}}} \langle |f| \rangle_Q.$$

It is clear that $\|\mathcal{M}_{\mathscr{D}} f\|_\infty \le \|f\|_\infty$, namely $\mathcal{M}_{\mathscr{D}}$ is bounded on $L^\infty(\mathbb{R}^d)$ with operator norm 1. It is also easy to see that it is of weak-type $(1,1)$, as we show in the next proposition.

Proposition 1.3.3. *Let $f \in L^1(\mathbb{R}^d)$. For every $\lambda > 0$ the following holds*

$$\big|\{x \in \mathbb{R}^d : \mathcal{M}_{\mathscr{D}} f(x) > \lambda\}\big| \le \frac{1}{\lambda}\|f\|_1.$$

Therefore, by interpolation it follows that $\mathcal{M}_{\mathscr{D}}$ is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty]$.

Proof. Fix $\lambda > 0$. Let $\mathcal{E}$ be the collection of dyadic cubes such that

$$\langle |f| \rangle_Q > \lambda$$

and consider the subcollection $\mathcal{F} \subseteq \mathcal{E}$ of cubes that are maximal in $\mathcal{E}$ with respect to inclusion. The family $\mathcal{F}$ is disjoint since if $Q, Q' \in \mathcal{F}$ are such that $Q \cap Q' \neq \varnothing$, then $Q \subseteq Q'$ or $Q' \subseteq Q$ by the nested property of dyadic sets, and then either $Q = Q'$ or one of them is not maximal. Write

$$E = \bigcup_{Q \in \mathcal{E}} Q, \quad F = \bigcup_{Q \in \mathcal{F}} Q.$$

It is clear that $\{\mathcal{M}_{\mathscr{D}} f > \lambda\} = E$, but also $E = F$ because for every $Q \in \mathcal{E}$ there exists $Q' \in \mathcal{F}$ such that $Q' \supseteq Q$. For every $Q \in \mathcal{F} \subseteq \mathcal{E}$, we have that $|Q| \le \lambda^{-1} \int_Q |f|$, therefore

$$\begin{aligned} |\{\mathcal{M}_{\mathscr{D}} f > \lambda\}| &= |F| = \sum_{Q \in \mathcal{F}} |Q| \\ &\le \frac{1}{\lambda} \sum_{Q \in \mathcal{F}} \int_Q |f| = \frac{1}{\lambda} \int_{\cup \mathcal{F}} |f| \le \frac{1}{\lambda}\|f\|_1, \end{aligned}$$

which concludes the proof. □

It is important to point out that the boundedness results for the dyadic maximal function actually follow from those of the classical maximal function, since the latter trivially pointwise domainates the dyadic one. Nevertheless, these properties are much easier to prove in the dyadic context. For example no covering lemma is required, unlike the usual arguments employed for $\mathcal{M}$. Notice that in the proof of the weak-type $(1,1)$ boundedness we only use the nested property of dyadic cubes.

The Euclidean context of $\mathbb{R}^d$ represents the most natural environment to consider dyadic sets for the first time. Nevertheless, the notion of dyadic sets can be extended to much more general spaces.

Definition 1.3.4. Let (X,σ) be a measure space. A collection $\mathfrak{D}$ of subsets of X is a *dyadic system* if

$$\mathfrak{D} = \bigcup_k \mathfrak{D}_k$$

with $k \in \mathbb{N}$ if X has finite measure and $k \in \mathbb{Z}$ otherwise, and

1. for every k, $\mathfrak{D}_k$ is a partition of X up to a set of measure zero, that is all sets in $\mathfrak{D}_k$ are disjoint and
$$\sigma\Big(X \setminus \bigcup_{Q\in\mathfrak{D}_k} Q\Big) = 0;$$
2. the dyadic sets are nested, namely two different $Q, Q' \in \mathfrak{D}$ are either disjoint or one of them contains the other;
3. for every $Q \in \mathfrak{D}_k$ there exists a unique $Q^{(1)} \in \mathfrak{D}_{k-1}$ such that $Q \subseteq Q^{(1)}$ and it is called its *dyadic parent* (with the convention that if X has finite measure and $Q \in \mathfrak{D}_0$, then $Q^{(1)} = Q$). Furthermore, Q can be partitioned into sets belonging to $\mathfrak{D}_{k+1}$, that are its *dyadic children*.

The definition that we just gave is as general as possible, in particular because no assumption is made on the space. In more restrictive contexts it is usually required that dyadic systems satisfy other properties as well. These extra conditions tipically concern the relation between the measure of a set and that of its parent, or ask for a control on the number of dyadic children. A classical reference is Theorem 11 in [11], where the author deals with a doubling space.

It is worth noticing that Definition 1.3.2 can be generalized to any dyadic system on any measure space, and that the proof of Proposition 1.3.3 works as well, so that also in this general setting the dyadic maximal function is of weak-type $(1,1)$ and bounded on L^p for every $p \in (1,\infty]$.

We now turn to some very basic definitions which come from probability. Let (Ω,Σ,μ) a probability space and $f \in L^1(\Omega)$. Given a σ-subalgebra Γ of Σ, the *conditional expectation* of f with respect to Γ is the only Γ-measurable function $\mathsf{E}_\Gamma f$ such that

$$\int_A f \,\mathrm{d}\mu = \int_A \mathsf{E}_\Gamma f \,\mathrm{d}\mu$$

for all $A \in \Gamma$. It is easy to see that when Γ is a countable partition of Ω, namely $\Omega = \cup_j A_j$, the conditional expectation is given by

$$\mathsf{E}_\Gamma f(x) = \sum_j \langle f \rangle_{A_j} \mathbb{1}_{A_j}.$$

Let (Ω, Σ, μ) be a σ-finite measure space. A *filtration* of Σ is an increasing sequence $\{\Sigma_n\}_{n\in\mathbb{N}}$ of σ-finite σ-subalgebras of Σ. For $p \in [1, \infty]$, a *martingale* in $L^p(\Omega)$ is a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions in $L^p(\Omega)$ such that, for every $n \in \mathbb{N}$, f_n is Σ_n-measurable and

$$\mathsf{E}_{\Sigma_{n-1}} f_n = f_{n-1}.$$

Given a martingale $\{f_n\}_n$, the *martingale differences* are defined as

$$\mathsf{D} f_n = f_n - f_{n-1} \quad \text{for } n \geq 1$$

and $\mathsf{D} f_0 = f_0$.

1.3.2 A dyadic filtration on general trees

As it was anticipated in Section 1.1.3, the Gromov metric ρ turns out to be the right choice as the natural metric on trees. Indeed, the set of all the balls arising from ρ induces a natural dyadic filtration on the tree. Recall that, by Proposition 1.1.10, this set of balls is given by

$$\Big\{\{x\} : x \in \mathfrak{X}\Big\} \cup \Big\{T_x : x \in \mathfrak{X}\Big\}.$$

Each σ-algebra of our martingale filtration is generated by a partition $\mathcal{D}_k$, which represents the k-th generation of our dyadic system, with $k \geq 0$ on rooted trees, and $k \in \mathbb{Z}$ on hanged trees.

In rooted trees, the first generation is trivial, namely $\mathcal{D}_0 = \{\mathfrak{X}\}$, while for $k > 0$ we define

$$\mathcal{D}_k = \Big\{\{x\} : |x|_o < k\Big\} \cup \Big\{T_x : |x|_o = k\Big\}.$$

Clearly, the entire system is

$$\mathcal{D} = \bigcup_{k=0}^{\infty} \mathcal{D}_k.$$

In hanged trees, the k-th generation for $k \in \mathbb{Z}$ is defined by

$$\mathcal{D}_k = \Big\{\{x\} : |x|_\omega < k\Big\} \cup \Big\{T_x : |x|_\omega = k\Big\}$$

and the whole system is

$$\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k.$$

In both cases, the system $\mathcal{D}$ coincides with the set of all balls with respect to the Gromov metric. In analogy with the classical setting, we will call *cubes* the sets belogning to $\mathcal{D}$, that is all singletons and sectors.

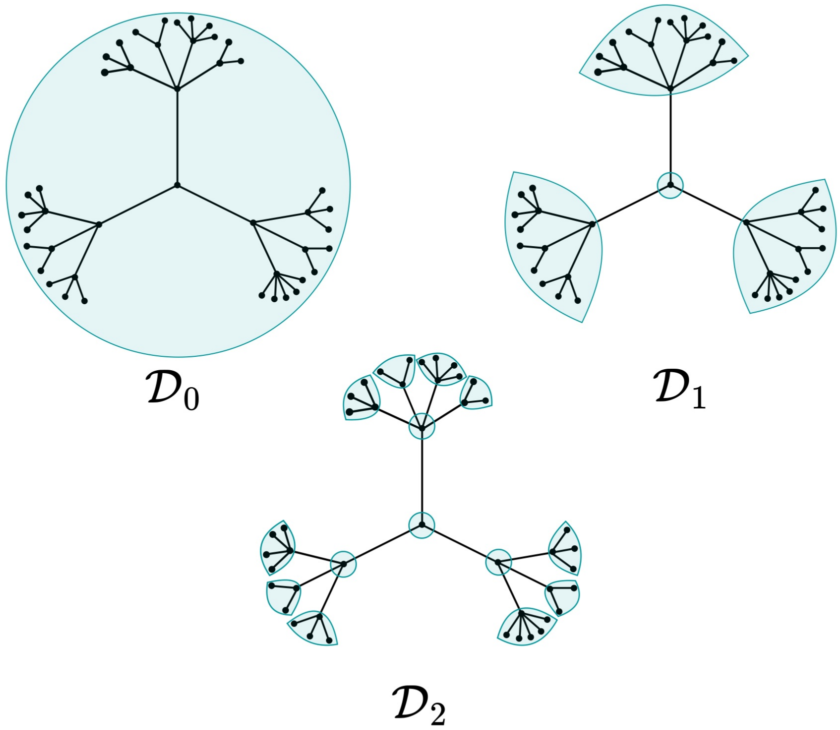

Figure 1.5: Graphic representation of the first generations $\mathcal{D}_0$, $\mathcal{D}_1$, $\mathcal{D}_2$ on a general rooted tree.

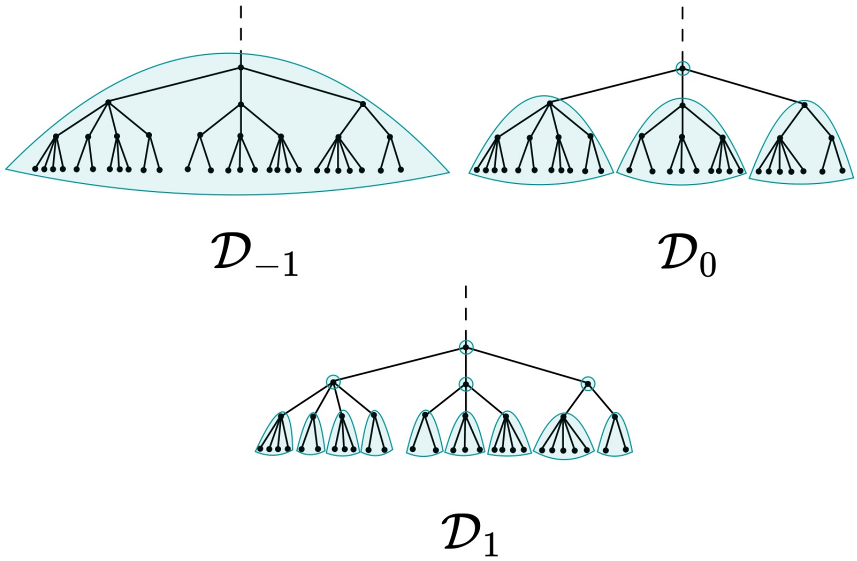

Figure 1.6: Graphic representation of the generations $\mathcal{D}_{-1}$, $\mathcal{D}_0$, $\mathcal{D}_1$ on a general hanged tree.

It is straightforward to see that the system $\mathcal{D}$ satisfies all the properties of Definition 1.3.4, so that is represents a dyadic system on $\mathfrak{X}$. However, note that a singleton $\{x\}$ belonging to $\mathcal{D}_k$ for some k will belong to $\mathcal{D}_m$ for every $m \geq k$. Therefore, if we follow the definition of dyadic parent given in 3 of Definition 1.3.4, a singleton will have two dyadic parents, namely itself and the sector that it generates, depending on which generation we are looking at it from. For this reason, it will be useful for us to think of singletons as point elements in the dyadic generation $\mathcal{D}_k$ with the smallest possible value of k. This signifies that while we will say that $\{x\} \in \mathcal{D}_k$, we will mean

$$k = \min\big\{m : \{x\} \in \mathcal{D}_m\big\}.$$

This convention does not imply that we are excluding $\{x\}$ from $\mathcal{D}_m$ for $m > k$. With this notation in mind we consider the dyadic parent $Q^{(1)}$ of a cube $Q \in \mathcal{D}_k$ as the unique cube containing it and belonging to $\mathcal{D}_{k-1}$. Remark that, as it is shown in the following picture, the dyadic parent turns out to always be a sector.

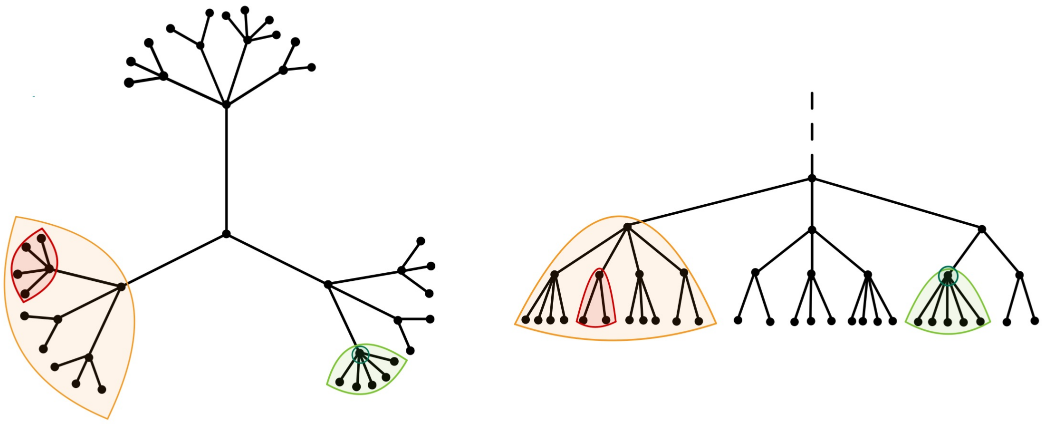

Figure 1.7: In both cases the dyadic parent of the red sector is the orange sector and the dyadic parent of the dark green singleton is the light green sector.

For every $Q \in \mathcal{D}_k$, we denote by $Q^{(n)}$ the only cube in $\mathcal{D}_{k-n}$ that contains Q and call it the n-th predecessor of Q (with $0 \leq n \leq k$ in the rooted tree and $n \in \mathbb{N}$ in the hanged one). The other way around we write

$$\mathcal{D}_\ell(Q) = \left\{ R \in \mathcal{D} : Q = R^{(\ell)} \right\}$$

for the subsets of Q that partition it and appear in the generation ℓ steps later. The successors of Q are its dyadic children, i.e. the sets in $\mathcal{D}_1(Q)$. For every cube $Q \in \mathcal{D}$ we will denote its *generator* $x_Q = x$ if $Q = \{x\}$ or $Q = T_x$. Note that for every $n \geq 0$, $x_{Q^{(n)}} = x_Q^{(n)}$.

In Section 3.1.2 we will need the following operators to define the spaces $H^1(\mathfrak{X})$ and $BMO(\mathfrak{X})$. We consider the filtration $\{\mathcal{D}_k\}_k$ on the tree (which is a double-sided filtration in the horocyclic case), the corresponding conditional expectations are

$$\mathsf{E}_k f = \sum_{Q \in \mathcal{D}_k} \langle f \rangle_Q \mathbb{1}_Q.$$

Moreover, we take into account the martingale defined by $f_k := \mathsf{E}_k f$, so that the martingale differences are

$$\mathsf{D}_k f = \mathsf{E}_k f - \mathsf{E}_{k-1} f$$

for every $k \in \mathbb{Z}$ in the horocyclic setting and for $k \geq 1$ in the rooted setting, where $\mathsf{D}_0 f = \mathsf{E}_0 f = \langle f \rangle_{\mathfrak{X}}$.

1.4 Sparse domination

Sparse domination is a powerful technique that provides a means to control operators and functions in terms of sparse collections of sets. Broadly speaking, a collection of sets is sparse if its elements have controlled overlap. The concept emerged as a refinement of classical inequalities, such as those involving singular integrals, and has played a crucial role in understanding the behavior of various operators in different L^p spaces, among others. This approach offers a modern perspective, enabling the development of new results in harmonic analysis.

This section aims to explore the fundamental concepts of sparse domination, which were recently developed in the literature in continuous settings such as in $\mathbb{R}^d$ or in spaces of homogeneous type. Similarly to the previous subsection we will present here the results on $\mathbb{R}^d$ equipped with the Lebesgue measure. Some useful references for this part are [41, 34, 29].

1.4.1 Sparse families of sets and sparse operators

Definition 1.4.1. Let $\mathcal{S}$ be a family of subsets of $\mathbb{R}^d$ and $\eta \in (0,1)$. The family $\mathcal{S}$ is said to be *η-sparse* if for every $Q \in \mathcal{S}$ there exists a subset $E_Q \subseteq Q$ such that

1. $|E_Q| \geq \eta |Q|$,
2. if $Q, R \in \mathcal{S}$, then $E_Q \cap E_R = \varnothing$ unless $Q = R$.

Note that a η-sparse family may not be disjoint but is in some sense close to being so. Indeed, every set of the family has a *core*, which is a subset that contains a significant part of its measure and such that the family of these *cores* actually is disjoint.

Example 1.4.2.
1. A disjoint family is sparse for every $\eta \in (0,1)$.
2. $\mathcal{S}_1 = \{[0, 2^k) : k \in \mathbb{Z}\}$ is $\frac{1}{2}$-sparse in $\mathbb{R}$: if $Q_k = [0, 2^k)$ then the sets $E_{Q_k} = [2^{k-1}, 2^k)$ are such that $E_{Q_k} \subseteq Q_k$, $|E_{Q_k}| = 2^{k-1} = \frac{1}{2}|Q_k|$ and $E_{Q_k} \cap E_{Q_{k'}} = \varnothing$ for $k \neq k'$.
3. $\mathcal{S}_2 = \{Q\} \cup \mathscr{D}_1(Q) \cup \mathscr{D}_2(Q)$, with $Q = [0,1)^2$ is $\frac{1}{4}$-sparse in $\mathbb{R}^2$, as one can see in Figure 1.8.

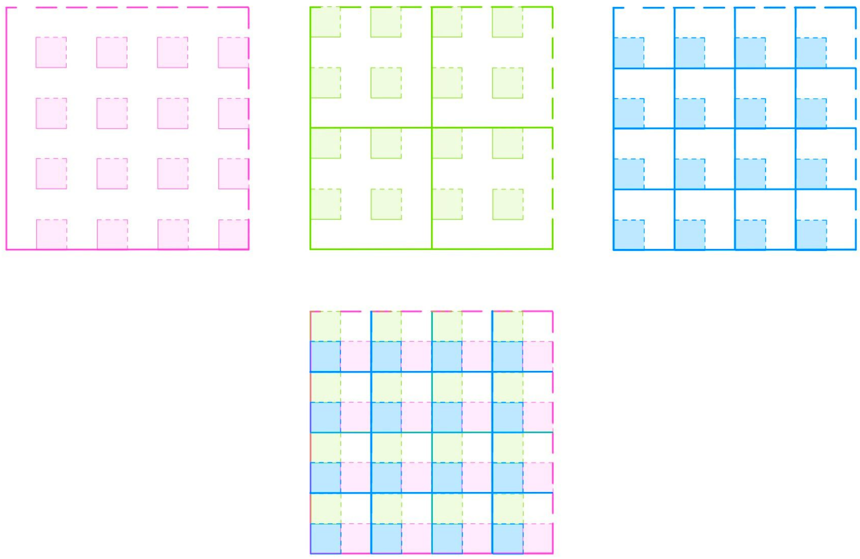

Figure 1.8: In pink Q and E_Q, in green $\mathscr{D}_1(Q)$ and the sets E_R for every $R \in \mathscr{D}_1(Q)$, in blue $\mathscr{D}_2(Q)$ and the sets E_P for every $P \in \mathscr{D}_2(Q)$.

A natural question that arises is whether the union of two different sparse families, say $\mathcal{S}_1$ which is η_1-sparse and $\mathcal{S}_2$ which is η_2-sparse, is still a sparse family. In order to answer we need to define Carleson families.

Definition 1.4.3. Let $\mathcal{S}$ be a family of subsets of $\mathbb{R}^d$ and $C > 1$. The family $\mathcal{S}$ is said to be C-Carleson if for every $Q \in \mathcal{S}$,

$$\sum_{\substack{R\in\mathcal{S}\\ R\subseteq Q}} |R| \leq C|Q|.$$

It is straightforward to see that if $\mathcal{S}$ is η-sparse, then it is $\frac{1}{\eta}$-Carleson, since

$$\sum_{\substack{R\in\mathcal{S}\\ R\subseteq Q}} |R| \leq \frac{1}{\eta}\sum_{\substack{R\in\mathcal{S}\\ R\subseteq Q}} |E_R| \leq \frac{1}{\eta}|Q|.$$

The converse statement, proved in [31], is much more delicate. An easier proof which only holds for families of dyadic cubes is provided in [34].

Theorem 1.4.4. *([31]) If $\mathcal{S}$ is a C-Carleson family, then it is $\frac{1}{C}$-sparse.*

Now, it is clear that if $\mathcal{S}_1$ is C_1-Carleson and $\mathcal{S}_2$ is C_2-Carleson, then $\mathcal{S}_1 \cup \mathcal{S}_2$ is $(C_1 + C_2)$-Carleson. Therefore the next Corollary is immediate.

Corollary 1.4.5. *If $\mathcal{S}_1$ is η_1-sparse and $\mathcal{S}_2$ is η_2-sparse, then $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ is $\left(\frac{1}{\eta_1} + \frac{1}{\eta_2}\right)^{-1}$-sparse.*

The starting point of sparse domination is to associate to a sparse family the corresponding sparse operator, that will turn out to satisfy many good properties, such as L^p boundedness, weak-type $(1, 1)$ and L^∞-BMO bounds, as well as weighted inequalities. Then the idea is to study other operators, which seem to be more complicated, by trying to bound them with a sparse operator in order to deduce the properties of the latter for the operator under study.

Definition 1.4.6. Let $\mathcal{S}$ be an η-sparse family, the *sparse operator* associated to $\mathcal{S}$ is the averaging operator defined by

$$\mathcal{A}_{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \mathbb{1}_Q.$$

Theorem 1.4.7. *If $\mathcal{S}$ is an η-sparse family of dyadic cubes in $\mathbb{R}^d$, then the sparse operator $\mathcal{A}_{\mathcal{S}}$ is bounded on $L^p(\mathbb{R}^d)$ for every $p \in (1, \infty)$.*

Proof. We compute the p-norm of $\mathcal{A}_{\mathcal{S}} f$ by duality. Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$ with $f, g \geq 0$,

$$\begin{aligned} \langle \mathcal{A}_{\mathcal{S}} f, g \rangle = \int_{\mathbb{R}^d} \mathcal{A}_{\mathcal{S}} f(x) g(x) \mathrm{d}x &= \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \int_Q g(x) \mathrm{d}x \\ &= \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \langle g \rangle_Q |Q| \leq \frac{1}{\eta} \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \langle g \rangle_Q |E_Q|. \end{aligned}$$

Since $E_Q \subseteq Q$, for each $x \in E_Q$ we have that

$$\langle f \rangle_Q \leq \mathcal{M}_{\mathscr{D}} f(x), \quad \langle g \rangle_Q \leq \mathcal{M}_{\mathscr{D}} g(x).$$

Therefore,

$$\begin{aligned} \langle \mathcal{A}_{\mathcal{S}} f, g \rangle \leq & \frac{1}{\eta} \int_{E_Q} \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \langle g \rangle_Q \mathrm{d}x \\ \leq & \frac{1}{\eta} \sum_{Q \in \mathcal{S}} \int_{E_Q} \mathcal{M}_{\mathscr{D}} f(x) \mathcal{M}_{\mathscr{D}} g(x) \mathrm{d}x \\ \leq & \frac{1}{\eta} \int_{\mathbb{R}^d} \mathcal{M}_{\mathscr{D}} f(x) \mathcal{M}_{\mathscr{D}} g(x) \mathrm{d}x \\ \leq & \frac{1}{\eta} \|\mathcal{M}_{\mathscr{D}} f\|_p \|\mathcal{M}_{\mathscr{D}} g\|_{p'} \\ \lesssim & \frac{1}{\eta} \|f\|_p \|g\|_{p'}, \end{aligned}$$

where we used Hölder's inequality, the disjointness of the family $\{E_Q\}_{Q\in\mathcal{S}}$ and the L^p and $L^{p'}$ boundedness of $\mathcal{M}_{\mathscr{D}}$. Hence,

$$\|\mathcal{A}_{\mathcal{S}} f\|_p = \sup_{\substack{g\in L^{p'}(\mathbb{R}^d)\\ \|g\|_{p'}=1}} \langle \mathcal{A}_{\mathcal{S}} f, g\rangle \lesssim_\eta \sup_{\substack{g\in L^{p'}(\mathbb{R}^d)\\ \|g\|_{p'}=1}} \|f\|_p\|g\|_{p'} = \|f\|_p.$$

□

In the proof of the next theorem we make use of the standard Calderón-Zygmund decomposition on $\mathbb{R}^d$, that we decided not to present in the Preliminaries since we will state it in a similar form in Theorem 2.1.1. Classical references are Theorem 2, Ch. I, [44] or Theorem 5.3.1, Ch. 5, [29].

Theorem 1.4.8. *If $\mathcal{S}$ is a sparse family of dyadic cubes in $\mathbb{R}^d$, then the sparse operator $\mathcal{A}_{\mathcal{S}}$ is of weak-type $(1,1)$.*

Proof. Let $\lambda > 0$ and $f \in L^1(\mathbb{R}^d)$, $f \geq 0$, the goal is to show that

$$\left|\left\{x\in\mathbb{R}^d : \mathcal{A}_{\mathcal{S}} f(x) > \lambda\right\}\right| \lesssim \frac{1}{\lambda}\|f\|_1.$$

Let $\Omega_\lambda := \{x\in\mathbb{R}^d : \mathcal{M}_{\mathscr{D}} f(x) > \lambda\}$ and let $\{Q_j\}_j$ be a covering of Ω_λ by dyadic cubes which are maximal with respect to inclusion and perform the Calderón-Zygmund decomposition to write $f = g + b$. The following holds

$$\begin{aligned}\left|\left\{x\in\mathbb{R}^d : \mathcal{A}_{\mathcal{S}} f(x) > \lambda\right\}\right| \leq& |\Omega_\lambda| + \left|\left\{x\in\mathbb{R}^d : \mathcal{A}_{\mathcal{S}} g(x) > \lambda/2\right\}\right| \\ &+ \left|\left\{x\in\mathbb{R}^d\setminus\Omega_\lambda : \mathcal{A}_{\mathcal{S}} b(x) > \lambda/2\right\}\right| \\ =&: \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

The weak-type of $\mathcal{M}_{\mathscr{D}}$ proven in Proposition 1.3.3 yields

$$\mathrm{I} = |\Omega_\lambda| \lesssim \frac{1}{\lambda}\|f\|_1.$$

The second term is bounded by using Chebyshev's inequality, the L^2-boundedness of $\mathcal{A}_{\mathcal{S}}$ proven in Theorem 1.4.7 and property 3) of Theorem 2.1.1

$$\mathrm{II} \leq \frac{4}{\lambda^2}\|\mathcal{A}_{\mathcal{S}} g\|_2^2 \lesssim \frac{1}{\lambda^2}\|g\|_2^2 \lesssim \frac{1}{\lambda}\|f\|_1.$$

Finally, we prove that $\mathrm{III} = 0$ by showing that $\mathrm{supp}(\mathcal{A}_{\mathcal{S}} b) \subseteq \Omega_\lambda$. Recall that for every x,

$$\mathcal{A}_{\mathcal{S}} b(x) = \sum_{R\in\mathcal{S}} \langle b\rangle_R \mathbb{1}_R(x) = \sum_{\substack{R\in\mathcal{S}\\ R\ni x}} \langle b\rangle_R.$$

Now let $x \in \mathbb{R}^d \setminus \Omega_\lambda$ and $R \in \mathcal{S}$ such that $x \in R$, it is clear that R cannot be contained in any Q_j. This implies that R and Q_j are either disjoint or Q_j is contained in R. Therefore,

$$\begin{aligned}
\mathcal{A}_{\mathcal{S}} b(x) =& \sum_{\substack{R \in \mathcal{S} \\ R \ni x}} \frac{1}{|R|} \int_R b(y) \mathrm{d}y \\
=& \sum_{\substack{R \in \mathcal{S} \\ R \ni x}} \frac{1}{|R|} \int_R \sum_j b_j(y) \mathrm{d}y \\
=& \sum_{\substack{R \in \mathcal{S} \\ R \ni x}} \frac{1}{|R|} \int_R \sum_{\{j : Q_j \subseteq R\}} b_j(y) \mathrm{d}y \\
=& \sum_{\substack{R \in \mathcal{S} \\ R \ni x}} \frac{1}{|R|} \sum_{\{j : Q_j \subseteq R\}} \int_{Q_j} b_j(y) \mathrm{d}y \\
=& 0
\end{aligned}$$

by 2) of Theorem 2.1.1. □

1.4.2 Weighted inequalities for sparse operators

The following proposition is a weighted boundedness result for sparse operators, see [22].

Proposition 1.4.9. *Let $\mathcal{S}$ be a sparse family and $p \in (1, \infty)$. If $w \in A_p(\mathbb{R}^d)$, then $\mathcal{A}_{\mathcal{S}}$ is bounded on $L^p(\mathbb{R}^d, w)$, in particular*

$$\|\mathcal{A}_{\mathcal{S}} f\|_{p,w} \lesssim [w]_{A_p(\mathbb{R}^d)}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{p,w}.$$

Proof. Let $f \in L^p(\mathbb{R}^d, w)$, $g \in L^{p'}(\mathbb{R}^d, w)$ with $f, g \geq 0$ and $\|g\|_{p',w} = 1$. For every $p \in (1, \infty)$, we consider the dual weight $\sigma = w^{-\frac{1}{p-1}}$ and we have

$$\begin{aligned}
\langle \mathcal{A}_{\mathcal{S}} f, gw \rangle =& \int_{\mathbb{R}^d} \mathcal{A}_{\mathcal{S}} f(x) g(x) w(x) \mathrm{d}x \\
=& \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \int_Q g(x) w(x) \mathrm{d}x \\
=& \sum_{Q \in \mathcal{S}} \langle f \sigma^{-1} \sigma \rangle_Q \langle gw \rangle_Q |Q| \\
=& \sum_{Q \in \mathcal{S}} \langle f \sigma^{-1} \rangle_{Q,\sigma} \langle g \rangle_{Q,w} |Q| \frac{\sigma(Q)}{|Q|} \frac{w(Q)}{|Q|}.
\end{aligned}$$

If $p \in [2, \infty)$ we proceed as follows

$$\langle \mathcal{A}_{\mathcal{S}} f, gw \rangle = \sum_{Q \in \mathcal{S}} \langle f \sigma^{-1} \rangle_{Q,\sigma} \langle g \rangle_{Q,w} \langle w \rangle_Q \langle \sigma \rangle_Q^{p-1} \langle \sigma \rangle_Q^{1-(p-1)} |Q|$$

$$\leq [w]_{A_p(\mathbb{R}^d)} \sum_{Q\in\mathcal{S}} \langle f\sigma^{-1}\rangle_{Q,\sigma}\langle g\rangle_{Q,w}\langle\sigma\rangle_Q^{2-p}|Q|$$
$$= [w]_{A_p(\mathbb{R}^d)} \sum_{Q\in\mathcal{S}} \langle f\sigma^{-1}\rangle_{Q,\sigma}\langle g\rangle_{Q,w}\big(\sigma(Q)\big)^{2-p}|Q|^{p-1}.$$

Notice that since $E_Q \subseteq Q$ and $p \in [2,\infty)$

$$\big(\sigma(Q)\big)^{2-p} = \big(\sigma(Q)\big)^{-(p-2)} \leq \big(\sigma(E_Q)\big)^{-(p-2)}.$$

Moreover, by Hölder's inequality

$$\begin{aligned} |E_Q| &= \int_{E_Q} w^{\frac{1}{p}}(x) w^{-\frac{1}{p}}(x)\mathrm{d}x \\ &\leq \left(\int_{E_Q} w(x)\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{E_Q} w^{-\frac{p'}{p}}(x)\mathrm{d}x\right)^{\frac{1}{p'}} \\ &= w(E_Q)^{\frac{1}{p}}\sigma(E_Q)^{\frac{1}{p'}} \end{aligned}$$

and by definition of sparse family

$$|Q|^{p-1} \lesssim |E_Q|^{p-1} \leq w(E_Q)^{\frac{1}{p'}}\sigma(E_Q)^{\frac{(p-1)^2}{p}}.$$

Therefore, by Hölder's inequality again

$$\begin{aligned} \langle \mathcal{A}_{\mathcal{S}} f, gw\rangle &\leq [w]_{A_p(\mathbb{R}^d)} \sum_{Q\in\mathcal{S}} \langle f\sigma^{-1}\rangle_{Q,\sigma}\langle g\rangle_{Q,w} w(E_Q)^{\frac{1}{p'}}\sigma(E_Q)^{\frac{1}{p}} \\ &\leq [w]_{A_p(\mathbb{R}^d)} \left(\sum_{Q\in\mathcal{S}} \langle f\sigma^{-1}\rangle_{Q,\sigma}^p \sigma(E_Q)\right)^{\frac{1}{p}} \left(\sum_{Q\in\mathcal{S}} \langle g\rangle_{Q,w}^{p'} w(E_Q)\right)^{\frac{1}{p'}} \\ &\leq [w]_{A_p(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} \big(\mathcal{M}_{\mathscr{D},\sigma}(f\sigma^{-1})(x)\big)^p\sigma(x)\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} \big(\mathcal{M}_{\mathscr{D},w}(g)(x)\big)^{p'} w(x)\mathrm{d}x\right)^{\frac{1}{p'}} \\ &= [w]_{A_p(\mathbb{R}^d)} \|\mathcal{M}_{\mathscr{D},\sigma}(f\sigma^{-1})\|_{p,\sigma}\|\mathcal{M}_{\mathscr{D},w}(g)\|_{p',w} \\ &\lesssim [w]_{A_p(\mathbb{R}^d)} \|f\sigma^{-1}\|_{p,\sigma}\|g\|_{p',w} \\ &= [w]_{A_p(\mathbb{R}^d)} \|f\|_{p,w}, \end{aligned}$$

and we have the claim in this case.

