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Locally trivial monodromy of moduli spaces of sheaves on Abelian surfaces

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Abstract

The locally trivial monodromy group is an important locally trivial deformation invariant for irreducible symplectic varieties and plays a fundamental role in their bimeromorphic classification, by Global Torelli Theorem. While this group has been determined for all known deformation classes in the smooth case, this problem has only been partially addressed in the singular setting. This Thesis contributes to completing this picture by explicitly computing the locally trivial monodromy group for a distinguished and rich class of irreducible symplectic varieties, namely singular moduli spaces of sheaves on Abelian surfaces. We establish a lattice-theoretic description of this group and provide a clear geometric interpretation of the latter: we prove that it is isomorphic to the classical monodromy group of a smooth moduli space of sheaves of the same kind, embedded within the most singular locus of the singular moduli space. Moreover, its generators are explicitly described as isometries induced by monodromy operators of the underlying surface and certain Fourier-Mukai equivalences on the derived category of the latter. Finally, as a main geometric application of the monodromy description, we prove the SYZ conjecture for this locally trivial deformation class of singular symplectic varieties, showing that any nef and isotropic line bundle induces a Lagrangian fibration.

Introduction

The classification of complex varieties is a central problem in Algebraic Geometry. In this context, a role of great relevance is played by varieties with numerically trivial canonical divisor. Bogomolov decomposition Theorem ([Bog74], [Bea83], [GKP11], [Dr18], [DG18], [GGK19], [HP19]) allows reducing the problem of their classification to the study of three classes of varieties: complex tori, irreducible Calabi-Yau varieties, and *irreducible symplectic varieties* (Chapter 1). This Thesis focuses on the study of the latter, and fits into a line of research that has been very active and in constant evolution over the last forty years.

The first case to be studied and which guided the development of this theory was the smooth case of *irreducible holomorphic symplectic manifolds*, also known as *compact hyperkähler manifolds*. This study led to the construction of several classes of examples of these manifolds and to a refined methodological framework for their classification.

Up to deformation, the currently known examples of irreducible holomorphic symplectic manifolds are K3 surfaces in dimension two, Hilbert schemes of points on K3 surfaces and generalized Kummer manifolds on Abelian surfaces in every even dimension ([Bea83]), and two sporadic examples in dimension six and ten, obtained as desingularizations of moduli spaces of sheaves on K3 and Abelian surfaces ([OG99], [OG03]). These have been the only families of examples known for several years. The compelling challenge of producing new examples of this type has animated a huge sector of this research field. However, the difficulty of this task has also led to an interest in their singular analogs, as removing the smoothness assumption makes it significantly easier to find new examples, and as they are more natural in view of the Minimal Model Program.

In the singular setting, we encounter various examples of irreducible symplectic varieties, arising from constructions of diverse nature: for instance, contractions of K3 surfaces and Hilbert schemes of points on these ([GPP24], [LBP25]), orbifolds obtained as quotients of smooth symplectic varieties by the action of specific automorphism groups (and their partial resolutions - see [BGMM25], [FM21], [MT07], [Men14], [Men18], [Men18], [Men20], [Men22b], [Mon13]), and moduli spaces of semistable sheaves on K3 and Abelian surfaces ([Mu84], [Mu87], [OG97], [Yos01a], [KLS06], [PR23], [PR24]) - the latter being a particularly rich source of examples and of central relevance to this work (Chapter 3). In this setting, a rigorous classification is currently far from being achieved, yet the methodological approach to this problem is well prescribed by Torelli Theorems ([Bea83], [Mar11], [Ver13], [BL21], [BL22]).

For an irreducible symplectic variety - both smooth and singular - Global Torelli Theorem (Chapter 2) asserts that most of its geometry is encoded in its second integral cohomology group, specifically in its lattice and weight two Hodge structure, through a *period map*. In the case of K3 surfaces, which inspired this construction, the period completely determines the geometry, leading to the classical formulation of Global Torelli Theorem: *"two K3 surfaces are bimeromorphic (hence isomorphic) if and only if there exists a Hodge isometry between their respective intersection lattices"*.

In higher dimensions, a new tool is introduced into this classification framework to measure how far the period is from fully determining the geometry of an irreducible symplectic variety: the *(locally trivial) monodromy group*. This special group of isometries arising from deformations represents the natural constraint that quantifies the distance between the formulation of Global Torelli Theorem for a deformation class of irreducible symplectic varieties and the classical one stated for K3 surfaces.

In the smooth case, the monodromy group has been computed for any known deformation class ([Mar08], [Mar22], [Mon16], [MR21], [Ono22]), while in the singular setting this problem has been addressed only partially. Indeed, the locally trivial monodromy group is only known for some special classes of orbifolds ([BMM24], [Nan25]) and is almost complete in the case of singular moduli spaces of sheaves. More precisely, after such computation for two sporadic examples in the case of moduli spaces of sheaves admitting a desingularization ([MR21], [Ono22]), a systematic computation for all singular moduli spaces of sheaves on K3 surfaces has recently been achieved ([OPR24]). The missing piece in this framework is a systematic computation of the locally trivial monodromy group of singular moduli spaces of sheaves on Abelian surfaces, which is precisely the subject of this work (Part II).

Main results

The research contribution of this work is presented in Part II and the main results are listed in the following.

* **Theorem A.1** (Theorem 5.2.1, Theorem 6.3.4 - second part, Corollary 6.3.6 - second part): we provide a lattice-theoretic description of the locally trivial monodromy group $\mathrm{Mon}^2_{\mathrm{lt}}(X)$ of an irreducible symplectic variety X that is locally trivial deformation equivalent to a singular moduli space of sheaves $K_v(S,H)$ on an Abelian surface S.

  The outcome is that, up to two exceptions discussed in Section 6.4, the group $\mathrm{Mon}^2_{\mathrm{lt}}(X)$ coincides with the group $\mathsf{N}(X)$ of orientation preserving isometries of $\mathrm{H}^2(X,\mathbb{Z})$ acting as $\pm\mathrm{id}$ on its discriminant group and belonging to the kernel of the character det $\cdot$ disc (Appendix B.4). The lattice-theoretic description coincides with that achieved by [Mar22], [Mon16] in the smooth case. We also point out that, in all of these cases, all deformations are locally trivial, hence $\mathrm{Mon}^2(X) = \mathrm{Mon}^2_{\mathrm{lt}}(X) = \mathsf{N}(X)$.

Moreover, we manage to establish an inclusion $\mathrm{Mon}^2_{\mathrm{lt}}(X) \supseteq \mathsf{N}(X)$, holding also in the exceptional cases.

* **Theorem B.1** (Corollary 6.3.3,Theorem 6.3.4 - first part, Corollary 6.3.6 - first part): we relate the monodromy group of a singular moduli space of sheaves $K_v(S,H)$ - with $v = mw$, $m > 1$ and w a primitive Mukai vector of square greater than 4 - to the monodromy group of a smooth moduli space of sheaves $K_w(S,H)$ by means of an isomorphism of groups arising from a geometric construction. We show that, for any irreducible symplectic variety X as above, each connected component Z of its most singular locus is an irreducible holomorphic symplectic manifold deformation of $K_w(S,H)$ and its closed embedding $i_{Z,X}\colon Z \hookrightarrow X$ induces an isomorphism of groups $i^{\sharp}_{Z,X}\colon \mathrm{Mon}^2(X) \to \mathrm{Mon}^2(Z)$.

* **Theorem A.2** (Theorem 7.2.4, Corollary 7.2.14): as a geometric application of Theorem B.1, we prove the SYZ conjecture for any irreducible symplectic variety X as above, by showing that, for any nef and isotropic line bundle L on X, there exists a Lagrangian fibration $f\colon X \to B$ such that $L = f^*\mathcal{O}_B(1)$ and we compute its polarization type.

Structure of the Thesis

The Thesis is organized into two Parts and an Appendix.

Part I provides the essential preliminary material, introducing the theory of irreducible symplectic varieties and outlining their classification problem, along with the methodological framework developed to address it. Moreover, some examples are presented, with main focus on the case of moduli spaces of sheaves on K3 and Abelian surfaces.

In **Chapter 1** we present the general theory of irreducible symplectic varieties. We recall the main definitions, describe their key properties and provide an overview of known examples. We begin by introducing the smooth case of irreducible holomorphic symplectic manifolds, in Section 1.1. We discuss their fundamental properties and we review the deformation classes known so far. In Section 1.2 we shift our focus to the singular setting. We introduce the notions of primitive and irreducible symplectic varieties, as generalizations of the notion of irreducible holomorphic symplectic manifold, we study their main features and their singularities type, and we present some key examples.

In **Chapter 2** we address the problem of the classification of this kind of varieties via Torelli Theorems and we point out the fundamental role played by the locally trivial monodromy group at this stage. After presenting Local and Global Torelli Theorems and carefully defining monodromy operators in the smooth setting, in Section 2.1, we introduce the technical machinery of locally trivial deformations for singular symplectic varieties, in Section 2.2. In Section 2.3 we state a generalization of Global Torelli Theorem for primitive symplectic varieties and we define the locally trivial monodromy group.

Chapter 3 is devoted to presenting the theory of moduli spaces of sheaves on K3 and Abelian surfaces. In Section 3.1 we outline their construction - starting from a triple (S, v, H) made of a K3 or Abelian surface S, a Mukai vector v of type (m, k), and a $v-$generic polarization H (see Definition 3.1.15) - and we explain how these provide examples of irreducible symplectic varieties, both smooth and singular. Their key properties, needed to approach their classification in light of the previous discussion, are presented in Section 3.2 and Section 3.3, respectively, which contain some insights on their locally trivial deformations and on their second integral cohomology group, respectively. A remarkable property is that both these features only depend on the numerical data (m, k) associated to the Mukai vector. Finally, in Section 3.4, a description of the locally trivial monodromy group of moduli spaces of sheaves is provided in all cases where this computation is known in the literature. This overview brings to our attention that the only missing piece is a systematic computation of the locally trivial monodromy group of singular moduli spaces of sheaves on Abelian surfaces.

Part II is entirely dedicated to the original research content of this work, consisting of the computation of the locally trivial monodromy group of singular moduli spaces of sheaves on Abelian surfaces. Each of the first three Chapters constitutes a step in the proof of the main result, while the final Chapter is devoted to its consequences.

In **Chapter 4** we introduce a groupoid representation designed to encode the monodromy information of a moduli space in a natural way. In Sections 4.1 and 4.2 we define a groupoid $\mathcal{G}^{m,k}$ of triples (S, v, H) of $(m, k)-$type, whose morphisms come from deformations of the triple itself, or from Fourier-Mukai equivalences, respectively. In Section 4.3 we construct two different representations of $\mathcal{G}^{m,k}$ with values in groupoids of free $\mathbb{Z}-$modules, formalizing the fact that morphisms in the above-mentioned groupoid induce locally trivial parallel transport operators. This is the technical tool that allows us to construct monodromy operators in the next Chapter.

In **Chapter 5** approach a first lattice theoretic description of the locally trivial monodormy group, by exhibiting an important subgroup, namely the group $\mathsf{N}(K_v(S, H))$ introduced in Appendix B.4 and Section 3.4. In Section 5.1 we show that monodromy operators of the underlying Abelian surface S induce locally trivial monodromy operators on $K_v(S, H)$ via the groupoid representation defined in

Section 4.3. In Section 5.2 we include the group $\mathsf{N}(K_v(S,H))$ in $\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H))$ by showing that its generators belong to the image of the above-mentioned representation.

In **Chapter 6** we relate the locally trivial monodromy group of a singular moduli space $K_v(S,H)$ with the monodromy group of a smooth moduli space by means of an isomorphism which arises from an explicit geometric construction. The key point is the study of the most singular locus of $K_v(S,H)$ and the action on the second integral cohomology group of its closed embedding, that will be addressed, respectively, in Sections 6.1 and 6.2. The latter will be used in Section 6.3 to produce an injective morphism on the respective locally trivial monodromy groups. Surjectivity of the latter will be shown building on the lattice theoretic description achieved in Chapter 5, especially the inclusion of Section 5.2. Finally, in Section 6.4, we discuss some necessary numerical assumptions on the Mukai vectors v considered.

In **Chapter 7**, we discuss some applications of the monodromy description achieved in Chapter 6. In Section 7.1, we deduce some straightforward applications of the main Theorem in its lattice-theoretic formulation (Theorem A.1), particularly in relation to the issue of the classification of irreducible symplectic varieties belonging to this locally trivial deformation class. In Section 7.2, we present an explicit geometric application of the main result in its intrinsic formulation (Theorem B.1), proving the SYZ conjecture for this class of irreducible symplectic varieties.

Finally, we collect in the **Appendix** further preliminary material and technical tools that support the main body of the Thesis.

In **Appendix A** we recall the general theory of moduli spaces of sheaves, as the solution of the problem of defining a scheme that parametrizes sheaves on a polarized varieties, with fixed numerical invariants and satisfying suitable stability conditions. In Section A.1 we introduce stability conditions for coherent sheaves and we set a natural way to fix their numerical invariants. In Section A.2 we state the moduli problem formalizing the question of the existence of such a parametrizing space and in Section A.3 we sketch the construction of the latter via Geometric Invariant Theory.

In **Appendix B** we gather technical results on lattice theory that are used throughout the computations in Chapter 5. In Section B.1 we introduce some special groups of isometries defined as kernels of natural characters and group morphisms. In Section B.2 we study some properties of primitive embeddings of lattices, together with some classical existence and uniqueness results for lattices, in Section B.3. We conclude the Appendix by introducing the special groups of reflections that arise in the lattice-theoretic description of the locally trivial monodromy groups of moduli spaces of sheaves and the monodromy group of the underlying surfaces, in Sections B.4 and B.5, respectively.

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Contents

Part I

Irreducible symplectic varieties: examples, classification and monodromy problems

Chapter 1

Irreducible symplectic varieties

Over the past decades, the study of *irreducible holomorphic symplectic manifolds* - also known as compact *hyperkähler* manifolds - has been a popular and steadily evolving research topic in complex algebraic geometry. Among the main motivations of this momentum we find foundational classification results such as Bogomolov Decomposition Theorem, which identifies the latter as a distinguished class of manifolds with numerically trivial first Chern class, and by the ambitious challenge of constructing new examples of such manifolds. This class of manifolds has been intensively studied and the theory of their bimeromorphic classification has reached a good level of refinement. More recently, attention has shifted towards the singular setting, with the development of a theory of singular symplectic varieties that generalizes the key features of hyperkähler geometry and its classification theory, while at the same time bringing a wide range of new examples into the picture.

This Chapter is devoted to introducing the theory of such varieties, collecting the main definitions, describing the key properties that characterize them and providing an overview of known examples. We will begin by dealing with the smooth case of *irreducible holomorphic symplectic manifolds*, in Section 1.1, and then move to the singular setting, in Section 1.2, focusing on the notions of *primitive* and *irreducible symplectic varieties*, that will be crucial for the second part of this work.

1.1 Irreducible holomorphic symplectic manifolds

We begin by considering the smooth case of irreducible holomorphic symplectic manifolds. This choice is motivated both by historical reasons, as these manifolds were the first to be studied, and by the fact that their theory provides a natural and technically simpler starting point for the development of the singular case through suitable generalizations. In Section 1.1.1, we introduce the main definitions and fundamental properties of irreducible holomorphic symplectic manifolds, together with the key tools needed to address the study of their geometry, while in Section

1.1.2, we will briefly review the known families of examples of this kind of manifolds.