Now if $p \in (1,2)$,

$$\begin{aligned} \langle \mathcal{A}_{\mathcal{S}} f, gw\rangle &= \sum_{Q\in\mathcal{S}} \langle f\sigma^{-1}\rangle_{Q,\sigma}\langle g\rangle_{Q,w}\langle\sigma\rangle_Q\langle w\rangle_Q^{\frac{1}{p-1}}\langle w\rangle_Q^{1-\frac{1}{p-1}}|Q| \\ &\leq [w]_{A_p(\mathbb{R}^d)}^{\frac{1}{p-1}} \sum_{Q\in\mathcal{S}} \langle f\sigma^{-1}\rangle_{Q,\sigma}\langle g\rangle_{Q,w}\langle w\rangle_Q^{1-\frac{1}{p-1}}|Q| \end{aligned}$$

$$=[w]_{A_p(\mathbb{R}^d)}^{\frac{1}{p-1}} \sum_{Q\in\mathcal{S}} \langle f\sigma^{-1}\rangle_{Q,\sigma}\langle g\rangle_{Q,w}\big(w(Q)\big)^{1-\frac{1}{p-1}}|Q|^{\frac{1}{p-1}}.$$

Note that $E_Q \subseteq Q$ and since $p \in (1,2)$ implies $1 - \frac{1}{p-1} < 0$

$$\big(w(Q)\big)^{1-\frac{1}{p-1}} \leq \big(w(E_Q)\big)^{1-\frac{1}{p-1}}.$$

As in the previous case $|E_Q| \leq w(E_Q)^{\frac{1}{p}}\sigma(E_Q)^{\frac{1}{p'}}$, so that

$$|Q|^{\frac{1}{p-1}} \lesssim |E_Q|^{\frac{1}{p-1}} \leq w(E_Q)^{\frac{1}{p(p-1)}}\sigma(E_Q)^{\frac{1}{p'(p-1)}}.$$

Finally, proceeding as in the previous case

$$\begin{aligned}\langle \mathcal{A}_{\mathcal{S}}f, gw\rangle \leq &[w]_{A_p(\mathbb{R}^d)}^{\frac{1}{p-1}} \sum_{Q\in\mathcal{S}} \langle f\sigma^{-1}\rangle_{Q,\sigma}\langle g\rangle_{Q,w} w(E_Q)^{\frac{1}{p'}}\sigma(E_Q)^{\frac{1}{p}}\\ \lesssim&[w]_{A_p(\mathbb{R}^d)}^{\frac{1}{p-1}}\|f\|_{p,w},\end{aligned}$$

which concludes the proof. □

Chapter 2

Classical Calderón-Zygmund theory and the Bergman projection on homogeneous trees

The study of the boundedness of operators on standard function spaces, such as L^p spaces, is a central topic in functional and harmonic analysis. A linear operator T is said to be bounded on a function space if there exists a constant $C > 0$ such that, for all functions f in the space,

$$\|Tf\| \leq C\|f\|.$$

This concept plays a crucial role in understanding how operators, such as integral or differential operators, behave and how they transform functions within these spaces.

This chapter is dedicated to exploring the Calderón-Zygmund theory within the context of doubling trees. This theory, initially developed for singular integrals and maximal operators in Euclidean spaces, provides powerful tools for studying the behavior of various operators on function spaces. In the setting of trees, it takes on a slightly distinct character, influenced by the discrete nature of the underlying structure. Indeed, some difficulties inherent to continuous cases, such as local singularities of kernels, disappear in our situation, while issues at infinity persist. In addition, the different geometry of the space sometimes requires different techniques. However, we will see that this discrete framework allows for the adaptation of many classical results in harmonic analysis.

As we showed in Proposition 1.1.16, we deal with a doubling setting whenever we assume the function q to be bounded from above, therefore all the results in this chapter apply to the general context of trees with bounded geometry. Nevertheless, here we will present the extension of Calderón-Zygmund theory only in the case of horocyclic homogeneous trees. There are three reasons that motivate this choice. First of all, this part is based on a joint work with Filippo De Mari and Matteo Monti ([24]), where we started to consider this topic and took into account exactly this case. Furthermore, if q is constant, then the notation is much more transparent. Finally, we state the results in the horocyclic version because the radial one was studied in the former work [25].

In Section 2.1 we present the Calderón-Zygmund decomposition and boundedness results for integral operators, both on L^p spaces and on the endpoints, so that we also introduce H^1 and BMO spaces. Then, Section 2.2 is devoted to the study of a particular operator, the Bergman projector, that represents the primary motivation for this work.

2.1 General theory

Consider a homogeneous horocyclic tree $\mathfrak{X}$ and a measure belonging to the family of measures μ_α introduced in Section 1.1.3, namely

$$\mu_\alpha(x) = q^{-\alpha|x|}.$$

We fix $\alpha > 1$ and omit it from the notation, so that we will always write μ instead of μ_α. The L^p spaces of functions on $\mathfrak{X}$ that are p-integrable with respect to μ will be denoted just by $L^p(\mathfrak{X})$, omitting the fixed measure, and analogously for all the other function spaces on $\mathfrak{X}$.

The goal of this section is to study the boundedness properties of integral operators whose kernels are defined on $\mathfrak{X}$. This means that the kernel is a function

$$K : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$$

and the action of the operator T on a function f is given by

$$Tf(x) = \sum_{y \in \mathfrak{X}} K(x, y) f(y) \mu(y).$$

To achieve the main results, we first need to write down how the standard Calderón-Zygmund decomposition for a function in $L^1(\mathfrak{X})$ reads here. The key idea behind this decomposition is to partition the function into two parts: the good part, which is well-behaved and manageable under the operator, and the bad part, which represents the irregular or singular components that may cause difficulties. The good part typically corresponds to a part of the function whose behavior under the action of the operator can be controlled by its L^2-bound, while the bad part requires more careful treatment, often needing additional estimates. By decomposing the function in these two summands, we gain insight into the boundedness properties of the operator and the role each part plays in ensuring that the operator is well-behaved in the desired function space. The proof follows the classical proof of the Calderón-Zygmund decomposition in a doubling space (see Theorem 5.3.1, Ch. 5, [29]), and makes use of the dyadic structure $\mathcal{D}$ on trees introduced in Section 1.3.2.

Theorem 2.1.1 (Calderón-Zygmund decomposition). *Let $f \in L^1(\mathfrak{X})$, $f \geq 0$, and $\lambda > 0$. There exists a family of disjoint sets $\{Q_j\}_j \subseteq \mathcal{D}$ such that, if*

$$b_j := f\mathbb{1}_{Q_j} - \langle f\mathbb{1}_{Q_j}\rangle_{Q_j}\mathbb{1}_{Q_j},$$

$b = \sum_j b_j$ and $g := f - b$, then

1) f=g+b;

2) the following localization properties and L^1-bound for b hold

$$\text{supp}(b_j) \subseteq Q_j, \quad \sum_{x\in Q_j} b_j(x)\mu(x) = 0, \quad \sum_j \|b_j\|_1 \leq 2\|f\|_1;$$

3) the following higher integrability properties for g hold

$$\|g\|_\infty \leq q\lambda, \quad \|g\|_2^2 \leq q\lambda\|f\|_1;$$

4) if $\Omega_\lambda := \cup_j Q_j$, then $\mu(\Omega_\lambda) \lesssim \frac{1}{\lambda}\|f\|_1$.

Proof. Let $f \in L^1(\mathfrak{X})$, $f \geq 0$ and $\lambda > 0$, consider

$$\Omega_\lambda := \Big\{x \in \mathfrak{X} : \mathcal{M}_{\mathcal{D}}f(x) > \lambda\Big\},$$

and cover Ω_λ with a family of pairwise disjoint sets $\{Q_j\}_j$ in $\mathcal{D}$ which are maximal with respect to inclusion.

By the definitions of b and g, 1) trivially holds. Moreover, it is clear that $\text{supp}(b_j) \subseteq Q_j$ and

$$\sum_{x\in Q_j} b_j(x)\mu(x) = \sum_{x\in Q_j} f(x)\mu_(x) - \langle f\mathbb{1}_{Q_j}\rangle_{Q_j}\mu(Q_j) = 0.$$

Furthermore,

$$\begin{aligned}
\sum_j \|b_j\|_1 &= \sum_j \sum_{x\in Q_j} |f\mathbb{1}_{Q_j}(x) - \langle f\mathbb{1}_{Q_j}\rangle_{Q_j}\mathbb{1}_{Q_j}(x)|\mu(x) \\
&\leq \sum_j \sum_{x\in Q_j} \big(f\mathbb{1}_{Q_j}(x) + \langle f\mathbb{1}_{Q_j}\rangle_{Q_j}\mathbb{1}_{Q_j}(x)\big)\,\mu(x) \\
&\leq \sum_{x\in\mathfrak{X}} f(x)\mu(x) + \sum_j \langle f\mathbb{1}_{Q_j}\rangle_{Q_j}\mu(Q_j) \\
&= \|f\|_1 + \sum_j \sum_{x\in Q_j} f\mathbb{1}_{Q_j}(x) \\
&\leq 2\|f\|_1,
\end{aligned}$$

hence 2) is proved. Concerning g, we have

$$g = f - \Big(\sum_j f \mathbb{1}_{Q_j} - \langle f \mathbb{1}_{Q_j} \rangle_{Q_j} \mathbb{1}_{Q_j} \Big) = f \mathbb{1}_{\mathfrak{X} \setminus \cup_j Q_j} + \sum_j \langle f \mathbb{1}_{Q_j} \rangle_{Q_j} \mathbb{1}_{Q_j},$$

which implies that g is positive. For $x \in \mathfrak{X} \setminus \cup_j Q_j$,

$$g(x) = f(x) = \langle f \rangle_{\{x\}} \leq \mathcal{M}_{\mathcal{D}} f(x) \leq \lambda.$$

For every $x \in \cup_j Q_j$, since the family $\{Q_j\}_j$ is disjoint, there exists exactly one set $Q_i \in \{Q_j\}_j$ such that $x \in Q_i$. By the maximality of the sets $\{Q_j\}_j$ the dyadic parent $Q_i^{(1)}$ satisfies

$$\frac{1}{\mu(Q_i^{(1)})} \sum_{y \in Q_i^{(1)}} f(y)\mu(y) \leq \lambda,$$

thus

$$\begin{aligned} g(x) =& \langle f \mathbb{1}_{Q_i} \rangle_{Q_i} \\ =& \frac{1}{\mu(Q_i)} \sum_{x \in Q_i} f(x) \\ \leq& \frac{1}{\mu(Q_i)} \sum_{x \in Q_i^{(1)}} f(x) \\ \leq& \frac{q}{\mu(Q_i^{(1)})} \sum_{x \in Q_i^{(1)}} f(x) \\ \leq& q\lambda, \end{aligned}$$

then $g \in L^\infty(\mathfrak{X})$ and $\|g\|_\infty \leq q\lambda$. Moreover,

$$\begin{aligned} \|g\|_2^2 =& \sum_{x \in \mathfrak{X}} |g(x)|^2 \mu(x) \\ \leq& \|g\|_\infty \sum_{x \in \mathfrak{X}} |g(x)| \mu(x) \\ \leq& q\lambda \left(\sum_{x \in \mathfrak{X} \setminus \cup_j Q_j} f(x)\mu(x) + \sum_j \sum_{x \in Q_j} \langle f \mathbb{1}_{Q_j} \rangle_{Q_j} \mu(x) \right) \\ =& q\lambda \left(\sum_{x \in \mathfrak{X} \setminus \cup_j Q_j} f(x)\mu(x) + \sum_j \sum_{y \in Q_j} f(y)\mu(y) \right) \\ =& q\lambda \|f\|_1, \end{aligned}$$

so that we obtain 3). Finally, by Proposition 1.3.3 $\mathcal{M}_{\mathcal{D}}$ is of weak-type $(1,1)$ which means that

$$\lambda \mu\Big(\Big\{ x \in \mathfrak{X} : \mathcal{M}_{\mathcal{D}} f(x) > \lambda \Big\} \Big) \leq \|\mathcal{M}_{\mathcal{D}}\|_{L^1 \to L^{1,\infty}} \|f\|_1$$

and 4) holds. □

As already mentioned, when analyzing the action of the operator on the bad part of the function, it is necessary to impose integrability conditions on the kernel to handle its behavior. In particular, the classical requirement is Hörmander's condition.

In a doubling measure metric space (X, δ, σ) the integral Hörmander's condition for a kernel $K : X \times X \to \mathbb{C}$ is given by

$$\sup_{v\in X, r>0} \sup_{x,y\in B_\delta(v,r)} \int_{X\setminus B_\delta(v,2r)} |K(z,x) - K(z,y)| \mathrm{d}\sigma(z) < +\infty.$$

In our setting we have $X = \mathfrak{X}$, $\delta = \rho$, $\sigma = \mu$ and $(\mathfrak{X}, \rho, \mu)$ is doubling. Recall that by Proposition 1.1.11 the Gromov balls are

$$B_\rho(x, r) = \begin{cases} \{x\} & \text{if } r \leq e^{-|x|}, \\ T_{x(|x|+\lfloor \log(r) \rfloor)} & \text{if } r > e^{-|x|}. \end{cases}$$

It easily follows that we can drop the supremum over the radii and that Hörmander's condition for a kernel $K \colon \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ may be rewritten as

$$\sup_{v\in\mathfrak{X}} \sup_{x,y\in T_v} \sum_{z\in\mathfrak{X}\setminus T_v} |K(z,x) - K(z,y)|\mu(z) < +\infty. \tag{2.1}$$

2.1.1 Weak-type $(1, 1)$ estimates

We are ready to state the classical theorem of weak-type $(1, 1)$ boundedness for operators in doubling settings, just rewritten in the discrete context of $\mathfrak{X}$.

Theorem 2.1.2. *Let T be an integral operator with kernel $K \colon \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$. If T is bounded on $L^2(\mathfrak{X})$ and K satisfies* (2.1)*, then T is of weak-type* $(1, 1)$.

Proof. Let $f \in L^1(\mathfrak{X})$, $f \geq 0$. The goal is to prove that for every $\lambda > 0$,

$$\lambda\mu\Big(\Big\{x \in \mathfrak{X} : |Tf(x)| > \lambda\Big\}\Big) \lesssim \|f\|_1.$$

Let $\lambda > 0$, we apply Theorem 2.1.1 to f and to the Calderón-Zygmund cubes $\{Q_j\}_j$, namely the maximal disjoint cubes in $\mathcal{D}$ that cover

$$\Omega_\lambda := \Big\{x \in \mathfrak{X} : \mathcal{M}_{\mathcal{D}} f(x) > \lambda\Big\}.$$

We can write

$$\begin{aligned} &\mu\Big(\Big\{x \in \mathfrak{X} : |Tf(x)| > \lambda\Big\}\Big) \\ &\quad \leq \mu(\Omega_\lambda) + \mu\Big(\Big\{x \in \mathfrak{X} : |Tg(x)| > \frac{\lambda}{2}\Big\}\Big) + \mu\Big(\Big\{x \in \mathfrak{X} \setminus \Omega_\lambda : |Tb(x)| > \frac{\lambda}{2}\Big\}\Big) \\ &\quad =: \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

The maximal function is of weak-type $(1,1)$, which gives

$$\mathrm{I} \lesssim \frac{1}{\lambda}\|f\|_1.$$

By Chebyshev's inequality, the L^2-boundedness of T, and 3) of Theorem 2.1.1 it follows that

$$\mathrm{II} \lesssim \frac{1}{\lambda^2}\|Tg\|_2^2 \lesssim \frac{1}{\lambda^2}\|g\|_2^2 \lesssim \frac{1}{\lambda}\|f\|_1.$$

Finally, Chebyshev's inequality again yields

$$\begin{aligned}\mathrm{III} &\lesssim \frac{1}{\lambda}\sum_{x\in\mathfrak{X}\setminus\Omega_\lambda} |Tb(x)|\mu(x)\\ &\leq \frac{1}{\lambda}\sum_j \sum_{x\in\mathfrak{X}\setminus\Omega_\lambda} |Tb_j(x)|\mu(x)\\ &\leq \frac{1}{\lambda}\sum_j \sum_{x\in\mathfrak{X}\setminus Q_j} |Tb_j(x)|\mu(x)\\ &=: \frac{1}{\lambda}\sum_j \mathrm{III}_j.\end{aligned}$$

Using the mean zero of b_j and Hörmander's condition (2.1), we get

$$\begin{aligned}\mathrm{III}_j &= \sum_{x\in\mathfrak{X}\setminus Q_j}\Big|\sum_{y\in Q_j} K(x,y)b_j(y)\mu(y)\Big|\mu(x)\\ &= \sum_{x\in\mathfrak{X}\setminus Q_j}\Big|\sum_{y\in Q_j} (K(x,y)-K(x,x_{Q_j}))b_j(y)\mu(y)\Big|\mu(x)\\ &\leq \sum_{y\in Q_j}|b_j(y)|\Big(\sup_{z\in Q_j}\sum_{x\in\mathfrak{X}\setminus Q_j}|K(x,y)-K(x,z)|\,\mu(x)\Big)\mu(y)\\ &\lesssim \|b_j\|_1.\end{aligned}$$

From 2) of Theorem 2.1.1 it follows that

$$\mathrm{III} \leq \frac{1}{\lambda}\sum_j \mathrm{III}_j \lesssim \frac{1}{\lambda}\sum_j \|b_j\|_1 \lesssim \frac{1}{\lambda}\|f\|_1$$

and, as claimed,

$$\mu\Big(\Big\{x\in\mathfrak{X} : |Tf(x)|>\lambda\Big\}\Big) \lesssim \frac{1}{\lambda}\|f\|_1.$$

□

A straightforward consequence of the previous theorem is a result of strong boundedness on L^p spaces for $p\in(1,\infty)$.

Corollary 2.1.3. *Let T be an integral operator with kernel $K\colon \mathfrak{X}\times\mathfrak{X}\to\mathbb{C}$. If T is bounded on $L^2(\mathfrak{X})$ and K satisfies* (2.1)*, then by interpolation it is bounded on $L^p(\mathfrak{X})$ for all $p\in(1,2)$. Furthermore, if the dual kernel $K^*(x,y)=\overline{K(y,x)}$ satisfies* (2.1)*, then by duality T is bounded on $L^p(\mathfrak{X})$ for all $p\in(2,\infty)$.*

2.1.2 H^1 and *BMO*

To complete our study, we focus on two other endpoints, namely those involving the atomic Hardy space H^1_{at}, which is strictly smaller than L^1, and the BMO space of functions with bounded mean oscillation, which is larger than L^∞. We start by introducing some suitable versions $H^1_{\text{at}}(\mathfrak{X})$ and $BMO(\mathfrak{X})$ of the classical spaces and then we prove the required results. For a deeper presentation of these spaces in more general settings we refer to [9] and [15], while for Hardy and BMO spaces on homogeneous trees, see [2], [3] and [37].

Definition 2.1.4. Let $p \in (1, \infty]$. A p-atom is a function $a \colon \mathfrak{X} \to \mathbb{C}$ such that

1. a is supported in Q for some $Q \in \mathcal{D}$;
2. $\|a\|_p \leq \mu(Q)^{\frac{1}{p}-1}$ if $p < \infty$ and $\|a\|_\infty \leq \mu(Q)^{-1}$ if $p = \infty$;
3. a has vanishing mean on Q, that is

$$\sum_{x \in Q} a(x)\mu(x) = 0.$$

Definition 2.1.5. Let $p \in (1, \infty]$. The atomic Hardy space $H^{1,p}_{\text{at}}(\mathfrak{X})$ is the space of functions $f \in L^1(\mathfrak{X})$ such that

$$f = \sum_i \lambda_i a_i,$$

where the a_i are p-atoms and $(\lambda_i)_i$ is a complex summable sequence. The norm of f in $H^{1,p}_{\text{at}}(\mathfrak{X})$ is given by

$$\|f\|_{H^{1,p}_{\text{at}}} := \inf\Big\{\sum_i |\lambda_i| : f = \sum_i \lambda_i a_i,\ a_i\ p\text{-atoms}\Big\}.$$

Note that by Hölder's inequality a p_2-atom is also a p_1-atom for all $p_1 \leq p_2$. It easily follows that the inclusions

$$H^{1,\infty}_{\text{at}}(\mathfrak{X}) \subseteq H^{1,p_1}_{\text{at}}(\mathfrak{X}) \subseteq H^{1,p_2}_{\text{at}}(\mathfrak{X})$$

hold for every $p_1 \leq p_2$.

Let $f \colon \mathfrak{X} \to \mathbb{C}$ and $Q \in \mathcal{D}$. Recall that we indicate by $\langle f \rangle_Q$ the mean of f on Q with respect to μ, that is

$$\langle f \rangle_Q = \frac{1}{\mu(Q)} \sum_{x \in Q} f(x)\mu(x).$$

Definition 2.1.6. Let $r \in [1, \infty)$. The bounded mean oscillation space $BMO^r(\mathfrak{X})$ is the space of functions $f : \mathfrak{X} \to \mathbb{C}$ such that

$$\sup_{Q \in \mathcal{D}} \frac{1}{\mu(Q)} \sum_{x \in Q} |f(x) - \langle f \rangle_Q|^r \mu(x) < \infty,$$

modulo the constant functions. The $BMO^r(\mathfrak{X})$-norm is given by

$$\|f\|_{BMO^r} := \sup_{Q \in \mathcal{D}} \Big(\frac{1}{\mu(Q)} \sum_{x \in Q} |f(x) - \langle f \rangle_Q|^r \mu(x) \Big)^{\frac{1}{r}}.$$

Again by Hölder inequality we obtain the following inclusions

$$BMO^{r_2}(\mathfrak{X}) \subseteq BMO^{r_1}(\mathfrak{X}) \subseteq BMO^1(\mathfrak{X})$$

for all $r_1 \leq r_2$.

Remark 2.1.7. In a doubling setting there are two outstanding results by Coifman and Weiss, presented in [15], concerning H^1_{at} and BMO spaces. Theorem A of [15] gives that $H^{1,p}_{\mathrm{at}}(\mathfrak{X}) = H^{1,\infty}_{\mathrm{at}}(\mathfrak{X})$ as vector spaces with equivalent norms for all $p \in (1, \infty]$. Furthermore by Theorem B of [15], the space $BMO^{p'}(\mathfrak{X})$ characterizes the dual of $H^{1,p}_{\mathrm{at}}(\mathfrak{X})$ for all $p \in (1, \infty]$ and p' its conjugate. As a byproduct of these results we also obtain that $BMO^r(\mathfrak{X}) = BMO^1(\mathfrak{X})$ as vector spaces with equivalent norms for all $r \in [1, \infty)$. Consequently, we will denote the space $H^{1,\infty}_{\mathrm{at}}(\mathfrak{X})$ by $H^1_{\mathrm{at}}(\mathfrak{X})$ and the space $BMO^1(\mathfrak{X})$ by $BMO(\mathfrak{X})$.

Remark 2.1.8. We introduced the Hardy spaces in their atomic form, as this version is the most convenient for proving endpoint estimates for operators failing to be bounded on L^1. However, they are usually initially defined using a square function associated with the martingale differences D_k or through the conditional expectations E_k. The martingale Hardy space $H^1(\mathfrak{X})$ is the space of functions in $L^1(\mathfrak{X})$ such that the norm defined below is finite

$$\|f\|_{H^1(\mathfrak{X})} := \Big\| \Big(\sum_{k=0}^{\infty} |\mathsf{D}_k f|^2 \Big)^{1/2} \Big\|_1,$$

while the little Hardy space $h^1(\mathfrak{X})$ is given by $L^1(\mathfrak{X})$ functions whose following norm

$$\|f\|_{h^1(\mathfrak{X})} := \Big\| \Big(\sum_{k=0}^{\infty} \mathsf{E}_{k-1} |\mathsf{D}_k f|^2 \Big)^{1/2} \Big\|_1$$

is finite, with the convention that, for $k = 0$, $\mathsf{E}_{k-1}|\mathsf{D}_k f|^2 = |\mathsf{E}_0 f|^2$.

When the filtration is regular, which is the case for us when we consider doubling trees, it turns out that $H^1(\mathfrak{X}) \simeq h^1(\mathfrak{X})$ (see [23, 28]). Moreover, as a combination of [32] and [49], $h^1(\mathfrak{X}) \simeq H^{1,p}_{\mathrm{at}}(\mathfrak{X})$ for all $p \in (1, \infty]$.

Concerning *BMO* spaces, it is also possible to give an equivalent definition that makes use of the martingale structure, just as we did for Hardy spaces. It is straightforward to see that

$$\|f\|_{BMO} \simeq \sup_{k\geq 0} \|\mathsf{E}_k |f - \mathsf{E}_k f|\|_\infty .$$

Indeed,

$$\begin{aligned}
\sup_{k\geq 0} \|\mathsf{E}_k |f - \mathsf{E}_k f|\|_\infty &= \sup_{y\in\mathfrak{X}} \sup_{k\geq 0} \mathsf{E}_k |f - \mathsf{E}_k f| (y) \\
&= \sup_{y\in\mathfrak{X}} \sup_{k\geq 0} \sum_{Q\in\mathcal{D}_k} \left\langle \left| f - \sum_{R\in\mathcal{D}_k} \langle f\rangle_R \mathbb{1}_R \right| \right\rangle_Q \mathbb{1}_Q(y) \\
&= \sup_{y\in\mathfrak{X}} \sup_{k\geq 0} \sum_{Q\in\mathcal{D}_k} \langle |f - \langle f\rangle_Q| \rangle_Q \mathbb{1}_Q(y) \\
&= \sup_{y\in\mathfrak{X}} \sup_{k\geq 0} \sum_{Q\in\mathcal{D}_k} \frac{1}{\mu(Q)} \sum_{x\in Q} |f(x) - \langle f\rangle_Q| \, \mu(x) \mathbb{1}_Q(y) \\
&= \sup_{Q\in\mathcal{D}} \frac{1}{\mu(Q)} \sum_{x\in Q} |f(x) - \langle f\rangle_Q| \, \mu(x).
\end{aligned}$$

We will explore this topic further, focusing on the nature and the usefulness of these equivalences, in Section 3.1.2.

Theorem 2.1.9. *Let T be an integral operator with kernel $K\colon \mathfrak{X}\times\mathfrak{X} \to \mathbb{C}$. If T is bounded on $L^2(\mathfrak{X})$ and K satisfies* (2.1)*, then T maps $H^1_{\mathrm{at}}(\mathfrak{X})$ into $L^1(\mathfrak{X})$.*

Proof. Let $f \in H^1_{\mathrm{at}}(\mathfrak{X})$ and let $f = \sum_i \lambda_i a_i$ be any decomposition of f into ∞-atoms. If we show that $\|Ta\|_1 \lesssim 1$ for every ∞-atom a, then

$$\|Tf\|_1 \lesssim \sum_i |\lambda_i| \|Ta_i\|_1 \lesssim \sum_i |\lambda_i|$$

and, passing to the infimum of $\sum_i |\lambda_i|$ on the possible decompositions of f into ∞-atoms, we get $\|Tf\|_1 \lesssim \|f\|_{H^1_{\mathrm{at}}}$.

Let a be an ∞-atom and let $Q \in \mathcal{D}$ be such that $\mathrm{supp}(a) \subseteq Q$ and $\|a\|_\infty \leq \mu(Q)^{-1}$, then

$$\|Ta\|_1 = \sum_{x\in Q} |Ta(x)|\mu(x) + \sum_{x\in\mathfrak{X}\setminus Q} |Ta(x)|\mu(x) =: \mathrm{I} + \mathrm{II}. \tag{2.2}$$

To handle I we use Cauchy-Schwartz inequality, the L^2-boundedness of T and the size estimate on a to obtain

$$\mathrm{I} = \sum_{x\in Q} |Ta(x)|\mu(x) \leq \mu(Q)^{1/2} \left(\sum_{x\in Q} |Ta(x)|^2 \mu(x) \right)^{1/2}$$

$$\leq \mu(Q)^{1/2}\|Ta\|_2 \lesssim \mu(Q)^{1/2}\|a\|_2 \leq 1.$$

Then, Hörmander's condition (2.1) and the mean zero of a yield

$$\begin{aligned}
\mathrm{II} &= \sum_{x\in\mathfrak{X}\setminus Q} |Ta(x)|\mu(x) \\
&= \sum_{x\in\mathfrak{X}\setminus Q} \Big|\sum_{y\in Q} K(x,y)a(y)\mu(y)\Big|\mu(x) \\
&= \sum_{x\in\mathfrak{X}\setminus Q} \Big|\sum_{y\in Q}(K(x,y)-K(x,x_Q))a(y)\mu(y)\Big|\mu(x) \\
&\leq \|a\|_1 \sup_{y\in Q} \sum_{x\in\mathfrak{X}\setminus Q} \Big|K(x,y)-K(x,x_Q)\Big|\mu(x) \\
&\lesssim \|a\|_1 \leq \mu(Q)\mu(Q)^{-1} = 1.
\end{aligned}$$

□

Theorem 2.1.10. *Let T be an integral operator with kernel $K\colon \mathfrak{X}\times\mathfrak{X}\to\mathbb{C}$. If T is bounded on $L^2(\mathfrak{X})$ and K satisfies* (2.1)*, then T maps $L^\infty(\mathfrak{X})$ into $BMO(\mathfrak{X})$.*

Proof. This result follows by duality from the previous theorem, but we prefer to give a direct proof. Let $f\in L^\infty(\mathfrak{X})$ and $Q\in\mathcal{D}$,

$$\begin{aligned}
\sum_{x\in Q} |Tf(x)-\langle f\rangle_Q|\mu(x) \leq & \sum_{x\in Q} |T(f\mathbb{1}_Q)(x)-\langle T(f\mathbb{1}_Q)\rangle_Q|\mu(x) \\
&+ \sum_{x\in Q} |T(f\mathbb{1}_{\mathfrak{X}\setminus Q})(x)-\langle T(f\mathbb{1}_{\mathfrak{X}\setminus Q})\rangle_Q|\mu(x) \\
=:& \ \mathrm{I}+\mathrm{II}.
\end{aligned}$$

To handle I we use Cauchy-Schwartz inequality and the L^2-boundedness of T to obtain

$$\begin{aligned}
\mathrm{I} &= \sum_{x\in Q} |T(f\mathbb{1}_Q)(x)-\langle T(f\mathbb{1}_Q)\rangle_Q|\mu(x) \\
&\leq 2\sum_{x\in Q} |T(f\mathbb{1}_Q)(x)|\mu(x) \\
&\lesssim \mu(Q)^{1/2}\Big(\sum_{x\in Q} |T(f\mathbb{1}_Q)(x)|^2\mu(x)\Big)^{1/2} \\
&\leq \mu(Q)^{1/2}\|T(f\mathbb{1}_Q)\|_2 \\
&\lesssim \mu(Q)^{1/2}\|(f\mathbb{1}_Q)\|_2 \\
&\leq \mu(Q)\|f\|_\infty.
\end{aligned}$$

To deal with II fix first $x\in Q$. Thus,

$$|T(f\mathbb{1}_{\mathfrak{X}\setminus Q})(x)-\langle T(f\mathbb{1}_{\mathfrak{X}\setminus Q})\rangle_Q|$$

$$
\begin{aligned}
&=\Big|\sum_{y\in\mathfrak{X}\setminus Q} K(x,y)f(y)\mu(y) - \frac{1}{\mu(Q)}\sum_{z\in Q}\sum_{y\in\mathfrak{X}\setminus Q} K(z,y)f(y)\mu(y)\mu(z)\Big|\\
&=\Big|\frac{1}{\mu(Q)}\sum_{z\in Q}\sum_{y\in\mathfrak{X}\setminus Q}\Big(K(x,y)-K(z,y)\Big)f(y)\mu(y)\mu(z)\Big|\\
&\leq \|f\|_\infty \sup_{z\in Q}\sum_{y\in\mathfrak{X}\setminus Q}\Big|K(x,y)-K(z,y)\Big|\mu(y)\\
&\lesssim \|f\|_\infty
\end{aligned}
$$

by Hörmander's condition. It follows that $\mathrm{II} \leq \mu(Q)\|f\|_\infty$. Therefore, for every $Q \in \mathcal{D}$

$$\frac{1}{\mu(Q)}\sum_{x\in Q}|Tf(x) - \langle f\rangle_Q|\mu(x) \lesssim \|f\|_\infty$$

which gives $\|Tf\|_{BMO} \lesssim \|f\|_\infty$. □

2.2 Example: the Bergman projection

As an application of the results presented in the previous section we introduce the Bergman projection $\mathcal{P}$, which projects the space $L^2(\mathfrak{X})$ of L^2-integrable functions with respect to the horocyclic measure μ into the subspace of $L^2(\mathfrak{X})$ of harmonic functions, i.e. the Bergman space $\mathcal{B}^2(\mathfrak{X})$. The main motivation for studying this object comes from complex analysis. Indeed, holomorphic Bergman spaces and projections are extensively studied in the continuous setting (see, for example, [27, 43, 51]) and variants thereof have been introduced on homogeneous trees only in recent developments ([13, 14]). Since a definition of holomorphic function on discrete structures is not clearly stated in the literature, in this context the role of holomorphic functions has been replaced by harmonic functions.