1.1.1 Definition and main properties

In the following, we will work only with smooth manifolds defined over the complex field $\mathbb{C}$, referring to these as *complex manifolds*.

Definition 1.1.1. Let X be a compact complex Kähler manifold.

(1) A *holomorphic symplectic form* on X is a closed holomorphic $2-$form $\sigma_X \in \mathrm{H}^0(X, \Omega^2_X)$ which is everywhere non-degenerate.

(2) The manifold X is an *irreducible holomorphic symplectic* (*IHS*) manifold if it is simply connected and there exists a holomorphic symplectic form such that $\mathrm{H}^0(X, \Omega^2_X) \simeq \mathbb{C}\sigma_X$ as complex vector spaces.

Remark 1.1.2. If X is an IHS manifold, then, by definition, the holomorphic symplectic form σ_X is unique up to scalar multiplication. Furthermore, from the existence of the latter, we can deduce the following fundamental properties:

(1) The (complex) dimension $\dim X$ of X is even. Indeed, the holomorphic symplectic form σ_X acts point-wisely as an alternating form on the tangent space. By non-degeneracy, we conclude that there exists a positive integer $n \in \mathbb{N}^*$ such that, for any $x \in X$, it holds $\dim(T_x X) = 2n$.

(2) The non-degeneracy of the symplectic form also induces an isomorphism between the tangent bundle $\mathcal{T}_X$ of X and the cotangent bundle Ω^1_X of X.

(3) The canonical bundle K_X of X is trivial, as, set $\dim X = 2n$, the closed holomorphic $n-$form $\sigma_X^n \in \mathrm{H}^0(X, \Omega^n_X)$ defines a trivializing section. In particular, we get $\mathrm{c}_1(X) = 0$.

In the more general context of compact complex Kähler manifolds with numerically trivial first Chern class, irreducible holomorphic symplectic manifolds play a fundamental role. Their classificatory importance stems from Bogomolov Decomposition Theorem, which identifies IHS manifolds as fundamental building blocks in this framework:

Theorem 1.1.3. *Let M be a compact, complex Kähler manifold such that $\mathrm{c}_1(M)_{\mathbb{R}} = 0 \in \mathrm{H}^2(M, \mathbb{R})$. Then there exists a finite étale covering $\widetilde{M} \to M$ such that $\widetilde{M}$ decomposes as a finite product*

$$\widetilde{M} = T \times \prod_i Y_i \times \prod_j X_j,$$

where T is a complex torus, Y_i are irreducible Calabi-Yau manifolds for every i and X_j are irreducible holomorphic symplectic manifolds for every j.

Proof. See [Bog74] for the original statement, see also [Bea83, Section 5, Théorème 2] for a complete proof. □

Remark 1.1.4. Irreducible holomorphic symplectic manifolds represent the algebraic counterpart of compact hyperkähler manifolds. We recall that a Riemannian manifold (M, g) is called *hyperkähler (HK)* if there exist three complex structures I, J and K on M satisfying the quaternionic relations and such that the metric g is Kähler with respect to I, J and K. Notice that this condition provides a family of complex structures, parametrized by $\mathbb{P}^1$, with respect to which the metric g is Kähler.

According to Berger's classification of Riemannian holonomy groups ([Ber55]), hyperkähler manifolds correspond to Riemannian manifolds (M, g) of real dimension $4n$ whose holonomy group equals to the symplectic group $Sp(n)$.

If (M, g) is a compact hyperkähler manifold and I, J and K are complex structures as above, then $X = (M, I)$ is an IHS manifold, with symplectic form $\sigma_I = \omega_J + i\omega_K$, where ω_J and ω_K are the Kähler forms associated to J and K, respectively. Conversely, as a consequence of Yau's solution of the Calabi conjecture, if X is an IHS manifold, then any Kähler class $\omega \in \mathrm{H}^2(X, \mathbb{R})$ determines a unique Ricci-flat metric g on the underlying real manifold M such that (M, g) is hyperkähler.

In light of Remark 1.1.4, with a slight abuse of terminology, we will refer to manifolds satisfying Definition 1.1.1 also as *hyperkähler manifolds*.

Second integral cohomology

We conclude this preliminary overview of foundational results on IHS manifolds by focusing on some key properties of the second integral cohomology group of the latter. Similarly as in the case of K3 surfaces, this object carries significant geometric information on the manifolds under study and will be used to approach their classification, as will be explained in the next Sections (1.1.2 and 2.1). In the following, we will let X be a IHS manifold.

(P1) $\mathrm{H}^2(X, \mathbb{Z})$ is a free $\mathbb{Z}-$module of rank $b_2(X)$.

Indeed, as X is simply connected by definition, Universal Coefficient Theorem for Cohomology implies that $\mathrm{H}^2(X, \mathbb{Z})$ is torsion free.

Remark 1.1.5. From simple connectedness we also deduce that $\mathrm{H}^1(X, \mathbb{Z})$ is trivial, hence the first Chern class provides an isomorphism between the Picard group $\mathrm{Pic}(X)$ and the Néron-Severi group $\mathrm{NS}(X)$ of X. Under this identification, we will consider the Picard group as a subgroup of $\mathrm{H}^2(X, \mathbb{Z})$.

(P2) $\mathrm{H}^2(X, \mathbb{Z})$ carries a non-degenerate integral symmetric form q_X of signature $(3, b_2(X) - 3)$, called *Beauville-Bogomolov-Fujiki (BBF) form*. By denoting by $q_X(\cdot, \cdot)$ the associated bilinear form, we will refer to the lattice $(\mathrm{H}^2(X, \mathbb{Z}), q_X)$ as the *Beauville-Bogomolov-Fujiki (BBF) lattice* of X.

The existence of the BBF form is due to Beauville, as stated in the following Theorem, together with some key properties of the latter.

Theorem 1.1.6. *Let n be a positive integer, let X be an IHS manifold of dimension $2n$ and let $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$ such that $\int_X (\sigma\overline{\sigma})^n = 1$.*

(1) The assignation, for any $\alpha \in \mathrm{H}^2(X, \mathbb{R})$,

$$f_X(\alpha) := \frac{n}{2}\int_X (\sigma\overline{\sigma})^{n-1}\alpha^2 + (1-n)\Big(\int_X \sigma^{n-1}\overline{\sigma}^n\alpha\Big)\cdot\Big(\int_X \sigma^n\overline{\sigma}^{n-1}\alpha\Big) \tag{1.1}$$

defines a quadratic form on $\mathrm{H}^2(X, \mathbb{R})$. Furthermore, there exists a positive constant $c \in \mathbb{R}$ such that $c \cdot f_X =: q_X$ is a primitive integral quadratic form on $\mathrm{H}^2(X, \mathbb{Z})$ of signature $(3, b_2(X) - 3)$.

(2) For any σ as above, it holds

$$q_X(\sigma) = 0 \quad \textit{and} \quad q_X(\sigma + \overline{\sigma}) > 0.$$

(3) For any Kähler class $\omega \in \mathrm{H}^2(X, \mathbb{R})$, it holds $q_X(\omega) > 0$.

Proof. See [Bea83, Section 8, Théorème 5] or [Huy99, Sections 1.4-1.9]. □

Another remarkable property of the BBF form was proved by Fujiki in [Fuj87], showing that there exists a positive rational constant $c_X \in \mathbb{Q}$, called *Fujiki constant* of X, such that, for any $\alpha \in \mathrm{H}^2(X, \mathbb{Z})$,

$$\int_X \alpha^{2n} = c_X \cdot q_X(\alpha)^n. \tag{1.2}$$

Remark 1.1.7. We point out that equality (1.2) implies that the BBF form q_X defines a differential and topological invariant of the manifold X.

Further details concerning the geometric information encoded in the BBF lattice structure will be addressed in the next Sections. We conclude this part with the last property of $\mathrm{H}^2(X, \mathbb{Z})$ that needs to be highlighted for classification purposes and, in contrast with the above-mentioned ones, strongly depends on the complex structure of X.

(P3) $\mathrm{H}^2(X, \mathbb{Z})$ admits a pure weight 2 Hodge structure, given by

$$\mathrm{H}^2(X, \mathbb{C}) = \mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{1,1}(X) \oplus \mathrm{H}^{0,2}(X),$$

where $\mathrm{H}^{2,0}(X) = \mathrm{H}^0(X, \Omega_X^2) \simeq \mathbb{C}\sigma_X$, $\mathrm{H}^{0,2}(X) = \mathrm{H}^2(X, \mathcal{O}_X) \simeq \mathbb{C}\overline{\sigma_X}$ and $\mathrm{H}^{1,1}(X) = \mathrm{H}^1(X, \Omega_X^1)$.

This is straightforward from Hodge Decomposition Theorem, as X is a compact Kähler manifold, and from Definition 1.1.1.

Remark 1.1.8. From (1.1), we deduce that $\mathrm{H}^{1,1}(X)$ is orthogonal to $\mathrm{H}^{2,0}(X) \oplus \mathrm{H}^{0,2}(X)$ with respect to the BBF form. Moreover, from (2) and (3) of Theorem 1.1.6, it follows that, for any Kähler class $\omega \in \mathrm{H}^{1,1}(X, \mathbb{R})$, the real vector space

$$\langle \omega, \mathrm{Re}(\sigma), \mathrm{Im}(\sigma) \rangle_{\mathbb{R}}$$

is positive with respect to $q_{X,\mathbb{R}}$ and it can be checked that the latter provides the prescribed signature $(3, b_2(X) - 3)$.

These considerations will be used in the next Chapter to provide a well defined notion of *period* for IHS manifolds and discuss their moduli theory.

1.1.2 Examples

We will now present some fundamental examples of IHS manifolds in every even dimension and briefly discuss their deformation theory, following [Huy99, Section 1.12]. In particular, we will see that deformations of the examples described below provide many further examples of IHS manifolds. On the other hand, the corresponding deformation types are the only ones known so far: every known example of an IHS manifold belongs to one of these families. Nonetheless, there is currently no evidence that these are the only possible deformation types in any fixed dimension, with the only exception of the case of surfaces.

We recall that a *deformation* of a compact complex manifold X is a smooth and proper holomorphic map $f\colon \mathcal{X} \to T$ (*family*) of analytic spaces onto a connected base T, such that there exists a distinguished point $t \in T$ such that the fiber $\mathcal{X}_t := f^{-1}(t)$ over t is isomorphic to X.

Remark 1.1.9. If X is an IHS manifold, as $\mathrm{H}^0(X, \mathcal{T}_X) \simeq \mathrm{H}^0(X, \Omega^1_X) = 0$, there exists a *universal deformation* $f\colon \mathcal{X} \to Def(X)$ of X, meaning that, for any deformation $f_T\colon \mathcal{X}_T \to T$ of X, there exists a unique holomorphic map $T \to Def(X)$ such that $\mathcal{X}_T \simeq \mathcal{X} \times_{Def(X)} T$. Here, the base $Def(X)$ is the germ of an analytic space and there exists a point $0 \in Def(X)$ such that $\mathcal{X}_0 \simeq X$.

Furthermore, by a result of Tian ([Tia87]) and Todorov ([Tod89]), the base $Def(X)$ is smooth of dimension $h^1(X, \mathcal{T}_X) = h^1(X, \Omega_X) = h^{1,1}(X)$ and in this case the deformations of X are said to be *unobstructed*.

From now on, given an IHS manifold X, up to shrinking the base, we will deal only with *small* deformations of X, i.e. families with base an analytic open neighborhood of 0 in $Def(X)$.

Theorem 1.1.10. *Let X be an IHS manifold and let $f\colon \mathcal{X} \to T$ be a small deformation of X. Then, for any $t \in T$, the fiber $\mathcal{X}_t$ over t is an IHS manifold.*

Proof. See [Bea83, Section 8, Proposition 9]. □

Hence, any small deformation of an IHS manifold is again an IHS manifold and the notion of *deformation of IHS manifolds* is well posed.

At this stage, we recall that deformation equivalent complex manifolds already share significant properties, for instance the underlying differentiable structures. Indeed, by Ehresman Fibration Theorem, if two such complex manifolds are deformation equivalent, then the underlying differentiable manifolds are diffeomorphic. In the case of IHS manifolds, this fact has a deeper implication on the respective BBF lattice structures, which is straightforward from Theorem 1.1.6 and Remark 1.1.7.

Corollary 1.1.11. *The Beauville-Bogomolov-Fujiki lattice is a deformation invariant, i.e., for any deformation equivalent IHS manifolds X_1 and X_2, there exists an isomorphisms of $\mathbb{Z}-$modules and isometry*

$$\varphi \in \mathrm{O}((\mathrm{H}^2(X_1, \mathbb{Z}), q_{X_1}), (\mathrm{H}^2(X_2, \mathbb{Z}), q_{X_2})).$$

In light of Theorem 1.1.10, we can provide a variety of examples of IHS manifolds as deformations of a selected example $\mathbf{X}$. Any IHS manifold arose in this way will be called *of deformation type* $\mathbf{X}$ and the BBF lattice will be studied as an invariant of the whole deformation type, providing an effective way to distinguish different deformation types in a fixed dimension. In particular, we will see that, for the deformation classes known so far, the comparison of the second Betti number $b_2(\mathbf{X})$ is already sufficient for this purpose, whilst the lattice and Hodge structure will come into play for more refined classification results.

Remark 1.1.12. In fact, there is no evidence for the fact that IHS manifolds with isometric BBF lattices must be deformation equivalent, but these lattices turn out to be different for every known deformation class. Additionally, whilst not all of these lattices are unimodular, they are all even, although, again, there is no general proof of the fact that the BBF lattice of an IHS manifold must be necessarily even.

1. K3 surfaces

We recall that a *K3 surface* is a compact, connected, complex surface S with irregularity $q(S) := h^1(\mathcal{O}_S) = 0$ and with trivial canonical bundle $K_S \simeq \mathcal{O}_S$. These define a distinguished class of minimal surfaces of Kodaira dimension 0 in the Enriques-Kodaira classification of compact complex analytic surfaces and, in particular, Kodaira ([Kod64]) showed that all K3 surfaces are deformation equivalent. Examples of K3 surfaces are given, for instance, by

* a generic quartic surface $\mathbb{V}(f_4)$ in $\mathbb{P}^3$, e.g. the *Fermat quartic*, defined by

$$f_4(x_0, \ldots, x_3) = x_0^4 + \cdots + x_3^4;$$

* more generally, complete intersections of k hypersurfaces $X_1, \ldots, X_k$ of degree, respectively, $d_1, \ldots, d_k \geq 2$, in $\mathbb{P}^{k+2}$, provided that $\sum_{i=1}^k d_i = k+3$. In particular, we deduce that $k \geq 3$, which yields the following options:
  - $k = 1$ and $d_1 = 3$, which is the previously mentioned case of a generic quartic surface in $\mathbb{P}^3$;
  - $k = 2$, $d_1 = 2$ and $d_2 = 3$, which corresponds to a complete intersection of a quadric and a cubic hypersurface in $\mathbb{P}^4$;
  - $k = 3$ and $d_1 = d_2 = d_3 = 2$, namely the complete intersection of three quadric hypersurfaces in $\mathbb{P}^5$;

* the *Kummer surface* $Kum(T)$ associated to a $2-$dimensional complex torus T, defined as the resolution of the singular surface $T/_{\iota}$ obtained as quotient of T under the action of the involution $\iota\colon p \mapsto -p$. The resolution is provided by blowing up the classes of the 16 fixed points.

These examples point out that not all K3 surfaces are projective - actually, the *generic* (in moduli) K3 surface is not projective.