We begin by defining harmonic functions on homogeneous trees, and then we introduce the Bergman space $\mathcal{B}^2(\mathfrak{X})$, including the explicit computation of its reproducing kernel. Finally, we use this kernel to investigate the boundedness properties of the projection operator $\mathcal{P}$.

2.2.1 Harmonic functions

Definition 2.2.1. Let f be a complex valued function on $\mathfrak{X}$. The *combinatorial Laplacian* of f is defined as

$$\Delta f(x) := \frac{1}{q+1}\sum_{y\sim x} f(y) - f(x), \quad x \in \mathfrak{X}. \tag{2.3}$$

The function f is said to be *harmonic in x* if $\Delta f(x) = 0$, namely if it satisfies the local mean property

$$f(x) = \frac{1}{q+1}\sum_{y\sim x} f(y).$$

We will say that f is *harmonic on* $Y \subseteq \mathfrak{X}$ if it is harmonic in x for every $x \in Y$ and that it is *harmonic* if it is harmonic on the whole of $\mathfrak{X}$.

Notice that the notion of harmonic function at a point or on a proper subset of $\mathfrak{X}$ involves its values on a larger subset of $\mathfrak{X}$. Furthermore, it is clear that the value of a harmonic function at a point x affects the values in the neighbors of x, which affect the values in their neighbors and so on, implying a certain stiffness of the function. Indeed, the next results, which are analogues of Lemma 4.1 in [13] and Proposition 2 in [25], show that the harmonicity of a function on a sector implies that certain averages (see (2.5)) of the function depend on the values at the generator and at its predecessor.

Lemma 2.2.2. *Let* $y \in \mathfrak{X}$. *If* $f : \mathfrak{X} \to \mathbb{C}$ *is harmonic on* T_y, *then for all* $n \in \mathbb{N}$

$$\sum_{x \in T_y \cap H_{|y|+n}} f(x) = \Big(\sum_{j=0}^{n} q^j\Big) f(y) - \Big(\sum_{j=0}^{n-1} q^j\Big) f(y^{(1)}). \tag{2.4}$$

Conversely, if f *is constant on* $T_y \cap H_{|y|+k}$ *and satisfies* (2.4) *for all* $k \in \mathbb{N}$, *then* f *is harmonic on* T_y.

Proof. Let $A_n := T_y \cap H_{|y|+n}$ for every $n \in \mathbb{N}$. It is clear that A_n is the set of points in T_y which have distance n from y and that $\#A_n = q^n$. We set $a_{-1} = f(y^{(1)})$ and for every $n \in \mathbb{N}$,

$$a_n = \frac{1}{q^n} \sum_{x \in A_n} f(x). \tag{2.5}$$

Since, for every $n \geq 1$, f is harmonic on $A_n \subseteq T_y$ we have that

$$\begin{aligned} q^n a_n = \sum_{x \in A_n} f(x) &= \sum_{x \in A_n} \frac{1}{q+1}\Big(f(x^{(1)}) + \sum_{z \in S(x)} f(z)\Big) \\ &= \frac{1}{q+1}\Big(\sum_{x \in A_n} f(x^{(1)}) + \sum_{x \in A_n} \sum_{z \in S(x)} f(z)\Big) \\ &= \frac{1}{q+1}\Big(q \sum_{x \in A_{n-1}} f(x) + \sum_{z \in A_{n+1}} f(z)\Big) \\ &= \frac{1}{q+1}(q q^{n-1} a_{n-1} + q^{n+1} a_{n+1}). \end{aligned}$$

Notice that $a_0 = f(y)$ and the harmonicity of f in y reads

$$a_0 = \frac{1}{q+1}(a_{-1} + q a_1).$$

Thus we obtained that for all $n \in \mathbb{N}$,

$$q^{n+1} a_{n+1} = (q+1) q^n a_n - q^n a_{n-1}. \tag{2.6}$$

Our claim is that for all $m \in \mathbb{N}$

$$q^m a_m = \Big(\sum_{j=0}^{m} q^j\Big) a_0 - \Big(\sum_{j=0}^{m-1} q^j\Big) a_{-1}. \tag{2.7}$$

We proceed by induction on m. The case $m = 0$ is trivial and the case $m = 1$ follows from (2.6) with $n = 0$. Now assume $m \geq 1$ and that (2.7) holds for every $i \leq m$, then by (2.6)

$$\begin{aligned}
q^{m+1} a_{m+1} &= (q+1) q^m a_m - q q^{m-1} a_{m-1} \\
&= (q+1)\Big(\sum_{j=0}^{m} q^j\Big) a_0 - (q+1)\Big(\sum_{j=0}^{m-1} q^j\Big) a_{-1} - q\Big(\sum_{j=0}^{m-1} q^j\Big) a_0 + q\Big(\sum_{j=0}^{m-2} q^j\Big) a_{-1} \\
&= a_0\Big(\sum_{j=0}^{m} q^{j+1} + \sum_{j=0}^{m} q^j - \sum_{j=0}^{m-1} q^{j+1}\Big) - a_{-1}\Big(\sum_{j=0}^{m-1} q^{j+1} + \sum_{j=0}^{m-1} q^j - \sum_{j=0}^{m-2} q^{j+1}\Big) \\
&= a_0\Big(\sum_{j=0}^{m+1} q^j\Big) - a_{-1}\Big(\sum_{j=0}^{m} q^j\Big).
\end{aligned}$$

Now if $f : \mathfrak{X} \to \mathbb{C}$ satisfies (2.4) and is constant on A_k for all $k \in \mathbb{N}$, say $f(x) = f_k$ for all $x \in A_k$, then (2.4) reads

$$q^k f(x) = q^k f_k = \Big(\sum_{j=0}^{k} q^j\Big) f(y) - \Big(\sum_{j=0}^{k-1} q^j\Big) f(y^{(1)})$$

and proceeding backwards in the first part of the proof gives that f is harmonic on T_y. □

Now we introduce a technique which exploits the previous Lemma to provide harmonic extensions of functions that are only harmonic on a horoball centered at the origin. If a function is harmonic on HB_n, then its harmonicity involves the values of the function on HB_{n+1}, and there are infinitely many ways to extend it to a function which coincides with it on HB_{n+1} but that is harmonic on the whole tree. The next result provides one way to do it, giving rise to functions which are also horocyclic on T_y for every $y \in H_{n+1}$ (i.e. constant on $T_y \cap H_k$ for all $k \geq n+1$).

Proposition 2.2.3. *Let $g \colon \mathfrak{X} \to \mathbb{C}$ be harmonic on HB_n. The harmonic extension of g at level n is*

$$g_n^H(x) = \begin{cases} g(x), & \text{if } x \in HB_{n+1}, \\ a_n(|x|) g(y) - (a_n(|x|) - 1) g(y^{(1)})), & \text{if } x \in \mathfrak{X} \setminus HB_{n+1}, \end{cases}$$

where $y = x^{(|x|-n-1)}$ and

$$a_n(|x|) = \sum_{j=0}^{|x|-n-1} q^{-j}.$$

The function g_n^H is harmonic, coincides with g on HB_{n+1} and is horocyclic on T_y for every $y \in H_{n+1}$.

Proof. It is clear that g and g_n^H coincide on HB_{n+1}. Furthermore, for every $y \in H_{n+1}$ and every $z, v \in T_y$ such that $|z| = |v|$ we have that $z^{(|z|-n-1)} = v^{(|v|-n-1)}$, hence g_n^H is radial on T_y. Finally, g_n^H is harmonic on HB_n, and in order to show that it is harmonic on $\mathfrak{X}$ we want to show that it satisfies (2.4) for every T_y, $y \in H_{n+1}$. Notice that for every $m \geq n+1$

$$\#\Big(T_y \cap H_m\Big) = q^{m-n-1}.$$

Thus we have

$$\begin{aligned}\sum_{\substack{x\in T_y\\|x|=m}} g_n^H(x) &= q^{m-n-1}\Big(a_n(m)g(y) - (a_n(m)-1)g(y^{(1)})\Big)\\ &= q^{m-n-1}\Big(a_n(m)g_n^H(y) - (a_n(m)-1)g_n^H(y^{(1)})\Big)\end{aligned}$$

since $y, y^{(1)} \in HB_{n+1}$. Moreover,

$$q^{m-n-1}a_n(m) = q^{m-n-1}\sum_{j=0}^{m-n-1} q^{-j} = \sum_{j=0}^{m-n-1} q^{m-n-j-1} = \sum_{j=0}^{m-n-1} q^j$$

and then

$$q^{m-n-1}(a_n(m)-1) = \sum_{j=0}^{m-n-2} q^j,$$

so that g_n^H satisfies (2.4) and by Lemma 2.2.2 it is harmonic on T_y. This holds for every $y \in H_{n+1}$, then g_n^H is harmonic on the whole of $\mathfrak{X}$. □

2.2.2 Bergman spaces

In what follows we study harmonic Bergman spaces with respect to the so-called Bergman measures introduced in Section 1.1.3, in analogy with [13, 25]. Notice that we keep on denoting by μ a measure μ_α for some fixed $\alpha > 1$.

Definition 2.2.4. Let $p \in [1, +\infty)$, the *horocyclic harmonic Bergman space* $\mathcal{B}^p(\mathfrak{X})$ is the space of harmonic functions $f : \mathfrak{X} \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{B}^p(\mathfrak{X})} := \Big(\sum_{x\in\mathfrak{X}} |f(x)|^p \mu(x)\Big)^{\frac{1}{p}} < +\infty.$$

Example 2.2.5. We explicitly build a function in $\mathcal{B}^p(\mathfrak{X})$. We will use many times this kind of construction in the rest of the work. Fix a vertex $y \in \mathfrak{X}$ and let $g : S(y) \to \mathbb{C}$ be such that

$$\sum_{x\in S(y)} g(x) = 0, \quad g \not\equiv 0.$$

Let f be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathfrak{X} \setminus S(y), \\ g(x), & \text{if } x \in S(y). \end{cases}$$

It is clear that f is harmonic on $HB_{|y|}$, so that we can consider its harmonic extension $f^H_{|y|}$. Since $x^{(|x|-|y|)} = y$ for every $x \in T_y \setminus \{y\}$ and $f(y) = 0$, we have

$$f^H_{|y|}(x) = \begin{cases} 0, & \text{if } x \notin T_y \setminus \{y\}, \\ \Big(\sum_{i=0}^{|x|-|y|-1} q^{-i}\Big) g(x^{(|x|-|y|-1)}, & \text{if } x \in T_y \setminus \{y\}. \end{cases}$$

Thus $f^H_{|y|}$ is harmonic and is bounded by

$$|f^H_{|y|}(x)| \leq \max_{z\in S(y)} |g(z)| \sum_{i=0}^{+\infty} q^{-i} = \max_{z\in S(y)} |g(z)| \frac{q}{q-1} < +\infty.$$

Furthermore, since $\operatorname{supp}(f^H_{|y|}) \subseteq T_y$ and $\mu(T_y) < +\infty$, $f^H_{|y|} \in \mathcal{B}^p(\mathfrak{X})$ for all $p \in [1, +\infty)$.

The next result is quite technical and will be a useful tool for many computations.

Lemma 2.2.6. *Let $n \in \mathbb{Z}$, $y \in H_n$ and $g : \mathfrak{X} \to \mathbb{C}$ be a function harmonic on HB_n and such that $g(x) = 0$ for all $x \in HB_{n+1}$ except on $S(y)$. There exist $b_n, b'_n > 0$ such that, for every $f \in \mathcal{B}^2(\mathfrak{X})$,*

$$\langle f, g^H_n \rangle_{\mathcal{B}^2} = \sum_{z\in S(y)} \overline{g(z)} \left(f(z)b_n - f(y)b'_n\right),$$

where in particular

$$b_n := q^{-\alpha n} \sum_{l=0}^{+\infty} q^{-\alpha(l+1)} \Big(\sum_{i=0}^{l} q^{-i}\Big) \Big(\sum_{j=0}^{l} q^{j}\Big),$$

$$b'_n := q^{-\alpha n} \sum_{l=0}^{+\infty} q^{-\alpha(l+1)} \Big(\sum_{i=0}^{l} q^{-i}\Big) \Big(\sum_{j=0}^{l-1} q^{j}\Big).$$

Proof. Let f, n and g be as in the statement. It is worth noticing that g has zero mean on $S(y)$ since it is harmonic in y and vanishes on HB_n, so that g_n^H belongs to $\mathcal{B}^2(\mathfrak{X})$ by Example 2.2.5. Furthermore, it is clearly supported in $T_y \setminus \{y\}$, hence

$$\begin{aligned}\langle f, g_n^H\rangle_{\mathcal{B}^2} &= \sum_{x\in T_y\setminus\{y\}} f(x)\overline{g_n^H(x)}\mu(x) = \sum_{k>n} q^{-\alpha k} \sum_{x\in H_k\cap T_y} f(x)\overline{g_n^H(x)} \\ &= \sum_{k>n} q^{-\alpha k} \sum_{x\in H_k\cap T_y} f(x)\Big(\Big(\sum_{i=0}^{k-n-1} q^{-i}\Big)\overline{g(z)} - \Big(\sum_{i=1}^{k-n-1} q^{-i}\Big)\overline{g(y)}\Big)\end{aligned}$$

where $z = x^{(k-n-1)}$ for every $x \in T_y \cap H_k$. Now, since $g(y) = 0$, we have

$$\begin{aligned}\langle f, g_n^H\rangle_{\mathcal{B}^2} &= \sum_{k>n} q^{-\alpha k}\Big(\sum_{i=0}^{k-n-1} q^{-i}\Big) \sum_{x\in H_k\cap T_y} f(x)\overline{g(x^{(k-n-1)})} \\ &= \sum_{k>n} q^{-\alpha k}\Big(\sum_{i=0}^{k-n-1} q^{-i}\Big) \sum_{z\in S(y)} \overline{g(z)} \sum_{x\in H_k\cap T_z} f(x) \\ &= \sum_{k>n} q^{-\alpha k}\Big(\sum_{i=0}^{k-n-1} q^{-i}\Big) \sum_{z\in S(y)} \overline{g(z)}\Big(\Big(\sum_{j=0}^{k-n-1} q^{j}\Big)f(z) - \Big(\sum_{j=0}^{k-n-2} q^{j}\Big)f(z^{(1)})\Big) \\ &= \sum_{z\in S(y)} \overline{g(z)}\Big(f(z)b_n - f(y)b_n'\Big),\end{aligned}$$

where we applied Lemma 2.2.2 to f. Also, we have

$$\begin{aligned}b_n :=& \sum_{k>n} q^{-\alpha k}\Big(\sum_{i=0}^{k-n-1} q^{-i}\Big)\Big(\sum_{j=0}^{k-n-1} q^{j}\Big) = \sum_{l=0}^{+\infty} q^{-\alpha(l+n+1)}\Big(\sum_{i=0}^{l} q^{-i}\Big)\Big(\sum_{j=0}^{l} q^{j}\Big) \\ =& q^{-\alpha n}\sum_{l=0}^{+\infty} q^{-\alpha(l+1)}\Big(\sum_{i=0}^{l} q^{-i}\Big)\Big(\sum_{j=0}^{l} q^{j}\Big)\end{aligned}$$

and analogously for b_n'. □

By the same proof we can also obtain that for every $g \in \mathcal{B}^2(\mathfrak{X})$ that vanishes in HB_n and every $f \in \mathcal{B}^2(\mathfrak{X})$,

$$\langle f, g_n^H\rangle_{\mathcal{B}^2} = \sum_{z\in H_{n+1}} \overline{g(z)}\left(f(z)b_n - f(z^{(1)})b_n'\right).$$

Here we show that, as we could expect, $\mathcal{B}^2(\mathfrak{X})$ is a Hilbert space. In the following section we will provide an explicit formula for the reproducing kernel of $\mathcal{B}^2(\mathfrak{X})$.

Proposition 2.2.7. *The horocyclic Bergman space $\mathcal{B}^2(\mathfrak{X})$ is a Hilbert space.*

Proof. We only have to show that $\mathcal{B}^2(\mathfrak{X})$ is closed in $L^2(\mathfrak{X})$. The combinatorial Laplacian Δ defined in (2.3) maps $L^2(\mathfrak{X})$ into $L^2(\mathfrak{X})$, indeed for $f \in L^2(\mathfrak{X})$

$$\begin{aligned}
\|\Delta f\|_2^2 &= \sum_{x\in\mathfrak{X}} |\Delta f(x)|^2 \mu(x) \\
&= \sum_{x\in\mathfrak{X}} \Big| \frac{1}{q+1} \sum_{y\sim x} f(y) - f(x) \Big|^2 \mu(x) \\
&\le \sum_{x\in\mathfrak{X}} \Big(\frac{1}{q+1} \sum_{y\sim x} |f(y)| + |f(x)| \Big)^2 \mu(x) \\
&\le \sum_{x\in\mathfrak{X}} (q+2) \Big(\frac{1}{(q+1)^2} \sum_{y\sim x} |f(y)|^2 + |f(x)|^2 \Big) \mu(x) \\
&= (q+2) \Big(\frac{1}{(q+1)^2} \sum_{x\in\mathfrak{X}} \sum_{y\sim x} |f(y)|^2 \mu(x) + \sum_{x\in\mathfrak{X}} |f(x)|^2 \mu(x) \Big),
\end{aligned}$$

where the inequality is given by the equivalence of the norms in $\mathbb{C}^{q+2}$, in particular by

$$\|v\|_1 \le \sqrt{q+2}\,\|v\|_2, \quad v \in \mathbb{C}^{q+2}.$$

The first term of the last equality can be rewritten as

$$\begin{aligned}
&\frac{1}{(q+1)^2} \sum_{x\in\mathfrak{X}} \sum_{y\sim x} |f(y)|^2 \mu(x) \\
&\qquad = \frac{1}{(q+1)^2} \Big(\sum_{x\in\mathfrak{X}} |f(x^{(1)})|^2 \mu(x^{(1)}) q^{-\alpha} + \sum_{x\in\mathfrak{X}} \sum_{y\in S(x)} |f(y)|^2 \mu(y) q^{\alpha} \Big) \\
&\qquad = \frac{1}{(q+1)^2} \left(q\|f\|_2^2 q^{-\alpha} + \|f\|_2^2 q^{\alpha} \right) \\
&\qquad = \frac{q^{1-\alpha} + q^{\alpha}}{(q+1)^2} \|f\|_2^2.
\end{aligned}$$

This gives

$$\|\Delta f\|_2^2 \le (q+2) \Big(\frac{q^{1-\alpha} + q^{\alpha}}{(q+1)^2} + 1 \Big) \|f\|_2^2$$

so that $\Delta f \in L^2(\mathfrak{X})$ and $\Delta : L^2(\mathfrak{X}) \to L^2(\mathfrak{X})$ is continuous. It is clear that $\mathcal{B}^2(\mathfrak{X}) = \Delta^{-1}(\{0\})$ and so it is closed in $L^2(\mathfrak{X})$ and it is a Hilbert space. □

2.2.3 The Bergman kernel

In this section we see that $\mathcal{B}^2(\mathfrak{X})$ is a reproducing kernel Hilbert space and find a formula for the kernel.

Fix $y \in \mathfrak{X}$ and consider the evaluation functional $\Phi_y : \mathcal{B}^2(\mathfrak{X}) \to \mathbb{C}$ given by $\Phi_y f := f(y)$. We have that

$$|\Phi_y(f)|^2 = |f(y)|^2 \le \frac{1}{\mu(y)} \sum_{x\in\mathfrak{X}} |f(x)|^2 \mu(x) = \frac{1}{\mu(y)} \|f\|_2^2.$$

Thus Φ_y is continuous and $\mathcal{B}^2(\mathfrak{X})$ is a reproducing kernel Hilbert space, that is for all $y \in \mathfrak{X}$ there exists $\mathcal{K}_y \in \mathcal{B}^2(\mathfrak{X})$ such that

$$f(y) = \langle f, \mathcal{K}_y \rangle_{\mathcal{B}^2}, \quad f \in \mathcal{B}^2(\mathfrak{X}).$$

The function $\mathcal{K} : \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ defined by $\mathcal{K}(y,x) := \mathcal{K}_y(x)$ is the *reproducing kernel* of $\mathcal{B}^2(\mathfrak{X})$.

In order to compute $\mathcal{K}_y$, we need a preliminary step, that is we have to find the reproducing kernels of the following auxiliary spaces. For all $v \in \mathfrak{X}$, let define the space

$$Z_v := \Big\{ \phi \colon S(v) \to \mathbb{C} : \sum_{x \in S(v)} \phi(x) = 0 \Big\} \simeq \mathbb{C}^{q-1}.$$

We fix $v \in \mathfrak{X}$ and consider the function

$$\Gamma_v(x,y) = \begin{cases} 0 & \text{if } \{x,y\} \not\subseteq T_v \setminus \{v\}, \\ \dfrac{q-1}{q} & \text{if } \{x,y\} \subseteq T_z \text{ for some } z \in S(v), \\ -\dfrac{1}{q} & \text{otherwise.} \end{cases}$$

Given $v \in \mathfrak{X}$, some observations about Γ_v are in order. First of all, Γ_v is symmetric, that is $\Gamma_v(x,y) = \Gamma_v(y,x)$ for all $x, y \in \mathfrak{X}$. Moreover, $\Gamma_v(x, \cdot) \equiv 0$ for all $x \notin T_v \setminus \{v\}$ and $\operatorname{supp}(\Gamma_v(x,\cdot)) = T_v \setminus \{v\}$ for all $x \in T_v \setminus \{v\}$ and, in this case, the values of $\Gamma_v(x,\cdot)$ are completely determined by the values on $S(v)$. Indeed, if $x \in T_v \setminus \{v\}$ then $x \in T_z$ for some $z \in S(v)$ and

$$\Gamma_v(x,y) = \begin{cases} \dfrac{q-1}{q} & \text{if } y \in T_z, \\ -\dfrac{1}{q} & \text{if } y \notin T_z. \end{cases}$$

A representation is given in Figure 2.1.

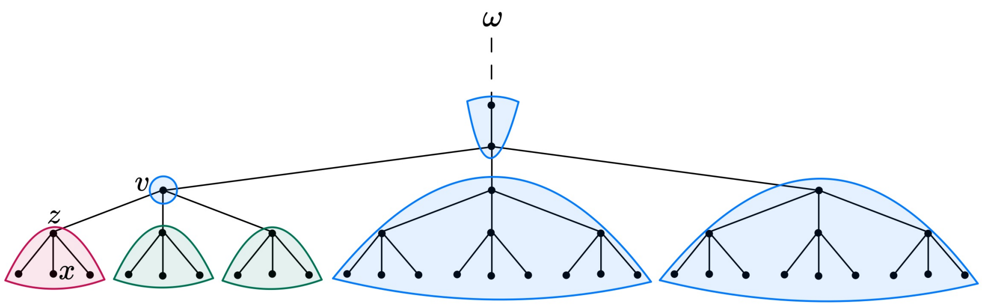

Figure 2.1: Realization of $\Gamma_v(x,\cdot)$ for $x \in T_z$, $z \in S(v)$. The function vanishes on the blue area, while its value is $\frac{q-1}{q}$ on the red sector and $-\frac{1}{q}$ on the green ones.

Lemma 2.2.8. *The function Γ_v is the reproducing kernel of Z_v, that is for every $\varphi \in Z_v$ and $x \in S(v)$,*

$$\varphi(x) = \langle\, \varphi,\, \Gamma_v(x,\cdot)|_{S(v)}\,\rangle_{Z_v}.$$

Proof. Firstly,

$$\sum_{y\in S(v)} \Gamma_v(x,y) = \frac{q-1}{q} - \frac{1}{q}(q-1) = 0$$

and then $\Gamma_v(x,\cdot)|_{S(v)} \in Z_v$. Now, if $\varphi \in Z_v$, then

$$\begin{aligned}\langle\, \varphi,\, \Gamma_v(x,\cdot)|_{S(v)}\,\rangle_{Z_v} &= \sum_{y\in S(v)} \varphi(y)\overline{\Gamma_v(x,y)}\\ &= \frac{q-1}{q}\varphi(x) - \frac{1}{q}\sum_{y\in S(v)\setminus\{x\}} \varphi(y)\\ &= \frac{q-1}{q}\varphi(x) - \frac{1}{q}(-\varphi(x)) = \varphi(x),\end{aligned}$$

where we used the fact that φ has vanishing mean on $S(v)$. □

Note that the function $\Gamma_v(x,\cdot)$ is harmonic on $HB_{|v|}$, indeed it vanishes there and it is sufficient to check the harmonicity at v in the case $x \in T_v \setminus \{v\}$, which follows by $\mathrm{supp}(\Gamma_v(x,\cdot)) = T_v \setminus \{v\}$ and $\Gamma_v(x,\cdot)|_{S(v)} \in Z_v$. This allows us to consider, for all $v \in \mathfrak{X}$, the harmonic extension $(\Gamma_{v^{(1)}}(v,\cdot))^H_{|v^{(1)}|}$, that is

$$(\Gamma_{v^{(1)}}(v,y))^H_{|v^{(1)}|} = \begin{cases} 0 & \text{if } y \notin T_{v^{(1)}} \setminus \{v^{(1)}\},\\ \Big(\sum_{i=0}^{|y|-|v|} q^{-i}\Big)\dfrac{q-1}{q} & \text{if } y \in T_v,\\ \Big(\sum_{i=0}^{|y|-|v|} q^{-i}\Big)\Big(-\dfrac{1}{q}\Big) & \text{if } y \in T_{v^{(1)}} \setminus \big(T_v \cup \{v^{(1)}\}\big).\end{cases}$$

It is clear that $|(\Gamma_{v^{(1)}}(v,\cdot))^H_{|v^{(1)}|}(y)| \le 1$ for all $y \in T_v \setminus \{v\}$ and this easily implies that $(\Gamma_{v^{(1)}}(v,\cdot))^H_{|v^{(1)}|}$ is in $\mathcal{B}^2(\mathfrak{X})$.

From now on, for $v \in \mathfrak{X}$ and $n \in \mathbb{N}$, we will denote

$$\begin{aligned}\Gamma_{n,v} &= \Gamma_{v^{(n+1)}}(v^{(n)},\cdot),\\ \Gamma^H_{n,v} &= \Big(\Gamma_{v^{(n+1)}}(v^{(n)},\cdot)\Big)^H_{|v^{(n+1)}|}.\end{aligned}$$

Lemma 2.2.9. *Let $v \in \mathfrak{X}$ and $f \in \mathcal{B}^2(\mathfrak{X})$. For every $n \in \mathbb{N}$ the following holds*

$$\langle\, f,\, \Gamma^H_{n,v}\,\rangle_{\mathcal{B}^2} = b_{|v^{(n+1)}|}\Big(f(v^{(n)}) - \frac{q+1}{q}f(v^{(n+1)}) + \frac{1}{q}f(v^{(n+2)})\Big).$$

Proof. Let $v \in \mathfrak{X}$ and $f \in \mathcal{B}^2(\mathfrak{X})$, by Lemma 2.2.6

$$\begin{aligned}\langle f, \Gamma^H_{0,v} \rangle_{\mathcal{B}^2} &= \sum_{y \in S(v^{(1)})} \overline{\Gamma_{0,v}(y)} \Big(f(y) b_{|v^{(1)}|} - f(v^{(1)}) b'_{|v^{(1)}|} \Big) \\ &= \sum_{y \in S(v^{(1)})} \overline{\Gamma_{0,v}(y)} f(y) b_{|v^{(1)}|} - f(v^{(1)}) b'_{|v^{(1)}|} \sum_{y \in S(v^{(1)})} \overline{\Gamma_{0,v}(y)}.\end{aligned}$$

We remind that $\Gamma_{0,v} \in Z_{v^{(1)}}$ and so the second term vanishes. Using that f is harmonic in $v^{(1)}$, we obtain

$$\begin{aligned}\langle f, \Gamma^H_{0,v} \rangle_{\mathcal{B}^2} &= \sum_{y \in S(v^{(1)})} \overline{\Gamma_{0,v}(y)} f(y) b_{|v^{(1)}|} \\ &= b_{|v^{(1)}|} \Big(\frac{q-1}{q} f(v) - \frac{1}{q} \sum_{y \in S(v^{(1)}) \setminus \{v\}} f(y) \Big) \\ &= b_{|v^{(1)}|} \Big(\frac{q-1}{q} f(v) - \frac{q+1}{q} f(v^{(1)}) + \frac{1}{q} f(v) + \frac{1}{q} f(v^{(2)}) \Big) \\ &= b_{|v^{(1)}|} \Big(f(v) - \frac{q+1}{q} f(v^{(1)}) + \frac{1}{q} f(v^{(2)}) \Big).\end{aligned}$$

Upon replacing v with $v^{(n)}$ we obtain the claim for every $n \in \mathbb{N}$. □

Theorem 2.2.10. *For every $v \in \mathfrak{X}$ the function*

$$\mathcal{K}_v(x) = \sum_{n=0}^{+\infty} d_{n,|v|} \Gamma^H_{n,v}(x)$$

is the kernel associated to v, where

$$d_{n,k} := \Big(\sum_{j=0}^{n} q^{-j} \Big) b^{-1}_{k-n-1}.$$

Before giving the proof of this Theorem we show that $\mathcal{K}_v$ belongs to $\mathcal{B}^2(\mathfrak{X})$.