All K3 surfaces are IHS manifolds. Indeed, by a result due to Siu ([Siu83]), all K3 surfaces are Kähler, and, by a deformation argument, we deduce that these are all simply connected. Finally, triviality of the canonical bundle K_S guarantees the existence of a unique holomorphic symplectic form σ_S.

Conversely, according to the Enriques-Kodaira classification of surfaces, any IHS manifold of dimension 2 is a K3 surface.

BBF lattice. For a K3 surface S, it holds $b_2(S) = 22$ and the Beauville-Bogomolov-Fujiki lattice $(\mathrm{H}^2(S,\mathbb{Z}), q_S)$ is isometric to the unimodular lattice

$$\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \tag{1.3}$$

of signature $(3,19)$ (see [Huy16, Chapter 14, Example 1.4]), called the *K3 lattice*, where U is the rank 2 unimodular hyperbolic lattice and $E_8(-1)$ is the unique even, unimodular, negative definite lattice of rank 8 (see Theorem B.3.1). The Fujiki constant satisfies $c_{K3} = 1$ and the BBF form q_S coincides with the intersection form on S. As will be made clear in the next Chapter, the theory of IHS manifolds builds a consistent generalization of the theory of K3 surfaces in higher dimension.

2. Hilbert schemes of points on a K3 surface - $K3^{[n]}$

Let $n \geq 2$ be a positive integer and let S be a projective K3 surface. The *Hilbert scheme of n points on S* is the scheme parametrizing $0-$dimensional closed subschemes $Z \subseteq S$ of length $l(\mathcal{O}_Z) = h^0(Z, \mathcal{O}_Z) = n$ and will be denoted by $Hilb^n(S)$ or by $S^{[n]}$, interchangeably.

Following [Bea83], in order to achieve a better understanding of the geometry of $Hilb^n(S)$, we relate it to the $n-$th symmetric product $\mathrm{Sym}^n(S)$ of S, defined as the quotient $S^n/_{\mathfrak{S}_n}$ of the $n-$th product of S under the action of the symmetric group $\mathfrak{S}_n$. The latter, often denoted also as $S^{(n)}$, is a singular complex variety of dimension $2n$, whose singular locus is defined by the classes of $n-$uples $(p_1, \ldots, p_n) \in S^n$ such that there exists $i, j \in \{1, \ldots, n\}$ such that $p_i = p_j$. The set of these singular points defines a closed subvariety $\Delta_n \subseteq S^{(n)}$, often called the *big diagonal*.

Mapping each class $[(Z, \mathcal{O}_Z)] \in S^{[n]}$ to the class of the $0-$cycle corresponding to its support $Supp(Z)$, counted with multiplicity, defines a morphism

$$\mathrm{HC}_n\colon S^{[n]} \longrightarrow S^{(n)}$$

called *Hilbert-Chow morphism*.

By a result due to Fogarty ([Fog68]), the Hilbert scheme $S^{[n]}$ is a smooth, connected manifold of dimension $2n$ and the Hilbert-Chow morphism is a resolution of singularities of $S^{(n)}$ corresponding to its blow-up along the diagonal Δ_n with reduced schematic structure (see [Bea83, Section 6] or [Ber12, Theorem 2.16]).

In [Fuj83], Fujiki showed that the Hilbert square $S^{[2]}$ of S is a IHS manifold of dimension 4 and Beauville generalized this result to any higher dimension ([Bea83, Section 6, Théorème 3]): for any $n \geq 2$, the Hilbert scheme of n points on S is an IHS manifold of dimension $2n$.

Consequence of the fact that all K3 surfaces are deformation equivalent is that, for a fixed $n \geq 2$, Hilbert schemes of n points on a K3 surface define a distinguished deformation class, called *of* $K3^{[n]}-$*type*.

Additionally, the fact the surface S is projective implies that the corresponding member $S^{[n]}$ of the family is projective. Nonetheless, the above-described machinery can also be adapted to non-projective K3 surfaces, by considering the corresponding *Douady space* of n points.

Notice that, in the case in which $n = 1$, the Hilbert scheme (or Douady space) of one point on a K3 surface S is isomorphic to S, and for this reason has been excluded from the discussion.

BBF lattice. Let us denote by $E := \mathrm{HC}_n^{-1}(\Delta_n) \in \mathrm{Pic}(S^{[n]})$ the exceptional divisor of the Hilbert-Chow morphism. By [Bea83, Proposition 6], there exists a primitive class $\delta \in \mathrm{H}^2(S^{[n]}, \mathbb{Z})$ of square $q_{S^{[n]}}(\delta) = -2(n-1)$ and such that $\mathrm{c}_1(E) = 2\delta$, and there is a natural primitive embedding $i\colon \mathrm{H}^2(S, \mathbb{Z}) \to \mathrm{H}^2(S^{[n]}, \mathbb{Z})$ such that

$$\mathrm{H}^2(S^{[n]}, \mathbb{Z}) = i(\mathrm{H}^2(S, \mathbb{Z})) \oplus \mathbb{Z}\delta$$

as lattices. In particular, we deduce that $b_2(S^{[n]}) = 23$. More precisely, the BBF lattice of an IHS manifold of type $K3^{[n]}$ is isometric to the lattice

$$\Lambda_{K3^{[n]}} := \Lambda_{K3} \oplus \langle -2(n-1) \rangle$$

of signature $(3, 20)$, called the $K3^{[n]}$ *lattice.* The Fujiki constant for this deformation type is given by $c_{S^{[n]}} = \frac{(2n)!}{n!2^n}$.

3. Generalized Kummer manifolds - Kum^n

The construction described in the previous Example (2) can be done as well starting from an Abelian surface A, namely a projective complex torus, which shares with K3 surfaces key properties such as triviality of the canonical bundle K_A and the existence of a holomorphic symplectic form. For any $n \geq 2$, the Hilbert scheme of n points $A^{[n]}$ on A provides again a connected complex Kähler manifold of dimension $2n$, but it fails both to be simply connected and to have a unique symplectic form.

In order to obtain a class of IHS manifolds, we proceed as follows. Let us consider instead the Hilbert scheme $A^{[n+1]}$ of $n+1$ points on A and let us compose the Hilbert-Chow morphism $\mathrm{HC_{n+1}}\colon A^{[n+1]} \to A^{(n+1)}$ with the natural summation map $\Sigma_{n+1}\colon A^{(n+1)} \to A$. The resulting morphism

$$\begin{aligned} a_n := \Sigma_{n+1} \circ \mathrm{HC_{n+1}}\colon A^{[n+1]} &\longrightarrow A \\ [(Z, \mathcal{O}_Z)] &\longmapsto \sum_{p\in A} l(\mathcal{O}_{Z,p})p \end{aligned} \tag{1.4}$$

is an isotrivial fibration. Again, by the work of Fujiki ([Fuj83]) for $n = 2$ and of Beauville ([Bea83, Section 7, Théorème 4]) for any $n \geq 2$, the fiber $Kum^n(A) := a_n^{-1}(0)$ over the 0 point of A is a projective IHS manifold of dimension $2n$, called *generalized Kummer manifold* on A.

To motivate the terminology, we point out that, for $n = 1$, the outcome of this construction is the Kummer surface $Kum(A)$ associated to A, which is a K3 surface.

Analogously as before, this machinery can be adapted to the case of a non-projective complex torus, providing a non-projective IHS manifold. For any $n \geq 2$, we get a deformation class of IHS manifolds of dimension $2n$, exactly as in the previous case. To distinguish them, we compare the BBF lattices.

BBF lattice. Paralleling the previous case, we consider the restriction F to $Kum^n(A)$ of the exceptional divisor E of the Hilbert-Chow morphism. By [Bea83, Section 7, Proposition 8], there exists a primitive class $\delta \in \mathrm{H}^2(Kum^n(A), \mathbb{Z})$ of square $-2(n+1)$ and such that $\mathrm{c}_1(F) = 2\delta$, together with a primitive embedding $i\colon \mathrm{H}^2(A, \mathbb{Z}) \to \mathrm{H}^2(Kum^n(A), \mathbb{Z})$ realizing the following identification of lattices:

$$\mathrm{H}^2(Kum^n(A), \mathbb{Z}) = i(\mathrm{H}^2(A, \mathbb{Z})) \oplus \mathbb{Z}\delta,$$

where $\mathrm{H}^2(A, \mathbb{Z})$, equipped with the intersection form, is abstractly isometric to three copies $U^{\oplus 3}$ of the unimodular hyperbolic plane U, of signature $(3, 3)$. We therefore deduce that $b_2(Kum^n(A)) = 7$ and that generalized Kummer manifolds realize a new distinguished deformation class, called *of Kummer type*. The BBF lattice of this deformation type is isometric to the lattice

$$\Lambda_{Kum^n} := U^{\oplus 3} \oplus \langle -2(n+1) \rangle$$

of signature $(3, 4)$ and the Fujiki constant equals to $c_{Kum^n} = \frac{(2n)!(n+1)}{n!2^n}$.

The deformation classes introduced above provide examples of IHS manifolds in each possible dimension, and for almost twenty years these have been the only known deformation classes.

4. O'Grady's sporadic examples - OG_6 and OG_{10}

In [OG99] and [OG03], O'Grady provided two examples of IHS manifolds realized as desingularizations of moduli spaces of sheaves on K3 and Abelian surfaces, respectively. As the theory of moduli spaces of sheaves will be subject of Chapter 3.1, at this stage we provide only a brief outline of the ingredients needed to introduce these examples, referring to Section 3.1 further details.

Let S be a projective K3 or Abelian surface, respectively, and let us consider the Mukai vector $w = (1, 0, -1) \in \widetilde{\mathrm{H}}(S, \mathbb{Z})$ (Section 3.1). Set $v = 2w$ and let H be a $v-$generic polarization on S (Definition 3.1.8). The moduli spaces $M_v(S, H)$ and $K_v(S, H)$, respectively, of Gieseker $H-$semistable sheaves on S with Mukai vector v are normal, compact, Kähler spaces (see Section 1.2.2) admitting a symplectic form, which are singular. In fact, they turn out to be *irreducible symplectic varieties* (see Definition 1.2.9(4)) admitting a *symplectic resolution* (see Definition 1.2.9 (2)), namely a resolution of singularities which inherits a holomorphic symplectic form.

∗ If S is a K3 surface, the symplectic resolution $OG_{10} := \widetilde{M_v(S, H)}$ of $M_v(S, H)$ is an IHS manifold of dimension 10 and $b_2(OG_{10}) \geq 24$, by [OG99]. Hence, O'Grady's 10−dimensional example defines a new deformation class of IHS manifolds, called of $OG_{10}-$*type*.

  BBF lattice. In [Rap08], Rapagnetta completed the computation of the second Betti number and provided a description of the BBF lattice of IHS manifolds of OG_{10}, showing that $b_2(OG_{10}) = 24$ and that $\mathrm{H}^2(OG_{10}, \mathbb{Z})$ is isometric to the lattice

$$\Lambda_{OG_{10}} := \Lambda_{K3} \oplus \left(\mathbb{Z}\widetilde{B} \oplus \mathbb{Z}\widetilde{\Sigma}, \begin{pmatrix} -2 & 3 \\ 3 & -6 \end{pmatrix} \right)$$

  of signature $(3, 21)$, where $\widetilde{B}$ and $\widetilde{\Sigma}$ are two divisors arising in the desingularization process. The Fujiki constant equals to $c_{OG_{10}} = 945$.

∗ If S is an Abelian surface, the symplectic resolution $OG_6 := \widetilde{K_v(S, H)}$ of $K_v(S, H)$ is an IHS manifold of dimension 6 and $b_2(OG_6) = 8$, by [OG03]. Analogously, O'Grady's 6−dimensional example defines a new deformation class of IHS manifolds, called of OG_6-*type*.

  BBF lattice. By [Rap07], the BBF lattice of an IHS manifold of OG_6 type is isometric to the lattice

$$\Lambda_{OG_6} := U^{\oplus 3} \oplus \left(\mathbb{Z}A \oplus \mathbb{Z}\widetilde{\Sigma}, \begin{pmatrix} -2 & 2 \\ 2 & -4 \end{pmatrix} \right),$$

  where $U^{\oplus 3}$ corresponds to the lattice $(\mathrm{H}^2(S, \mathbb{Z}), \cdot)$ of the Abelian surface and the divisors A and $\widetilde{\Sigma}$ are again a product of the resolution. The Fujiki constant is $c_{OG_6} = 60$.

Remark 1.1.13. These sporadic examples only appear in dimension 6 and 10 and these are the only IHS manifolds that can be obtained as desingularizations of moduli spaces of sheaves.

In [PR13], Perego and Rapagnetta proved that IHS manifolds of type OG_6 and OG_{10} can be constructed as symplectic resolutions of moduli spaces as above with a different choice of Mukai vector $v' = 2w'$, provided that w' is a primitive Mukai vector of square $(w')^2 = w^2 = 2$. As will be explained in more details in Section 3.2, any other choice of Mukai vector - hence, of the invariants fixed for the parametrized sheaves - the moduli spaces of sheaves as above, when non-empty, are either smooth - and of deformation type $K3^{[n]}$ or Kum^n - or singular symplectic varieties which do not admit symplectic resolutions (see Theorem 3.1.24 and Tables 3.1 and 3.2).

Anyway, there are several other constructions, not involving moduli spaces of sheaves, that realize IHS manifolds belonging to the previous deformation types, including Examples 2 and 3.

Unfortunately, any example of IHS manifold that has been constructed so far falls into these deformation classes, but it is not known whether this list is complete, as there is no proof of the fact that these are the only admissible deformation types for any dimension. For this reason as well, attention has shifted towards the singular setting, where a vast range of new examples comes into play.

1.2 Singular symplectic varieties

Up to this point, the discussion has been carried out only in the smooth setting. Nonetheless, even keeping an eye on the classification of smooth complex manifolds of Kodaira dimension zero, singularities become unavoidable. Indeed, for a projective variety of this kind, the minimal model - whose conjectural existence is predicted by the Minimal Model Program - is a projective variety, birational to the latter, with terminal singularities and nef canonical divisor, which is also numerically trivial - assuming the abundance conjecture. In this direction, a generalization of Bogomolov Decomposition Theorem to the setting of projective varieties with klt singularities and trivial canonical bundle was provided by [GKP11], [Dr18], [DG18], [GGK19] and [HP19], identifying a special class of singular symplectic varieties playing the same role as IHS manifolds.

In the last years, a growing research effort in hyperkähler geometry has been directed towards the development of a general theory for singular symplectic varieties tailored to provide the most meaningful generalization of the theory of irreducible holomorphic symplectic manifolds to the singular setting. As a product, a generalization of most of the foundational results presented in the previous Sections has been achieved, in particular in the case of irreducible and primitive symplectic varieties, whose theory represents the main focus of this Section.

Before starting, in Section 1.2.1 we give a quick tour of those notions related to singularities that will come into play when dropping the smoothness assumption, with a view towards singular symplectic varieties. In Section 1.2.2 we will provide an overview of definitions and main properties of the latter, compared to those given in Section 1.1, and we will carry out a first study of their singular locus. In Section 1.2.3, we will gain a first insight into the fact that allowing singularities gives rise to several and diverse classes of new examples.