Lemma 2.2.11. *$\mathcal{K}_v \in \mathcal{B}^2(\mathfrak{X})$ for every $v \in \mathfrak{X}$.*

Proof. Define $d_{n,k}$ as above and consider for every $m \in \mathbb{N}$ the partial sums given by

$$s^m_v(x) = \sum_{n=0}^{m} d_{n,|v|} \Gamma^H_{n,v}(x).$$

It is clear that $s^m_v \in \mathcal{B}^2(\mathfrak{X})$ for every $m \in \mathbb{N}$ because it is a finite sum of functions in $\mathcal{B}^2(\mathfrak{X})$. Our goal is to show that $\|\mathcal{K}_v - s^m_v\|_2 \to 0$. Let $x \in T_{v^{(m+1)}}$,

$$|(\mathcal{K}_v - s^m_v)(x)| = \Big| \sum_{n=m+1}^{+\infty} d_{n,|v|} \Gamma^H_{n,v}(x) \Big| \le \sum_{n=m+1}^{+\infty} d_{n,|v|} \Big| \Gamma^H_{n,v}(x) \Big|$$

$$\leq \sum_{n=m+1}^{+\infty} d_{n,|v|} \lesssim \sum_{n=m+1}^{+\infty} q^{\alpha(|v|-n-1)}$$
$$\simeq q^{\alpha|v|} \sum_{n=m+1}^{+\infty} q^{-\alpha n} \lesssim q^{\alpha(|v|-m)}.$$

Now let $x \notin T_{v^{(m+1)}}$, in this case the first sector of the form $T_{v^{(n)}}$ which contains x is $T_{v^{(|v|-|x\wedge v|)}} = T_{x\wedge v}$. It follows that $\Gamma^H_{n,v}(x) = 0$ for every $n = m+1, \ldots, |v| - |x \wedge v| + 1$. As in the previous computation we obtain

$$|(\mathcal{K}_v - s_v^m)(x)| = \Big| \sum_{n=|v|-|x\wedge v|}^{+\infty} d_{n,|v|} \Gamma^H_{n,v}(x) \Big| \lesssim q^{\alpha|x\wedge v|}.$$

Then we get

$$\|\mathcal{K}_v - s_v^m\|_2^2 = \sum_{x\in\mathfrak{X}} |(\mathcal{K}_v - s_v^m)(x)|^2 \mu(x)$$
$$= \sum_{x\in T_{v^{(m+1)}}} |(\mathcal{K}_v - s_v^m)(x)|^2 \mu(x) + \sum_{x\notin T_{v^{(m+1)}}} |(\mathcal{K}_v - s_v^m)(x)|^2 \mu(x)$$
$$= \sum_{x\in T_{v^{(m+1)}}} |(\mathcal{K}_v - s_v^m)(x)|^2 \mu(x) + \sum_{i=|v^{(m+2)}|}^{-\infty} \sum_{|x\wedge v|=i} |(\mathcal{K}_v - s_v^m)(x)|^2 \mu(x)$$
$$\lesssim \sum_{x\in T_{v^{(m+1)}}} q^{2\alpha(|v|-m)} \mu(x) + \sum_{i=|v|-m-2}^{-\infty} \sum_{|x\wedge v|=i} q^{2\alpha|x\wedge v|} \mu(x)$$
$$\lesssim q^{2\alpha(|v|-m)} \mu(T_{v^{(m+1)}}) + \sum_{i=m+2-|v|}^{+\infty} q^{-2\alpha i} \mu(\{x : |x\wedge v| = -i\}).$$

Notice that $\{x : |x \wedge v| = -i\} = T_{v^{(|v|+i)}} \setminus T_{v^{(|v|+i-1)}}$. By the proof of Lemma 1.1.13 it is easy to see that for all $z \in \mathfrak{X}$,

$$\mu(T_z) = \frac{q^{-\alpha|z|}}{1-q^{1-\alpha}}.$$

Therefore,

$$\mu(\{x : |x\wedge v| = -i\}) = \frac{1}{1-q^{1-\alpha}} \left(q^{-\alpha|v^{(|v|+i)}|} - q^{-\alpha|v^{(|v|+i-1)}|} \right)$$
$$= \frac{1}{1-q^{1-\alpha}} \left(q^{\alpha i} - q^{\alpha(i-1)} \right) = q^{\alpha i} \frac{1-q^{-\alpha}}{1-q^{1-\alpha}} \simeq q^{\alpha i}.$$

Therefore we obtain

$$\|\mathcal{K}_v - s_v^m\|_2^2 \lesssim q^{2\alpha(|v|-m)} q^{-\alpha|v^{(m+1)}|} + \sum_{i=m+2-|v|}^{+\infty} q^{-2\alpha i} q^{\alpha i}$$

$$\lesssim q^{-\alpha(2m+4-2|v|)}q^{-\alpha(|v|-m-1)} + q^{-2\alpha}(1-q^{-\alpha})\sum_{i=m+2-|v|}^{+\infty} q^{-2\alpha i}q^{\alpha i}$$
$$\lesssim q^{\alpha(|v|-m)} + \sum_{i=m+2-|v|}^{+\infty} q^{-\alpha i} \xrightarrow[m\to+\infty]{} 0.$$

This proves that $\mathcal{K}_v \in \mathcal{B}^2(\mathfrak{X})$ since it is closed in $L^2(\mathfrak{X})$. □

We are in a position to prove Theorem 2.2.10.

Proof of Theorem 2.2.10. Let $f \in \mathcal{B}^2(\mathfrak{X})$, we claim that

$$f(v) = \langle f\,,\,\mathcal{K}_v\rangle_{\mathcal{B}^2}.$$

By Lemma 2.2.9 we have

$$\begin{aligned}\langle f\,,\sum_{n=0}^{+\infty} d_{n,|v|}\Gamma^H_{n,v}\rangle_{\mathcal{B}^2} = &\sum_{n=0}^{+\infty} d_{n,|v|}\langle f\,,\,\Gamma^H_{n,v}\rangle_{\mathcal{B}^2}\\ =&\sum_{n=0}^{+\infty}\Big(\sum_{j=0}^{n} q^{-j}\Big)\Big(f(v^{(n)}) - \frac{q+1}{q}f(v^{(n+1)}) + \frac{1}{q}f(v^{(n+2)})\Big)\\ =&f(v) - \frac{q+1}{q}f(v^{(1)}) + \Big(1+\frac{1}{q}\Big)f(v^{(1)})\\ &+\sum_{m=2}^{+\infty} f(v^{(m)})\Big(\sum_{j=0}^{m} q^{-j} - \frac{q+1}{q}\sum_{j=0}^{m-1} q^{-j} + \frac{1}{q}\sum_{j=0}^{m-2} q^{-j}\Big)\\ =&f(v).\end{aligned}$$

since

$$\begin{aligned}&\frac{1}{q}\sum_{j=0}^{m-2} q^{-j} - \frac{q+1}{q}\sum_{j=0}^{m-1} q^{-j} + \sum_{j=0}^{m} q^{-j}\\ &=\sum_{j=0}^{m-2} q^{-j}\left(\frac{1}{q} - \frac{q+1}{q} + 1\right) - q^{1-m}\frac{q+1}{q} + q^{1-m} + q^{-m}\\ &= -q^{1-m} - q^{-m} + q^{1-m} + q^{-m}\\ &= 0.\end{aligned}$$

Therefore $\mathcal{K}_v$ has the reproducing property and it is the kernel associated to v. □

Now we give an explicit formula for the kernel $\mathcal{K}$ which exhibits its symmetry.

Theorem 2.2.12. *The kernel $\mathcal{K}(v,\cdot) = \mathcal{K}_v$ may be rewritten as*

$$\mathcal{K}(v,x) = \sum_{m=-|x\wedge v|-1}^{+\infty} b^{-1}_{-m-1}\Big(\sum_{j=0}^{m+|v|} q^{-j}\Big)\Big(\sum_{i=0}^{m+|x|} q^{-i}\Big)\Gamma_{m+|v|,v}(x^{(m+|x|)}).$$

In particular, $\mathcal{K}$ is real and symmetric.

Proof. By Theorem 2.2.10

$$\mathcal{K}(v,x) = \mathcal{K}_v(x) = \sum_{n=0}^{+\infty} d_{n,|v|} \Gamma^H_{n,v}(x).$$

Recall that $\Gamma^H_{n,v}(x) \neq 0$ if and only if $x \in T_{v^{(n+1)}} \setminus \{v^{(n+1)}\}$. The smaller $n \in \mathbb{N}$ such that $x \in T_{v^{(n+1)}} \setminus \{v^{(n+1)}\}$ is given by $v^{(n+1)} = x \wedge v$. It follows that $|v| - n - 1 = |x \wedge v|$, thus $n = |v| - |x \wedge v| - 1$. We obtain

$$\begin{aligned}
\mathcal{K}(v,x) &= \sum_{n=|v|-|x\wedge v|-1}^{+\infty} d_{n,|v|} \Gamma^H_{n,v}(x) \\
&= \sum_{n=|v|-|x\wedge v|-1}^{+\infty} d_{n,|v|} \Big(\sum_{i=0}^{|x|-|v^{(n+1)}|-1} q^{-i} \Big) \Gamma_{n,v}(x^{(|x|-|v^{(n+1)}|-1)}) \\
&= \sum_{n=|v|-|x\wedge v|-1}^{+\infty} d_{n,|v|} \Big(\sum_{i=0}^{|x|-|v|+n} q^{-i} \Big) \Gamma_{n,v}(x^{(|x|-|v|+n)}).
\end{aligned}$$

Now we put $m = n - |v|$ and notice that

$$d_{m+|v|,|v|} = \Big(\sum_{j=0}^{m+|v|} q^{-j} \Big) b^{-1}_{-m-1}.$$

In particular we have that

$$v^{(n+1)} = x \wedge v^{(m+1+|x\wedge v|)}$$

since $|v^{(n+1)}| = |v| - n - 1 = -m - 1 = |x \wedge v^{(m+1+|x\wedge v|)}|$ and both $v^{(n+1)}$ and $x \wedge v^{(m+1+|x\wedge v|)}$ belong to $[v, \omega)$. It follows that

$$\begin{aligned}
\mathcal{K}(v,x) = \sum_{m=-|x\wedge v|-1}^{+\infty} & b^{-1}_{-m-1} \Big(\sum_{j=0}^{m+|v|} q^{-j} \Big) \Big(\sum_{i=0}^{m+|x|} q^{-i} \Big) \\
& \Gamma_{x\wedge v^{(m+1+|x\wedge v|)}}(v^{(m+|v|)}, x^{(m+|x|)}).
\end{aligned}$$

It is clear that $x \wedge v^{(m+1+|x\wedge v|)} = v^{(m+1+|v|)}$ so that

$$\Gamma_{m+|v|,v}(x^{(m+|x|)}) = \Gamma_{x\wedge v^{(m+1+|x\wedge v|)}}(v^{(m+|v|)}, x^{(m+|x|)})$$

and the rest of the statement follows. □

2.2.4 Boundedness of the Bergman projection

By Proposition 2.2.7, the horocyclic harmonic Bergman space $\mathcal{B}^2(\mathfrak{X})$ is closed in $L^2(\mathfrak{X})$. Hence it is natural to consider the orthogonal projection $\mathcal{P}$ of $L^2(\mathfrak{X})$ onto $\mathcal{B}^2(\mathfrak{X})$, that is clearly given by integration against the Bergman kernel, namely

$$\mathcal{P}f = \sum_{x\in\mathfrak{X}} \mathcal{K}(\cdot, x)f(x)\mu(x), \quad f \in L^2(\mathfrak{X}).$$

We refer to it as the *horocyclic Bergman projection.* The aim is to prove that the extension of $\mathcal{P}$ to $L^p(\mathfrak{X})$ is of weak-type $(1,1)$, is bounded for all $p \in (1,\infty)$ and to obtain limiting results involving H^1 and BMO. To do that we obviously just want to apply the results obtained in Section 2.1 exploiting the formula we found for $\mathcal{K}$.

Theorem 2.2.13. *The Bergman kernel* $\mathcal{K}$ *of* $\mathcal{B}^2(\mathfrak{X})$ *satisfies Hörmander's condition* (2.1) *for* μ.

Proof. We start by proving (2.1) with $y = v$. In this case, for every $x \in T_v$ and $z \notin T_v$ we have that $x \wedge z = v \wedge z$. Furthermore,

$$x^{(m+|x|)} = v^{(m+|v|)} \tag{2.8}$$

for every $m \geq -|v|$ since they both have horocyclic index $-m$ and lie on $[v,\omega)$. It is clear that $|v| \geq |v \wedge z| + 1$ because $z \notin T_v$, thus (2.8) holds true in particular for all $m \geq -|v \wedge z| - 1$. Hence for all $m \geq -|v \wedge z| - 1$

$$\Gamma_{m+|z|,z}(x^{(m+|x|)}) = \Gamma_{m+|z|,z}(v^{(m+|v|)})$$

and by Theorem 2.2.12

$$\begin{aligned}
&|\mathcal{K}(z,x) - \mathcal{K}(z,v)| \\
&\leq \sum_{m=-|v\wedge z|-1}^{+\infty} b_{-m-1}^{-1} \Big(\sum_{j=0}^{m+|z|} q^{-j}\Big)\Big(\sum_{i=m+1+|v|}^{m+|x|} q^{-i}\Big) \left|\Gamma_{m+|z|,z}(v^{(m+|v|)})\right| \\
&\leq \sum_{m=-|v\wedge z|-1}^{+\infty} b_{-m-1}^{-1} \Big(\sum_{j=0}^{+\infty} q^{-j}\Big)\Big(\sum_{i=m+1+|v|}^{+\infty} q^{-i}\Big) \\
&\lesssim \sum_{m=-|v\wedge z|-1}^{+\infty} q^{-\alpha m} q^{-m-|v|}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\sum_{z\in\mathfrak{X}\setminus T_v} |\mathcal{K}(z,x) - \mathcal{K}(z,v)|\mu(z) &\lesssim q^{-|v|} \sum_{z\in\mathfrak{X}\setminus T_v} q^{-\alpha|z|} \sum_{m=-|v\wedge z|-1}^{+\infty} q^{-m(\alpha+1)} \\
&\simeq q^{-|v|} \sum_{z\in\mathfrak{X}\setminus T_v} q^{-\alpha|z|} q^{(\alpha+1)|v\wedge z|}
\end{aligned}$$

$$
\begin{aligned}
&= q^{-|v|} \sum_{i=-|v|+1}^{+\infty} \sum_{|v\wedge z|=-i} q^{-\alpha|z|} q^{-(\alpha+1)i} \\
&= q^{-|v|} \sum_{i=-|v|+1}^{+\infty} q^{-(\alpha+1)i} \mu(\{z : |v \wedge z| = -i\}) \\
&\simeq q^{-|v|} \sum_{i=-|v|+1}^{+\infty} q^{-(\alpha+1)i} q^{\alpha i} \simeq q^{-|v|} q^{|v|} = 1,
\end{aligned}
$$

where the last line follows by the computation in the proof of Lemma 2.2.11. It follows that

$$
\sup_{v\in\mathfrak{X}} \sup_{x\in T_v} \sum_{z\in\mathfrak{X}\setminus T_v} |\mathcal{K}(z,x) - \mathcal{K}(z,v)|\mu(z) < +\infty,
$$

that is (2.1) holds in the case $y = v$. The general case easily follows by the triangular inequality,

$$
\begin{aligned}
&\sup_{v\in\mathfrak{X}} \sup_{x,y\in T_v} \sum_{z\in\mathfrak{X}\setminus T_v} |\mathcal{K}(z,x) - \mathcal{K}(z,y)|\mu(z) \\
&\quad \le \sup_{v\in\mathfrak{X}} \sup_{x,y\in T_v} \sum_{z\in\mathfrak{X}\setminus T_v} \Big(|\mathcal{K}(z,x) - \mathcal{K}(z,v)| + |\mathcal{K}(z,v) - \mathcal{K}(z,y)| \Big)\mu(z) \\
&\quad = \sup_{v\in\mathfrak{X}} \sup_{x\in T_v} \sum_{z\in\mathfrak{X}\setminus T_v} |\mathcal{K}(z,x) - \mathcal{K}(z,v)|\mu(z) \\
&\qquad\qquad + \sup_{v\in\mathfrak{X}} \sup_{y\in T_v} \sum_{z\in\mathfrak{X}\setminus T_v} |\mathcal{K}(z,y) - \mathcal{K}(z,v)|\mu(z).
\end{aligned}
$$

□

Noticing that $\mathcal{K}$ is self-adjoint, the next result is a straightforward consequence of Theorem 2.2.13 and provides an example of application for Theorems 2.1.2, 2.1.9, 2.1.10 and Corollary 2.1.3.

Corollary 2.2.14. *The following results hold for the horocyclic Bergman projection $\mathcal{P}$:*

1) $\mathcal{P}$ is of weak-type $(1,1)$;

2) $\mathcal{P}$ is bounded on $L^p(\mathfrak{X})$ for all $p \in (1,\infty)$;

3) $\mathcal{P}$ is bounded from $H^1_{at}(\mathfrak{X})$ to $L^1(\mathfrak{X})$;

4) $\mathcal{P}$ is bounded from $L^\infty(\mathfrak{X})$ to $BMO(\mathfrak{X})$.

This conclusion is consistent with what one might have expected for $\mathcal{P}$, following the analogy with the well-known continuous case, where equivalent behaviors and outcomes are observed.

Chapter 3

Nondoubling Calderón-Zygmund theory on general trees and the radial Bergman projection

The study of Calderón-Zygmund theory in nondoubling contexts is crucial for extending classical harmonic analysis results to spaces that do not fulfill the measure growth condition typical of doubling spaces. Classical methods may not apply directly in general settings where the measure could be very irregular, and significant modifications are required to account for these more complicated structures. These spaces arise in various areas of mathematics, including geometric analysis, harmonic analysis on non Euclidean spaces, and the study of graphs and trees. Indeed, trees with unbounded geometry are among them.

In the literature, several approaches have been developed to deal with different nondoubling settings, while usually still imposing some conditions on the measure, or exploiting the peculiarities of a specific measure. In particular, measures satisfying a polynomial growth condition are broadly investigated. Some classical references in this direction are [38, 39, 45, 46, 36, 47].

This chapter relies in part of the results presented in [16], a joint work with José Conde Alonso, Filippo De Mari, Matteo Monti and Maria Vallarino. It is devoted to extending Calderón-Zygmund theory to the specific nondoubling setting of general trees. We will explore the necessary adjustments to classical results, focusing on how to handle the lack of a doubling condition. Unlike the previous chapter, we will adopt a dyadic perspective, which will play a key role in enabling the extension of important results. In Section 3.1 we exhibit a nondoubling Calderón-Zygmund decomposition and sufficient conditions to guarantee usual boundedness properties for integral operators on any tree equipped with any measure. Concerning the endpoints, we introduce suitable definitions of H^1 and BMO spaces adapted to the dyadic system $\mathcal{D}$ defined in Section 1.3.2. The results presented there can be thought of as a general Calderón-Zygmund theory for non locally doubling trees. Afterwards, in Section 3.2 we show that the Bergman projector is an example of application in the case of radial trees.

3.1 General theory

The goal of this section is to extend the results of Section 2.1 to L^2-bounded integral operators T defined on a generic tree endowed with any measure. Just as in the doubling context, in order to analyse the action of the operator on a function f, we need a decomposition of f as a sum of a good and a bad part. The decomposition proved in Theorem 2.1.1 is not useful in this case and must be adapted to the nondoubling setting by means of the dyadic structure that we found for general trees. Unfortunately, in this new context the bad part turns out to be even more problematic than before, since the supports of its components are no longer disjoint, although we can still recover the other properties on the mean and the L^1-norm. Concerning the good part, we do lose some regularity as well, since we can no longer control the L^∞-norm. However, we can replace it with an analogous, but weaker, bound on its BMO norm.

The next theorem is inspired by the first nondoubling adjustment of the Calderón-Zygmund decomposition, stated in [35]. It is an adaptation of Theorem 3.2 in [18] and Section 1 of [6], while the BMO estimate was first proved in [19]. The BMO norm considered there is given by

$$\|g\|_{BMO} \simeq \sup_{Q\in\mathcal{D}} \frac{1}{\mu(Q)} \sum_{x\in Q} |g(x) - \langle g\rangle_Q|\mu(x) + \sup_{\substack{Q\in\mathcal{D}\\ Q\neq\mathfrak{X}}} |\langle g\rangle_Q - \langle g\rangle_{Q^{(1)}}| + \left|\sum_{x\in\mathfrak{X}} g(x)\mu(x)\right|,$$

for which we will provide more details in Section 3.1.2.

Theorem 3.1.1 (Nondoubling Calderón-Zygmund decomposition)**.** *Let $Q \in \mathcal{D}$, $f \in L^1(Q)$, $f \geq 0$ and $\lambda > \|f\|_{L^1(Q)}/\mu(Q)$. There exists a family of disjoint sets $\{Q_j\}_j \subseteq \mathcal{D}$ such that, if*

$$b_j := f\mathbb{1}_{Q_j} - \langle f\mathbb{1}_{Q_j}\rangle_{Q_j^{(1)}}\mathbb{1}_{Q_j^{(1)}},$$

$b = \sum_j b_j$ and $g := f - b$, then

1) f=g+b;

2) the following localization properties and L^1-bound for b hold

$$\operatorname{supp}(b_j) \subseteq Q_j^{(1)}, \quad \sum_{x\in Q_j^{(1)}} b_j(x)\mu(x) = 0, \quad \sum_j \|b_j\|_1 \leq 2\|f\|_1;$$

3) the following higher integrability properties for g hold

$$\|g\|_{BMO} \lesssim \lambda, \quad \|g\|_2^2 \lesssim \lambda \|f\|_1;$$

4) if $\Omega_\lambda := \cup_j Q_j$, then $\mu(\Omega_\lambda) \lesssim \frac{1}{\lambda}\|f\|_1$.

Proof. Let $Q \in \mathcal{D}$, $f \in L^1(Q)$, $f \geq 0$ and $\lambda > \|f\|_{L^1(Q)}/\mu(Q)$. Consider

$$\Omega_\lambda := \Big\{ x \in \mathfrak{X} : \mathcal{M}_{\mathcal{D}} f(x) > \lambda \Big\},$$

and cover Ω_λ with a family of pairwise disjoint sets $\{Q_j\}_j$ in $\mathcal{D}$ which are maximal with respect to inclusion, that is $\Omega_\lambda \subseteq \cup_j Q_j$ and $Q_j^{(1)} \not\subseteq \bigcup_\ell Q_\ell$ for any j.

By the definitions of b and g, 1) trivially holds. Moreover, it is clear that $\operatorname{supp}(b_j) \subseteq Q_j^{(1)}$ and

$$\sum_{x \in Q_j^{(1)}} b_j(x)\mu(x) = \sum_{x \in Q_j} f(x)\mu(x) - \langle f \mathbb{1}_{Q_j} \rangle_{Q_j^{(1)}} \mu(Q_j^{(1)}) = 0.$$

Furthermore

$$\begin{aligned}
\sum_j \|b_j\|_1 &= \sum_j \sum_{x \in Q_j^{(1)}} |f \mathbb{1}_{Q_j}(x) - \langle f \mathbb{1}_{Q_j} \rangle_{Q_j^{(1)}} \mathbb{1}_{Q_j^{(1)}}(x)| \mu(x) \\
&\leq \sum_j \sum_{x \in Q_j^{(1)}} \Big(f \mathbb{1}_{Q_j}(x) + \langle f \mathbb{1}_{Q_j} \rangle_{Q_j^{(1)}} \mathbb{1}_{Q_j^{(1)}}(x) \Big) \mu(x) \\
&\leq \sum_{x \in \mathfrak{X}} f(x)\mu(x) + \sum_j \langle f \mathbb{1}_{Q_j} \rangle_{Q_j^{(1)}} \mu(Q_j^{(1)}) \\
&= \|f\|_1 + \sum_j \sum_{x \in Q_j} f \mathbb{1}_{Q_j}(x) \\
&\leq 2\|f\|_1,
\end{aligned}$$

hence 2) is proved. Concerning g, we have

$$\begin{aligned}
g &= f - \Big(\sum_j f \mathbb{1}_{Q_j} - \langle f \mathbb{1}_{Q_j} \rangle_{Q_j^{(1)}} \mathbb{1}_{Q_j^{(1)}} \Big) \\
&= f \mathbb{1}_{\mathfrak{X} \setminus \cup_j Q_j} + \sum_j \langle f \mathbb{1}_{Q_j} \rangle_{Q_j^{(1)}} \mathbb{1}_{Q_j^{(1)}} \\
&=: g_1 + g_2,
\end{aligned}$$

which implies that g_1, g_2 and g are positive. For $x \in \mathfrak{X} \setminus \cup_j Q_j$,

$$g_1(x) = f(x) = \langle f \rangle_{\{x\}} \leq \mathcal{M}_{\mathcal{D}} f(x) \leq \lambda,$$

so $\|g_1\|_\infty \le \lambda$ and $\|f\|_{BMO} \le 2\lambda$. If $Q = Q_j$ is a selected cube, we have

$$\begin{aligned}
\langle g_2\rangle_{Q_j^{(1)}} \mathbb{1}_{Q_j} =& \Big(\frac{1}{\mu(Q_j^{(1)})}\sum_{x\in Q_j^{(1)}}\sum_{\ell}\langle f\mathbb{1}_{Q_\ell}\rangle_{Q_\ell^{(1)}}\mathbb{1}_{Q_\ell^{(1)}}(x)\mu(x)\Big)\mathbb{1}_{Q_j}\\
=&\Big(\frac{1}{\mu(Q_j^{(1)})}\sum_{x\in Q_j^{(1)}}\sum_{\ell:Q_\ell^{(1)}\subsetneq Q_j^{(1)}}\langle f\mathbb{1}_{Q_\ell}\rangle_{Q_\ell^{(1)}}\mathbb{1}_{Q_\ell^{(1)}}(x)\mu(x)\\
&+\frac{1}{\mu(Q_j^{(1)})}\sum_{x\in Q_j^{(1)}}\sum_{\ell:Q_j^{(1)}\subseteq Q_\ell^{(1)}}\langle f\mathbb{1}_{Q_\ell}\rangle_{Q_\ell^{(1)}}\mathbb{1}_{Q_\ell^{(1)}}(x)\mu(x)\Big)\mathbb{1}_{Q_j}\\
=&\Big(\frac{1}{\mu(Q_j^{(1)})}\sum_{\ell:Q_\ell^{(1)}\subsetneq Q_j^{(1)}}\mu(Q_\ell^{(1)})\langle f\mathbb{1}_{Q_\ell}\rangle_{Q_\ell^{(1)}}\\
&+\frac{1}{\mu(Q_j^{(1)})}\sum_{\ell:Q_j^{(1)}\subseteq Q_\ell^{(1)}}\mu(Q_j^{(1)})\langle f\mathbb{1}_{Q_\ell}\rangle_{Q_\ell^{(1)}}\Big)\mathbb{1}_{Q_j}\\
=&\frac{\sum_{x\in A_j} f(x)\mu(x)}{\mu(Q_j^{(1)})}\mathbb{1}_{Q_j}+\Big(\sum_{\ell:Q_j^{(1)}\subseteq Q_\ell^{(1)}}\langle f\mathbb{1}_{Q_\ell}\rangle_{Q_\ell^{(1)}}\Big)\mathbb{1}_{Q_j},
\end{aligned}$$

where

$$A_j = \bigcup_{\ell:Q_\ell^{(1)}\subsetneq Q_j^{(1)}} Q_\ell.$$

By the definition and disjointness of the Calderón-Zygmund cubes, we have

$$g_2\mathbb{1}_{Q_j} = \Big(\sum_\ell \langle f\mathbb{1}_{Q_\ell}\rangle_{Q_\ell^{(1)}}\mathbb{1}_{Q_\ell^{(1)}}\Big)\mathbb{1}_{Q_j} = \Big(\sum_{\ell:Q_j^{(1)}\subseteq Q_\ell^{(1)}}\langle f\mathbb{1}_{Q_\ell}\rangle_{Q_\ell^{(1)}}\Big)\mathbb{1}_{Q_j}.$$

Hence,

$$\begin{aligned}
\frac{1}{\mu(Q_j)}\sum_{x\in Q_j}|g_2(x)-\langle g_2\rangle_{Q_j^{(1)}}|\mu(x) =& \frac{\sum_{x\in A_j} f(x)\mu(x)}{\mu(Q_j^{(1)})}\\
=&\frac{1}{\mu(Q_j^{(1)})}\sum_{\ell:Q_\ell^{(1)}\subsetneq Q_j^{(1)}}\sum_{x\in Q_\ell} f(x)\mu(x)\\
\le&\frac{1}{\mu(Q_j^{(1)})}\sum_{x\in Q_j^{(1)}} f(x)\mu(x)\\
=&\langle f\rangle_{Q_j^{(1)}} \le \lambda.
\end{aligned}$$

Now if Q is not a selected interval, we similarly obtain

$$\langle g_2\rangle_{Q^{(1)}}\mathbb{1}_Q = \frac{\sum_{x\in A_Q} f(x)\mu(x)}{\mu(Q^{(1)})}\mathbb{1}_Q + \Big(\sum_{\ell:Q^{(1)}\subseteq Q_\ell^{(1)}}\langle f\mathbb{1}_{Q_\ell}\rangle_{Q_\ell^{(1)}}\Big)\mathbb{1}_Q,$$

where

$$A_Q = \bigcup_{\ell : Q_\ell^{(1)} \subsetneq Q^{(1)}} Q_\ell.$$

In this case it is possible that $Q = Q_\ell^{(1)}$ for some ℓ, so that

$$g_2 \mathbb{1}_Q = \Big(\sum_{\ell : Q^{(1)} \subseteq Q_\ell^{(1)}} \langle f \mathbb{1}_{Q_\ell} \rangle_{Q_\ell^{(1)}} \Big) \mathbb{1}_Q + \Big(\sum_{\ell : Q = Q_\ell^{(1)}} \langle f \mathbb{1}_{Q_\ell} \rangle_{Q_\ell^{(1)}} \Big) \mathbb{1}_Q.$$

Therefore,

$$\begin{aligned}
&\frac{1}{\mu(Q)} \sum_{x \in Q} |g_2(x) - \langle g_2 \rangle_{Q^{(1)}}| \mu(x) \\
&= \frac{1}{\mu(Q)} \sum_{x \in Q} \Big| \sum_{\ell : Q = Q_\ell^{(1)}} \langle f \mathbb{1}_{Q_\ell} \rangle_{Q_\ell^{(1)}} - \frac{1}{\mu(Q^{(1)})} \sum_{x \in A_Q} f(x) \mu(x) \Big| \mu(x) \\
&\leq \sum_{\ell : Q = Q_\ell^{(1)}} \langle f \mathbb{1}_{Q_\ell} \rangle_{Q_\ell^{(1)}} + \frac{\sum_{x \in A_Q} f(x) \mu(x)}{\mu(Q^{(1)})}.
\end{aligned}$$

The second summand is bounded by λ as in the previous case, while as for the first

$$\begin{aligned}
\sum_{\ell : Q = Q_\ell^{(1)}} \langle f \mathbb{1}_{Q_\ell} \rangle_{Q_\ell^{(1)}} &= \frac{1}{\mu(Q)} \sum_{\ell : Q = Q_\ell^{(1)}} \sum_{x \in Q_\ell} f(x) \mu(x) \\
&\leq \frac{1}{\mu(Q)} \sum_{x \in Q} f(x) \mu(x) \\
&= \langle f \rangle_Q \leq \lambda,
\end{aligned}$$

so that $\|g\|_{BMO} \lesssim \lambda$. The L^2-estimate $\|g\|_2^2 \lesssim \lambda \|f\|_1$ follows by interpolation (see Ch.5, [50]). Finally, by Proposition 1.3.3, $\mathcal{M}_\mathcal{D}$ is of weak-type $(1,1)$ which means that

$$\lambda \mu\Big(\Big\{ x \in \mathfrak{X} : \mathcal{M}_\mathcal{D} f(x) > \lambda \Big\} \Big) \lesssim \|f\|_1$$

and 4) holds. □

Remark 3.1.2. The selected family $\{Q_j\}_j$ is usually referred to as the *Calderón-Zygmund cubes*. Under the same hypothesis as the previous theorem, for any disjoint family $\{R_j\}_j \subseteq \mathcal{D}$ such that

$$\Big\{ \mathcal{M}_\mathcal{D} f > \lambda \Big\} \subseteq \bigcup_j R_j$$

and $R_j^{(1)} \not\subseteq \bigcup_\ell R_\ell$ for any j, if we define b_j, b and g as in the theorem, then statements 1), 2) and 3) hold true.

The operators under study are of the form

$$Tf(x) = \sum_{y \in \mathfrak{X}} K(x,y) f(y) \mu(y),$$

for some kernel $K\colon \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$ which satisfies $K(x,y) = K(y,x)$ for all $x, y \in \mathfrak{X}$. Analogously to the definition given in Chapter 2, K is said to satisfy Hörmander's condition if

$$\sup_{Q \in \mathcal{D}} \sup_{x,y \in Q} \sum_{z \in \mathfrak{X} \setminus Q} |K(z,x) - K(z,y)| \mu(z) < \infty. \tag{3.1}$$

While in the doubling setting, requiring K to satisfy Hörmander's condition was sufficient to obtain the usual endpoint bounds, in this case an additional condition on K is necessary. The following estimate can be viewed as an integral size condition of the same flavor of that in [18] and already present in [46]:

$$\sup_{Q \in \mathcal{D}} \sup_{x \in Q} \sum_{z \in Q^{(1)} \setminus Q} |K(x,z)| \mu(z) < \infty. \tag{3.2}$$

We will say that K satisfies the size condition if it satisfies (3.2).

3.1.1 Weak-type $(1,1)$ estimates

We can now state the nondoubling theorem for the weak-type $(1,1)$ boundedness.

Theorem 3.1.3. *Let $(\mathfrak{X}, \mu)$ be a general tree equipped with any measure. Let T be an integral operator with symmetric kernel $K\colon \mathfrak{X} \times \mathfrak{X} \to \mathbb{C}$. If T is bounded on $L^2(\mathfrak{X})$ and K satisfies* (3.1) *and* (3.2)*, then T is of weak-type* $(1,1)$*. By interpolation and duality $\mathcal{P}$ is also bounded on $L^p(\mathfrak{X})$ for all $p \in (1, \infty)$.*

Proof. Let $f \in L^1(\mathfrak{X})$, $f \geq 0$. If $\mu(\mathfrak{X}) < \infty$ and $0 < \lambda \leq \|f\|_1 / \mu(\mathfrak{X})$, then

$$\lambda \mu \left(\{ x \in \mathfrak{X} : |Tf(x)| > \lambda \} \right) \leq \lambda \mu(\mathfrak{X}) \leq \|f\|_1,$$

and there is nothing to prove. Otherwise, if $\mu(\mathfrak{X}) = \infty$ fix $\lambda > 0$ and if $\mu(\mathfrak{X}) < \infty$ fix $\lambda > \|f\|_1 / \mu(\mathfrak{X})$. We apply Theorem 3.1.1 to the Calderón-Zygmund cubes $\{Q_j\}_j$, that is, to the smallest disjoint cubes in $\mathcal{D}$ that cover $\{\mathcal{M}_{\mathcal{D}} f > \lambda\}$. Denote $\Omega_\lambda = \cup_j Q_j$ and write

$$\begin{aligned} \mu\Big(\Big\{x \in \mathfrak{X} : |Tf(x)| > \lambda\Big\}\Big) &\leq \mu(\Omega_\lambda) + \mu\Big(\Big\{x \in \mathfrak{X} : |Tg(x)| > \frac{\lambda}{2}\Big\}\Big) \\ &\quad + \mu\Big(\Big\{x \in \mathfrak{X} \setminus \Omega_\lambda : |Tb(x)| > \frac{\lambda}{2}\Big\}\Big) \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

The weak-type of $\mathcal{M}_{\mathcal{D}}$ immediately gives

$$\mathrm{I} \lesssim \frac{\|f\|_1}{\lambda}.$$

Chebyshev's inequality, the L^2-boundedness of T, and property (3) of Theorem 3.1.1 yield

$$\mathrm{II} \lesssim \frac{1}{\lambda^2}\|Tg\|_2^2 \lesssim \frac{1}{\lambda^2}\|g\|_2^2 \lesssim \frac{1}{\lambda}\|f\|_1.$$

Finally, by Chebyshev's inequality again,

$$\begin{aligned}\mathrm{III} &\leq \frac{1}{\lambda}\sum_{x\in\mathfrak{X}\setminus\Omega_\lambda}|Tb(x)|\mu(x) \leq \frac{1}{\lambda}\sum_j\sum_{x\in\mathfrak{X}\setminus\Omega_\lambda}|Tb_j(x)|\mu(x)\\ &\leq \frac{1}{\lambda}\sum_j\sum_{x\in\mathfrak{X}\setminus Q_j}|Tb_j(x)|\mu(x) =: \frac{1}{\lambda}\sum_j \mathrm{III}_j.\end{aligned}$$

To estimate each term, we first split into two regions:

$$\begin{aligned}\mathrm{III}_j &= \sum_{x\in\mathfrak{X}\setminus Q_j}\Big|\sum_{y\in Q_j^{(1)}}K(x,y)b_j(y)\mu(y)\Big|\mu(x)\\ &= \sum_{x\in\mathfrak{X}\setminus Q_j^{(1)}}\Big|\sum_{y\in Q_j^{(1)}}K(x,y)b_j(y)\mu(y)\Big|\mu(x) + \sum_{x\in Q_j^{(1)}\setminus Q_j}\Big|\sum_{y\in Q_j^{(1)}}K(x,y)b_j(y)\mu(y)\Big|\mu(x).\end{aligned}$$

For the first one, we use the mean zero of b_j and (3.1):

$$\begin{aligned}\sum_{x\in\mathfrak{X}\setminus Q_j^{(1)}}\Big|\sum_{y\in Q_j^{(1)}}K(x,y)b_j(y)\mu(y)\Big|\mu(x) &= \sum_{x\in\mathfrak{X}\setminus Q_j^{(1)}}\Big|\sum_{y\in Q_j^{(1)}}(K(x,y)-K(x,x_{Q_j}))b_j(y)\mu(y)\Big|\mu(x)\\ &\leq \sum_{y\in Q_j^{(1)}}|b_j(y)|\Big(\sup_{z\in Q_j^{(1)}}\sum_{x\in\mathfrak{X}\setminus Q_j^{(1)}}|K(x,y)-K(x,z)|\,\mu(x)\Big)\mu(y) \lesssim \|b_j\|_1.\end{aligned}$$

For the remaining term, we further use the definition of b_j:

$$\begin{aligned}&\sum_{x\in Q_j^{(1)}\setminus Q_j}\Big|\sum_{y\in Q_j^{(1)}}K(x,y)b_j(y)\mu(y)\Big|\mu(x)\\ &\quad\leq \sum_{x\in Q_j^{(1)}\setminus Q_j}\Big|\sum_{y\in Q_j}K(x,y)b_j(y)\mu(y)\Big|\mu(x) + \sum_{x\in Q_j^{(1)}}\Big|T\Big(\langle f\mathbb{1}_{Q_j}\rangle_{Q_j^{(1)}}\mathbb{1}_{Q_j^{(1)}}\Big)(x)\Big|\mu(x)\\ &\quad=: \alpha_j+\beta_j.\end{aligned}$$

We deal with β_j using Hölder's inequality and the L^2-bound:

$$\beta_j \leq \mu(Q_j^{(1)})^{\frac{1}{2}}\left\|T\left(\langle f\mathbb{1}_{Q_j}\rangle_{Q_j^{(1)}}\mathbb{1}_{Q_j^{(1)}}\right)\right\|_2 \lesssim \mu(Q_j^{(1)})^{\frac{1}{2}}\left\|\langle f\mathbb{1}_{Q_j}\rangle_{Q_j^{(1)}}\mathbb{1}_{Q_j^{(1)}}\right\|_2 = \|f\mathbb{1}_{Q_j}\|_1.$$

For α_j we use Fubini's theorem and (3.2):

$$\alpha_j \leq \sum_{x\in Q_j^{(1)}\setminus Q_j}\sum_{y\in Q_j}|K(x,y)||f(y)|\mu(y)\mu(x)$$

$$\leq \sum_{y\in Q_j} |f(y)|\mu(y) \sup_{y'\in Q_j} \sum_{x\in Q_j^{(1)}\setminus Q_j} |K(x,y')|\mu(x) \lesssim \|f\mathbb{1}_{Q_j}\|_1.$$

Finally, we collect all estimates together and by property (2) of Theorem 3.1.1,

$$\mathrm{III} \lesssim \frac{1}{\lambda}\sum_j \left(\|b_j\|_1 + \|f\mathbb{1}_{Q_j}\|_1\right) \lesssim \frac{1}{\lambda}\|f\|_1,$$

as desired. □

Remark 3.1.4. Notice that the proof of the weak-type $(1,1)$ boundedness does not make use of the fact that K is symmetric.

3.1.2 H^1 and BMO

The goal of this section is to prove endpoint results for general integral operators on trees. When the tree has unbounded geometry, and the resulting metric measure space is nondoubling, new definitions of Hardy and bounded mean oscillation spaces are necessary to address the lack of regularity of the underlying filtration. We refer to [45, 17].

Regarding Hardy spaces, we briefly review the definitions provided in Section 2.1.2, which can be stated in exactly the same way as in the nondoubling case. For $p \in (1,\infty]$, a p-atom is a function $a\colon \mathfrak{X} \to \mathbb{C}$ that is supported in Q for some $Q \in \mathcal{D}$, has zero mean and $\|a\|_p \leq \mu(Q)^{1/p-1}$ if $p < \infty$ and $\|a\|_\infty \leq \mu(Q)^{-1}$ if $p = \infty$. In case the total measure of the tree is finite, then the function $a = \mu(\mathfrak{X})^{-1}$ is said to be a p-atom as well. The atomic Hardy space $H^{1,p}_{\mathrm{at}}(\mathfrak{X})$ is the space of functions $f \in L^1(\mathfrak{X})$ that admit an atomic decomposition of the form $f = \sum_i \lambda_i a_i$, where the a_i are p-atoms and $(\lambda_i)_i$ is a complex summable sequence, and whose norm in $H^{1,p}_{\mathrm{at}}(\mathfrak{X})$ is

$$\|f\|_{H^{1,p}_{\mathrm{at}}} := \inf\Big\{\sum_i |\lambda_i| : f = \sum_i \lambda_i a_i,\ a_i\ p\text{-atoms}\Big\}.$$

The martingale Hardy space $H^1(\mathfrak{X})$ is the space of functions in $L^1(\mathfrak{X})$ such that the norm defined below is finite

$$\|f\|_{H^1(\mathfrak{X})} := \Big\|\Big(\sum_{k=0}^{\infty} |\mathsf{D}_k f|^2\Big)^{1/2}\Big\|_1,$$

while the little Hardy space $h^1(\mathfrak{X})$ is given by $L^1(\mathfrak{X})$ functions whose following norm

$$\|f\|_{h^1(\mathfrak{X})} := \Big\|\Big(\sum_{k=0}^{\infty} \mathsf{E}_{k-1}|\mathsf{D}_k f|^2\Big)^{1/2}\Big\|_1$$

is finite, with the convention that, for $k = 0$, $\mathsf{E}_{k-1}|\mathsf{D}_k f|^2 = |\mathsf{E}_0 f|^2$. As noted in Remark 2.1.8, the equivalence $h^1(\mathfrak{X}) \simeq H^{1,p}_{\mathrm{at}}(\mathfrak{X})$ holds in general for all $p \in (1,\infty]$.

Nevertheless, in this situation the equivalence $H^1(\mathfrak{X}) \simeq h^1(\mathfrak{X})$ no longer holds, and the atomic decomposition previously presented does not apply to $H^1(\mathfrak{X})$. Since $H^1(\mathfrak{X})$, rather than $h^1(\mathfrak{X})$, is the endpoint interpolation space, a new suitable atomic decomposition is required. It is constructed in [17], to which we refer for a comprehensive and exhaustive presentation of the problem.

Definition 3.1.5. Let $p \in (1,\infty]$, a p-atomic block is a function $b\colon \mathfrak{X} \to \mathbb{C}$ such that

1. b is supported in $Q^{(1)}$ for $Q^{(1)} \in \mathcal{D}_k$ for some $k \geq 0$;
2. b has vanishing mean on $Q^{(1)}$, that is
$$\sum_{x \in Q^{(1)}} b(x)\mu(x) = 0;$$
3. $b = \sum_j \lambda_j a_j$, where $(\lambda_j)_j$ is a complex summable sequence and a_j is a p-subatom, namely $\mathrm{supp}(a_j) \subseteq Q_j$ for $Q_j \in \mathcal{D}_{k_j}$ for some $k_j \geq k$ and
$$\|a\|_p \leq \mu(Q)^{\frac{1}{p}-1} \frac{1}{k_j - k + 1} \text{ if } p < \infty,$$
$$\|a\|_\infty \leq \mu(Q)^{-1} \frac{1}{k_j - k + 1} \text{ if } p = \infty.$$

For an atomic block b, define

$$|b|_{\mathrm{ab}}^{1,p} := \inf_{\substack{b = \sum_j \lambda_j a_j \\ a_j\, p-\text{subatoms}}} \sum_{j=1}^{\infty} |\lambda_j|.$$

Definition 3.1.6. Let $p \in (1,\infty]$, the atomic Hardy space $H_{\mathrm{ab}}^{1,p}(\mathfrak{X})$ is the space of functions $f \in L^1(\mathfrak{X})$ such that
$$f = \sum_i b_i,$$
where the b_i are p-atomic blocks. The norm of f in $H_{\mathrm{ab}}^{1,p}(\mathfrak{X})$ is given by

$$\begin{aligned} \|f\|_{H_{\mathrm{ab}}^{1,p}} &:= \inf\Big\{ \sum_{i=1}^{\infty} |b_i|_{\mathrm{at}}^{1,p} : f = \sum_i b_i,\ b_i\ p\text{-atomic blocks} \Big\} \\ &= \inf\Big\{ \sum_{i,j=1}^{\infty} |\lambda_{i,j}| : f = \sum_i b_i,\ b_i = \sum_j \lambda_{i,j} a_j,\ a_j\ p\text{-subatoms} \Big\}. \end{aligned}$$

By Theorem A in [17], for all $p \in (1,\infty]$

$$H^1(\mathfrak{X}) \simeq H_{\mathrm{ab}}^{1,p}(\mathfrak{X}),$$

and we have the desired atomic decomposition. In the following we will use decompositions both with ∞-atomic blocks and 2-atomic blocks.

As we show in the next lemma, atomic blocks can be assumed to adopt a simpler form. We say that a function $b \in L^1(\mathfrak{X})$ is an *algebraic atom* if it is supported in $Q^{(1)}$ for $Q^{(1)} \in \mathcal{D}_k$ for some $k \geq 0$ and has vanishing mean. Of course, atomic blocks are algebraic atoms. An algebraic atom s is said to be a *simple algebraic atom* if there exists a function $a : \mathfrak{X} \to \mathbb{C}$ such that $\operatorname{supp}(a) \subseteq Q$ and

$$s = a - \langle a \rangle_{Q^{(1)}} \mathbb{1}_{Q^{(1)}}.$$

Finally, a *simple p-atomic block* is a function $b : \mathfrak{X} \to \mathbb{C}$ supported in $Q^{(1)} \in \mathcal{D}_k$, with vanishing mean and such that there exists a supported in Q with

$$\|a\|_p \leq \frac{\mu(Q)^{\frac{1}{p}-1}}{2} \text{ if } p < \infty,$$
$$\|a\|_\infty \leq \frac{\mu(Q)^{-1}}{2} \text{ if } p = \infty,$$

and such that

$$b = a - \langle a \rangle_{Q^{(1)}} \mathbb{1}_{Q^{(1)}}.$$

Basically, a simple atomic block is an atomic block which is also a simple algebraic atom.

We prove next lemma for $p = \infty$ and omit it from the notation, but it holds true for every $p \in (1, \infty]$.

Lemma 3.1.7. *Every function $f \in H^1(\mathfrak{X})$ can be written as*

$$f = \sum_i \lambda_i s_i,$$

where the s_i are simple atomic blocks. Moreover, we have

$$\|f\|_{H^1} \simeq \inf_{\substack{f=\sum_i \lambda_i s_i \\ s_i \text{ simple atomic block}}} \sum_{i=1}^{\infty} |\lambda_i| =: \|f\|_{H^1_{\mathrm{sat}}}.$$

Proof. Clearly, $\|f\|_{H^1} \simeq \|f\|_{H^1_{\mathrm{ab}}} \lesssim \|f\|_{H^1_{\mathrm{sat}}}$, for simple atomic blocks are atomic blocks. Indeed, if $b = a - \langle a \rangle_{Q^{(1)}} \mathbb{1}_{Q^{(1)}}$ is a simple atomic block, then $a_1 = a$ and $a_2 = -\langle a \rangle_Q \mathbb{1}_Q$ are subatoms satisfying the right size condition, since

$$\|a_1\|_\infty = \|a\|_\infty \leq \frac{\mu(Q)^{-1}}{2}$$

and

$$\|a_2\|_\infty = |\langle a \rangle_{Q^{(1)}}| \leq \frac{1}{\mu(Q^{(1)})} \|a\|_\infty \mu(Q) \lesssim \mu(Q^{(1)})^{-1}.$$

Therefore, the infimum defining $\|\cdot\|_{H^1_{\mathrm{ab}}}$ ranges over a larger set than that defining $\|\cdot\|_{H^1_{\mathrm{sat}}}$.

Conversely, let b be an atomic block supported on $Q \in \mathcal{D}_k$ with $|b|^1_{\mathrm{at}} = 1$. It is enough to show that $\|b\|_{H^1_{\mathrm{sat}}} \lesssim 1$. Let $b = \sum_i \lambda_i a_i$ be a decomposition of b into subatoms such that $\sum_i |\lambda_i| \simeq 1$. For each i, denote the support of a_i by $Q_i \in \mathcal{D}_{k_i}$ and write

$$a_i = a_i - \langle a_i \rangle_{Q_i^{(1)}} \mathbb{1}_{Q_i^{(1)}} + \sum_{\ell=1}^{k_i-k-1} \Big(\langle a_i \rangle_{Q_i^{(\ell)}} \mathbb{1}_{Q_i^{(\ell)}} - \langle a_i \rangle_{Q_i^{(\ell+1)}} \mathbb{1}_{Q_j^{(\ell+1)}} \Big) + \langle a_i \rangle_Q \mathbb{1}_Q.$$

Since

$$\sum_i \lambda_i \langle a_i \rangle_Q = 0$$

because of the mean 0 of b, we can write

$$\begin{aligned} b = & \sum_i \lambda_i \Big(a_i - \langle a_i \rangle_{Q_i^{(1)}} \mathbb{1}_{Q_i^{(1)}} \Big) \\ & + \sum_i \lambda_i \frac{1}{k_i - k + 1} \sum_{\ell=1}^{k_i-k-1} (k_i - k + 1) \Big(\langle a_i \rangle_{Q_j^{(\ell)}} \mathbb{1}_{Q_i^{(\ell)}} - \langle a_i \rangle_{Q_i^{(\ell+1)}} \mathbb{1}_{Q_i^{(\ell+1)}} \Big), \end{aligned}$$

which we claim is a decomposition of b into simple atomic blocks. Indeed, for each i and $1 \le \ell \le k_i - k + 1$,

$$\Big\| (k_i - k + 1) \langle a_i \rangle_{Q_i^{(\ell)}} \mathbb{1}_{Q_i^{(\ell)}} \Big\|_\infty \le \frac{(k_i - k + 1)\mu(Q_i)}{\mu(Q_i^{(\ell)})} \|a_i\|_\infty \le \frac{1}{\mu(Q_i^{(\ell)})},$$

as required. An entirely similar estimate holds for the terms of the form $a_i - \langle a_i \rangle_{Q_i^{(1)}} \mathbb{1}_{Q_i^{(1)}}$. Therefore,

$$\|b\|_{H^1_{\mathrm{sat}}} \lesssim \sum_i |\lambda_i| \left(1 + \sum_{\ell=1}^{k_i-k-1} \frac{1}{k_i - k + 1} \right) \lesssim \sum_i |\lambda_i| \simeq 1,$$

which concludes the proof. □

Let us now address the *BMO* spaces, which, like the Hardy spaces, will require appropriate modifications to be adapted to the nondoubling case. In a similar way, though not identical, to the definitions provided in Section 2.1.2, we define the *BMO* norm by means of the martingale filtration as follows. If the total measure of the tree is not finite, we say that f belongs to $BMO(\mathfrak{X})$ if

$$\|f\|_{BMO} := \sup_{k \ge 0} \|\mathsf{E}_k |f - \mathsf{E}_{k-1} f|\|_\infty < \infty.$$

while when the total measure is finite the *BMO*-norm is given by

$$\|f\|_{BMO} := \sup_{k \ge 0} \|\mathsf{E}_k |f - \mathsf{E}_{k-1} f|\|_\infty + \|\mathsf{E}_0 f\|_\infty < \infty.$$

Notice that the new term $\|\mathsf{E}_0 f\|_\infty$ is just the absolute value of the mean of f on $\mathfrak{X}$, it only appears in finite measure spaces and it is not concerned with the lack of the doubling property. Conversely, the real difference with respect to the doubling case is due to the fact that the conditional expectations are taken in different generations. This expresses the necessity of taking into account the interactions between dyadic sets and their dyadic parents. Exactly as in Section 2.1.2 one can see that in the case of finite total measure

$$\|f\|_{BMO} \simeq \sup_{Q\in\mathcal{D}} \frac{1}{\mu(Q)} \sum_{x\in Q} \left|f(x) - \langle f\rangle_{Q^{(1)}}\right| \mu(x) + \left|\sum_{x\in\mathfrak{X}} f(x)\mu(x)\right|,$$

and it immediately follows that

$$\begin{aligned} \|f\|_{BMO} \simeq \sup_{Q\in\mathcal{D}} & \frac{1}{\mu(Q)} \sum_{x\in Q} |f(x) - \langle f\rangle_Q|\mu(x) \\ & + \sup_{\substack{Q\in\mathcal{D}\\ Q\neq\mathfrak{X}}} |\langle f\rangle_Q - \langle f\rangle_{Q^{(1)}}| + \left|\sum_{x\in\mathfrak{X}} f(x)\mu(x)\right|, \end{aligned} \tag{3.3}$$

and we just drop the last summand when the total measure is infinite.

Theorem 3.1.8. *Let $(\mathfrak{X}, \mu)$ be a general tree equipped with any measure. Let T be an integral operator with symmetric kernel $K\colon \mathfrak{X}\times\mathfrak{X}\to\mathbb{C}$. If T is bounded on $L^2(\mathfrak{X})$ and K satisfies* (3.1) *and* (3.2)*, then T maps $H^1_{\mathrm{ab}}(\mathfrak{X})$ into $L^1(\mathfrak{X})$.*

Proof. By Lemma 3.1.7 and arguing as in Theorem 2.1.9, it is enough to prove that $\|Tb\|_1 \lesssim 1$ for every simple atomic block b. Fix one such b, so that

$$b = a - \langle a\rangle_{Q^{(1)}} \mathbb{1}_{Q^{(1)}},$$

where $\operatorname{supp}(a) \subseteq Q$ and $\|a\|_\infty \lesssim \mu(Q)^{-1}$. We write

$$\|Tb\|_1 \le \|Tb\|_{L^1(Q^{(1)})} + \|Tb\|_{L^1(\mathfrak{X}\setminus Q^{(1)})},$$

and handle each term in turn. For the first one, we further split

$$\|Tb\|_{L^1(Q^{(1)})} \le \|Ta\|_{L^1(Q)} + \|Ta\|_{L^1(Q^{(1)}\setminus Q)} + |\langle a\rangle_{Q^{(1)}}|\|T\mathbb{1}_{Q^{(1)}}\|_{L^1(Q^{(1)})}.$$

The first term on the right hand side above is dealt with using the L^2-bound of T and the size estimate on a:

$$\|Ta\|_{L^1(Q)} \le \mu(Q)^{\frac12}\|Ta\|_{L^2(Q)} \lesssim \mu(Q)^{\frac12}\|a\|_{L^2(Q)} \le 1.$$

Similarly, for the third term on the right hand side above we obtain

$$|\langle a\rangle_{Q^{(1)}}|\|T\mathbb{1}_{Q^{(1)}}\|_{L^1(Q^{(1)})} \le \|a\|_\infty \frac{\mu(Q)}{\mu(Q^{(1)})}\|T\mathbb{1}_{Q^{(1)}}\|_{L^2(Q^{(1)})}\mu(Q^{(1)})^{\frac12} \lesssim 1,$$

using $\operatorname{supp}(a) \subseteq Q$. The second term on the right hand side above requires (3.2):

$$\begin{aligned}\|Ta\|_{L^1(Q^{(1)}\setminus Q)} &\leq \sum_{x\in Q^{(1)}\setminus Q} \sum_{y\in Q} |K(x,y)||a(y)|\mu(y)\mu(x)\\ &\leq \|a\|_1 \sup_{y\in Q} \sum_{x\in Q^{(1)}\setminus Q} |K(x,y)|\mu(x)\\ &\lesssim \|a\|_1 \leq \|a\|_\infty \mu(Q) \leq 1.\end{aligned}$$

Finally, using (3.1) and the mean zero of b in Q yields

$$\begin{aligned}\|Tb\mathbb{1}_{\mathfrak{X}\setminus Q^{(1)}}\|_1 &= \sum_{x\in\mathfrak{X}\setminus Q^{(1)}} \Big| \sum_{y\in Q^{(1)}} K(x,y)b(y)\mu(y)\Big|\mu(x)\\ &= \sum_{x\in\mathfrak{X}\setminus Q^{(1)}} \Big| \sum_{y\in Q^{(1)}} (K(x,y) - K(x,x_Q^{(1)}))b(y)\mu(y)\Big|\mu(x)\\ &\leq \|b\|_1 \sup_{y\in Q^{(1)}} \sum_{x\in\mathfrak{X}\setminus Q^{(1)}} \big|K(x,y) - K(x,x_Q^{(1)})\big|\mu(x)\\ &\lesssim \|b\|_1 \leq \|b\|_{H^1_{\mathrm{ab}}} \leq 1,\end{aligned}$$

which finishes the proof. □

Theorem 3.1.9. *Let $(\mathfrak{X},\mu)$ be a general tree equipped with any measure. Let T be an integral operator with symmetric kernel $K\colon \mathfrak{X}\times\mathfrak{X}\to\mathbb{C}$. If T is bounded on $L^2(\mathfrak{X})$ and K satisfies* (3.1) *and* (3.2)*, then T maps $L^\infty(\mathfrak{X})$ into $BMO(\mathfrak{X})$.*

Proof. We use the characterization of the BMO-norm given in (3.3). Let $f \in L^\infty(\mathfrak{X})$, we want to bound

$$\sup_{Q\in\mathcal{D}} \frac{1}{\mu(Q)} \sum_{x\in Q} |Tf(x) - \langle Tf\rangle_Q|\mu(x) + \sup_{\substack{Q\in\mathcal{D}\\ Q\neq\mathfrak{X}}} |\langle Tf\rangle_Q - \langle Tf\rangle_{Q^{(1)}}| + \left|\sum_{x\in\mathfrak{X}} Tf(x)\mu(x)\right|$$

by $\|f\|_\infty$ up to a constant. The estimate on the last term is trivial. Consider the first summand and use the kernel representation to write

$$\begin{aligned}&\frac{1}{\mu(Q)} \sum_{x\in Q} |Tf(x) - \langle Tf\rangle_Q|\mu(x)\\ &=\frac{1}{\mu(Q)} \sum_{x\in Q}\Big| \frac{1}{\mu(Q)} \sum_{y\in Q} (Tf(x) - Tf(y))\mu(y)\Big|\mu(x)\\ &\leq \frac{1}{\mu(Q)^2} \sum_{x,y\in Q} |Tf(x) - Tf(y)|\mu(y)\mu(x)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mu(Q)^2} \sum_{x,y\in Q} \sum_{z\in\mathfrak{X}} |(K(x,z)-K(y,z))f(z)|\mu(z)\mu(y)\mu(x) \\
&\le \sup_{x,y\in Q} \sum_{z\in\mathfrak{X}\setminus Q} |K(x,z)-K(y,z)||f(z)|\mu(z) \\
&\quad + \frac{1}{\mu(Q)^2} \sum_{x,y\in Q} \sum_{z\in Q} |K(x,z)f(z)|\mu(z)\mu(y)\mu(x) \\
&\quad + \frac{1}{\mu(Q)^2} \sum_{x,y\in Q} \sum_{z\in Q} |K(y,z)f(z)|\mu(z)\mu(y)\mu(x) \\
&= \sup_{x,y\in Q} \sum_{z\in\mathfrak{X}\setminus Q} |K(x,z)-K(y,z)||f(z)|\mu(z) + 2\langle |T(f\mathbb{1}_Q)|\rangle_Q =: \mathrm{I} + 2\mathrm{II}.
\end{aligned}
$$

The estimate $\mathrm{I} \lesssim \|f\|_\infty$ follows directly from (3.1). Next, by the L^2-bound for T

$$
\mathrm{II} \le \mu(Q)^{-\frac{1}{2}} \|Tf\|_{L^2(Q)} \lesssim \mu(Q)^{-\frac{1}{2}} \|f\|_{L^2(Q)} \lesssim \|f\|_\infty,
$$

and the first term is bounded. On the other hand, for each $Q \in \mathcal{D}$ we have

$$
\begin{aligned}
&\left| \langle Tf\rangle_Q - \langle Tf\rangle_{Q^{(1)}} \right| \\
&\quad = \frac{1}{\mu(Q)\mu(Q^{(1)})} \left| \mu(Q^{(1)}) \sum_{x\in Q} Tf(x)\mu(x) - \mu(Q) \sum_{z\in Q^{(1)}} Tf(z)\mu(z) \right| \\
&\quad = \frac{1}{\mu(Q)\mu(Q^{(1)})} \left| \sum_{\substack{x\in Q \\ z\in Q^{(1)}}} \sum_{y\in\mathfrak{X}} (K(x,y)-K(z,y))f(y)\mu(y)\mu(x)\mu(z) \right| \\
&\quad \le \sup_{\substack{x\in Q \\ z\in Q^{(1)}}} \sum_{y\in\mathfrak{X}\setminus Q^{(1)}} |K(x,y)-K(z,y))|\,|f(y)|\mu(y) \\
&\qquad + \langle |T(f\mathbb{1}_Q)|\rangle_Q + \langle |T(f\mathbb{1}_{Q^{(1)}})|\rangle_{Q^{(1)}} \\
&\qquad + \sup_{x\in Q} \sum_{y\in Q^{(1)}\setminus Q} |K(x,y)||f(y)|\mu(y) \\
&\quad =: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}
$$

The estimates for I and II are the same as for the first summand, and the estimate for III is similar to that for II. Finally, (3.2) implies $\mathrm{IV} \lesssim \|f\|_\infty$ and completes the proof. □

3.2 Example on radial trees: the Bergman projection

Until the end of this Chapter, and throughout the next one, the tree $\mathfrak{X}$ is assumed to be radial. Similarly to what was seen in Chapter 2, we can define the combinatorial Laplacian of a function defined on any tree as

$$\Delta f(x) = \frac{1}{q(x)+1} \sum_{y \sim x} f(y) - f(x)$$

and the harmonic functions are those for which the Laplacian is identically zero. The harmonic Bergman spaces with respect to the Bergman measures introduced in Section 1.1.3 (and that we omit from the notation) are defined, for $p \in [1, \infty)$ as

$$\mathcal{B}^p(\mathfrak{X}) = \Big\{ f \in L^p(\mathfrak{X}) : \; f \text{ is harmonic} \Big\}.$$

We are interested in studying the Hilbert space $\mathcal{B}^2(\mathfrak{X})$ and in particular the behavior of the Bergman projection $\mathcal{P} : L^2(\mathfrak{X}) \to \mathcal{B}^2(\mathfrak{X})$ in the case of radial trees. The approach used in this context is different from that employed in the previous Chapter. Indeed, instead of computing an explicit formula for the kernel, we exhibit an orthonormal basis for the Bergman space $\mathcal{B}^2(\mathfrak{X})$. Then, we exploit it to achieve many good estimates on the reproducing kernel of $\mathcal{B}^2(\mathfrak{X})$, that coincides with the kernel of the projection. These estimates allow to prove that Hörmander's condition (3.1) and the size condition (3.2) are satisfied, as well as to prove sharper endpoint results for $\mathcal{P}$. They will also be particularly useful in Chapter 4, that is entirely dedicated to the Bergman projection.

3.2.1 An orthonormal basis for $\mathcal{B}^2(\mathfrak{X})$

In order to find an orthonormal basis for $\mathcal{B}^2(\mathfrak{X})$, we start defining a family of auxiliary functions φ_k, which are in turn characterized by means of some essential coefficients ν_j^k. Concerning harmonic functions on radial trees, it is possible to recover results equivalent to the ones stated in Subsection 2.2.1, and that lead to the construction of the functions φ_k.

Lemma 3.2.1. *Let $y \in \mathfrak{X}$. If $f : \mathfrak{X} \to \mathbb{C}$ is harmonic on T_y, then for all $n \in \mathbb{N}$*

$$\sum_{\substack{x \in T_y \\ |x| = |y|+n}} f(x) = \Big(\sum_{j=0}^{n} \prod_{k=1}^{j} q(|y|+n-k) \Big) f(y) - \Big(\sum_{j=0}^{n-1} \prod_{k=1}^{j} q(|y|+n-k) \Big) f(y^{(1)}). \quad (3.4)$$

Conversely, if f is constant on $T_y \cap \{x : |x| = |y| + k\}$ and satisfies (3.4) *for all $k \in \mathbb{N}$, then f is harmonic on T_y.*

Proposition 3.2.2. *Let $g\colon \mathfrak{X} \to \mathbb{C}$ be null and harmonic on $B(o,n)$. The harmonic extension of g at level n is*

$$g_n^H(x) = \begin{cases} g(x), & \text{if } x \in B(o,n+1), \\ a_n(|x|)g(y), & \text{if } x \in \mathfrak{X} \setminus B(o,n+1), \end{cases}$$

where $y = x^{(|x|-n-1)}$ and

$$a_n(|x|) = \sum_{j=0}^{|x|-n-1} \prod_{j=1}^{k} q(n+j)^{-1}.$$

The function g_n^H is harmonic, coincides with g on $B(o,n+1)$ and is radial on T_y for every $y \in S(o,n+1)$.