1.2.1 A prelude to singularities

In the following, we will drop the smoothness assumption, and we will work with normal complex analytic varieties. We recall that a variety X is *normal* if, for any point $x \in X$, its local ring $\mathcal{O}_{X,x}$ is an integrally closed domain. This ensures, in particular, that the singular locus X^{sing} of X is a closed subvariety of codimension at least 2. Furthermore, any Cartier divisor on X is a Weil divisor, whilst the opposite inclusion is, without any further assumption, in general, false. In this case, Weil divisors that are not Cartier provide informations on the singularities of X.

Definition 1.2.1. A normal projective complex variety X is called:

(1) $\mathbb{Q}-$*factorial* if, for any Weil divisor D on X, there exists $m \in \mathbb{N}$ such that mD is Cartier;

(2) *Gorenstein* if the canonical divisor K_X is Cartier;

(3) $\mathbb{Q}-$*Gorenstein* if there exists $m \in \mathbb{N}$ such that mK_X is Cartier.

The above definitions are then generalized to the non-projective setting by [BL22, Section 2.12].

The first singularity type we are going to discuss are those naturally arising in the Minimal Model Program ([BCHM10]).

Definition 1.2.2. Let X be a $\mathbb{Q}-$Gorenstein variety and let $\rho\colon \widetilde{X} \to X$ be a resolution of singularities, so that

$$K_{\widetilde{X}} = \rho^* K_X + \sum_{i=1}^{l} a_i E_i,$$

where $E_1, \dots, E_l$ are the exceptional divisors of ρ and $a_1, \dots, a_l \in \mathbb{Q}$ are rational numbers called *discrepancies*. Then X is said to be

(1) *terminal* (or to have *terminal singularities*) if $a_i > 0$ for all i;

(2) *canonical* (or to have *canonical singularities*) if $a_i \geq 0$ for all i;

(3) *klt* (or to have *klt singularities*) if $a_i > -1$ for all i and ρ is a *log resolution* (see [BCHM10, Section 3.1]).

The resolution ρ is called *crepant* if $a_i = 0$ for all $i = 1, \dots, l$, i.e. $\rho^* K_X = K_{\widetilde{X}}$.

Remark 1.2.3. The above definitions are clearly ordered from the strongest to the weakest and are independent on the choice of the resolution.

A singularity type arising in a slightly different context and that will play a role in the next discussion is the following.

Definition 1.2.4. A normal projective complex variety X has *rational singularities* if there exists a resolution of singularities $\rho\colon \widetilde{X} \to X$ such that, for any $i > 0$, it holds $R^i\rho_*\mathcal{O}_{\widetilde{X}} = 0$.

Remark 1.2.5. We point out that all singularities introduced in Definition 1.2.2 are rational, by [Elk78] (see also [KM98, Corollary 5.22]).

Having rational singularities has important implications at the level of cohomology, in particular on Hodge structures, in a context in which classical Hodge Decomposition Theorem fails and the cohomology of varieties of this type usually carries a *mixed Hodge structure*, by [Del72] and [Del75].

Proposition 1.2.6. *Let X be a normal complex variety with rational singularities and let $\rho\colon \widetilde{X} \to X$ be a resolution of singularities.*

(1) *The pullback of ρ induces an isomorphism $\mathrm{H}^1(X,\mathbb{Z}) \simeq \mathrm{H}^1(\widetilde{X},\mathbb{Z})$ and an injection $\mathrm{H}^2(X,\mathbb{Z}) \hookrightarrow \mathrm{H}^2(\widetilde{X},\mathbb{Z})$.*

(2) *In particular, if X is compact and $\widetilde{X}$ is compact and Kähler, then $\mathrm{H}^k(X,\mathbb{Z})$ admits a pure weight k Hodge structure, for $k = 1, 2$.*

Proof. See [BL21, Lemma 2.1]. □

1.2.2 Definitions and main properties

In this Section we will approach the study of singular symplectic varieties and introduce the notions of *irreducible* and *primitive symplectic variety* as generalizations of that of IHS manifold, following [GKP11], [BL22, Section 2, Section 3] and [OPR24, Section 1.1], to which we refer for a more detailed discussion.

Definition 1.2.7. Let X be a normal complex analytic variety, let X_{reg} be its smooth locus and let $j\colon X_{\mathrm{reg}} \to X$ be the corresponding open embedding.

(1) For any $0 \le p \le \dim(X)$, we define the *sheaf of reflexive $p-$forms* as

$$\Omega_X^{[p]} := j_*\Omega^p_{X_{\mathrm{reg}}} = (\wedge^p\Omega_X)^{**}$$

and we call a global section $\sigma \in \mathrm{H}^0(X, \Omega_X^{[p]})$ a *reflexive $p-$form* on X, which corresponds naturally to a holomorphic $p-$form on X_{reg}.

(2) If $f\colon Y \to X$ is a finite, dominant morphism between two irreducible normal varieties, the usual pull-back morphism on the smooth loci induces a morphism of reflexive sheaves $f^*\colon \Omega_X^{[p]} \to \Omega_Y^{[p]}$. The induced morphism

$$f^{[*]}\colon \mathrm{H}^0(X, \Omega_X^{[p]}) \to \mathrm{H}^0(Y, \Omega_Y^{[p]})$$

on reflexive $p-$forms will be called *reflexive pull-back morphism*.

In the following, we will also assume that X admits a *Kähler form* ω and we will refer to [BL22, Section 2.3] and [Var89, II, Section 2.1, Section 2.2] for its precise definition and main properties. Any Kähler form ω on X defines an element $[\omega] \in \mathrm{H}^2(X, \mathbb{R})$ and the set of all classes obtained in this way forms an open cone $\mathcal{K}_X$, called the *Kähler cone of* X, in

$$\mathrm{H}^{1,1}(X, \mathbb{R}) = F^1\mathrm{H}^2(X, \mathbb{C}) \cap \mathrm{H}^2(X, \mathbb{R}),$$

by [BL22, Proposition 2.8], where we are using the mixed Hodge structure of $\mathrm{H}^2(X, \mathbb{C})$ (see [BL22, (2.4)]). A normal complex analytic variety X admitting a Kähler form will be called a *Kähler space*. In particular, it satisfies the following properties:

Proposition 1.2.8. *Let X be a Kähler space.*

(1) If X is smooth, then it is a Kähler manifold with respect to the usual definition.

(2) Any subspace of X is Kähler.

(3) If X is reduced, then it admits a resolution of singularities by a Kähler manifold.

Proof. See [Var89, II, Proposition 1.3.1]. □

We are finally in the position to introduce the notions of singular symplectic varieties that will be the focus of this Section.

Definition 1.2.9. Let X be a compact Kähler space.

(1) A *symplectic form* on X is a closed reflexive $2-$form $\sigma \in \mathrm{H}^0(X, \Omega_X^{[2]})$ which is non-degenerate at each point of X_{reg}.

(2) A *symplectic variety* is a pair (X, σ) where σ is a symplectic form on X, such that for every resolution $\rho \colon \widetilde{X} \to X$, the form $\rho^*(\sigma_{|X_{\mathrm{reg}}})$ extends to a holomorphic $2-$form $\widetilde{\sigma}$ on $\widetilde{X}$. In particular, if the latter is a holomorphic symplectic form, the resolution ρ is called a *symplectic resolution*.

(3) A *primitive symplectic variety (PSV)* is a symplectic variety (X, σ) such that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$ and $\mathrm{H}^0(X, \Omega_X^{[2]}) = \mathbb{C}\sigma$.

(4) An *irreducible symplectic variety (ISV)* is a symplectic variety (X, σ) such that, for every finite quasi-étale morphism $f \colon Y \to X$, the exterior algebra of reflexive forms on Y is generated by $f^{[*]}\sigma$.

We recall that a *quasi-étale* morphism is a morphism that is étale in codimension 1, and that a quasi-étale morphism onto a smooth manifold is étale.

With a slight abuse of notation, in the cases above we will refer to X as a symplectic (respectively, primitive symplectic and irreducible symplectic) variety, omitting the reference to the symplectic form.

For later use, we highlight some properties of singularities of symplectic varieties, in relation with the notions introduced in Section 1.2.1, and deduce some useful consequences.

Remark 1.2.10. If X is a symplectic variety, then X is Gorenstein and with canonical - hence rational - singularities, by [Bea00].

In particular, by Proposition 1.2.8 and Proposition 1.2.6, the $k-$th cohomology cohomology groups $\mathrm{H}^k(X,\mathbb{Z})$, for $k=1,2$, carry a pure weight k Hodge structure, and for any resolution of singularities $\widetilde{X}\to X$, it holds $\mathrm{H}^1(\widetilde{X},\mathbb{Z})\simeq \mathrm{H}^1(X,\mathbb{Z})$.

1. Comparison of definitions

We now compare the different notions of symplectic variety introduced so far.

(PSV) An irreducible symplectic variety is primitive symplectic, whilst the converse is, in general, false.

Indeed, condition (4) of Definition 1.2.9 applied to the identity morphism implies $\mathrm{H}^0(X,\Omega_X^{[2]})=\mathbb{C}\sigma$. The vanishing of $\mathrm{H}^1(X,\mathcal{O}_X)$ follows - non trivially - from the fact that, due to [GGK19, Corollary 13.3], irreducible symplectic varieties are simply connected, combined with the fact that these have rational singularities (see Remark 1.2.10). For a rigorous proof of this fact, we refer to [PR23v1, Proposition 1.10].

Examples of primitive symplectic varieties that are not irreducible symplectic will be provided in the next Section (Exampls 1.2.17, 1.2.18), see also [PR23, Example 1.5]). For a more detailed discussion on the interplay of different notions of singular sympletic varieties, we refer to [Per20].

(IHS) An irreducible symplectic variety is smooth if and only if it is an irreducible holomorphic symplectic manifold.

This is a consequence of Bogomolov Decomposition Theorem and simple connectedness of irreducible symplectic varieties.

In fact, the following generalization of Bogomolov Decomposition Theorem to the setting of projective varieties with klt singularities and trivial canonical bundle identifies irreducible symplectic varieties as fundamental building blocks, playing the same role of IHS manifolds in the smooth setting.

Theorem 1.2.11. *(Bogomolov Decomposition Theorem, projective klt setting) Let V be a normal, projective variety with klt singularities and numerically trivial canonical bundle. Then there exists a finite quasi-étale covering $\widetilde{V}\to V$ such that $\widetilde{V}$ decomposes as a finite product*

$$\widetilde{V}=T\times\prod_i Y_i\times\prod_j X_j,$$

where T is a complex torus, Y_i are irreducible Calabi-Yau varieties for every i and X_j are irreducible symplectic varieties for every j.

Proof. See [GKP11], [Dr18], [DG18], [GGK19] and [HP19]. □

2. BBF lattice

If X is an irreducible symplectic varietiy, then, as previously remarked, it is simply connected and, therefore, the second integral cohomology group $\mathrm{H}^2(X,\mathbb{Z})$ of X is a free $\mathbb{Z}-$module of rank $b_2(X)$. In the more general case of primitive symplectic varieties, the latter is replaced by its torsion-free part $\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}}$, for which the following holds.

Proposition 1.2.12. *If X is a primitive symplectic variety, then $\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}}$ admits a non-degenerate symmetric bilinear form q_X of signature $(3, b_2(X)-3)$.*

Proof. The existence of q_X is proved by Namikawa (in the projective $\mathbb{Q}-$factorial case, see [Nam01a, Theorem 8]; see [BL22, Section 5.1] for a more general statement) by defining $q_{X_{\mathrm{reg}}}$ on the smooth locus X_{reg} via (1.1), using the symplectic form σ of X, as $\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}}$ admits a pure weight 2 Hodge structure, by Remark 1.2.10. For any resolution $\rho\colon \widetilde{X}\to X$, an analogous quadratic form $q_{\widetilde{X}}$ is induced, by the same expression, involving the extension $\widetilde{\sigma}$ of σ. The latter induces, by restriction (see Remark 1.2.10), a quadratic form q_X on $\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}}$. As in Remark 1.1.8, the prescribed signature is given by the positive definite real vector space $\langle\omega, \mathrm{Re}(\sigma), \mathrm{Im}(\sigma)\rangle_{\mathbb{R}}$, for any choice of a Kähler class $\omega\in \mathrm{H}^{1,1}(X,\mathbb{R})$ (see [Sch20, Theorem 2], [BL22, Remark 6.3 (1), Theorem 6.8]). For further details, we refer to [BL22, Section 5.1]. □

Again, we will refer to the quadratic form q_X defined in Proposition 1.2.12 as the *Beauville-Bogomolov-Fujiki form* of X and to the pair $(\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}}, q_X)$ as the *Beauville-Bogomolov-Fujiki lattice* of X.

3. Singular locus

We conclude this preliminary discussion on symplectic varieties by highlighting some properties of their singular loci.

Remark 1.2.13 (On the codimension of X^{sing})**.** Let X be a symplectic variety.

(1) By [Nam01c], the singular locus X^{sing} of X admits no irreducible components of odd codimension, as the smooth locus is symplectic. Hence, $\mathrm{codim}(X^{sing}) = 2n$ for some $n\in\mathbb{N}^*$.

(2) Furthermore, again by [Nam01c, Corollary 1], it holds $\mathrm{codim}(X^{sing})\geq 4$ if and only if X has terminal singularities. Under the assumption of $\mathbb{Q}-$factoriality, this notion is related to the existence of a symplectic resolution in the following way: if X is $\mathbb{Q}-$factorial and admits a symplectic resolution $\rho\colon \widetilde{X}\to X$, then X cannot be terminal, as ρ is a crepant resolution. In particular, a $\mathbb{Q}-$factorial symplectic variety admits a symplectic resolution only if $\mathrm{codim}(X^{sing}) = 2$, namely if and only if it is not terminal.

The last remarkable property of the singular locus of symplectic varieties we want to discuss concerns its symplectic structure. Indeed, the following result shows that not only X^{sing} admits a symplectic structure, but that it can be can be sliced into a finite sequence of strata which carry again a symplectic structure. This feature will be crucial in the second part of this work.

Proposition 1.2.14. *Let X be a symplectic variety. Then there exists a finite stratification by closed subvarieties*

$$X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_l,$$

such that, for every $i = 0, \dots m-1$, the stratum X_{i+1} is the singular locus with reduced structure $(X_i^{sing})_{red}$ of X_i and the normalisation of each irreducible component of X_i is a symplectic variety.

Proof. See [Kal06, Theorem 2.3] and [BL22, Theorem 3.4 (2)]. □

The stratification of Proposition 1.2.14 will be called *the stratification of singularities of* X and its smaller stratum X_l will be called *the most singular locus* X^{ms} *of* X.

Remark 1.2.15. Notice that, by definition, each irreducible component of the most singular locus must be a smooth manifold, which by Proposition 1.2.14, is symplectic, hence, even dimensional. Furthermore, iterated applications of Remark 1.2.13 (1) imply that each stratum of the stratification must be even dimensional.

1.2.3 Examples

We conclude this section by introducing some examples of singular symplectic varieties, leaving outside the discussion those arising as moduli spaces of sheaves, which will be the actual focus of this work and to which is devoted Chapter 3.1. We will sketch the construction of different classes of symplectic varieties, referring to [Per20] for a more detailed exposition. As the deformation theory for this kind of varieties will be carefully treated in Section 2.2, we limit ourselves to introducing the examples, postponing the justification of the fact that deformation of these are still symplectic varieties. Moreover, for the majority of these examples, it is known that they define different deformation classes, by a comparison of the second Betti numbers or of the singularities type, both defining deformation invariants, once provided a suitable deformation theory.

The first example is an application of the following result.