The proofs follow from the proofs of Lemma 2.2.2 and Proposition 2.2.3, respectively, just by adapting them to the radial context.

For each $k \geq 0$, we denote by φ_k the only radial function on $\mathfrak{X}$ such that $\varphi_k(x) = 0$ if $|x| < k$, $\varphi_k(x) = 1$ if $|x| = k$ and is harmonic at all points x such that $|x| > k$. The values of φ_k for $|x| \geq k$ are given by the explicit formula

$$\varphi_k(x) = \varphi_k(|x|) = \sum_{\ell=1}^{|x|-k+1} \nu_k^{|x|-\ell}, \tag{3.5}$$

where the coefficients are given by

$$\nu_k^j := \prod_{\ell=k}^{j} \frac{1}{q(\ell)}, \quad j \geq k.$$

To see that φ_k is harmonic for $|x| > k$, notice that

$$\begin{aligned} \varphi_k(|x|+1) &= \sum_{\ell=1}^{|x|-k+2} \nu_k^{|x|-\ell+1} = \nu_k^{|x|} + \sum_{\ell=2}^{|x|-k+2} \nu_k^{|x|-\ell+1} \\ &= \nu_k^{|x|} + \sum_{\ell=1}^{|x|-k+1} \nu_k^{|x|-\ell} = \nu_k^{|x|} + \varphi_k(|x|), \end{aligned}$$

that also implies

$$\varphi_k(|x|-1) = \varphi_k(|x|) - \nu_k^{|x|-1}.$$

Therefore

$$\begin{aligned} &\varphi_k(|x|-1) + q(|x|)\varphi_k(|x|+1) \\ &\quad = \varphi_k(|x|) - \nu_k^{|x|-1} + q(|x|)\varphi_k(|x|) + q(|x|)\nu_k^{|x|} \\ &\quad = \varphi_k(|x|)(1 + q(|x|)) \end{aligned}$$

that is to say that φ_k is harmonic in x.

Remark 3.2.3. It is just a matter of computation to check that $\varphi_k(|x|) = a_{k-1}(|x|)$. We chose to rephrase these functions in this way for convenience of notation and to emphasize the importance of the coefficients ν_k^j involved. Indeed, the function φ_k encodes, through the quantities ν_k^j, the key decay properties of the kernel studied in the next section. We adopt the convention that $\nu_k^j = 1$ if $k > j$. Many times, the trivial estimate

$$\nu_k^j \leq 2^{-(j-k+1)}$$

will be enough for our purposes, but the unboundedness of q will often force us to use the larger decay captured by ν_k^j. We will sometimes use the multiplicative identity

$$\nu_k^j = \nu_k^m \cdot \nu_{m+1}^j \text{ for } k \leq m \leq j.$$

Formula (3.5) immediately implies that $1 \leq \varphi_k(x) \leq 2$ for all $x \in \mathrm{supp}(\varphi_k)$, a fact that we will use repeatedly.

Given a sector $Q \in \mathcal{D}_{k-1}$ and a function $e : Q \cap \{|x| = k\} \to \mathbb{R}$ with mean zero, consider the function

$$E_Q(e)(x) = \begin{cases} \varphi_k(x)e(y) & \text{if } x \in T_y \text{ for some } y \in S(x_Q), \\ 0 & \text{otherwise,} \end{cases}$$

that is radial and harmonic on $\mathfrak{X}$. Indeed, if we look at e as a function on the whole of $\mathfrak{X}$, then $E_Q(e)$ coincides with its harmonic extension (as it was defined in Proposition 3.2.2) at level $k-1$. For all $x \in \mathfrak{X}$ let denote $Q = T_x$, consider the space

$$\begin{aligned} Z_Q :=& \Big\{ e\colon S(x) \to \mathbb{C} : \sum_{y \in S(x)} e(y) = 0 \Big\} \\ \simeq& \Big\{ v = (v_j) \in \mathbb{C}^{q(x)} : \sum_j v_j = 0 \Big\} \simeq \mathbb{C}^{q(x)-1} \end{aligned}$$

and pick an orthonormal basis $\{e^\ell\}_{\ell=1}^{q(x)-1}$ of Z_Q. The spaces Z_Q are clearly the analogue of the spaces Z_v introduced in Section 2.2.3.

Let $S(x) = \{y_j : j = 1, \ldots, q(x)\}$ and, for each $\ell = 1, \ldots, q(x) - 1$, define the function

$$e_Q^\ell : S(x) \to \mathbb{C}, \quad e_Q^\ell(y_j) = e_j^\ell.$$

Finally, for each $x \in \mathfrak{X}$, $\ell = 1, \ldots, q(x) - 1$ and $Q = T_x$ set

$$h_Q^\ell(z) = \frac{E_Q(e_Q^\ell)(z)}{\|E_Q(e_Q^\ell)\|_2}, \quad z \in \mathfrak{X}.$$

When $Q \in \mathcal{D}$ is a singleton, define by convention $h_Q \equiv 0$.

Lemma 3.2.4. *For every sector $Q \in \mathcal{D}_{k-1}$ and every $\ell = 1, \dots, q(x_Q) - 1$, the following properties hold*

1. $E_Q(e_Q^\ell)$ *and* h_Q^ℓ *have zero mean;*
2. $\|E_Q(e_Q^\ell)\|_2 = \|\varphi_k \mathbb{1}_Q\|_2 / q(x_Q)$.

Proof. Let Q be a sector and $\ell \in \{1, \dots, q(x_Q) - 1\}$,

$$\begin{aligned}\sum_{x\in\mathfrak{X}} E_Q(e_Q^\ell)(x)\mu(x) &= \sum_{x\in Q\setminus\{x_Q\}} E_Q(e_Q^\ell)(x)\mu(x) \\ &= \sum_{x\in S(x_Q)} e_Q^\ell(x) \sum_{y\in T_x} \varphi_k(y)\mu(y) \\ &=: C_k^{(1)} \sum_{x\in S(x_Q)} e_Q^\ell(x) = 0,\end{aligned}$$

and clearly the same holds for h_Q^ℓ. Moreover,

$$\begin{aligned}\|E_Q(e_Q^\ell)\|_2^2 &= \sum_{x\in S(x_Q)} \sum_{y\in T_x} |\varphi_k(y) e_Q^\ell(x)|^2 \mu(y) \\ &= \sum_{x\in S(x_Q)} |e_Q^\ell(x)|^2 \sum_{y\in T_x} \varphi_k(y)^2 \mu(y) \\ &= \sum_{x\in S(x_Q)} |e_Q^\ell(x)|^2 \|\varphi_k \mathbb{1}_{T_x}\|_2^2 \\ &= \|\varphi_k \mathbb{1}_{T_x}\|_2^2 \\ &= \Big(\frac{\|\varphi_k \mathbb{1}_Q\|_2}{q(x_Q)}\Big)^2\end{aligned}$$

since $\varphi_k(x_Q) = 0$ and $\|\varphi_k \mathbb{1}_{T_x}\|_2^2 = \|\varphi_k \mathbb{1}_{T_{x'}}\|_2^2$ for all $x, x' \in S(x_Q)$. □

Theorem 3.2.5. *The family*

$$\mathcal{U} = \big\{h_Q^\ell : \quad Q \in \mathcal{D}, \ell = 1, \dots, q(x_Q) - 1\big\} \cup \left\{\frac{\mathbb{1}_{\mathfrak{X}}}{\mu(\mathfrak{X})}\right\}$$

is an orthonormal basis of $\mathcal{B}^2(\mathfrak{X})$.

Proof. The functions under consideration are normalized in $L^2(\mathfrak{X})$, so that we have to check orthogonality and completeness. First, if $Q \in \mathcal{D}_{k-1}$ is a sector and $\ell \neq \ell'$ then

$$\begin{aligned}\langle h_Q^\ell, h_Q^{\ell'}\rangle &= \sum_{x\in S(x_Q)} \sum_{y\in T_x} e_Q^\ell(x) e_Q^{\ell'}(x) \varphi_k(y)^2 \mu(y) \\ &= \sum_{x\in S(x_Q)} e_Q^\ell(x) e_Q^{\ell'}(x) \sum_{y\in T_x} \varphi_k(y)^2 \mu(y)\end{aligned}$$

$$=: C_k^{(2)} \sum_{x\in S(x_Q)} e_Q^\ell(x) e_Q^{\ell'}(x)$$
$$= C_k^{(2)} \sum_{j=1}^{q(x_Q)} e_j^\ell e_j^{\ell'} = 0,$$

by the orthogonality of the vectors that generate h_Q^ℓ and $h_Q^{\ell'}$. Second, since $\mathrm{supp}(h_Q^\ell) \subseteq Q$ for all Q, we have that $\langle h_Q^\ell, h_R^{\ell'}\rangle \neq 0$ if and only if Q and R are different and contained one into the other, say $Q \subsetneq R$. Therefore, the following computation for $\mathcal{D}_{r-1} \ni Q \subsetneq R \in \mathcal{D}_{k-1}$ deals with all the remaining cases:

$$\langle h_Q^\ell, h_R^{\ell'}\rangle = \sum_{x\in S(x_Q)} \sum_{y\in T_x} e_Q^\ell(x) e_R^{\ell'}(x_0) \varphi_r(y)\varphi_k(y)\mu(y)$$
$$= e_R^{\ell'}(x_0) \sum_{x\in S(x_Q)} e_Q^\ell(x) \sum_{y\in T_x} \varphi_r(y)\varphi_k(y)\mu(y)$$
$$=: C_{k,r}\, e_R^{\ell'}(x_0) \sum_{x\in S(x_Q)} e_Q^\ell(x) = 0.$$

Above, the point $x_0 \in S(x_R)$ is the only one such that $Q \subseteq T_{x_0}$. Finally, all functions h_Q^ℓ have mean zero by 1 of Lemma 3.2.4 so they are orthogonal to constants.

It remains to show that $\mathcal{U}$ generates the whole space. Suppose that $f \in \mathcal{B}^2(\mathfrak{X})$ is orthogonal to every element of $\mathcal{U}$. In particular, f is orthogonal to constants and this implies that $f(o) = 0$. Indeed,

$$0 = \langle f, \mathbb{1}_{\mathfrak{X}}\rangle = \sum_{x\in\mathfrak{X}} f(x)\mu(x) = \sum_{n=0}^{\infty} \mu(n) \sum_{x\in S(o,n)} f(x) = f(o) \sum_{n=0}^{\infty} \mu(n),$$

where we used the fact that $\sum_{x\in S(o,n)} f(x) = f(o)$, which follows from Lemma 3.2.1 for $n = 0$. We assume by induction that $f(x) = 0$ if $|x| < n$ and we prove that $f(y) = 0$ for all y with $|y| = n$. Fix x with $|x| = n-1$ and denote $Q = T_x$. Since $f(x) = f(x^{(1)}) = 0$ and f is harmonic in x, we have

$$\sum_{y\in S(x)} f(y) = 0.$$

But then, for all ℓ

$$0 = \langle h_Q^\ell, f\rangle = \sum_{y\in S(x)} \sum_{z\in T_y} e_Q^\ell(y) f(z)\varphi_n(z)\mu(z)$$
$$= \sum_{y\in S(x)} e_Q^\ell(y) \sum_{z\in T_y} f(z)\varphi_n(z)\mu(z)$$

$$
\begin{aligned}
&= \sum_{y\in S(x)} e_Q^\ell(y) \sum_{j=0}^{\infty} \sum_{\substack{z\in T_y \\ |z|=|y|+j}} f(z)\varphi_n(z)\mu(z) \\
&= \sum_{y\in S(x)} e_Q^\ell(y) \sum_{j=0}^{\infty} \varphi_n(|y|+j)\mu(|y|+j) \sum_{\substack{z\in T_y \\ |z|=|y|+j}} f(z) \\
&= \sum_{y\in S(x)} e_Q^\ell(y) \sum_{j=1}^{\infty} K_{j,n} f(y) \\
&= K_n \sum_{y\in S(x)} e_Q^\ell(y) f(y),
\end{aligned}
$$

using Lemma 3.2.1. This implies that $f(y) = 0$ for all $y \in S(x)$. □

The previous result uses a key property of the functions h_Q^ℓ, namely that they are not only orthogonal to constants, but they are orthogonal to all radial functions and they themselves behave like radial functions in almost all sectors. Moreover, we strongly use the radiality of the measure and of the functions φ_k.

3.2.2 Kernel estimates

Since pointwise evaluation is a bounded functional, $\mathcal{B}^2(\mathfrak{X})$ is a closed subspace of $L^2(\mathfrak{X})$, as in the continuous case and it is actually a reproducing kernel Hilbert space with kernel $\mathcal{K}$. The Bergman projector $\mathcal{P} : L^2(\mathfrak{X}) \to \mathcal{B}^2(\mathfrak{X})$ is an integral operator whose kernel coincides with $\mathcal{K}$. It would be possible to find a formula for $\mathcal{K}$, as we did in the homogeneous context, but in this case we rather exploit the explicit construction of the orthonormal basis. Indeed, because of Theorem 3.2.5, $\mathcal{K}$ can be written as

$$
\mathcal{K}(x,y) = \frac{1}{\mu(\mathfrak{X})^2}\mathbb{1}_{\mathfrak{X}}(x)\mathbb{1}_{\mathfrak{X}}(y) + \sum_{Q\in\mathcal{D}} \sum_{\ell=1}^{q(x_Q)-1} h_Q^\ell(x) h_Q^\ell(y)
$$

and $\mathcal{P}$ has the explicit expression

$$
\mathcal{P}f(x) = \frac{1}{\mu(\mathfrak{X})^2}\langle f\rangle_{\mathfrak{X}}\mathbb{1}_{\mathfrak{X}}(x) + \sum_{Q\in\mathcal{D}} \sum_{\ell=1}^{q(x_Q)-1} \langle f, h_Q^\ell\rangle h_Q^\ell(x).
$$

The choice to work with $\mathcal{K}$ adopting this different point of view not only eases our notation in the sequel, but it also encodes the right way of manipulating the pieces of the kernel of $\mathcal{P}$ that we introduce in the following definition. For each $Q \in \mathcal{D}$ that is not a singleton, we set

$$
\mathcal{D}_Q f(x) := \sum_{\ell=1}^{q(x_Q)-1} \langle f, h_Q^\ell\rangle h_Q^\ell(x),
$$

while when Q is a singleton we set $\mathcal{D}_Q f = 0$. By the definition and the properties of the functions h_Q^ℓ, for any f we have $\operatorname{supp}(\mathcal{D}_Q f) \subseteq Q \setminus \{x_Q\}$ and $\mathcal{D}_Q f$ has vanishing mean. We will sometimes work with truncated versions of $\mathcal{P}$, which we denote

$$\mathcal{P}^R f := \sum_{\substack{T\in\mathcal{D}\\ T\subseteq R}} \mathcal{D}_T f,$$

in the case of the localized truncation inside $R \in \mathcal{D}$, and for a disjoint family $\mathcal{F} \subseteq \mathcal{D}$

$$\mathcal{P}_{\mathcal{F}} f := \frac{1}{\mu(\mathfrak{X})^2}\langle f\rangle_{\mathfrak{X}} \mathbb{1}_{\mathfrak{X}} + \sum_{\substack{T\in\mathcal{D}\\ T\not\subseteq R \text{ if } R\in\mathcal{F}}} \mathcal{D}_T f.$$

We next collect two key estimates involving the operators $\mathcal{D}_R$ that will be useful in the following. By definition, $\mathcal{D}_R$ is an integral operator with kernel $\mathcal{K}_R$ described by

$$\mathcal{K}_R(x,y) = \sum_{j=1}^{q(x_R)-1} h_R^j(x) h_R^j(y).$$

On the other hand, the space Z_R introduced above is a reproducing kernel Hilbert space, whose kernel is clearly given by

$$k_R(w,w') = \sum_{j=1}^{q(x_R)-1} e_R^j(w) e_R^j(w').$$

For $R \in \mathcal{D}_{k-1}$, by 2 of Lemma 3.2.4 we have

$$\begin{aligned}
\mathcal{K}_R(x,y) &= \sum_{j=1}^{q(x_R)-1} h_R^j(x) h_R^j(y) \\
&= \frac{\varphi_k(x)\varphi_k(y)}{\|E_R(e_R^j)\|_2^2} \sum_{j=1}^{q(x_R)-1} e_R^j(x^{(|x|-|x_R|-1)}) e_R^j(y^{(|y|-|x_R|-1)}) \\
&= \frac{q(x_R)\varphi_k(x)\varphi_k(y)}{\|\varphi_k \mathbb{1}_R\|_2^2} \sum_{j=1}^{q(x_R)-1} e_R^j(x^{(|x|-|x_R|-1)}) e_R^j(y^{(|y|-|x_R|-1)}) \\
&= \frac{q(x_R)\varphi_k(x)\varphi_k(y)}{\|\varphi_k \mathbb{1}_R\|_2^2} k_R(x^{(|x|-|x_R|-1)}, y^{(|y|-|x_R|-1)}).
\end{aligned} \tag{3.6}$$

Finally, an analogous computation to the one performed in Subsection 2.2.3, shows that k_R is uniquely determined by

$$k_R(w,w') = \begin{cases} -\dfrac{1}{q(x_R)}, & \text{if } w \neq w', \\ 1 - \dfrac{1}{q(x_R)}, & \text{if } w = w'. \end{cases} \qquad (w,w') \in Z_R \times Z_R. \tag{3.7}$$

Lemma 3.2.6. *Let $k \geq 0$. If x, y are such that $|x| \geq |y| = m \geq k$, then*

$$\varphi_k(x) - \varphi_k(y) \lesssim \nu_k^m.$$

Proof. Indeed,

$$\begin{aligned}
\varphi_k(x) - \varphi_k(y) &= \sum_{\ell=1}^{|x|-k+1} \nu_k^{|x|-\ell} - \sum_{\ell=1}^{|y|-k+1} \nu_k^{|y|-\ell} \\
&= \sum_{\ell=|x|-|y|+1}^{|x|-k+1} \nu_k^{|x|-\ell} - \sum_{\ell=1}^{|y|-k+1} \nu_k^{|y|-\ell} + \sum_{\ell=1}^{|x|-|y|} \nu_k^{|x|-\ell} \\
&= \nu_k^m \sum_{\ell=1}^{|x|-|y|} \nu_{m+1}^{|x|-\ell} \lesssim \nu_k^m,
\end{aligned}$$

using $\nu_{m+1}^{|x|-\ell} \lesssim 2^{-|x|+\ell+m}$. □

Lemma 3.2.7. *If $Q \in \mathcal{D}_k$ is not a single point, the following two estimates hold:*

1. *For all $\ell \geq 1$ and every $z \in Q^{(\ell)}$ we have*

$$\sup_{x,y \in Q} |\mathcal{K}_{Q^{(\ell)}}(z,x) - \mathcal{K}_{Q^{(\ell)}}(z,y)| \lesssim \begin{cases} \dfrac{1}{\mu(Q^{(\ell-1)})} \nu_{k-\ell+1}^k, & \text{if } z \in Q^{(\ell-1)}, \\ \dfrac{1}{\mu(Q^{(\ell-1)})} \nu_{k-\ell}^k, & \text{otherwise.} \end{cases}$$

2. *For all $x, y \in Q^{(1)} \setminus \{x_Q^{(1)}\}$,*

$$|\mathcal{K}_{Q^{(1)}}(x,y)| \lesssim \begin{cases} \dfrac{1}{\mu(Q)}, & \text{if } x \wedge y \neq x_Q^{(1)}, \\ \dfrac{1}{q(x_Q^{(1)})\mu(Q)}, & \text{otherwise.} \end{cases}$$

Proof. 1. By the triangle inequality, it is enough to prove the assertion for $y = x_Q$. From (3.6) we have that

$$\begin{aligned}
&\mathcal{K}_{Q^{(\ell)}}(z,x) - \mathcal{K}_{Q^{(\ell)}}(z,x_Q) \\
&\qquad = \frac{q(x_Q^{(\ell)})\varphi_{k-\ell+1}(z)(\varphi_{k-\ell+1}(x) - \varphi_{k-\ell+1}(x_Q))}{\|\varphi_{k-\ell+1}\mathbb{1}_{Q^{(\ell)}}\|_2^2} k_{Q^{(\ell)}}(z^{(|z|-k+\ell-1)}, x_Q^{(\ell-1)}).
\end{aligned}$$

Note that $\operatorname{supp}(\varphi_{k-\ell+1}) \subseteq Q^{(\ell)} \setminus \{x_Q^{(\ell)}\}$ and $1 \leq \varphi_{k-\ell+1} < 2$, so $\|\varphi_{k-\ell+1}\mathbb{1}_{Q^{(\ell)}}\|_2^2 \simeq q(x_Q^{(\ell)})\mu(Q^{(\ell-1)})$. Therefore, using Lemma 3.2.6 and (3.7) yields

$$|\mathcal{K}_{Q^{(\ell)}}(z,x) - \mathcal{K}_{Q^{(\ell)}}(z,x_Q)|$$

$$
\begin{aligned}
&= \left| \frac{q(x_Q^{(\ell)})\varphi_{k-\ell+1}(z)(\varphi_{k-\ell+1}(x) - \varphi_{k-\ell+1}(x_Q))}{\|\varphi_{k-\ell+1}\mathbb{1}_{Q^{(\ell)}}\|_2^2} k_{Q^{(\ell)}}(z^{(|z|-k+\ell-1)}, x_Q^{(\ell-1)}) \right| \\
&\lesssim \nu_{k-\ell+1}^k \frac{1}{\mu(Q^{(\ell-1)})} |k_{Q^{(\ell)}}(z^{(|z|-k+\ell-1)}, x_Q^{(\ell-1)})| \\
&\simeq \begin{cases} \dfrac{1}{\mu(Q^{(\ell-1)})}\nu_{k-\ell+1}^k, & \text{if } z \in Q^{(\ell-1)}, \\ \dfrac{1}{q(x_Q^{(\ell)})\mu(Q^{(\ell-1)})}\nu_{k-\ell+1}^k, & \text{otherwise.} \end{cases}
\end{aligned}
$$

2. Again by (3.6), (3.7) and the fact that $\|\varphi_k \mathbb{1}_{Q^{(1)}}\|_2^2 \simeq q(x_Q^{(1)})\mu(Q)$,

$$
\begin{aligned}
\mathcal{K}_{Q^{(1)}}(x,y) &= \frac{q(x_Q^{(1)})\varphi_k(x)\varphi_k(y)}{\|\varphi_k \mathbb{1}_{Q^{(1)}}\|_2^2} k_{Q^{(1)}}(x^{(|x|-k)}, y^{(|y|-k)}) \\
&\simeq \begin{cases} \dfrac{1}{\mu(Q)}, & \text{if } x \wedge y \neq x_Q^{(1)}, \\ -\dfrac{1}{q(x_Q^{(1)})\mu(Q)}, & \text{otherwise.} \end{cases}
\end{aligned}
$$

□

We next prove an estimate for truncations of $\mathcal{P}$. Given a family $\mathcal{F}$ of disjoint sets in $\mathcal{D}$ contained in a sector Q_0, we denote the kernel of the truncation $\mathcal{P}_{\mathcal{F}}^{Q_0}$ by $\mathcal{K}_{\mathcal{F}}^{Q_0}$, which can be written as

$$
\mathcal{K}_{\mathcal{F}}^{Q_0} = \sum_{\substack{Q \in \mathcal{D},\, Q \subseteq Q_0 \\ Q \not\subseteq R \in \mathcal{F}}} \mathcal{K}_Q. \tag{3.8}
$$

In the particular case when $Q_0 = \mathfrak{X}$, we slightly abuse notation and exclude $\mathcal{K}_{\mathfrak{X}}$ from the sum and will deal with it separately below.

Proposition 3.2.8. *Let $Q \in \mathcal{D}_k$ be such that $Q \subseteq Q_0$ and $Q \not\subseteq R$ for any $R \in \mathcal{F}$. For every $z \in Q^{(\ell)} \setminus Q^{(\ell-1)} \subseteq Q_0$, we have*

$$
\sup_{x,y \in Q} |\mathcal{K}_{\mathcal{F}}^{Q_0}(z,x) - \mathcal{K}_{\mathcal{F}}^{Q_0}(z,y)| \lesssim \begin{cases} \dfrac{1}{\mu(Q^{(\ell)})}\nu_{k-\ell}^k, & \text{if } z = x_Q^{(\ell)}, \\ \dfrac{1}{\mu(Q^{(\ell-1)})}\nu_{k-\ell}^k, & \text{if } z \neq x_Q^{(\ell)}. \end{cases}
$$

Proof. We have

$$
|\mathcal{K}_{\mathcal{F}}^{Q_0}(z,x) - \mathcal{K}_{\mathcal{F}}^{Q_0}(z,y)| \leq \sum_{j=0}^{k-\ell} |\mathcal{K}_{Q^{(\ell+j)}}(z,x) - \mathcal{K}_{Q^{(\ell+j)}}(z,y)|,
$$

where the first term of the sum vanishes whenever $z = x_Q^{(\ell)}$. By part (1) of Lemma 3.2.7, one has

$$|\mathcal{K}_{\mathcal{F}}^{Q_0}(z,x) - \mathcal{K}_{\mathcal{F}}^{Q_0}(z,y)| \lesssim \begin{cases} \displaystyle\sum_{j=1}^{k-\ell} \frac{1}{\mu(Q^{(\ell+j-1)})} \nu_{k-\ell-j+1}^k, & \text{if } z = x_Q^{(\ell)} \\ \displaystyle\frac{1}{\mu(Q^{(\ell-1)})} \nu_{k-\ell}^k + \sum_{j=1}^{k-\ell} \frac{1}{\mu(Q^{(\ell+j-1)})} \nu_{k-\ell-j+1}^k, & \text{if } z \neq x_Q^{(\ell)}. \end{cases}$$

The statement follows from the fact that both sums are geometric and hence controlled by their first term, and the one appearing in the term below is in turn controlled by the one on its left. □

The following notation will sometimes be used in the sequel: if $Q \in \mathcal{D}$, $\overline{Q}$ is the smallest sector that contains it. Notice that it is only different from Q if the latter is a singleton. This last estimate is particularly useful when controlling local terms.

Proposition 3.2.9. *Let b be a simple algebraic atom associated to $Q \in \mathcal{D}_k$. For every $\ell \in \{1, \ldots, |x_Q|\}$,*

$$\mathcal{D}_{\overline{Q}^{(\ell)}}(b)(x) \lesssim \begin{cases} \dfrac{\nu_{k-\ell+1}^k}{\mu(Q^{(\ell-1)})} \|b\|_1, & \text{if } x \in \overline{Q}^{(\ell-1)}, & \text{(3.9a)} \\ \dfrac{\nu_{k-\ell}^k}{\mu(Q^{(\ell-1)})} \|b\|_1, & \text{if } x \in \overline{Q}^{(\ell)} \setminus \overline{Q}^{(\ell-1)}. & \text{(3.9b)} \end{cases}$$

Furthermore, if Q is a single point, then $\mathcal{D}_{\overline{Q}} b \equiv 0$.

Proof. The case $\ell > 1$ follows from the fact that b has null mean and Lemma 3.2.7 applied to $\overline{Q}^{(1)}$. Indeed,

$$\begin{aligned} |\mathcal{D}_{\overline{Q}^{(\ell)}} b(x)| &= \left| \sum_{y \in Q^{(1)}} \Big(\mathcal{K}_{\overline{Q}^{(\ell)}}(x,y) - \mathcal{K}_{\overline{Q}^{(\ell)}}(x, x_{\overline{Q}^{(1)}}) \Big) b(y) \mu(y) \right| \\ &\leq \sup_{y \in Q^{(1)}} \left| \mathcal{K}_{\overline{Q}^{(\ell)}}(x,y) - \mathcal{K}_{\overline{Q}^{(\ell)}}(x, x_{\overline{Q}^{(1)}}) \right| \|b\|_1 \\ &\leq \begin{cases} \dfrac{\nu_{k-\ell+1}^k}{\mu(Q^{(\ell-1)})} \|b\|_1, & \text{if } x \in \overline{Q}^{(\ell-1)}, \\ \dfrac{\nu_{k-\ell}^k}{\mu(Q^{(\ell-1)})} \|b\|_1, & \text{if } x \in \overline{Q}^{(\ell)} \setminus \overline{Q}^{(\ell-1)}, \end{cases} \end{aligned}$$

regardless of whether $Q = \overline{Q}$ or not. For $\ell = 1$, if Q is a single point one can argue similarly as above. Otherwise, denote by R the element so that $x \in R \in \mathcal{D}_1(Q^{(1)})$. By (2) of Lemma 3.2.7, one has

$$|\mathcal{D}_{Q^{(1)}} b(x)| \leq \sum_{y \in Q^{(1)} \setminus \{x_Q^{(1)}\}} |\mathcal{K}_{Q^{(1)}}(x,y)| |b(y)| \mu(y)$$

$$\lesssim \frac{1}{\mu(Q)}\Big(\|b\mathbb{1}_R\|_1 + \frac{1}{q(x_Q^{(1)})}\|b\mathbb{1}_{Q^{(1)}\setminus(R\cup\{x_Q^{(1)}\})}\|_1\Big).$$

If $R = Q$, then clearly $|\mathcal{D}_{Q^{(1)}}b(x)| \lesssim \|b\|_1\mu(Q)^{-1}$. Otherwise, since $b = a - \langle a\rangle_{Q^{(1)}}\mathbb{1}_{Q^{(1)}}$ we have that

$$|\mathcal{D}_{Q^{(1)}}b(x)| \lesssim \frac{1}{\mu(Q)}\Big(\mu(R)|\langle a\rangle_{Q^{(1)}}| + \frac{1}{q(x_Q^{(1)})}\|b\|_1\Big) \lesssim \frac{\|b\|_1}{q(x_Q^{(1)})\mu(Q)}.$$

Finally, the fact that $\mathcal{D}_{\overline{Q}}b \equiv 0$ when $Q = \{v\}$ follows from

$$\begin{aligned}\mathcal{D}_{\overline{Q}}b(x) &= -\frac{a(v)}{\mu(\overline{Q})}\sum_{y\in\overline{Q}\setminus\{v\}}\mathcal{K}_{\overline{Q}}(x,y)\mu(y)\\ &= -\frac{a(v)}{\mu(\overline{Q})}\sum_{z\in\mathcal{D}_1(v)}k_{\overline{Q}}(x^{|x|-|v|-1},z)\sum_{y\in T_z}\frac{\varphi_{|v|+1}(x)\varphi_{|v|+1}(y)}{\|\varphi_{|v|+1}\mathbb{1}_{\overline{Q}}\|_2^2}\mu(y) = 0, \qquad (3.10)\end{aligned}$$

by the definition of $k_{\overline{Q}}$ and the radiality of $\varphi_{|v|+1}$. □

Remark 3.2.10. The key to obtaining the estimates above is to exploit the joint cancellation enjoyed by the family $\{h_Q^\ell\}_\ell$ for each fixed Q. We do that by studying the operators $\mathcal{D}_Q$ as a whole instead of considering the individual products $\langle f, h_Q^\ell\rangle h_Q^\ell$, of which there can be arbitrarily many within each cube Q, and for which size estimates are not precise enough. This is a key difference with respect to the usual point of view that one adopts in classical dyadic harmonic analysis, in which the action of each function in the Haar basis is dealt with separately. In a sense, one can interpret this point of view as a way to dealing with the fact that $\mathfrak{X}$ may be infinite-dimensional.

3.2.3 H^1 and BMO boundedness via cancellation

We start checking the Hörmander and size conditions for the kernel of $\mathcal{P}$. As one could have expected, they are weaker than the estimates in Subsection 3.2.2 and can be deduced from them.

Proposition 3.2.11. *The Bergman kernel $\mathcal{K}$ satisfies both Hörmander's condition* (3.1) *and the size condition* (3.2)*.*

Proof. Hörmander's condition can be checked as follows. First of all, notice that if $Q \in \mathcal{D}$ is a singleton, then there is nothing to prove. Hence, fix a sector $Q \in \mathcal{D}$ and $x \neq y \in Q$. Summing in annuli and using Proposition 3.2.8 with $Q_0 = \mathfrak{X}$ and $\mathcal{F} = \emptyset$, yields

$$\sum_{z\in\mathfrak{X}\setminus Q}|\mathcal{K}(z,x) - \mathcal{K}(z,y)|\mu(z)$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{|x_Q|} \sum_{z \in Q^{(\ell)} \setminus Q^{(\ell-1)}} |\mathcal{K}(z,x) - \mathcal{K}(z,y)| \mu(z) \\
&\leq \sum_{\ell=1}^{|x_Q|} \mu(x_Q^{(\ell)}) \frac{\nu_{|x_Q|-\ell}^{|x_Q|}}{\mu(Q^{(\ell)})} + \sum_{\ell=1}^{|x_Q|} \mu\Big(Q^{(\ell)} \setminus (Q^{(\ell-1)} \cup \{x_Q^{(\ell)}\})\Big) \frac{\nu_{|x_Q|-\ell}^{|x_Q|}}{\mu(Q^{(\ell-1)})} \\
&\lesssim \sum_{\ell=1}^{|x_Q|} \nu_{|x_Q|-\ell}^{|x_Q|} + \sum_{\ell=1}^{|x_Q|} \mu(Q^{(\ell-1)}) q(x_Q^{(\ell)}) \frac{\nu_{|x_Q|-\ell+1}^{|x_Q|}}{q(x_Q^{(\ell)}) \mu(Q^{(\ell-1)})} \lesssim 1.
\end{aligned}
$$

Concerning the size condition a similar computation using Lemma 3.2.7, part (2) gives (3.2) for $\mathcal{K}$. □

As a consequence, we can apply Theorem 3.1.3 to $\mathcal{P}$ and obtain the following result.