Proposition 1.2.16. *If X is a connected symplectic variety admitting a symplectic resolution of singularities that is an irreducible holomorphic symplectic manifold, then X is a primitive symplectic variety.*

Proof. See [PR23v1, Proposition 1.9]. □

Example 1.2.17 (Symmetric product of a K3 surface)**.** For any $n \geq 2$, the $n-$th symmetric product $S^{(n)}$ of a K3 surface is a primitive symplectic variety, by Proposition

1.2.16, as the Hilbert-Chow morphism $\mathrm{HC}_n\colon Hilb^n(S) \to S^{(n)}$ is a symplectic resolution. Anyway, the quotient by the action of the symmetric group $\mathfrak{S}_n$ defines a finite quasi-étale covering $S^n \to S^{(n)}$, and $h^0(S^n, \Omega^2_{S^n}) = n$, hence X is not an irreducible symplectic variety.

Example 1.2.18 (Singular Kummer varieties)**.** Analogously, let us consider the symmetric product $A^{(n+1)}$ of an Abelian surface A and the natural summation map $\Sigma_{n+1}\colon A^{(n+1)} \to A$ (see Section 1.1.2, Example 3). Then, the *singular Kummer variety* given by the fiber $\Sigma^{-1}_{n+1}(0) \subseteq A^{(n+1)}$ over 0 of the summation morphism is a symplectic variety of dimension $2n$ admitting the generalized Kummer manifold $Kum^n(A)$ as symplectic resolution. Hence, by Proposition 1.2.16, it is a primitive symplectic variety, but, exactly as in Example 1.2.17, it is not an irreducible symplectic variety.

Example 1.2.19 (Quotients of IHS manifolds by the action of finite symplectic groups of automorphisms and their terminalizations)**.** These classes of examples are provided by partial resolutions of quotients of IHS manifolds of dimension $2n \geq 4$ by the action of *symplectic automorphisms*, i.e. whose pullback action preserves the holomorphic symplectic form. Indeed, this kind of quotients preserve the symplectic structure, and the blow-up of the (possibly empty) codimension 2 irreducible components of the singular locus is a symplectic resolution providing a symplectic variety with at most terminal singularities (see Remark 1.2.13 (2)).

This is a particular case of *terminalization* of a symplectic variety, namely a crepant resolution of the latter via a terminal variety, for which the following result holds.

Proposition 1.2.20. *Let X be a symplectic (respectively, irreducible symplectic) variety and G a symplectic group of automorphisms of X.*

(1) Any irreducible component of the fixed locus of G has even codimension.

(2) If G is finite, then X/G is a symplectic (respectively, irreducible symplectic) variety.

(3) Any projective terminalization of X is a symplectic (respectively, irreducible symplectic) variety.

Proof. See [BGMM25, Lemma 3.16, Proposition 3.17], [Per20, Proposition 2.15], or [Bea00]. □

As an application of this principle, we obtain several examples of irreducible symplectic varieties starting from quotients of IHS manifolds. Among these, we find, for instance:

(1) (terminalizations of) quotients of $Hilb^2(S)$, when S is a projective K3 surface, by the action of $G = \langle \phi \rangle$, with ϕ an automorphism of order $2, 3, 5, 7, 11$ ([MT07], [Men14], [Men18], [Mon13]);

(2) (terminalizations of) quotients of $Kum^2(S)$, when S is an Abelian surface, by the action of $G = \langle \phi \rangle$, with ϕ an automorphism of order $2, 3$ ([KM18], [Men20], [FM21]);

(3) more generally, terminalizations of quotients of $Hilb^2(S)$ when S is a projective K3 surface, or of $Kum^n(S)$, for $n = 2, 3$, when S is an Abelian surface, under the action of a finite group G of symplectic automorphisms induced by the underlying surface ([BGMM25]).

Moreover, in [BGMM25], is presented a classification of the deformation type of the varieties obtained, as well as the admissible groups G and the deformation type - among those known - of the IHS manifolds considered for this construction. Indeed, the assumptions made in view of an efficient classification of terminalizations $Y \to X/G$, lead to exclude O'Grady's examples as candidates for X (see [BGMM25, Remark 3.19]). Reducing to treat the case of X belonging to the deformation types $K3^{[n]}$ and Kum^n and assuming that G is a group of symplectic automorphism induced by the underlying surface, it turns out that the only candidates for the deformation type of X in this construction are precisely $K3^{[2]}$, Kum^2 and Kum^3, and the groups G of induced automorphisms are listed.

Additionally, their smooth terminalizations are fully characterized, showing that they are all of $K3^{[n]}-$type, for $n = 2, 3$.

Example 1.2.21 (Fujiki's examples)**.** In a similar fashion, Fujiki ([Fuj83]) had already provided several constructions of symplectic varieties as quotients of the product S^2 of a K3 surface S under special actions induced by symplectic automorphisms of S. These have been classified and shown to be irreducible symplectic varieties by Menet ([Men22b]). This construction has been generalized to products S^n, with $n \geq 2$, and the terminalizations of these quotients, known as *Fujiki varieties* (see [Men22b, Definition 1.2]), are irreducible symplectic varieties with simply connected smooth locus (see [Men22b, Corollary 1.8]).

The above-mentioned constructions in Examples 1.2.19 and 1.2.21 provide irreducible symplectic varieties, some of which belong to a special class, namely the one of *irreducible symplectic orbifolds*. This notion, first introduced in [Cam04], provides another generalization of the notion of IHS manifold in the singular setting.

Definition 1.2.22. A compact Kähler orbifold X is an *irreducible symplectic orbifold (ISO)* if its smooth locus X_{reg} is simply connected and admits a holomorphic symplectic form σ such that $\mathrm{H}^0(X_{\mathrm{reg}}, \Omega^2_{X_{\mathrm{reg}}}) \simeq \mathbb{C}\sigma$.

In fact, it is immediate to notice that an irreducible symplectic orbifold is smooth if and only if it is an IHS manifold. Furthermore, by [Per20, Proposition 2.14], irreducible symplectic orbifolds are irreducible symplectic varieties.

Example 1.2.23. Terminalizations of quotients of $S^{[2]}$, where S is a K3 surface, under the action of a symplectic automorphism of order 2 and all the above-mentioned terminalizations of quotients of generalized Kummer fourfolds and sixfolds are irreducible symplectic orbifolds. By [Men22b], all Fujiki's examples are all irreducible symplectic orbifolds.

Example 1.2.24 (Orbifolds of Nikulin type)**.** A remarkable distinguished class of irreducible symplectic orbifolds is given by *orbifolds of Nikulin type*, namely irreducible symplectic orbifolds obtained as deformations of terminalizations of $S^{[2]}/\langle\iota\rangle$, where S is a projective K3 surface and ι is a symplectic involution, called *Nikulin orbifolds*. The first example of orbifolds of this type, already mentioned in Example 1.2.19 (1), was introduced by in [MT07]. In the last years, the theory of orbifolds of Nikulin type has been intensively studied, and a complete family of such orbifolds has been determined and fully described in the work of [CGKK24].

Although we intentionally omitted proper criteria to distinguish the classes of examples introduced above, by those we get a glimpse of the vastitude and diversity of constructions of symplectic varieties arising when singularities are allowed. From the above-mentioned constructions we get several examples of fourfolds and sixfolds, and in Chapter 3.1 examples in every even dimension will complete this overview. At this stage, a classification of primitive (or irreducible) symplectic varieties is far from being obtained. Nonetheless, analogously as in the smooth setting, the only case in which the classification is complete is the one of surfaces.

Example 1.2.25 (Singular symplectic surfaces)**.** In [GPP24] it is shown that primitive symplectic surfaces are all and only contractions of ADE configurations of rational curves on a K3 surface ([GPP24, Corollary 2.3]), where we recall that an ADE configuration on a K3 surface is a configuration of $(-2)-$curves whose associated intersection lattice is given by the direct sum of some copies of the Dynkin diagrams A_n for $n \geq 1$, D_n for $n \geq 4$ and E_n for $n = 6,7,8$. As these correspond to all the possible configurations of smooth rational curves of negative intersection on a K3 surface, a classification of the ADE configurations provides a classification of all the possible primitive symplectic surfaces obtained via their contraction. The outcome of this classification is the following ([GPP24, Theorem 1.8, Theorem 1.9]):

(1) there are exactly 5836 different ADE configurations on K3 surfaces whose contraction is a primitive symplectic surface;

(2) there are exactly 5826 different ADE configurations on K3 surfaces whose contraction is an irreducible symplectic surface;

(3) there are exactly 4697 different ADE configurations on K3 surfaces whose contraction is a irreducible symplectic orbifold of dimension 2.

All the configurations are listed and their geometric properties are described. In particular, as different ADE configurations provide different singularities, this construction provides 5826 different deformation classes (Section 2.2) of irreducible symplectic surfaces.

Example 1.2.26 (Hilbert schemes of points on irreducible symplectic surfaces)**.** Applying the same philosophy that guided Fujiki's Examples 2 and 3, Hilbert squares on singular symplectic surfaces have been studied and shown to provide examples of singular symplectic fourfolds belonging to the three above-introduced classes. More precisely:

(1) If X is a primitive symplectic surface, then $Hilb^2(X)$ is a primitive symplectic variety of dimension 4, by [GPP24, Theorem 1.10 (1)].

(2) If X is a projective irreducible symplectic surface, then $Hilb^2(X)$ is a projective irreducible symplectic variety of dimension 4, by [LBP25, Theorem 3.3].

(3) If X is an irreducible symplectic orbifold of dimension 2, then $Hilb^2(X)$ is an irreducible symplectic orbifold of dimension 4, by [GPP24, Theorem 1.10 (2)].

Remark 1.2.27. A result by [FM21] shows that, if X is an irreducible symplectic orbifold of dimension 4 with terminal singularities, then $3 \leq b_2(X) \leq 23$. Among the constructions presented in Examples 1.2.19 and 1.2.21 we find examples of such orbifolds with $b_2 \in \{4, 5, 6, 7, 8, 10, 11, 14, 16, 23\}$ (see [BGMM25, Table 1]). These examples, together with the new constructions in [GPP24], show that, for any integer $3 \leq b \leq 23$, there exists an irreducible symplectic orbifold X of dimension 4 such that $b_2(X) = b$. Nevertheless, this statement does not answer to the question about filling the gaps in the list of possible second Betti numbers found in [FM21]. Indeed, the irreducible symplectic orbifolds of Example 1.2.26 (3) have canonical but not terminal singularities.

We conclude, for now, this overview on examples of singular symplectic varieties. We will return to it and further expand it in Chapter 3.1, devoted to moduli spaces of sheaves, which will provide a wealth of (new) examples of varieties of this type, in all dimensions and with non-quotient singularities.

Chapter 2

Monodromy and Torelli Theorems

One of the central questions in the study of compact complex Kähler manifolds is whether their geometry can be recovered from Hodge-theoretic data. For some classes of such manifolds, this principle is formalized by Global Torelli theorem, which describes to what extent the geometry of a manifold is determined by its integral Hodge structure. This is the case, for instance, for compact complex curves, complex tori, and K3 surfaces. The starting point of this discussion is a suitable higher-dimensional generalization of Torelli Theorem for K3 surfaces, stating that most of the geometry of IHS manifolds is encoded in the lattice and weight 2 Hodge structure of their second integral cohomology group, via a period map. This topic is the focus of Section 2.1. A key tool in decoding this information is provided by the monodromy group of the manifold under consideration. After introducing an efficient deformation theory for singular symplectic varieties in Section 2.2, we turn in Section 2.3 to a generalization of the above-mentioned framework to the singular setting, culminating in Global Torelli Theorem for primitive symplectic varieties and the study of their locally trivial monodromy group.

The starting point is the following result:

Theorem 2.0.1 (Global Torelli Theorem for K3 surfaces)**.** *Two K3 surfaces S_1 and S_2 are isomorphic if and only if there exists an isomorphism of Hodge structures $g\colon \mathrm{H}^2(S_1,\mathbb{Z}) \to \mathrm{H}^2(S_2,\mathbb{Z})$ which is an isometry with respect to the intersection pairing.*

Proof. See [PS71] and [BR75]. $\square$

Building on the latter, it became natural to ask whether an analogous higher-dimensional statement could hold for IHS manifolds, and in particular whether their geometry might be determined by their Beauville-Bogomolov-Fujiki lattice and their weight two Hodge structure. This question led to the following first formulation of Global Torelli Theorem for IHS manifolds.

Conjecture 2.0.2 (Classic Torelli for IHS manifolds)**.** Two IHS manifolds X_1 and X_2 are isomorphic if and only if there exists an integral Hodge isometry

$$g \in \mathrm{O}((\mathrm{H}^2(X_1, \mathbb{Z}), q_{X_1}), (\mathrm{H}^2(X_2, \mathbb{Z}), q_{X_2})).$$

Such formulation was early proven false in [Deb84], starting from the Hilbert scheme of points on a K3 surface. Nonetheless, the two non-isomorphic IHS manifolds considered turned out to be bimeromorphic. This consideration, together with the fact that, by minimality, bimeromorphic K3 surfaces are necessarily isomorphic, led to the following new formulation.

Conjecture 2.0.3 (Classic bimeromorphic Torelli for IHS manifolds)**.** Two IHS manifolds X_1 and X_2 are bimeromorphic if and only if there exists an integral Hodge isometry

$$g \in \mathrm{O}((\mathrm{H}^2(X_1, \mathbb{Z}), q_{X_1}), (\mathrm{H}^2(X_2, \mathbb{Z}), q_{X_2})).$$

This expectation was eventually dashed by a counterexample constructed by Namikawa in [Nam02b], in the case of generalized Kummer fourfolds. The reason for these apparent inconsistencies can be understood by studying an important deformation invariant of IHS manifolds: the *monodromy group*. In the following, a generalization of the Global Torelli theorem for irreducible symplectic varieties, both smooth and singular, is properly stated. We will also clarify that, in simple terms, the formulation of the Global Torelli theorem is shaped by the monodromy group of the corresponding deformation class.

2.1 Moduli spaces of IHS manifolds and Torelli Theorems

In the following we will discuss how to codify the geometry of an IHS manifold by means of lattice and Hodge-theoretic data. As a first step, we will investigate how these properties vary along deformations of the manifold under study.

2.1.1 Local systems and parallel transports along families

We recall that, if X is an IHS manifold, then it admits a universal deformation $p\colon \mathcal{X} \to Def(X)$ onto a smooth and connected base of dimension $h^{1,1}(X) = b_2(X) - 2$ (see Remark 1.1.9) and that any small deformation of X is again an IHS manifold (see Theorem 1.1.10). Additionally, the Beauville-Bogomolov-Fujiki lattice is a deformation invariant, by Corollary 1.1.11.

In order to formalize the concept of preserving a geometric property along a deformation, we introduce the following notion.

Definition 2.1.1. Let X be a path-connected topological space, R a unitary commutative ring and M an $R-$module.

(1) A *local system* over X with fiber M is a sheaf of $R-$modules $\mathcal{L}$ over X that is locally isomorphic to the constant sheaf M, namely, for any $x \in X$ there exists an open neighborhood $U_x \subseteq X$ of X such that $\mathcal{L}_{|U_x} \simeq M$. In particular, for any $x \in X$, one has $\mathcal{L}_x \simeq M$.

Let $\gamma\colon [0,1] \to X$ be a continuous path on X. Then $\gamma^{-1}\mathcal{L}$ is a constant sheaf on $[0,1]$ with fiber M. Consequently, $\Gamma([0,1], \gamma^{-1}\mathcal{L}) \simeq M$ and the evaluation morphisms induce an automorphism of M of the form $\mu_{\mathcal{L}}(\gamma)\colon \mathcal{L}_{\gamma(0)} \to \mathcal{L}_{\gamma(1)}$.

(2) Let $x_1, x_2 \in X$ and let γ be a path in X from x_1 to x_2. An isomorphism $\mu_{\mathcal{L}}(\gamma)\colon \mathcal{L}_{x_1} \to \mathcal{L}_{x_2}$ as above is called *parallel transport operator along γ in the local system $\mathcal{L}$.*

A parallel transport defined as above does not depend on the homotopy class of γ. If, additionally, γ is a loop centered at $x_1 = x_2$, the associated parallel transport operator is called *monodromy operator* and this association defines a group morphism

$$\begin{aligned} \mu_{\mathcal{L}}\colon \pi_1(X) &\longrightarrow \mathrm{Aut}(M) \\ [\gamma] &\longmapsto \mu_{\mathcal{L}}(\gamma) \end{aligned} \tag{2.1}$$

called *monodromy representation of M.*

Remark 2.1.2. In the next discussion, we will be particularly interested in the following local systems, arising in the context of smooth deformations of compact Kähler manifolds. Let X be an IHS manifold and let us consider a deformation $p\colon \mathcal{X} \to T$, which, up to shrinking the base, is a family of IHS manifolds. By Ehresman fibration Theorem, for any $t \in T$, there exists an open neighborhood $V_t \subseteq T$ of t and a diffeomorphism $h_t\colon \mathcal{X}_t \times V_t \to p^{-1}(V_t)$ fitting in the following commutative diagram:

$$\begin{tikzcd} \mathcal{X}_t \times V_t \arrow[rr, "h_t"] \arrow[dr, "\pi_2"'] & & p^{-1}(V_t) \arrow[dl, "p"] \\ & V_t & \end{tikzcd} \tag{2.2}$$

(1) Let us consider the constant sheaves $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$ on X. Commutativity of diagram (2.2), together with Leray's spectral sequence, shows that $R^i p_* \mathbb{Z}$, $R^i p_* \mathbb{Q}$ and $R^i p_* \mathbb{C}$ are local systems on T, for any $i = 0, \dots, 2\dim X$, with fiber, respectively,

$$(R^i p_* \mathbb{Z})_t \simeq \mathrm{H}^i(\mathcal{X}_t, \mathbb{Z}) \simeq \mathrm{H}^i(X, \mathbb{Z}),$$
$$(R^i p_* \mathbb{Q})_t \simeq \mathrm{H}^i(\mathcal{X}_t, \mathbb{Q}) \simeq \mathrm{H}^i(X, \mathbb{Q}),$$
$$(R^i p_* \mathbb{C})_t \simeq \mathrm{H}^i(\mathcal{X}_t, \mathbb{C}) \simeq \mathrm{H}^i(X, \mathbb{C}).$$

In particular, any path $\gamma \in \Omega(T, t_1, t_2)$ in T connecting any two base points $t_1, t_2 \in T$ defines a parallel transport operator $\mathrm{H}^2(\mathcal{X}_{t_1}, \mathbb{Z}) \simeq \mathrm{H}^2(\mathcal{X}_{t_2}, \mathbb{Z})$ respecting the BBF forms, as the latter is a topological data, by Theorem 1.2.

Isometries of this type will play a fundamental role in the next discussion, and we will come back to these in Section 2.1.3.

(2) For any $t \in T$, any fiber $\mathcal{X}_t$ of $p\colon \mathcal{X} \to T$ is an IHS manifold, hence there exists a holomorphic symplectic form σ_t on $\mathcal{X}_t$ such that $\mathrm{H}^0(\mathcal{X}_t, \Omega^2_{\mathcal{X}_t}) \simeq \mathbb{C}\sigma_t$. For later use, we point out that this feature deforms together with the manifold $\mathcal{X}_t$. More precisely, let us consider the relative cotangent sheaf $\Omega_{\mathcal{X}/T}$. Up to shrinking the base, the sheaf $p_*\Omega^2_{\mathcal{X}/T}$ is a local system with fiber $(p_*\Omega^2_{\mathcal{X}/T})_t \simeq \mathrm{H}^0(\mathcal{X}_t, \Omega^2_{\mathcal{X}_t})$, for any $t \in T$, and there exists a smooth section $\sigma \in \mathrm{H}^0(T, p_*\Omega^2_{\mathcal{X}/T})$ such that $\sigma_t \in \mathrm{H}^0(\mathcal{X}_t, \Omega^2_{\mathcal{X}_t})$ is a holomorphic symplectic form on $\mathcal{X}_t$ for any $t \in T$ (see [Bea83, Section 8, Proposition 9, Remarque 1]).

2.1.2 Local Torelli Theorem

Following [Bea83, Section 8], we will use the facts collected above to define a *period map* tailored to codify the fundamental geometric properties of an IHS manifold and its deformations. We recall that, for any IHS manifold X, its Beauville-Bogomolov-Fujiki lattice $(\mathrm{H}^2(X,\mathbb{Z}), q_X)$ is a lattice of signature $(3, b_2(X) - 3)$, which is a deformation invariant (Corollary 1.1.11). We will denote by $(\Lambda, \cdot)$ the abstract lattice of signature $(3, b_2(X) - 3)$ representing the BBF lattice of any IHS manifold of the same deformation type $\mathbf{X}$ of X.

Definition 2.1.3. Let X be an IHS manifold and Λ a lattice of signature $(3, b_2(X) - 3)$ as above.

(1) A *marking* on X is an isometry $\eta \in \mathrm{O}((\mathrm{H}^2(X,\mathbb{Z}), q_X), (\Lambda, q_\Lambda))$.

(2) A *marked IHS manifold* is a pair (X, η) made of an IHS manifold X, equipped with a marking η.

(3) Two marked IHS manifolds (X_1, η_1) and (X_2, η_2) are *isomorphic* if there exists an isomorphism $f\colon X_1 \to X_2$ such that $\eta_2 = \eta_1 \circ f^*$.

The (coarse) *moduli space of* $\Lambda-$*marked IHS manifolds* is defined as

$$\mathfrak{M}_\Lambda := \{(X, \eta) \text{ marked IHS manifold}\}/\simeq,$$

where $\simeq$ is the equivalence relation given by the isomorphism of marked IHS manifolds. The local structure of the latter is described by Local Torelli Theorem.

Let (X, η) be a marked IHS manifold and let $p\colon \mathcal{X} \to Def(X)$ be the universal deformation family, with $0 \in Def(X)$ such that $\mathcal{X}_0 \simeq X$.

(1) By Remark 2.1.2 (1) we can extend the marking η, via parallel transport operators in the local system $R^2p_*\mathbb{Z}$, to a family of markings $\{\eta_t\colon \mathrm{H}^2(\mathcal{X}_t,\mathbb{Z}) \to \Lambda\}_{t\in Def(X)}$ such that $\eta_0 = \eta$. For any member η_t of the family, we will denote by

$$\eta_{t,\mathbb{C}}\colon \mathrm{H}^2(\mathcal{X}_t,\mathbb{C}) \to \Lambda \otimes_{\mathbb{Z}} \mathbb{C} =: \Lambda_{\mathbb{C}}$$

its $\mathbb{C}-$linear extension, namely the isometry obtained by deforming the marking $\eta_{\mathbb{C}}$ via parallel transports in the local system $R^2 p_* \mathbb{C}$.

(2) By Remark 2.1.2 (2) there is a holomorphic family of holomorphic symplectic forms $\{\sigma_t \in \mathrm{H}^0(\mathcal{X}_t, \Omega^2_{\mathcal{X}_t})\}_{t \in Def(X)}$. We define the *period* of $\mathcal{X}_t$ as

$$[\eta_{t,\mathbb{C}}(\sigma_t)] \in \mathbb{P}(\Lambda_{\mathbb{C}}). \tag{2.3}$$

By the properties of the BBF lattice (Theorem 1.1.6 (2)) the period of a marked IHS manifold lies in the *period domain*

$$D_\Lambda := \{[x] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \colon x^2 = 0,\ (x, \overline{x}) > 0\}, \tag{2.4}$$

which is an open analytic subvariety on a quadric hypersurface in $\mathbb{P}(\Lambda_{\mathbb{C}})$ and is simply connected (see [Huy12, Proposition 3.1]).

Theorem 2.1.4 (Local Torelli Theorem)**.** *The local period map*

$$\begin{aligned} P_X \colon Def(X) &\longrightarrow D_\Lambda \\ t &\longmapsto [\eta_{t,\mathbb{C}}(\sigma_t)] \end{aligned}$$

is a local biholomorphism.

Proof. See [Bea83, Section 8, Théorème 5 (b)]. □

Local Torelli Theorem allows us to endow the moduli space $\mathfrak{M}_\Lambda$ with a complex structure, by using the local period maps as local charts.

Theorem 2.1.5. *Let X be an IHS manifold and let $\eta \colon \mathrm{H}^2(X, \mathbb{Z}) \to \Lambda$ be a marking on X.*

(1) The moduli space of $\Lambda-$marked IHS manifolds $\mathfrak{M}_\Lambda$ is a non-Hausdorff complex manifold of dimension $b_2(X) - 2$.

(2) The local period map induces a natural holomorphic embedding $Def(X) \hookrightarrow \mathfrak{M}_\Lambda$ identifying $Def(X)$ with an open neighborhood of (X, η) in $\mathfrak{M}_\Lambda$.

Proof. See [Huy12, Proposition 4.3]. □

By construction of the complex structure on $\mathfrak{M}_\Lambda$, the local period maps $P_X \colon Def(X) \to D_\Lambda$ glue to the *global period map*

$$P_\Lambda \colon \mathfrak{M}_\Lambda \to D_\Lambda, \tag{2.5}$$

which is a local biholomorphism, by construction ([Huy12, Corollary 4.5]). In order to study the properties of the fibers of the latter, in relation with the global geometry of $\mathfrak{M}_\Lambda$ - namely, identify its connected components and characterize the inseparable points - we need to introduce a notion that will be central throughout this work.

2.1.3 Monodromy and Global Torelli Theorem

As explained in Remark 2.1.2, any deformation of IHS manifolds induces natural isomorphisms at the level of cohomology, arising as parallel transport operators in suitable local systems. Among theese, we identify a special class that will play a fundamental role in the next discussion.

Definition 2.1.6. (1) Let X_1 and X_2 be two IHS manifolds. An isometry $g \in \mathrm{O}(\mathrm{H}^2(X_1,\mathbb{Z}),\mathrm{H}^2(X_2,\mathbb{Z}))$ is a *parallel transport operator from* X_1 *to* X_2 if there exists a family of IHS manifolds $p\colon \mathcal{X} \to T$, two points $t_1, t_2 \in T$ such that $\mathcal{X}_{t_i} \simeq X_i$, for $i = 1,2$, and a continuous path γ in T from t_1 to t_2 such that g is the parallel transport along γ in the local system $R^2p_*\mathbb{Z}$, which we denote by $\mathsf{PT}_p(\gamma)$. We denote by

$$\mathsf{PT}^2(X_1,X_2) \subseteq \mathrm{O}(\mathrm{H}^2(X_1,\mathbb{Z}),\mathrm{H}^2(X_2,\mathbb{Z}))$$

the set of parallel transport operators from X_1 to X_2.

(2) Let X be an IHS manifold. An isometry $g \in \mathrm{O}(\mathrm{H}^2(X,\mathbb{Z}))$ is a *monodromy operator* on X if it is a parallel transport operator from X to itself. We denote by

$$\mathrm{Mon}^2(X) := \mathsf{PT}^2(X,X) \subseteq \mathrm{O}(\mathrm{H}^2(X,\mathbb{Z})) \tag{2.6}$$

the set of monodromy operators of X.

Remark 2.1.7. As explained in [Mar11, footnote 3], one can easily prove that the set $\mathrm{Mon}^2(X)$ of monodromy operators of an IHS manifold X is actually a subgroup of $\mathrm{O}(\mathrm{H}^2(X,\mathbb{Z}))$.

Indeed, if $f,g \in \mathrm{Mon}^2(X)$, there exist two families $p\colon \mathcal{X} \to T$ and $p'\colon \mathcal{X}' \to T'$ of IHS manifolds, two points $t \in T$ and $t' \in T'$ such that $\mathcal{X}_t \simeq X \simeq \mathcal{X}'_{t'}$ and two loops γ in T and γ' in T' centered, respectively, in t and t', such that f is the monodromy operator in the family p associated to γ and g is the monodromy operator in the family p' associated to γ'. We can therefore define a new deformation $p''\colon \mathcal{X}'' \to T''$ of X by gluing $\mathcal{X}$ and $\mathcal{X}'$ via the isomorphism $\mathcal{X}_t \simeq X \simeq \mathcal{X}'_{t'}$ and by gluing T and T' by means of the relation $t = t' =: t''$. The loop $\gamma'' := \gamma * \gamma'$ in the new reducible base is centered in t'', by construction, and we define $g \circ f$ as the monodromy operator associated to γ'' in the family p''.

Definition 2.1.8. Let X be an IHS manifold. The group $\mathrm{Mon}^2(X)$ defined in (2.6) is called the *monodromy group of* X (see also [Mar11, Definition 1.1]).

Remark 2.1.9. Let X be an IHS manifold and let Λ be an abstract lattice of signature $(3, b_2(X) - 3)$. By [Ver13, Theorem 3.4], the monodromy group $\mathrm{Mon}^2(X)$ of X is a subgroup of finite index of $\mathrm{O}(\mathrm{H}^2(X,\mathbb{Z}))$. The index $[\mathrm{O}(\mathrm{H}^2(X,\mathbb{Z}))\colon \mathrm{Mon}^2(X)]$ already carries important information concerning the geometry of the moduli space $\mathfrak{M}_\Lambda$ of $\Lambda-$marked pairs of IHS manifolds. Indeed, two marked IHS manifolds

(X_1, η_1) and (X_2, η_2) are deformation equivalent if and only if they belong to the same connected component of $\mathfrak{M}_\Lambda$, which, by [Mar11, Lemma 7.5], is equivalent to requiring that $\eta_2^{-1} \circ \eta_1$ is a parallel transport operator. Indeed, the orthogonal group $\mathrm{O}(\Lambda) \simeq \mathrm{O}(\mathrm{H}^2(X, \mathbb{Z}))$ acts on $\mathfrak{M}_\Lambda$ via composition on the markings, and the monodromy group $\mathrm{Mon}^2(X)$ coincides with the stabilizer of any connected component under this action. As a consequence, the number of connected components of $\mathfrak{M}_\Lambda$ containing marked pairs of IHS manifolds deformation equivalent to X to is equal to $[\mathrm{O}(\mathrm{H}^2(X, \mathbb{Z})) \colon \mathrm{Mon}^2(X)]$.

We can therefore proceed by restricting to a connected component $\mathfrak{M}^0_\Lambda$ of $\mathfrak{M}_\Lambda$, e.g. the one containing a fixed marked IHS manifold (X, η). Surprisingly, the restriction of the global period map to each of these turns out to be surjective.

Theorem 2.1.10 (Surjectivity of the global period map)**.** *Let $\mathfrak{M}^0_\Lambda$ be a connected component of the moduli space of $\Lambda-$marked IHS manifolds. Then the restriction*

$$P^0_\Lambda \colon \mathfrak{M}^0_\Lambda \longrightarrow D_\Lambda$$

of the global period map defined in (2.5) is surjective.