Corollary 3.2.12. *The Bergman projection $\mathcal{P}$ is of weak-type $(1,1)$. By interpolation and duality $\mathcal{P}$ is also bounded on $L^p(\mathfrak{X})$ for all $p \in (1,\infty)$.*

We could also apply Theorem 3.1.9 and 3.1.8, but all the estimates on $\mathcal{K}$ that we just proved, allow us to finish proving the stronger $H^1 - H^1$ and $BMO - BMO$ endpoint bounds for the Bergman projector. Even if the $H^1 - H^1$ estimates implies the $BMO - BMO$ one, we prove it directly without using duality.

Theorem 3.2.13. *The Bergman projection $\mathcal{P}$ maps $BMO(\mathfrak{X})$ into itself.*

Proof. Let $f \in BMO(\mathfrak{X})$. Since $\mathcal{P}\mathbb{1}_{\mathfrak{X}} = \mathbb{1}_{\mathfrak{X}}$

$$
\left| \sum_{x \in \mathfrak{X}} \mathcal{P}f(x)\mu(x) \right| = \left| \sum_{z \in \mathfrak{X}} f(z)\mu(z) \right| \leq \|f\|_{BMO},
$$

and we just have to bound the two suprema in (3.3). Fix $Q \in \mathcal{D}$ and split f as $f = f - \langle f \rangle_{Q^{(1)}} + \langle f \rangle_{Q^{(1)}} =: \tilde{f} + \langle f \rangle_{Q^{(1)}}$. Since $\mathcal{P}$ sends constants to constants, it is enough estimate the BMO norm of $\mathcal{P}\tilde{f}$. By the fact that $\tilde{f}$ has mean 0 over $Q^{(1)}$, we have

$$
|\langle \tilde{f} \rangle_{Q^{(\ell)}}| + \sup_{R \in \mathcal{D}_\ell(Q)} |\langle \tilde{f} \rangle_R| \lesssim (\ell+1)\|f\|_{BMO}, \quad \ell \geq 0. \tag{3.11}
$$

To estimate the norm of $\mathcal{P}\tilde{f}$, we study separately the two suprema in (3.3). For the first, we write $\tilde{f} = \tilde{f}\mathbb{1}_Q + \tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q}$ and we apply triangle inequality as follows

$$
\begin{aligned}
&\frac{1}{\mu(Q)} \sum_{x \in Q} |\mathcal{P}\tilde{f}(x) - \langle \mathcal{P}\tilde{f} \rangle_Q| \mu(x) \\
&\quad = \frac{1}{\mu(Q)} \sum_{x \in Q} |\mathcal{P}(\tilde{f}\mathbb{1}_Q(x) + \tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q}(x)) - \langle \mathcal{P}(\tilde{f}\mathbb{1}_Q + \tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q}) \rangle_Q| \mu(x)
\end{aligned}
$$

$$\begin{aligned}
\leq & \frac{1}{\mu(Q)} \sum_{x \in Q} |\mathcal{P}(\tilde{f}\mathbb{1}_Q)(x) - \langle \mathcal{P}(\tilde{f}\mathbb{1}_Q)\rangle_Q| \mu(x) \\
& + \frac{1}{\mu(Q)} \sum_{x \in Q} |\mathcal{P}(\tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q})(x) - \langle \mathcal{P}(\tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q})\rangle_Q| \mu(x) \\
=: & \, \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Concerning the first part, John-Nirenberg inequality and the L^2-boundedness of $\mathcal{P}$ give

$$\begin{aligned}
\mathrm{I} &= \frac{1}{\mu(Q)} \sum_{x \in Q} |\mathcal{P}(\tilde{f}\mathbb{1}_Q)(x) - \langle \mathcal{P}(\tilde{f}\mathbb{1}_Q)\rangle_Q| \mu(x) \\
&\lesssim \left(\frac{1}{\mu(Q)} \sum_{x \in Q} |\mathcal{P}(\tilde{f}\mathbb{1}_Q)(x)|^2 \mu(x) \right)^{\frac{1}{2}} \\
&\lesssim \left(\frac{1}{\mu(Q)} \sum_{x \in \mathfrak{X}} |\tilde{f}\mathbb{1}_Q(x)|^2 \mu(x) \right)^{\frac{1}{2}} \\
&= \left(\frac{1}{\mu(Q)} \sum_{x \in Q} |f(x) - \langle f \rangle_{Q^{(1)}}|^2 \mu(x) \right)^{\frac{1}{2}} \\
&\leq \|f\|_{BMO}.
\end{aligned}$$

For II, if Q is a singleton there is nothing to do. Otherwise, we decompose it into annuli:

$$\begin{aligned}
\mathrm{II} &= \frac{1}{\mu(Q)} \sum_{x \in Q} |\mathcal{P}(\tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q})(x) - \langle \mathcal{P}(\tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q})\rangle_Q| \mu(x) \\
&= \frac{1}{\mu(Q)} \sum_{x \in Q} \left| \sum_{z \in \mathfrak{X}\setminus Q} \mathcal{K}(z,x)\tilde{f}(z)\mu(z) - \frac{1}{\mu(Q)} \sum_{y \in Q} \sum_{z \in \mathfrak{X}\setminus Q} \mathcal{K}(z,y)\tilde{f}(z)\mu(z)\mu(y) \right| \mu(x) \\
&\leq \frac{1}{\mu(Q)^2} \sum_{x \in Q} \sum_{y \in Q} \sum_{z \in \mathfrak{X}\setminus Q} |\mathcal{K}(z,x) - \mathcal{K}(z,y)||\tilde{f}(z)|\mu(z)\mu(y)\mu(x) \\
&\leq \sup_{x,y \in Q} \sum_{z \in \mathfrak{X}\setminus Q} |\mathcal{K}(z,x) - \mathcal{K}(z,y)||\tilde{f}(z)|\mu(z) \\
&\leq \sum_{\ell=1}^{|x_Q|} \sup_{x,y \in Q} \sum_{z \in Q^{(\ell)} \setminus Q^{(\ell-1)}} |\mathcal{K}(z,x) - \mathcal{K}(z,y)||\tilde{f}(z)|\mu(z) =: \sum_{\ell=1}^{|x_Q|} \mathrm{II}_\ell.
\end{aligned}$$

We now estimate each term using Lemma 3.2.8 and (3.11):

$$\mathrm{II}_\ell \lesssim |\tilde{f}(x_Q^{(\ell)})|\mu(x_Q^{(\ell)}) \frac{\nu^{|x_Q|}_{|x_Q|-\ell+1}}{\mu(Q^{(\ell)})} + \frac{\nu^{|x_Q|}_{|x_Q|-\ell}}{\mu(Q^{(\ell-1)})} \sum_{z \in Q^{(\ell)} \setminus (Q^{(\ell-1)} \cup \{x_Q^{(\ell)}\})} |\tilde{f}(z)|\mu(z)$$

$$
\begin{aligned}
&\leq \nu^{|x_Q|}_{|x_Q|-\ell+1}\langle|\tilde{f}|\rangle_{Q^{(\ell)}} + \nu^{|x_Q|}_{|x_Q|-\ell+1}\frac{1}{q(x_Q^{(\ell)})}\sum_{R\in\mathcal{D}_1(Q^{(\ell)})}\langle|\tilde{f}|\rangle_R \\
&\leq 2\cdot\nu^{|x_Q|}_{|x_Q|-\ell+1}\langle|\tilde{f}|\rangle_{Q^{(\ell)}} + \nu^{|x_Q|}_{|x_Q|-\ell+1}\frac{1}{q(x_Q^{(\ell)})}\sum_{R\in\mathcal{D}_1(Q^{(\ell)})}\langle|\tilde{f}|\rangle_R - \langle|\tilde{f}|\rangle_{Q^{(\ell)}} \\
&\lesssim \ell\cdot\nu^{|x_Q|}_{|x_Q|-\ell+1}\||\tilde{f}|\|_{BMO} \lesssim \ell\cdot\nu^{|x_Q|}_{|x_Q|-\ell+1}\|f\|_{BMO}.
\end{aligned}
$$

The resulting quantity is acceptable due to the geometric decay of $\nu^{|x_Q|}_{|x_Q|-\ell}$. To bound the second term in (3.3), we split

$$
\begin{aligned}
|\langle\mathcal{P}\tilde{f}\rangle_Q - \langle\mathcal{P}\tilde{f}\rangle_{Q^{(1)}}| \leq &|\langle\mathcal{P}(\tilde{f}\mathbb{1}_Q)\rangle_Q| + |\langle\mathcal{P}(\tilde{f}\mathbb{1}_{Q^{(1)}})\rangle_{Q^{(1)}}| \\
&+ |\langle\mathcal{P}(\tilde{f}\mathbb{1}_{Q^{(1)}\setminus Q})\rangle_Q| \\
&+ |\langle\mathcal{P}(\tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q^{(1)}})\rangle_Q - \langle\mathcal{P}(\tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q^{(1)}})\rangle_{Q^{(1)}}|.
\end{aligned}
$$

The first two terms above are estimated similarly. Indeed, by the L^2-boundedness of $\mathcal{P}$,

$$
|\langle\mathcal{P}(\tilde{f}\mathbb{1}_{Q^{(1)}})\rangle_{Q^{(1)}}| \lesssim \left(\langle|\tilde{f}|^2\rangle_{Q^{(1)}}\right)^{\frac{1}{2}} = \left(\langle|\tilde{f} - \langle\tilde{f}\rangle_{Q^{(1)}}|^2\rangle_{Q^{(1)}}\right)^{\frac{1}{2}} \leq \|\tilde{f}\|_{BMO},
$$

and a similar computation applies to $|\langle\mathcal{P}(\tilde{f}\mathbb{1}_Q)\rangle_Q|$. The third term above is estimated using Lemma 3.2.7, part (2), as follows:

$$
\begin{aligned}
|\langle\mathcal{P}(\tilde{f}\mathbb{1}_{Q^{(1)}\setminus Q})\rangle_Q| &\leq \frac{1}{\mu(Q)}\sum_{x\in Q}\sum_{y\in Q^{(1)}\setminus Q}|\mathcal{K}(x,y)||\tilde{f}(y)|\mu(y) \\
&\leq \sup_{x\in Q}\sum_{y\in Q^{(1)}\setminus Q}|\mathcal{K}(x,y)||\tilde{f}(y)|\mu(y) \\
&\lesssim \sum_{y\in Q^{(1)}\setminus Q}|\mathcal{K}(x_Q,y)||\tilde{f}(y)|\mu(y) \\
&\leq \sum_{y\in Q^{(1)}\setminus Q}\sum_{\ell=1}^{|x_Q|}|\mathcal{K}_{Q^{(\ell)}}(x_Q,y)||\tilde{f}(y)|\mu(y) \\
&\leq \left(\sum_{\ell=1}^{|x_Q|}\frac{1}{\mu(Q^{(\ell)})}\right) f(x_Q^{(1)})\mu(x_Q^{(1)}) \\
&\quad + \left(\frac{1}{q(x_Q^{(1)})\mu(Q)} + \sum_{\ell=1}^{|x_Q|}\frac{1}{\mu(Q^{(\ell)})}\right)\sum_{y\in Q^{(1)}\setminus(Q\cup\{x_Q^{(1)}\})}|\tilde{f}(y)|\mu(y) \\
&\lesssim \langle|\tilde{f}|\rangle_{Q^{(1)}} + \frac{1}{q(x_Q^{(1)})}\sum_{R\in\mathcal{D}_1(Q^{(1)})}\langle|\tilde{f}|\rangle_R \\
&\lesssim \|f\|_{BMO},
\end{aligned}
$$

finishing the computation as with II_ℓ before. Finally, for the last term we write

$$\begin{aligned}
&|\langle \mathcal{P}(\tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q^{(1)}})\rangle_Q - \langle \mathcal{P}(\tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q^{(1)}})\rangle_{Q^{(1)}}| \\
&\leq \frac{1}{\mu(Q)}\frac{1}{\mu(Q^{(1)})}\sum_{x\in Q}\sum_{y\in Q^{(1)}} |\mathcal{P}(\tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q^{(1)}})(x) - \mathcal{P}(\tilde{f}\mathbb{1}_{\mathfrak{X}\setminus Q^{(1)}})(y)|\mu(y)\mu(x) \\
&\leq \frac{1}{\mu(Q)}\frac{1}{\mu(Q^{(1)})}\sum_{x\in Q}\sum_{y\in Q^{(1)}}\sum_{z\in\mathfrak{X}\setminus Q^{(1)}} |\tilde{f}(z)||\mathcal{K}(z,x)-\mathcal{K}(z,y)|\mu(z)\mu(y)\mu(x) \\
&\leq \sup_{x,y\in Q^{(1)}} \sum_{z\in\mathfrak{X}\setminus Q^{(1)}} |\tilde{f}(z)||\mathcal{K}(z,x)-\mathcal{K}(z,y)|\mu(z),
\end{aligned}$$

and we split into annuli as with term I above to conclude. □

Theorem 3.2.14. *The Bergman projection $\mathcal{P}$ maps $H^1(\mathfrak{X})$ into itself.*

Proof. Let $b \in H^1(\mathfrak{X})$ be a simple atomic block supported on $Q^{(1)} \in \mathcal{D}$, i.e. $b = a - \langle a\rangle_{Q^{(1)}} 1_{Q^{(1)}}$, with a supported in Q satisfying the right size condition. It is enough to check that $\|\mathcal{P}b\|_{H^1} \lesssim 1$, and we do it by finding a decomposition of it into atomic blocks that need not be simple, by the equivalence in Lemma 3.1.7. Because of cancellation, we have

$$\begin{aligned}
\mathcal{P}b &= \sum_{R\in\mathcal{D}} \mathcal{D}_R b = \sum_{\substack{R\in\mathcal{D}\\ R\subsetneq Q^{(1)}}} \mathcal{D}_R b + \sum_{\substack{R\in\mathcal{D}\\ R\supseteq Q^{(1)}}} \mathcal{D}_R b \\
&= \sum_{\substack{R\in\mathcal{D}\\ R\subsetneq Q^{(1)}}} \sum_{\ell} \Big\langle a - \langle a\rangle_{Q^{(1)}}, h_R^\ell \Big\rangle h_R^\ell + \sum_{k=1}^{|Q|} \mathcal{D}_{Q^{(k)}} b = \mathcal{P}^Q a + \sum_{k=1}^{|Q|} \mathcal{D}_{Q^{(k)}} b.
\end{aligned}$$

Clearly, $\mathrm{supp}(\mathcal{P}^Q a) \subseteq Q$ and has mean zero, so it is an atomic block. By the L^2-boundedness,

$$\|\mathcal{P}^Q a\|_2 \lesssim \|a\|_2 \leq \mu(Q)^{-1/2},$$

so $\|\mathcal{P}^Q a\|_{H^1} \lesssim 1$.

We next claim that, for each k, $\mathcal{D}_{Q^{(k)}} b$ is an atomic block. It is supported on $Q^{(k)}$, where it has zero mean, and we can write it as

$$\mathcal{D}_{Q^{(k)}} b = \mathbb{1}_{Q^{(k-1)}} \mathcal{D}_{Q^{(k)}} b + \sum_{\substack{T: T^{(1)} = Q^{(k)}\\ T \neq Q^{(k-1)}}} \mathbb{1}_T \mathcal{D}_{Q^{(k)}} b.$$

We have

$$\begin{aligned}
\|\mathbb{1}_{Q^{(k-1)}} \mathcal{D}_{Q^{(k)}} b\|_2 &\leq \mu(Q^{(k-1)})^{\frac{1}{2}} \|\mathbb{1}_{Q^{(k-1)}} \mathcal{D}_{Q^{(k)}} b\|_\infty \\
&\lesssim \frac{\mu(Q^{(k-1)})^{\frac{1}{2}} \nu_{|x_Q|-k+1}^{|x_Q|}}{\mu(Q^{(k-1)})} \|b\|_1
\end{aligned}$$

$$\leq \frac{\nu_{|x_Q|-k+1}^{|x_Q|}}{\mu(Q^{(k-1)})^{\frac{1}{2}}},$$

by Hölder's inequality, Proposition 3.2.9 and the fact that $\|b\|_1 \leq \|b\|_{H^1} \leq 1$. Analogously we get

$$\|\mathbb{1}_T \mathcal{D}_{Q^{(k)}} b\|_2 \lesssim \frac{\nu_{|x_Q|-k}^{|x_Q|}}{\mu(Q^{(k-1)})^{\frac{1}{2}}}$$

for every $T \in \mathcal{D}$ such that $T^{(1)} = Q^{(k)}$, $T \neq Q^{(k-1)}$. It follows that $\mathcal{D}_{Q^{(k)}} b$ is an atomic block for every $k = 1, \dots, |Q|$ and

$$\|\mathcal{D}_{Q^{(k)}} b\|_2 \lesssim \frac{\nu_{|x_Q|-k+1}^{|x_Q|}}{\mu(Q^{(k-1)})^{\frac{1}{2}}} + (q(x_Q^{(k)}) - 1)\frac{\nu_{|x_Q|-k}^{|x_Q|}}{\mu(Q^{(k-1)})^{\frac{1}{2}}} \simeq \frac{\nu_{|x_Q|-k+1}^{|x_Q|}}{\mu(Q^{(k-1)})^{\frac{1}{2}}}.$$

Finally, we get

$$\|\mathcal{P}b\|_{H^1} \lesssim 1 + \sum_{k=1}^{|Q|} \nu_{|x_Q|-k+1}^{|x_Q|} \lesssim 1.$$

□

Chapter 4

Sparse domination for the radial Bergman projection

Sparse domination in harmonic analysis is a potent approach to the study of various operators such as singular integrals and maximal functions. It provides a way to control the behavior of these operators using sparse sets, which are subsets of the domain that might not be disjoint but that are not too dense either. The key idea is that certain operators, which could be difficult to handle directly, can be dominated by simpler operators acting on these sparse sets, allowing for better control and estimation of their behavior. More precisely, the goal is to dominate the operator T under study by a sparse operator

$$\langle Tf_1, f_2\rangle \lesssim \langle \mathcal{A}_{\mathcal{S}} f_1, f_2\rangle, \tag{4.1}$$

where

$$\mathcal{A}_{\mathcal{S}} f(x) = \sum_{Q\in\mathcal{S}} \langle f\rangle_Q \mathbb{1}_Q(x)$$

and $\mathcal{S}$ is a family of sparse sets. This technique is particularly useful for establishing bounds and proving inequalities. Indeed, $\mathcal{A}_{\mathcal{S}}$ is an averaging operator that enjoys nice boundedness properties and the bilinear form domination (4.1) in turn implies the same bounds for the operator of interest T. Sparse domination also plays a crucial role in the study of weighted norm inequalities, by entailing weighted estimates in a really transparent way. Some basics and references about sparse domination can be found in Section 1.4.

The work presented in this chapter appears in [16] and is dedicated to establishing a sparse-like domination result and weighted inequalities for the Bergman projection on radial trees. Specifically, we provide a standard sparse domination in the simpler case of the doubling setting, where the structure of the measure allows for more straightforward control. However, for the more involved nondoubling case, we encounter additional challenges that require the inclusion of an extra term to add to the sparse operator to properly account for the behavior of the Bergman projection. This distinction highlights the increased difficulty in handling the nondoubling case, where additional nuances must be considered. The idea behind this result is inspired by the work done in [19], where an extra term first appears that allows for a sparse-type domination where a pure sparse domination cannot be achieved.

4.1 Sparse domination

To start, we define the form $\mathcal{E}_{\mathcal{S}}$ that represents the extra term needed in the nondoubling context. If $\mathcal{S} \subseteq \mathcal{D}$ is a sparse family we put

$$\mathcal{E}_{\mathcal{S}}(f_1, f_2) := \sum_{Q=\overline{Q}\in\mathcal{S}} \langle f_1\rangle_Q \left(\frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{S}\\ R^{(1)}=Q^{(1)}}} \langle f_2\rangle_R \right) \mu(Q). \tag{4.2}$$

Remark 4.1.1. From the orthogonality and normalization of the operators $\mathcal{D}_Q$ it follows immediately that the truncations of $\mathcal{P}$ are uniformly L^2-bounded

$$\sup_{Q_0\in\mathcal{D},\, \mathcal{F}\subseteq\mathcal{D}} \left\| \mathcal{P}_{\mathcal{F}}^{Q_0} \right\|_{L^2(\mathfrak{X})\to L^2(\mathfrak{X})} < \infty. \tag{4.3}$$

Theorem 4.1.2. *Let $\mathfrak{X}$ be any radial tree and let $f_1, f_2 \in L^1(\mathfrak{X})$. There exists a sparse collection $\mathcal{S}$ such that*

$$|\langle \mathcal{P} f_1, f_2\rangle| \lesssim \langle \mathcal{A}_{\mathcal{S}} f_1, f_2\rangle + \mathcal{E}_{\mathcal{S}}(f_1, f_2).$$

Proof. Since we will have $\mathfrak{X} \in \mathcal{S}$ for all pairs (f_1, f_2), it is enough to prove sparse domination for the operator $\mathcal{P} - \langle f\rangle_{\mathfrak{X}} \mathbb{1}_{\mathfrak{X}}$. Let $f_1, f_2 \in L^1(\mathfrak{X})$ be nonnegative, and set $Q_0 = \mathfrak{X}$. We have

$$\begin{aligned}\langle \mathcal{P} f_1, f_2\rangle =& \langle f_1\rangle_{Q_0} \|f_2\|_1 + \Big\langle \sum_{Q\in\mathcal{D}} \mathcal{D}_Q f_1, f_2 \Big\rangle \\ =& \langle f_1\rangle_{Q_0} \langle f_2\rangle_{Q_0} \mu(Q_0) + \langle \mathcal{P}^{Q_0} f_1, f_2\rangle.\end{aligned}$$

Since $Q_0 \in \mathcal{S}$, the first term in the right hand side above is acceptable, and we focus on estimating the second for the remainder of the proof. We denote

$$\Omega_i^0 = \Big\{ \mathcal{M}_{\mathcal{D}} f_i > 4\|\mathcal{M}_{\mathcal{D}}\|_{L^1(\mathfrak{X})\to L^{1,\infty}(\mathfrak{X})} \langle f_i\rangle_{Q_0} \Big\}, \qquad i \in \{1, 2\},$$

and $\Omega^0 = \Omega_1^0 \cup \Omega_2^0$. We cover Ω^0 with a disjoint family $\mathcal{F}^1 \subseteq \mathcal{D}$, maximal with respect to inclusion. For $i = 1, 2$, we apply the Calderón-Zygmund decomposition from Theorem 3.1.1 to f_i and Remark 3.1.2 at height $\lambda_i = 4\|\mathcal{M}_{\mathcal{D}}\|_{L^1(\mathfrak{X})\to L^{1,\infty}(\mathfrak{X})} \langle f_i\rangle_{Q_0}$. We next estimate the non localizing part of $\mathcal{P}$, namely the truncation

$$\mathcal{P}_{\mathcal{F}^1}^{Q_0} f = \mathcal{P}^{Q_0} f - \sum_{Q\in\mathcal{F}^1} \mathcal{P}^Q f.$$

We claim

$$|\langle \mathcal{P}_{\mathcal{F}^1}^{Q_0} f_1, f_2\rangle| \lesssim \langle f_1\rangle_{Q_0}\langle f_2\rangle_{Q_0}\mu(Q_0) + \sum_{Q\in\mathcal{F}^1}\langle f_1\rangle_Q\langle f_2\rangle_Q\mu(Q)$$
$$+ \sum_{Q=\overline{Q}\in\mathcal{F}^1}\langle f_1\rangle_Q\Bigg(\frac{1}{q(x_Q^{(1)})}\sum_{\substack{R=\overline{R}\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}}\langle f_2\rangle_R\Bigg)\mu(Q). \tag{4.4}$$

We split according to the Calderón-Zygmund pieces:

$$\begin{aligned}|\langle \mathcal{P}_{\mathcal{F}^1}^{Q_0} f_1, f_2\rangle| &\le |\langle \mathcal{P}_{\mathcal{F}^1}^{Q_0} g_1, g_2\rangle| + |\langle \mathcal{P}_{\mathcal{F}^1}^{Q_0} b_1, g_2\rangle| + |\langle \mathcal{P}_{\mathcal{F}^1}^{Q_0} g_1, b_2\rangle| + |\langle \mathcal{P}_{\mathcal{F}^1}^{Q_0} b_1, b_2\rangle|\\ &=: \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV},\end{aligned}$$

and we analyze separately the four cases. For term I, Cauchy-Schwarz's inequality and (4.3) yield

$$\begin{aligned}\mathrm{I} &\le \|\mathcal{P}_{\mathcal{F}^1}^{Q_0} g_1\|_2\|g_2\|_2\\ &\lesssim \|g_1\|_2\|g_2\|_2\\ &\lesssim (\langle f_1\rangle_{Q_0}\|f_1\|_1\,\langle f_2\rangle_{Q_0}\|f_2\|_1)^{1/2}\\ &= \langle f_1\rangle_{Q_0}\langle f_2\rangle_{Q_0}\mu(Q_0).\end{aligned}$$

To analyze the terms involving the bad parts in the Calderón-Zygmund decomposition, we write

$$\begin{aligned}\mathcal{P}_{\mathcal{F}^1}^{Q_0}(b_{1,Q}) &= \sum_{\substack{D=\overline{D}\in\mathcal{D}\\ D\cap Q^{(1)}\neq\emptyset}}\mathcal{D}_D b_{1,Q} - \sum_{\substack{R=\overline{R}\in\mathcal{F}^1\\ R\cap Q^{(1)}\neq\emptyset}}\sum_{\substack{T=\overline{T}\in\mathcal{D}\\ T\subseteq R\\ T\cap Q^{(1)}\neq\emptyset}}\mathcal{D}_T b_{1,Q}\\ &= \sum_{k=1}^{|x_Q|}\mathcal{D}_{Q^{(k)}} b_{1,Q} + \sum_{\substack{T=\overline{T}\in\mathcal{D}\\ T\subseteq Q^{(1)}\setminus\Omega^0}}\mathcal{D}_T b_{1,Q}.\end{aligned} \tag{4.5}$$

For every $T\subseteq Q^{(1)}\setminus\Omega^0$, we have $b_{1,Q}|_T\equiv\langle f\mathbb{1}_Q\rangle_{Q^{(1)}}$, and so the last sum in (4.5) vanishes. Using that $\mathcal{D}_{\overline{Q}}b_{1,Q}\equiv 0$ when Q is a singleton, the above formula reduces to

$$\mathcal{P}_{\mathcal{F}^1}^{Q_0}(b_{1,Q}) = \sum_{k=1}^{|x_Q|}\mathcal{D}_{\overline{Q}^{(k)}} b_{1,Q}. \tag{4.6}$$

Since $\mathcal{P}_{\mathcal{F}^1}^{Q_0}$ is self-adjoint, it is enough to study II and the estimate for III will follow. For every $k\in\{1,\dots,|x_Q|\}$, we put $g_2^{(k)} = g_2 - \langle g_2\rangle_{\overline{Q}^{(k)}}$. By (3.3), we have $\langle g_2^{(k)}\rangle_R \lesssim \|g_2\|_{BMO}$ for every $R\in\mathcal{D}^1(\overline{Q}^{(k)})$. Then, by the vanishing mean of $\mathcal{D}_{\overline{Q}^{(k)}}(b_{1,Q})$ and Proposition 3.2.9 we have

$$\langle \mathcal{D}_{\overline{Q}^{(k)}}(b_{1,Q}), g_2\rangle = \langle \mathcal{D}_{\overline{Q}^{(k)}}(b_{1,Q}), g_2 - \langle g_2\rangle_{\overline{Q}^{(k)}}\rangle$$

$$
\begin{aligned}
=&\|\mathcal{D}_{\overline{Q}^{(k)}}(b_{1,Q})\mathbb{1}_{\overline{Q}^{(k-1)}}\|_\infty \|g_2^{(k)}\mathbb{1}_{\overline{Q}^{(k-1)}}\|_1 \\
&+\sum_{z\in S(x_Q^{(k)})\setminus\{x_Q^{(k-1)}\}} \|\mathcal{D}_{\overline{Q}^{(k)}}(b_{1,Q})\mathbb{1}_{T_z}\|_\infty \|g_2^{(k)}\mathbb{1}_{T_z}\|_1 \\
\lesssim&\nu_{|x_Q|-k+1}^{|x_Q|}\|b_{1,Q}\|_1\|g_2\|_{BMO}\Big(1+\frac{1}{q(x_Q^{(k)})}\sum_{z\in S(x_Q^{(k)})\setminus\{x_Q^{(k-1)}\}} 1\Big) \\
\lesssim&\nu_{|x_Q|-k+1}^{|x_Q|}\|b_{1,Q}\|_1\|g_2\|_{BMO}.
\end{aligned}
$$

Summing over k yields

$$
\begin{aligned}
\sum_{Q\in\mathcal{F}^1}\langle\mathcal{P}_{\mathcal{F}^1}^{Q_0}(b_{1,Q}),g_2\rangle \lesssim& \sum_{Q\in\mathcal{F}^1}\|b_{1,Q}\|_1\|g_2\|_{BMO}\sum_{k=1}^{|x_Q|}\nu_{|x_Q|-k+1}^{|x_Q|} \\
\lesssim&\|f_1\|_{1,Q_0}\|f_2\|_{1,Q_0}\mu(Q_0).
\end{aligned}
$$

We are left with term IV, that we need to further split as follows:

$$
\begin{aligned}
\langle\mathcal{P}_{\mathcal{F}^1}^{Q_0}b_1,b_2\rangle =& \sum_{Q,R\in\mathcal{F}^1}\langle\mathcal{P}_{\mathcal{F}^1}^{Q_0}b_{1,Q},b_{2,R}\rangle \\
\le&2\sum_{Q\in\mathcal{F}^1}\sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}}\langle\mathcal{P}_{\mathcal{F}^1}^{Q_0}b_{1,Q},b_{2,R}\rangle \\
=&2\sum_{Q\in\mathcal{F}^1}\mathrm{IV}_Q,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{IV}_Q =&\sum_{x\in\mathfrak{X}}\mathcal{P}_{\mathcal{F}^1}^{Q_0}(b_{1,Q})(x)\sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} b_{2,R}(x)\mu(x) \\
=&\sum_{x\in Q^{(1)}}\mathcal{P}_{\mathcal{F}^1}^{Q_0}(b_{1,Q})(x)\sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} b_{2,R}(x)\mu(x) \\
&+\sum_{k=2}^{|x_Q|}\sum_{x\in Q^{(k)}\setminus Q^{(k-1)}}\mathcal{P}_{\mathcal{F}^1}^{Q_0}(b_{1,Q})(x)\sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} b_{2,R}(x)\mu(x) \\
=:&\,\mathrm{IV}_{Q,1}+\sum_{k=2}^{|x_Q|}\mathrm{IV}_{Q,k}.
\end{aligned}
$$