Proof. See [Huy99, Theorem 8.1] and [Huy12, Theorem 5.5]. □

Remark 2.1.11. For later use, we briefly sketch the two main ingredients on which the proof of Theorem 2.1.10 relies:

(1) The description of the Kähler cone $\mathcal{K}_X$ of a *generic* IHS manifold X, i.e. such that $\mathrm{Pic}(X) = 0$ (see [Huy12, Section 5.1]). In that case, by [Huy12, Theorem 5.1], it coincides with the *positive cone* $\mathcal{C}_X$ of X, namely the connected component of the open cone $\mathcal{C}'_X = \{\alpha \in \mathrm{H}^{1,1}(X, \mathbb{R}) \colon q_X(\alpha) > 0\}$ containing a Kähler class.

(2) The existence of special global deformations, called *twistor spaces*, in the period domain (see [Huy12, Sections 3.2, 4.3, 4.4]). In simple terms, any Kähler class $\alpha = [\omega_I] \in \mathrm{H}^{1,1}(X, \mathbb{R})$ corresponding to a hyperkähler structure $X = (M, I, g)$ (see Remark 1.1.4), identifies a positive $3-$space (see Remark 1.1.8)

$$W_\alpha = \langle [\omega_I], \mathrm{Re}(\sigma_I), \mathrm{Im}(\sigma_I) \rangle_{\mathbb{R}} \subseteq \mathrm{H}^2(X, \mathbb{R})$$

and a complex line $T_\alpha := \mathbb{P}(\eta_{\mathbb{C}}(W_\alpha \otimes \mathbb{C})) \cap D_\Lambda$ in the period domain, called *twistor line*, for any choice of a marking η on X. A local surjectivity argument ([Huy12, Proposition 5.4]) provides a lift for a distinguished class of twistor lines - called *generic* - whose generic points correspond to periods of marked generic IHS manifolds. Finally, any two points $x, y \in D_\Lambda$ can be connected by means of a finite chain of generic twistor lines ([Huy12, Proposition 3.7]).

The global period map fails to be injective. Nonetheless, an efficient description of its fibers, in relation with the non-Hausdorff structure of $\mathfrak{M}^0_\Lambda$, is provided by Global Torelli Theorem.

Theorem 2.1.12 (Global Torelli Theorem)**.** *Let $\mathfrak{M}^0_\Lambda$ be a connected component of the moduli space of $\Lambda-$marked IHS manifolds.*

(1) The restriction $P^0_\Lambda \colon \mathfrak{M}^0_\Lambda \longrightarrow D_\Lambda$ of the global period map is surjective and generically injective. For any $x \in D$, the fiber $(P^0_\Lambda)^{-1}(x)$ consists of pairwise inseparable points.

(2) For any pair of inseparable points $(X_1, \eta_1), (X_2, \eta_2) \in \mathfrak{M}^0_\Lambda$, the IHS manifolds X_1 and X_2 are bimeromorphic. Conversely, if X_1 and X_2 are two bimeromorphic IHS manifolds, there exists two markings η_1 and η_2 such that (X_1, η_1) and (X_2, η_2) are inseparable points in $\mathfrak{M}^0_\Lambda$.

(3) The marked pair (X, η) is a Hausdorff point if and only if $\mathcal{C}_X = \mathcal{K}_X$.

Proof. (1) Surjectivity has been addressed in Theorem 2.1.10. Generic injectivity is the content of [Ver13] and [Ver20], see [Huy12] for further details. More precisely, all fibers over points in the countable union of hyperplane sections $D_\Lambda \cap \bigcup_{0 \neq \alpha \in \Lambda} \alpha^\perp$ consist of exactly one point.

(2) See [Huy99, Theorem 4.3] or [Huy12, Proposition 4.7] for the first part of the statement and [Huy99, Theorem 4.6, 4.6′] and [Huy03a, Theorem 2.5] for the second part.

(3) Follows from (1) and (2), together with Huybrechts' characterization of the Kähler cone of a generic IHS manifold (Remark 2.1.11). See [Mar11, Proposition 5.14] for further details. □

The above-described characterization of the fibers of the global period map is taken further in the following reformulation of Global Torelli Theorem ([Mar11]), providing an efficient Hodge-theoretic method to approach the bimeromorphic - and even isomorphic - classification if IHS manifolds in the same deformation class.

Theorem 2.1.13 (Hodge Theoretic Global Torelli Theorem)**.** *Let X_1 and X_2 be two deformation equivalent irreducible holomorphic symplectic manifolds.*

(1) X_1 and X_2 are bimeromorphic if and only if there exists a parallel transport operator $g \colon \mathrm{H}^2(X_1, \mathbb{Z}) \to \mathrm{H}^2(X_2, \mathbb{Z})$ which is an isomorphism of Hodge structures.

(2) Let $g \colon \mathrm{H}^2(X_1, \mathbb{Z}) \to \mathrm{H}^2(X_2, \mathbb{Z})$ be a parallel transport operator which is an isomorphism of Hodge structures. There exists an isomorphism $f \colon X_1 \to X_2$ such that $g = f_*$ if and only if g maps a Kähler class on X_1 to a Kähler class on X_2.

Proof. This is the content of Theorem 1.3 of [Mar11]. We only sketch the proof of part (1), as it contains some insights that will be useful in the next discussion.

(1) If $f \colon X_1 \to X_2$ is a bimeromorphism, then the existence of a parallel transport which is a Hodge isometry follows from part (2) of Theorem 2.1.12. Indeed, [Huy03a, Theorem 2.5, Corollary 2.7] shows that the correspondence

$[\Gamma_f]\colon \mathrm{H}^*(X_1,\mathbb{Z}) \to \mathrm{H}^*(X_2,\mathbb{Z})$ induced by the graph Γ_f of f is a graded ring isomorphism respecting the Hodge structures, whose restriction

$$[\Gamma_f]_2\colon \mathrm{H}^2(X_1,\mathbb{Z}) \to \mathrm{H}^2(X_2,\mathbb{Z})$$

coincides with f_* and respects the BBF forms. Moreover, it is shown that there exists two deformations $p_1\colon \mathcal{X}_1 \to B$ and $p_2\colon \mathcal{X}_2 \to B$ of X_1 and X_2, respectively, over the same one dimensional disk B, with $0 \in B$ such that $\mathcal{X}_{i,0} \simeq X_i$ for $i = 1,2$, and a relative isomorphism $\widetilde{f}\colon \mathcal{X}_1 \setminus p_1^{-1}(0) \to \mathcal{X}_2 \setminus p_2^{-1}(0)$ induced by f. The latter allows us to glue the two families along any point $b \in B \setminus \{0\}$, to define, as in Remark 2.1.7, a parallel transport operator from X_1 to X_2, induced by $[\Gamma_f]$, which coincides with f_*, as made explicit in [Mar11, Section 3.2].

Conversely, if $g\colon \mathrm{H}^2(X_1,\mathbb{Z}) \to \mathrm{H}^2(X_2,\mathbb{Z})$ is a parallel transport operator which is an isomorphism of Hodge structures, let $\eta_2\colon \mathrm{H}^2(X_2,\mathbb{Z}) \to \Lambda$ be a marking on X_2. Then $\eta_1 := \eta_2 \circ g$ is a marking on X_1 and, by Remark 2.1.9, the marked pairs (X_1,η_1) and (X_2,η_2) belong to the same connected component $\mathfrak{M}^0_\Lambda$ of $\mathfrak{M}_\Lambda$. As g is a Hodge isometry, they have the same period

$$P^0_\Lambda(X_1) = \eta_1(\sigma_{X_1}) = \eta_2(g(\sigma_{X_1})) = \eta_2(\sigma_{X_2}) = P^0_\Lambda(X_2),$$

hence, by part (1) of Theorem 2.1.12, they are inseparable points and, by part (2), they are defined by bimeromorphic IHS manifolds.

(2) See [Mar11, Section 3.2]. □

Remark 2.1.14 (Back to K3 surfaces)**.** Theorem 2.1.12 and Theorem 2.1.13 provide a consistent generalization of Global Torelli Theorem for K3 surfaces (Theorem 2.0.1). The moduli space of marked K3 surfaces $\mathfrak{M}_{K3}$ consists of two connected components, interchanged by the involution sending a marked pair (S,η) to $(S,-\eta)$ and, in particular, the fiber of the global period map over a generic point in D_{K3} consists exactly of two points. In fact (see Remark 2.1.9), the monodromy group $\mathrm{Mon}^2(S)$ of a K3 surface is an index 2 subgroup of $\mathrm{O}(\mathrm{H}^2(S,\mathbb{Z}))$, by [Bor72] - more precisely, it coincides with the group $\mathrm{O}^+(\mathrm{H}^2(S,\mathbb{Z}))$ of *orientation preserving isometries* (see Remark 2.1.15 below). Hence, any Hodge isometry between two K3 surfaces is, up to sign, a parallel transport operator that can be lift to an isomorphism, and Theorem 2.1.13 yields the original formulation of Global Torelli Theorem for K3 surfaces. For further details, we refer to [Huy12, Section 6.1 and Remark 6.7].

Let X be an IHS manifold and consider de BBF lattice structure on $\mathrm{H}^2(X,\mathbb{Z})$. In order to refine the inclusion of groups $\mathrm{Mon}^2(X) \subseteq \mathrm{O}(\mathrm{H}^2(X,\mathbb{Z}))$ stated in Remark 2.1.7 and to examine in depth the geometric information carried by the monodromy index (see Remark 2.1.9), we introduce a notion of *orientation* on $\mathrm{H}^2(X,\mathbb{Z})$. We refer to Appendix B.1.2 for general definitions and basic facts, or to [Mar11, Section 4] and [Mar08, Section 4.1] for a more detailed discussion.

Remark 2.1.15 (Orientations I)**.** Let X be an IHS manifold, let σ_X be the holomorphic symplectic form and let $\omega \in \mathrm{H}^{1,1}(X, \mathbb{R})$ be a Kähler class. As already pointed out in Remark 1.1.8, by the properties of the BBF lattice (see Theorem 1.1.6 (2) and (3)) these span a positive 3−space

$$W_\omega = \langle \omega, \mathrm{Re}(\sigma_X), \mathrm{Im}(\sigma_X) \rangle_{\mathbb{R}} \subseteq \mathrm{H}^2(X, \mathbb{R})$$

that defines an orientation on the indefinite lattice $\mathrm{H}^2(X, \mathbb{Z})$ of signature $(3, b_2(X) - 3)$. In particular, if $\eta\colon \mathrm{H}^2(X, \mathbb{Z}) \to \Lambda$ is a marking towards a lattice Λ of the same signature, the orientation W_ω induces an orientation on Λ and defines a twistor line T_{W_ω} in the period domain D_Λ via the period map (see Remark 2.1.11 (2)). Hence, the orientation on Λ is preserved along generic twistor lines and, by [Huy12, Proposition 5.4], the corresponding orientation on $\mathrm{H}^2(X, \mathbb{Z})$ is constant on marked pairs (X, η) belonging to the same connected component $\mathfrak{M}^0_\Lambda$ of $\mathfrak{M}_\Lambda$ (see also [Mar11, Section 4]). By [Mar11, Lemma 7.5] (see also Remark 2.1.9), we get that the chosen orientation is preserved by parallel transport operators. More explicitly, both σ_X and ω extend to smooth sections of the respective local systems (see Remark 2.1.2), defining a family of positive spaces $\{W_t = \langle \omega_t, \mathrm{Re}(\sigma_t), \mathrm{Im}(\sigma_t) \rangle_{\mathbb{R}}\}_{t \in Def(X)}$, whose orientation is constant along parallel transport operators.

The previous Remark provides a useful lattice-theoretic constraint for the study of the monodromy group of an IHS manifold.

Corollary 2.1.16. *Let X be an irredicible holomorphic symplectic manifold. Then*

$$\mathrm{Mon}^2(X) \subseteq \mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z})).$$

Remark 2.1.17. We recall that the group of orientation preserving isometries $\mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z}))$ is an index 2 subgroup of $\mathrm{O}(\mathrm{H}^2(X, \mathbb{Z}))$. The $\mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z}))$−action on $\mathfrak{M}_\Lambda$ provides two isomorphic orbits interchanged by the involution sending a marked pair (X, η) to $(X, -\eta)$, as the morphism $-\mathrm{id}_{\mathrm{H}^2(X,\mathbb{Z})}$ is orientation reversing. If (X, η) is a generic marked IHS manifold, by Remark 2.1.9 and Theorem 2.1.13, the index $[\mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z}))\colon \mathrm{Mon}^2(X)]$ counts the number of IHS manifolds deformation equivalent to X that are bimeromorphic to X. Equivalently, the monodromy index counts the number of IHS manifolds bimeromorphic to X in a fixed deformation class, for any generic weight 2 Hodge isometry class.

In other words, the monodromy group measures how far is the period of an IHS manifold from determining its bimeromorphic type.

Corollary 2.1.18. *If X is an IHS manifold, then the classic formulation of Bimeromorphic Torelli Theorem (Conjecture 2.0.3) holds if and only if its monodromy group is maximal, i.e. equality holds in Corollary 2.1.16.*

This is the case, for instance, for K3 surfaces. In higher dimensions, we have the following description.

Remark 2.1.19 (Monodromy of IHS manifolds - state of the art)**.** The monodromy group has been completely described for any known deformation class of IHS manifold, as follows:

(1) If X is of $K3^{[n]}-$type, for $n \geq 2$, then $\mathrm{Mon}^2(X)$ is the group $\mathsf{W}(X)$ of orientation preserving isometries of $\mathrm{H}^2(X,\mathbb{Z})$ acting as $\pm\mathrm{id}$ on its discriminant group (see Appendix B.4). The monodromy index $[\mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z}))\colon \mathsf{W}(X)]$ is equal to $2^{\rho(n-1)-1}$, where $\rho(k)$ is the number of primes occurring in the factorization of any integer k. This result has been proved in [Mar08] and [Mar10].

(2) If X is of Kum^n-type, for $n \geq 2$, then $\mathrm{Mon}^2(X)$ is the index 2 subgroup $\mathsf{N}(X)$ of $\mathsf{W}(X)$ of isometries in the kernel of the character $\det\cdot\mathrm{disc}$ (see (B.4)), and the monodromy index equals to $2^{\rho(n)}$. This result has been proved by [Mar22] and [Mon16].

(3) If X is of OG_6-type, then $\mathrm{Mon}^2(X) = \mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z}))$, by [MR21].

(4) If X is of $OG_{10}-$type, then $\mathrm{Mon}^2(X) = \mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z}))$, by [Ono22].

Corollary 2.1.20. *Classic Bimeromorphic Torelli holds for any IHS manifold X of deformation type OG_6, OG_{10} and $K3^{[n]}$, if and only if $n = p^r + 1$ for some prime number p and positive integer $r \in \mathbb{N}^*$. On the other hand, it never holds for IHS manifolds of Kum^n-type, for any $n \geq 2$.*

In the next Section, we will discuss a generalization of the theory above introduced to the singular setting, where the monodromy problem has been addressed only partially, mainly due to the existence of a rich variety of new examples (recall Section 1.2.3).

2.2 Locally trivial deformations of symplectic varieties

This Section is devoted to introduce an efficient notion of deformation, designed to preserve the essential properties of a variety, in a context in which we allow singularities (see Section 1.2.1). For further details, we refer to [BL22, Section 4] and [OPR24, Section 1.2] (see also [Nam01a], [Nam01b]).