For $k\ge 2$ we have $b_{2,R}(x_Q^{(k)})=0$ because $\mu(R)\le\mu(Q)$. Therefore,

$$
|\mathrm{IV}_{Q,k}| = \Big|\sum_{x\in Q^{(k)}\setminus Q^{(k-1)}}\mathcal{P}_{\mathcal{F}^1}^{Q_0}(b_{1,Q})(x)\sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} b_{2,R}(x)\mu(x)\Big|
$$

$$
\begin{aligned}
&= \Big| \sum_{x\in Q^{(k)}\setminus(Q^{(k-1)}\cup\{x_Q^{(k)}\})} \sum_{y\in Q^{(1)}} (\mathcal{K}^{Q_0}_{\mathcal{F}^1}(x,y) - \mathcal{K}^{Q_0}_{\mathcal{F}^1}(x,x_Q)) b_{1,Q}(y)\mu(y) \sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} b_{2,R}(x)\mu(x)\Big| \\
&\le \|b_{1,Q}\|_1 \sup_{y\in Q^{(1)}} \sum_{x\in Q^{(k)}\setminus(Q^{(k-1)}\cup\{x_Q^{(k)}\})} |\mathcal{K}^{Q_0}_{\mathcal{F}^1}(x,y) - \mathcal{K}^{Q_0}_{\mathcal{F}^1}(x,x_Q)| \sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} |b_{2,R}(x)|\mu(x) \\
&\le \|b_{1,Q}\|_1 \left(\sum_{\substack{Q^{(k-1)}\neq S=\overline{S}\in\mathcal{D}\\ S^{(1)}=Q^{(k)}}} \sum_{x\in S} \sup_{y\in Q^{(1)}} |\mathcal{K}^{Q_0}_{\mathcal{F}^1}(x,y) - \mathcal{K}^{Q_0}_{\mathcal{F}^1}(x,x_Q)| \sum_{\substack{R\in\mathcal{F}^1,\, R\subsetneq S\\ \mu(R)\le\mu(Q)}} |b_{2,R}(x)|\mu(x) \right) \\
&\lesssim \|b_{1,Q}\|_1 \left(\sum_{\substack{Q^{(k-1)}\neq S=\overline{S}\in\mathcal{D}\\ S^{(1)}=Q^{(k)}}} \frac{1}{\mu(Q^{(k-1)})} \nu^{|x_Q|}_{|x_Q|-k} \sum_{x\in S} \sum_{\substack{R\in\mathcal{F}^1,\, R\subsetneq S\\ \mu(R)\le\mu(Q)}} |b_{2,R}(x)|\mu(x) \right) \\
&\le \|b_{1,Q}\|_1 \nu^{|x_Q|}_{|x_Q|-k} \left(\sum_{\substack{Q^{(k-1)}\neq S=\overline{S}\in\mathcal{D}\\ S^{(1)}=Q^{(k)}}} \langle f_2\rangle_S \right) \lesssim \|b_{1,Q}\|_1 \nu^{|x_Q|}_{|x_Q|-k+1} \langle f_2\rangle_{Q_0},
\end{aligned}
$$

by Proposition 3.2.8 and the maximality of the Calderón-Zygmund cubes Q. For $\mathrm{IV}_{Q,1}$ we use (4.6) to get

$$
\begin{aligned}
\mathrm{IV}_{Q,1} &= \sum_{x\in Q^{(1)}} \mathcal{P}^{Q_0}_{\mathcal{F}^1}(b_{1,Q})(x) \sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} b_{2,R}(x)\mu(x) \\
&= \sum_{x\in Q^{(1)}} \sum_{k=1}^{|x_Q|} \mathcal{D}_{\overline{Q}^{(k)}} b_{1,Q}(x) \sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} b_{2,R}(x)\mu(x) \\
&= \sum_{x\in Q^{(1)}} \mathcal{D}_{\overline{Q}^{(1)}} b_{1,Q}(x) \sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} b_{2,R}(x)\mu(x) \\
&\quad + \sum_{x\in Q^{(1)}} \sum_{k=2}^{|x_Q|} \mathcal{D}_{\overline{Q}^{(k)}} b_{1,Q}(x) \sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} b_{2,R}(x)\mu(x) \\
&=: \mathrm{IV}^a_{Q,1} + \mathrm{IV}^b_{Q,1}.
\end{aligned}
$$

We next use Proposition 3.2.9 to get

$$
\begin{aligned}
|\mathrm{IV}^b_{Q,1}| \le& \|b_{1,Q}\|_1 \sum_{k=2}^{|x_Q|} \frac{\nu^{|x_Q|}_{|x_Q|-k+1}}{\mu(Q^{(k-1)})} \sum_{x\in Q^{(1)}} \sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} b_{2,R}(x)\mu(x) \\
\le& \|b_{1,Q}\|_1 \sum_{k=2}^{|x_Q|} \nu^{|x_Q|}_{|x_Q|-k+1} \langle f_2\rangle_{Q^{(1)}}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \|b_{1,Q}\|_1 \langle f_2\rangle_{Q_0} \sum_{k=2}^{|x_Q|} \nu^{|x_Q|}_{|x_Q|-k+1}\\
&\lesssim \|b_{1,Q}\|_1 \langle f_2\rangle_{Q_0}.
\end{aligned}
$$

To estimate $\mathrm{IV}^a_{Q,1}$ we proceed differently depending on whether $Q = \overline{Q}$ or not. In the latter case, we use Proposition 3.2.9 and the fact that $\mu(Q) \simeq \mu(Q^{(1)})$ to get

$$
\begin{aligned}
|\mathrm{IV}^a_{Q,1}| &\le \frac{\nu^{|x_Q|}_{|x_Q|}}{\mu(Q)} \|b_{1,Q}\|_1 \sum_{x\in Q^{(1)}} \sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} |b_{2,R}(x)|\mu(x)\\
&\lesssim \|b_{1,Q}\|_1 \frac{1}{\mu(Q^{(1)})} \sum_{\substack{R\in\mathcal{F}^1\\ R\subsetneq Q^{(1)}}} \|b_{2,R}\|_1\\
&\lesssim \|b_{1,Q}\|_1 \langle f_2\rangle_{Q^{(1)}} \lesssim \|b_{1,Q}\|_1 \langle f_2\rangle_{Q_0}.
\end{aligned}
$$

If $Q = \overline{Q}$, then we split again as

$$
\begin{aligned}
\mathrm{IV}^a_{Q,1} =& \sum_{x\in Q} \mathcal{D}_{\overline{Q}^{(1)}} b_{1,Q}(x) \sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} b_{2,R}(x)\mu(x)\\
&+ \sum_{x\in Q^{(1)}\setminus(Q\cup\{x_Q^{(1)}\})} \mathcal{D}_{\overline{Q}^{(1)}} b_{1,Q}(x) \sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} b_{2,R}(x)\mu(x)\\
=&: \mathrm{IV}^{a,1}_{Q,1} + \mathrm{IV}^{a,2}_{Q,1}.
\end{aligned}
$$

Hence Proposition 3.2.9 yields

$$
\begin{aligned}
|\mathrm{IV}^{a,1}_{Q,1}| \le& \sum_{x\in Q} |\mathcal{D}_{\overline{Q}^{(1)}} b_{1,Q}(x)| \sum_{\substack{R\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}} |b_{2,R}(x)|\mu(x)\\
\lesssim& \|b_{1,Q}\|_1 \frac{1}{\mu(Q)} \sum_{x\in Q} \sum_{\substack{R\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}} |b_{2,R}(x)|\mu(x)\\
\le& \|b_{1,Q}\|_1 \left(\sum_{\substack{S\in\mathcal{D}_1(Q^{(1)})\\ S\ne Q}} \langle f_2 \mathbb{1}_S\rangle_{Q^{(1)}} + \langle f_2\rangle_Q \right)\\
\le& \|b_{1,Q}\|_1 \big(\langle f_2\rangle_{Q^{(1)}} + \langle f_2\rangle_Q\big) \lesssim \|b_{1,Q}\|_1 \big(\langle f_2\rangle_{Q_0} + \langle f_2\rangle_Q\big).
\end{aligned}
$$

Finally, for the last term, again by applying Proposition 3.2.9 and using the fact that if $\mu(R) \le \mu(Q)$ and $Q^{(1)} = R^{(1)}$ then $R = \overline{R}$,

$$
|\mathrm{IV}^{a,2}_{Q,1}| \le \sum_{x\in Q^{(1)}\setminus(Q\cup\{x_Q^{(1)}\})} |\mathcal{D}_{\overline{Q}^{(1)}} b_{1,Q}(x)| \sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} |b_{2,R}(x)|\mu(x)
$$

$$
\begin{aligned}
&\lesssim \|b_{1,Q}\|_1 \frac{1}{\mu(Q)} \frac{1}{q(x_Q^{(1)})} \sum_{x\in Q^{(1)}\setminus(Q\cup\{x_Q^{(1)}\})} \sum_{\substack{R\in\mathcal{F}^1\\ \mu(R)\le\mu(Q)}} |b_{2,R}(x)|\mu(x)\\
&\le \|b_{1,Q}\|_1 \frac{1}{\mu(Q)} \frac{1}{q(x_Q^{(1)})}\\
&\quad\times\Bigg(\sum_{\substack{S\in\mathcal{D}_1(Q^{(1)})\\ S\notin\mathcal{F}^1}} \sum_{\substack{R\in\mathcal{F}^1\\ R^{(1)}\subseteq S}} \sum_{x\in S} |b_{2,R}(x)|\mu(x) + \sum_{\substack{R=\overline{R}\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}} \sum_{x\in Q^{(1)}\setminus Q} |b_{2,R}(x)|\mu(x)\Bigg)\\
&\le \|b_{1,Q}\|_1 \Bigg(\frac{1}{q(x_Q^{(1)})} \sum_{\substack{S\in\mathcal{D}_1(Q^{(1)})\\ S\notin\mathcal{F}^1}} \langle \sum_{\substack{R\in\mathcal{F}^1\\ R^{(1)}\subseteq S}} |b_{2,R}|\rangle_S + \sum_{\substack{R=\overline{R}\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}} \frac{1}{q(x_Q^{(1)})} \frac{\|b_{2,R}\|_1}{\mu(Q)}\Bigg)\\
&\lesssim \|b_{1,Q}\|_1 \langle f_2\rangle_{Q_0} \Bigg(\frac{1}{q(x_Q^{(1)})} \sum_{\substack{S\in\mathcal{D}_1(Q^{(1)})\\ S\notin\mathcal{F}^1}} 1 \Bigg) + \|b_{1,Q}\|_1 \frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}} \langle f_2\rangle_R\\
&\le \|b_{1,Q}\|_1 \Bigg(\langle f_2\rangle_{Q_0} + \frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}} \langle f_2\rangle_R \Bigg).
\end{aligned}
$$

So, when Q is a sector, we control the term $\mathrm{IV}_{Q,1}$ by

$$
|\mathrm{IV}_{Q,1}| \lesssim \|b_{1,Q}\|_1 \Big(\langle f_2\rangle_{Q_0} + \langle f_2\rangle_Q + \frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}} \langle f_2\rangle_R \Big).
$$

Collecting all the estimates together yields

$$
\begin{aligned}
|\mathrm{IV}_Q| &\le |\mathrm{IV}_{Q,1}| + \sum_{k=2}^{|x_Q|} |\mathrm{IV}_{Q,k}|\\
&\lesssim \|b_{1,Q}\|_1 \Bigg(\langle f_2\rangle_{Q_0} \Big(1 + \sum_{k=2}^{|x_Q|} \nu_{|x_Q|-k+1}^{|x_Q|}\Big) + \langle f_2\rangle_Q + \frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}} \langle f_2\rangle_R \Bigg)\\
&\lesssim \|b_{1,Q}\|_1 \langle f_2\rangle_{Q_0} + \|b_{1,Q}\|_1 \langle f_2\rangle_Q + \|b_{1,Q}\|_1 \frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}} \langle f_2\rangle_R,
\end{aligned}
$$

whenever $Q = \overline{Q}$, and

$$
|\mathrm{IV}_Q| \lesssim \|b_{1,Q}\|_1 \langle f_2\rangle_{Q_0}
$$

in the singleton case. Finally, we obtain

$$
|\mathrm{IV}| \lesssim \sum_{Q\in\mathcal{F}^1} |\mathrm{IV}_Q|
$$

$$\begin{aligned}
&\lesssim \sum_{Q\in\mathcal{F}^1} \|b_{1,Q}\|_1\big(\langle f_2\rangle_{Q_0} + \langle f_2\rangle_Q\big) + \sum_{Q=\overline{Q}\in\mathcal{F}^1} \|b_{1,Q}\|_1 \frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}} \langle f_2\rangle_R \\
&\leq \langle f_1\rangle_{Q_0}\langle f_2\rangle_{Q_0}\mu(Q_0) + \sum_{Q\in\mathcal{F}^1} \langle f_1\rangle_Q\langle f_2\rangle_Q\mu(Q) \\
&\quad + \sum_{Q=\overline{Q}\in\mathcal{F}^1} \langle f_1\rangle_Q\Bigg(\frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}} \langle f_2\rangle_R\Bigg)\mu(Q).
\end{aligned}$$

This concludes the estimate for IV and yields (4.4), which in turn implies

$$\begin{aligned}
|\langle \mathcal{P}^{Q_0} f_1, f_2\rangle| \leq{} & C\Bigg(\langle f_1\rangle_{Q_0}\langle f_2\rangle_{Q_0}\mu(Q_0) + \sum_{Q\in\mathcal{F}^1} \langle f_1\rangle_Q\langle f_2\rangle_Q\mu(Q) \\
&+ \sum_{Q=\overline{Q}\in\mathcal{F}^1} \langle f_1\rangle_Q\Bigg(\frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{F}^1\\ R^{(1)}=Q^{(1)}}} \langle f_2\rangle_R\Bigg)\mu(Q)\Bigg) \\
&+ \sum_{R\in\mathcal{F}^1} \langle \mathcal{P}^R(f_1\mathbb{1}_R), f_2\mathbb{1}_R\rangle,
\end{aligned}$$

because $\operatorname{supp}(\mathcal{P}^R f) \subseteq R$ for each $R\in\mathcal{D}$. On the other hand,

$$\begin{aligned}
\mu(\Omega^0) \leq{} & \mu(\Omega_1^0) + \mu(\Omega_2^0) \\
\leq{} & \|\mathcal{M}_{\mathcal{D}}\|_{L^1(\mathfrak{X})\to L^{1,\infty}(\mathfrak{X})} \\
& \times \Big(\frac{\|f_1\|_1}{4\|\mathcal{M}_{\mathcal{D}}\|_{L^1(\mathfrak{X})\to L^{1,\infty}(\mathfrak{X})}\langle f_1\rangle_{Q_0}} + \frac{\|f_2\|_1}{4\|\mathcal{M}_{\mathcal{D}}\|_{L^1(\mathfrak{X})\to L^{1,\infty}(\mathfrak{X})}\langle f_2\rangle_{Q_0}}\Big) \\
={} & \frac{\mu(Q_0)}{2}.
\end{aligned}$$

We may then declare that $Q_0\in\mathcal{S}$ and choose $E_{Q_0} = Q_0\setminus\Omega^0$, because we have just proved that $\mu(E_{Q_0}) \geq \frac{1}{2}\mu(Q_0)$. The proof then ends by iteration. Indeed, for every $R\in\mathcal{F}^1$ we can now estimate

$$\begin{aligned}
|\langle \mathcal{P}^R(f_1\mathbb{1}_R), f_2\mathbb{1}_R\rangle| \leq{} & C\Bigg(\langle f_1\rangle_R\langle f_2\rangle_R\mu(R) + \sum_{T\in\mathcal{F}^1(R)} \langle f_1\rangle_T\langle f_2\rangle_T\mu(T) \\
&+ \sum_{T=\overline{T}\in\mathcal{F}^2(R)} \langle f_1\rangle_T\Bigg(\frac{1}{q(x_T^{(1)})} \sum_{\substack{S=\overline{S}\in\mathcal{F}^2(R)\\ S^{(1)}=T^{(1)}}} \langle f_2\rangle_S\Bigg)\mu(T)\Bigg) \\
&+ \sum_{T\in\mathcal{F}^2(R)} \langle \mathcal{P}^T(f_1\mathbb{1}_T), f_2\mathbb{1}_T\rangle,
\end{aligned}$$

applying (4.4). We then proclaim that $\mathcal{F}^1\subseteq\mathcal{S}$ and construct the corresponding sets E_R for each $R\in\mathcal{F}^1$ as before. The procedure can be iterated and yields the assertion after summing all the estimates for each cube. □

Remark 4.1.3. The proof of Theorem 4.1.2 was written down specifically for $\mathcal{P}$, but the result could be formulated in a much more general way. Clearly, the iterative scheme of proof and the construction of $\mathcal{S}$ do not rely on the specific definition of $\mathcal{P}$ and could have been carried over with any operator. Then, the core of the proof consists in estimating the action of the truncations $\mathcal{P}_{\mathcal{F}^1}^{Q_0}$ over the parts of the Calderón-Zygmund decomposition of f_1 and f_2. Those estimates in turn rely on uniform L^2-boundedness of truncations (good-good term), which can be formulated in a general way, and kernel estimates (good-bad and bad-bad terms). The kernel estimates and the form $\mathcal{E}_{\mathcal{S}}$ are what is specific of $\mathcal{P}$ and of the way in which we choose to split our operator into scales. It is conceivable that the same scheme of proof can be applied to the study of other operators whose structure is tied to that of a martingale filtration in any filtered space, finding wider applicability. In the nondoubling situation, however, one should expect that additional terms like $\mathcal{E}_{\mathcal{S}}$ play a role in the study of most sparse domination results to compensate the lack of regularity of the underlying filtration.

In the next corollary we show that Theorem 4.1.2 recovers the standard sparse domination result for $\mathcal{P}$ when $q : \mathfrak{X} \to \mathbb{N}$ is bounded.

Corollary 4.1.4. *Let $\mathfrak{X}$ be a radial tree such that q is bounded and let $f_1, f_2 \in L^1(\mathfrak{X})$. There exists a sparse collection $\mathcal{S}$ such that*

$$|\langle \mathcal{P} f_1, f_2 \rangle| \lesssim \langle \mathcal{A}_{\mathcal{S}} f_1, f_2 \rangle.$$

Proof. Assume that $q(x) \leq M$ for some $M > 0$ and every $x \in \mathfrak{X}$. In that case $\mu(Q^{(1)}) \lesssim \mu(Q)$ for all Q and so

$$\begin{aligned}
\mathcal{E}_{\mathcal{S}}(f_1, f_2) &= \sum_{Q \in \mathcal{S}} \langle f_1 \rangle_Q \frac{1}{q(x_Q^{(1)})} \left(\sum_{\substack{R = \overline{R} \in \mathcal{S} \\ R^{(1)} = Q^{(1)}}} \langle f_2 \rangle_R \right) \mu(Q) \\
&\leq \sum_{Q \in \mathcal{S}} \langle f_1 \rangle_Q \frac{1}{q(x_Q^{(1)})} \langle f_2 \rangle_{Q^{(1)}} \mu(Q^{(1)}) \\
&\lesssim \sum_{Q \in \mathcal{S}} \langle f_1 \rangle_{Q^{(1)}} \langle f_2 \rangle_{Q^{(1)}} \mu(Q^{(1)}).
\end{aligned}$$

But

$$\mu(E_Q) \gtrsim \mu(Q) \gtrsim \mu(Q^{(1)}),$$

which means that if $\mathcal{S}$ is sparse, then the family $\tilde{\mathcal{S}} = \{Q^{(1)} : Q \in \mathcal{S}\}$ is also sparse (with a different sparse constant), and that the bound given by Theorem 4.1.2 becomes

$$\langle \mathcal{P} f_1, f_2 \rangle \lesssim \langle \mathcal{A}_{\mathcal{S}} f_1, f_2 \rangle + \langle \mathcal{A}_{\tilde{\mathcal{S}}} f_1, f_2 \rangle.$$

This is a sparse estimate, because the union of two sparse families is sparse by Corollary 1.4.5. □

4.2 Weights and the B_p condition

Let now move to the study of the weighted inequalities that follow from Theorem 4.1.2. Keeping in mind Definition 1.2.4 of weights on $\mathbb{R}$, we define a weight on the tree as a function $w : \mathfrak{X} \to (0, \infty)$ and for $E \subseteq \mathfrak{X}$ denote

$$w(E) = \sum_{x \in E} w(x)\mu(x).$$

In the doubling setting, the natural class of weights is the Muckenhaupt class, consisting of the weights for which the maximal function is bounded on the corresponding weighted space. Notice that in our context the standard Hardy-Littlewood maximal function and the dyadic one coincide, since the family of all balls and the family of dyadic sets are the same. Then the dyadic Muckenhaupt class of weight on $\mathfrak{X}$ can be defined in analogy with the Euclidean setting by means of Theorem 1.2.6. For $p \in (1, \infty)$, let

$$B_p(\mathfrak{X}, \mu) := \Big\{ w \colon \mathfrak{X} \to (0, \infty) \colon [w]_{B_p(\mathfrak{X},\mu)} := \sup_{Q \in \mathcal{D}} \langle w \rangle_Q \langle w^{-\frac{1}{p-1}} \rangle_Q^{p-1} < \infty \Big\}$$

where $\langle w \rangle_Q = w(Q)/\mu(Q)$. In order to lighten the notation, the class $B_p(\mathfrak{X}, \mu)$ will be just denoted by $B_p(\mu)$. For $w \in B_p(\mathfrak{X}, \mu)$ and $p \in (1, \infty)$, we consider

$$L^p(\mathfrak{X}, w) := \Big\{ f \colon \mathfrak{X} \to \mathbb{C} \colon \|f\|_{p,w} := \Big(\sum_{x \in \mathfrak{X}} |f(x)|^p w(x)\mu(x) \Big)^{1/p} < \infty \Big\}.$$

Remark 4.2.1. Exactly as in Proposition 1.4.9, it can be shown that $\mathcal{A}_{\mathcal{S}}$ satisfies weighted inequalities for weights belonging to the Muckenhaupt class, regardless of whether the context is doubling or not. Therefore, for any radial tree $\mathfrak{X}$ and $w \in B_p(\mu)$, $\mathcal{A}_{\mathcal{S}}$ is bounded on $L^p(\mathfrak{X}, w)$ and

$$\|\mathcal{A}_{\mathcal{S}}\|_{L^p(\mathfrak{X},w) \to L^p(\mathfrak{X},w)} \lesssim [w]_{B_p(\mu)}^{\max\{1, \frac{1}{p-1}\}}.$$

Corollary 4.2.2. *Let $\mathfrak{X}$ be a radial tree with q bounded and let $p \in (1, \infty)$. If $w \in B_p(\mu)$, then for $f \in L^p(\mathfrak{X}, w)$*

$$\|\mathcal{P} f\|_{L^p(\mathfrak{X},w)} \lesssim [w]_{B_p(\mu)}^{\max\{1, \frac{1}{p-1}\}} \|f\|_{L^p(\mathfrak{X},w)}.$$

Remark 4.2.3. Observe that the sufficient condition we identified for the weights to ensure the weighted boundedness of the Bergman projection can be viewed as the discrete analogue of the condition found in [4] by Bekollé and Bonami for the same problem in the context of the hyperbolic disk. Indeed, our supremum is taken over all dyadic sets, but it is immediate to see that whenever Q is a singleton, the product of the two w-measures equals 1. Thus, we can simply take the supremum over the sectors, which correspond to the discrete equivalent of the Carleson boxes on the disk, over which the supremum is taken in [4]. In addition, our quantitative results (in particular the linear dependence on $[w]_{B_2(\mu)}$ in the L^2-bound) are in line with their continuous counterpart from [42]. This correspondence justifies the choice of referring to the class as B_p rather than A_p.

Now, let us focus on the more challenging problem of finding a suitable class of weights for the nondoubling case. The presence of $\mathcal{E}_{\mathcal{S}}$ in the sparse domination introduces additional complexities, and the Muckenhaupt class B_p turns out to be too large for our purposes there. To address this issue, we introduce a new class of weights, which is more refined and tailored to handle the extra term $\mathcal{E}_{\mathcal{S}}$,

$$\tilde{B}_p(\mu) := \Big\{ w\colon \mathfrak{X} \to (0,\infty)\colon [w]_{\tilde{B}_p(\mu)} := \sup_{\substack{Q,R\in\mathcal{D} \\ Q^{(1)}=R^{(1)}}} \langle w\rangle_Q \langle w^{-\frac{1}{p-1}}\rangle_R^{p-1} < \infty \Big\}.$$

It is straightforward that $[w]_{B_p(\mu)} \le [w]_{\tilde{B}_p(\mu)}$, hence $\tilde{B}_p(\mu) \subseteq B_p(\mu)$.

Before turning to the proof of the weighted estimates, we prove the following fact satisfied by $B_p(\mu)$ weights.

Lemma 4.2.4. *If $E \subseteq S$ and $\mu(E) \simeq \mu(S)$, then*

$$w(S) \lesssim [w]_{B_p(\mu)} w(E). \tag{4.7}$$

Proof. By Hölder's inequality

$$\begin{aligned} \mu(E) =& \sum_{x\in E} w^{\frac{1}{p}}(x) w^{-\frac{1}{p}}(x)\mu(x) \\ \le& \left(\sum_{x\in E} w(x)\mu(x)\right)^{\frac{1}{p}} \left(\sum_{x\in E} w^{-\frac{p'}{p}}(x)\mu(x)\right)^{\frac{1}{p'}} \\ =& w(E)^{\frac{1}{p}} \big(w^{-\frac{p'}{p}}(E)\big)^{\frac{1}{p'}}. \end{aligned}$$

By definition of $[w]_{B_p(\mu)}$ we have

$$[w]_{B_p(\mu)} \ge \frac{w(S)}{\mu(S)} \frac{\big(w^{-\frac{p'}{p}}(S)\big)^{\frac{p}{p'}}}{\mu(S)^{p-1}} \simeq \frac{w(S)\big(w^{-\frac{p'}{p}}(S)\big)^{\frac{p}{p'}}}{\mu(E)^p} \ge \frac{w(S)\big(w^{-\frac{p}{p'}}(S)\big)^{\frac{p}{p'}}}{w(E)\big(w^{-\frac{p'}{p}}(E)\big)^{\frac{p}{p'}}}$$

so that

$$\frac{w(S)}{w(E)} \lesssim [w]_{B_p(\mu)} \frac{\big(w^{-\frac{p'}{p}}(E)\big)^{\frac{p}{p'}}}{\big(w^{-\frac{p'}{p}}(S)\big)^{\frac{p}{p'}}} \le [w]_{B_p(\mu)}.$$

□

Theorem 4.2.5. *Let $\mathfrak{X}$ be any radial tree and let $p \in (1,\infty)$. If $w \in \tilde{B}_p(\mu)$, then for $f \in L^p(\mathfrak{X}, w)$*

$$\|\mathcal{P}f\|_{L^p(\mathfrak{X},w)} \lesssim_{[w]_{\tilde{B}_p(\mu)}} \|f\|_{L^p(\mathfrak{X},w)}.$$

Proof. Let $p \in (1,\infty)$, $f \in L^p(\mathfrak{X}, w)$ and $g \in L^{p'}(\mathfrak{X}, w)$. By Theorem 4.1.2 we have that there exists a sparse collection $\mathcal{S}$ such that

$$\langle \mathcal{P}f, gw\rangle \lesssim \langle \mathcal{A}_{\mathcal{S}} f, gw\rangle + \mathcal{E}_{\mathcal{S}}(f, gw).$$

Furthermore, by Remark 4.2.1, we already know that $\mathcal{A}_{\mathcal{S}}$ is bounded on $L^p(\mathfrak{X}, w)$ and we only have to deal with $\mathcal{E}_{\mathcal{S}}(f, gw)$:

$$\begin{aligned}
\mathcal{E}_{\mathcal{S}}(f, gw) =& \sum_{Q\in\mathcal{S}} \langle f\rangle_Q \left(\frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{S}\\ R^{(1)}=Q^{(1)}}} \langle gw\rangle_R \right) \mu(Q) \\
=& \sum_{Q\in\mathcal{S}} \langle f w^{\frac{p'}{p}}\rangle_{Q,w^{-\frac{p'}{p}}} (w^{-\frac{p'}{p}}(Q))^{\frac{1}{p}} \\
& \times \left(\frac{1}{q(x_Q^{(1)})} (w^{-\frac{p'}{p}}(Q))^{1-\frac{1}{p}} \sum_{\substack{R=\overline{R}\in\mathcal{S}\\ R^{(1)}=Q^{(1)}}} \langle gw\rangle_R \right) \\
\leq& \left(\sum_{Q\in\mathcal{S}} \langle f w^{\frac{p'}{p}}\rangle^p_{Q,w^{-\frac{p'}{p}}} w^{-\frac{p'}{p}}(Q) \right)^{\frac{1}{p}} \\
& \times \left(\sum_{Q\in\mathcal{S}} w^{-\frac{p'}{p}}(Q) \left(\frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{S}\\ R^{(1)}=Q^{(1)}}} \langle gw\rangle_R \right)^{p'} \right)^{\frac{1}{p'}}.
\end{aligned}$$

Concerning the first factor, by (4.7) and 4 of Proposition 1.2.7, i.e that $[w^{-\frac{p'}{p}}]_{B_{p'}(\mu)} = [w]^{\frac{p'}{p}}_{B_p(\mu)}$, we obtain

$$\begin{aligned}
\left(\sum_{Q\in\mathcal{S}} \langle f w^{\frac{p'}{p}}\rangle^p_{Q,w^{-\frac{p'}{p}}} w^{-\frac{p'}{p}}(Q) \right)^{\frac{1}{p}} \lesssim& [w^{-\frac{p'}{p}}]^{\frac{1}{p}}_{B_{p'}(\mu)} \left(\sum_{Q\in\mathcal{S}} \langle f w^{\frac{p'}{p}}\rangle^p_{Q,w^{-\frac{p'}{p}}} w^{-\frac{p'}{p}}(E_Q) \right)^{\frac{1}{p}} \\
\leq& [w]^{\frac{p'}{p^2}}_{B_p(\mu)} \left(\sum_{x\in\mathfrak{X}} \left(\mathcal{M}_{\mathcal{D},w^{-\frac{p'}{p}}} (f w^{\frac{p'}{p}}) \right)^p (x)\, w^{-\frac{p'}{p}}(x) \mu(x) \right)^{\frac{1}{p}} \\
\leq& [w]^{\frac{p'}{p^2}}_{B_p(\mu)} \left(\sum_{x\in\mathfrak{X}} f^p(x) w^{p'-\frac{p'}{p}}(x) \mu(x) \right)^{\frac{1}{p}} \\
=& [w]^{\frac{p'}{p^2}}_{B_p(\mu)} \|f\|_{L^p(\mathfrak{X},w)}.
\end{aligned}$$

For the second one we use Jensen's inequality to get

$$\left(\sum_{Q\in\mathcal{S}} w^{-\frac{p'}{p}}(Q) \left(\frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{S}\\ R^{(1)}=Q^{(1)}}} \langle gw\rangle_R \right)^{p'} \right)^{\frac{1}{p'}}$$

$$
\begin{aligned}
&= \left(\sum_{Q\in\mathcal{S}} \frac{w^{-\frac{p'}{p}}(Q)}{\mu(Q)^{p'}} \left(\frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{S}\\ R^{(1)}=Q^{(1)}}} \langle gw\rangle_{R,w} w(R) \right)^{p'} \right)^{\frac{1}{p'}} \\
&\leq \left(\sum_{Q\in\mathcal{S}} \frac{w^{-\frac{p'}{p}}(Q)}{\mu(Q)^{p'}} \frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{S}\\ R^{(1)}=Q^{(1)}}} \langle gw\rangle_{R,w}^{p'} w(R) w(R)^{p'-1} \right)^{\frac{1}{p'}} \\
&\lesssim \left(\sup_{\substack{Q,R\in\mathcal{D}\\ Q^{(1)}=R^{(1)}}} \frac{w^{-\frac{p'}{p}}(Q)}{\mu(Q)} \frac{w(R)^{p'-1}}{\mu(R)^{p'-1}} \right)^{\frac{1}{p'}} \\
&\quad \times \left(\sum_{Q\in\mathcal{S}} \frac{1}{q(x_Q^{(1)})} \sum_{\substack{R=\overline{R}\in\mathcal{S}\\ R^{(1)}=Q^{(1)}}} \langle gw\rangle_{R,w}^{p'} w(R) \right)^{\frac{1}{p'}} \\
&\leq \left(\sup_{\substack{Q,R\in\mathcal{D}\\ Q^{(1)}=R^{(1)}}} \langle w^{-\frac{p'}{p}}\rangle_Q^{\frac{1}{p'}} \langle w\rangle_R^{\frac{1}{p}} \right) \left(\sum_{Q\in\mathcal{S}} \langle gw\rangle_{Q,w}^{p'} w(Q) \right)^{\frac{1}{p'}} \\
&\lesssim [w]_{\tilde{B}_p(\mu)}^{\frac{1}{p}} [w]_{B_p(\mu)}^{\frac{1}{p'}} \|g\|_{L^{p'}(\mathfrak{X},w)},
\end{aligned}
$$

where in the last step we proceeded as we did for the first factor. Finally, gathering all the estimates together

$$
\mathcal{E}_{\mathcal{S}}(f,gw) \lesssim [w]_{\tilde{B}_p(\mu)}^{\frac{1}{p}} [w]_{B_p(\mu)}^{\frac{p'}{p^2}+\frac{1}{p'}} \|f\|_{L^p(\mathfrak{X},w)} \|g\|_{L^{p'}(\mathfrak{X},w)}
$$

and then

$$
\langle \mathcal{P}f, gw\rangle \lesssim [w]_{\tilde{B}_p(\mu)}^{\frac{1}{p}} [w]_{B_p(\mu)}^{\frac{p'}{p^2}+\frac{1}{p'}} \|f\|_{L^p(\mathfrak{X},w)} \|g\|_{L^{p'}(\mathfrak{X},w)}.
$$

□

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