We recall that, a *family* or *deformation* of complex analytic spaces is a flat and proper morphism $f\colon \mathcal{X} \to T$ of complex analytic spaces onto a connected base. Among these, we identify the following special class of deformations.

Definition 2.2.1. (1) A *locally trivial family* is a proper morphism $f\colon \mathcal{X} \to T$ of complex analytic spaces such that the base T is connected and, for every point $x \in \mathcal{X}$, there exist open neighborhoods $V_x \subseteq \mathcal{X}$ of x and $V_{f(x)} \subseteq T$ of $f(x)$ and an open subset $U_x \subseteq f^{-1}(f(x))$ such that

$$V_x \simeq U_x \times V_{f(x)},$$

where the isomorphism is an isomorphism of complex analytic spaces commuting with the projections over T.

(2) If X is a complex analytic variety, a *locally trivial deformation of* X is a locally trivial family $f\colon \mathcal{X} \to T$ for which there is $t \in T$ such that $\mathcal{X}_t := f^{-1}(t) \simeq X$.

(3) A *locally trivial family of primitive (resp. irreducible) symplectic varieties* is a locally trivial family whose fibers are all primitive (resp. irreducible) symplectic varieties.

(4) Two primitive (resp. irreducible, see Definition 1.2.9) symplectic varieties are said to be *locally trivial deformation equivalent* if there exists a locally trivial family of primitive (resp. irreducible) symplectic varieties having both of them as fibers.

Remark 2.2.2. If X is a primitive symplectic variety, then, by [BL22, Lemma 4.6, Section 4.4], it holds $\mathrm{H}^0(X, \mathcal{T}_X) = 0$ and there exists a universal deformation $f\colon \mathcal{X} \to Def_{\mathrm{lt}}(X)$, where $Def_{\mathrm{lt}}(X)$ is the closed complex subspace of the base $Def(X)$ of a universal deformation (a priori *miniversal*, often called *Kuranishi family*) parametrizing locally trivial deformations of X. By [BL22, Theorem 4.7], the space $Def_{\mathrm{lt}}(X)$ is smooth of dimension $h^{1,1}(X)$.

In the following, given a complex analytic space X, we will deal only with *small* locally trivial deformations of X, i.e. locally trivial families with base an analytic open neighborhood U of X in the base $Def_{\mathrm{lt}}(X)$. In the case of primitive symplectic varieties, this assumption guarantees that the family under study is a familiy of primitive symplectic varieties, as stated below.

Proposition 2.2.3. *Every small locally trivial deformation of a primitive symplectic variety X is a primitive symplectic variety. In particular, the locally trivial Kuranishi family $f\colon \mathcal{X} \to Def_{\mathrm{lt}}(X)$ is universal for all of its fibers.*

Proof. See [BL22, Corollary 4.11]. □

An analogous result, with the same generality, is missing in the case of irreducible symplectic varieties. Nonetheless, the following sufficient criteria are provided in [OPR24, Section 1.2], which will work in the case of our interest, in the next discussion.

Proposition 2.2.4. *Let X be an irreducible symplectic variety and let $f\colon \mathcal{X} \to T$ be a small locally trivial deformation of X, with $0 \in T$ such that $\mathcal{X}_0 \simeq X$.*

(1) If the smooth locus X_{reg} of X is simply connected, then $\mathcal{X}_t$ is an irreducible symplectic variety with simply connected smooth locus for any $t \in T$.

(2) If X has at most terminal singularities, then $\mathcal{X}_t$ is an irreducible symplectic variety with at most terminal singularities for any $t \in T$.

(3) If X is projective, f is projective and T is quasi-projective, then there exists an analytic open neighborhood $U \subseteq T$ of 0 such that $\mathcal{X}_t$ is a projective irreducible symplectic variety for any $t \in U$.

Proof. See, respectively, Proposition 1.7, Proposition 1.8 and Proposition 1.9 of [OPR24]. □

Hence, when studying the locally trivial deformation class of an irreducible symplectic variety X satisfying the hypotheses above, up to shrinking the base, we can deal with locally trivial deformations of irreducible symplectic varieties.

Moreover, locally trivial deformations of primitive symplectic varieties preserve $\mathbb{Q}-$factoriality as well, by the following result due to [BL22].

Proposition 2.2.5. *Let X be a primitive symplectic variety. Then every small locally trivial deformation of X is $\mathbb{Q}$-factorial if and only if X is $\mathbb{Q}$-factorial.*

Proof. See [BL22, Lemma 5.20]. □

Remark 2.2.6. Working with locally trivial deformations is less restrictive than it seems. For instance, in the case of $\mathbb{Q}-$factorial and terminal symplectic varieties, which will be the main focus of this work, all flat deformations are locally trivial, by [Nam06].

Furthermore, as smooth deformations are locally trivial, the notions introduced so far define a consistent generalization of the classical deformation theory for irreducible holomorphic symplectic manifolds.

We conclude this Subsection by dealing with the behavior of the singular locus of a symplectic varieties along a locally trivial deformation, as it will be a crucial point in Section 6.1.

Remark 2.2.7. (1) We point out that, if $f\colon \mathcal{X} \to T$ is a locally trivial family of symplectic varieties, then there is a natural relative stratification

$$\mathcal{X} = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_l,$$

such that, for every $t \in T$, set $\mathcal{X}_{i,t} := \mathcal{X}_i \cap \mathcal{X}_t$ for $i = 0, \dots, l$, the stratification

$$\mathcal{X}_t \supseteq \mathcal{X}_{1,t} \supseteq \cdots \supseteq \mathcal{X}_{l,t}$$

is the stratification of singularities of $\mathcal{X}_t$ as in Proposition 1.2.14 and the restriction $f_i\colon \mathcal{X}_i \to T$ is again a locally trivial family.

(2) We also remark that, by construction, the restriction $f_l\colon \mathcal{X}_l \to T$ of f defines a smooth and proper morphism and, by local triviality, it preserves connected components of $\mathcal{X}_l$. Hence, the restriction $f_l^0 := f_{|\mathcal{X}_l^0}\colon \mathcal{X}_l^0 \to T$ of f_l to any connected component $\mathcal{X}_l^0$ of $\mathcal{X}_l$ defines a smooth and proper morphism, such that the connected components of each fiber consist of smooth symplectic manifolds.

(3) Anyway, this does not guarantee, in general, that $f_l^0\colon \mathcal{X}_l^0 \to T$ is a smooth deformation of IHS manifolds, as its fiber may be disconnected. Nonetheless, since f_l^0 is smooth and proper, by Stein's factorization Theorem, there exists a commutative diagram

$$\begin{tikzcd} & \mathcal{X}_l^0 \arrow[dl, "f_l^0"'] \arrow[dr, "q"] & \\ T & & \widetilde{T} \arrow[ll, "h"] \end{tikzcd}$$

where $q\colon \mathcal{X}_l^0 \to \widetilde{T}$ is a smooth and proper morphism with connected fibers and $h\colon \widetilde{T} \to T$ is a finite étale morphism, thus $\widetilde{T}$ is connected.

2.3 Locally trivial monodromy operators and Torelli Theorems

In this Section we will present a generalization of the framework introduced in Section 2.1 to the singular setting, following [BL22]. The outcome of this discussion is a version of Local and Global Torelli Theorem for primitive symplectic varieties, with surjectivity of the global period map in the $\mathbb{Q}-$factorial terminal case.

In the following we will quickly collect the main facts on which such generalization relies, while keeping the main focus on the definition and the characterization of the *locally trivial monodromy group* of a primitive symplectic variety. Indeed, in this setting, the latter plays the role of the monodromy group, carrying the same geometric interpretation in light of Global Torelli Theorem. Moreover, the second Part of this work will be devoted to the computation of the locally trivial monodromy group of a distinguished locally trivial deformation class of irreducible symplectic varieties.

2.3.1 Torelli Theorems for primitive symplectic varieties

We start by recalling that, by Proposition 1.2.12 (see also part 2 of Section 1.2.2), if X is a primitive symplectic variety, then its Beauville-Bogomolov-Fujiki lattice $(\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}}, q_X)$ is an indefinite lattice of signature $(3, b_2(X)-3)$. By [BL22, Lemma 5.7], the latter is a locally trivial deformation invariant, meaning that, if X_1 and X_2 are two locally trivial deformation equivalent primitive symplectic varieties, then there exists an isometry $g \in \mathrm{O}(\mathrm{H}^2(X_1,\mathbb{Z})_{\mathrm{tf}}, \mathrm{H}^2(X_2,\mathbb{Z})_{\mathrm{tf}})$.

Remark 2.3.1. Let X be a primitive symplectic variety and let $p\colon \mathcal{X} \to T$ a small locally trivial deformation of X.

(1) Local triviality of p ensures the existence of a commutative diagram as in (2.2), endowing the locally trivial family with local systems $R^i p_*\mathbb{Z}$, $R^i p_*\mathbb{Q}$, $R^i p_*\mathbb{C}$ with fibers, respectively, $\mathrm{H}^i(X,\mathbb{Z})$, $\mathrm{H}^i(X,\mathbb{Q})$, $\mathrm{H}^i(X,\mathbb{C})$ (see Remark 2.1.2 (1)), for $i = 0,\ldots,2\dim X$.

(2) In particular, any path $\gamma \in \Omega(T,t_1,t_2)$ in T connecting any two base points $t_1, t_2 \in T$ defines a parallel transport operator $\mathrm{H}^2(\mathcal{X}_{t_1},\mathbb{Z})_{\mathrm{tf}} \simeq \mathrm{H}^2(\mathcal{X}_{t_2},\mathbb{Z})_{\mathrm{tf}}$ respecting the BBF forms, as the latter is a topological data, by [BL22, Lemma 5.7].

(3) Arguing similarly as in the smooth case (see Remark 2.1.2 (2), [BL22, Proposition 5.5 and its proof] and [Nam01a, Theorem 7]) and up to shrinking the base, we get that the symplectic form $\sigma_X \in \mathrm{H}^0(X,\Omega_X^{[2]})$ extends to a flat section σ of the local system $p_*\Omega^{[2]}_{\mathcal{X}/T}$, with $\sigma_t \in \mathrm{H}^0(\mathcal{X}_t,\Omega^{[2]}_{\mathcal{X}_t})$ a symplectic form for any $t \in T$.

Analogously as in the smooth case, we can give the following definitions.

Definition 2.3.2. (1) A *marked primitive symplectic variety* is a pair (X,η) made of primitive symplectic variety X, equipped with a marking $\eta\colon \mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}} \to \Lambda$ onto a lattice Λ of signature $(3, b_2(X)-3)$.

(2) Two marked primitive symplectic varieties (X_1,η_1) and (X_2,η_2) are *isomorphic* if there exists an isomorphism $f\colon X_1 \to X_2$ such that $\eta_2 = \eta_1 \circ f^*$.

We can therefore define the (coarse) *moduli space of* $\Lambda-$*marked IHS manifolds* is defined as

$$\mathfrak{M}_\Lambda := \{(X,\eta) \text{ marked primitive symplectic variety}\}/\simeq, \tag{2.7}$$

where $\simeq$ is the equivalence relation given by the isomorphism of marked primitive symplectic varieties. Again, the local structure of non-Hausdorff complex manifold on $\mathfrak{M}_\Lambda$ is defined by using the universal bases $Def_{\mathrm{lt}}(X)$ as local charts, identifying points over which the fibers are isomorphic as marked primitive symplectic varieties, by means of a local period map.

Theorem 2.3.3 (Local Torelli Theorem for primitive symplectic varieties)**.** *Let X be a primitive symplectic variety and let Λ be a lattice of signature $(3, b_2(X)-3)$. The local period map*

$$\begin{aligned} P_X\colon Def_{\mathrm{lt}}(X) &\longrightarrow D_\Lambda \\ t &\longmapsto [\eta_{t,\mathbb{C}}(\sigma_t)] \end{aligned}$$

defined as in (2.3) and where D_Λ is the period domain defined in (2.4), is a local biholomorphism.

Proof. See [BL22, Proposition 5.5]. □

Again, the local period map induces a holomorphic embedding $Def_{\mathrm{lt}}(X) \hookrightarrow \mathfrak{M}_\Lambda$ identifying $Def_{\mathrm{lt}}(X)$ with an open neighborhood of (X, η) in $\mathfrak{M}_\Lambda$, and the local period maps $P_X\colon Def_{\mathrm{lt}}(X) \to D_\Lambda$ glue to the *global period map*

$$P_\Lambda\colon \mathfrak{M}_\Lambda \to D_\Lambda, \tag{2.8}$$

which is a local biholomorphism by construction. The characterization of inseparable points in $\mathfrak{M}_\Lambda$ works similarly as in the smooth case, as stated below.

Proposition 2.3.4. *Let $\mathfrak{M}_\Lambda^0$ be a connected component of the moduli space $\mathfrak{M}_\Lambda$ of $\Lambda-$marked primitive symplectic varieties and let $(X_1, \eta_1), (X_2, \eta_2) \in \mathfrak{M}_\Lambda^0$ be two inseparable points. Then, there exists a bimeromorphic map $f\colon X_1 \dashrightarrow X_2$.*

Proof. This is the content of [BL22, Theorem 6.14], which is an adaptation to the singular setting of [Huy99, Theorem 4.3] (see Theorem 2.1.12 (2)). □

The converse statement, with the same kind of generality, does not hold anymore in the singular setting. In fact, two bimeromorphic (or even birational projective) primitive symplectic varieties are not necessarily locally trivial deformation equivalent (see [BL21, Theorem 4.9]). A slightly weaker statement, which is sufficient for our purposes, follows from the following non trivial result of [BL22], which is an adaptation of the strategy of [Huy99, Theorem 4.6, 4.6′] and [Huy03a, Theorem 2.5] (see also the proof of Theorem 2.1.13 (1)) to the projective singular setting.

Lemma 2.3.5. *Let X_1 and X_2 be two projective primitive symplectic varieties and let $f\colon X_1 \dashrightarrow X_2$ be a birational map which is an isomorphism in codimension 1 and such that $f_*\colon \mathrm{Pic}(X_1)_\mathbb{Q} \to \mathrm{Pic}(X_2)_\mathbb{Q}$ is well defined and an isomorphism. Then, there exist two locally trivial deformations $p_1\colon \mathcal{X}_1 \to B$ and $p_2\colon \mathcal{X}_2 \to B$ of X_1 and X_2, respectively, over the same one dimensional disk B, with $0 \in B$ such that $\mathcal{X}_{i,0} \simeq X_i$ for $i = 1, 2$, such that $\mathcal{X}_1$ and $\mathcal{X}_2$ are birational over B and $p_1^{-1}(B \setminus \{0\}) \simeq p_2^{-1}(B \setminus \{0\})$.*

Proof. See [BL22, Theorem 6.16]. □

The statement above generalizes to the non-projective case as well, provided that the two bimeromorphic primitive symplectic varieties are $\mathbb{Q}-$factorial and terminal.

Lemma 2.3.6. *Let X_1 and X_2 be two $\mathbb{Q}-$factorial and terminal primitive symplectic varieties and let $f\colon X_1 \dashrightarrow X_2$ be a bimeromorphism. Then, there exist two locally trivial deformations $p_1\colon \mathcal{X}_1 \to B$ and $p_2\colon \mathcal{X}_2 \to B$ of X_1 and X_2, respectively, over the same one dimensional disk B, with $0 \in B$ such that $\mathcal{X}_{i,0} \simeq X_i$ for $i = 1, 2$, such that $\mathcal{X}_1$ and $\mathcal{X}_2$ are bimeromorphic over B and $p_1^{-1}(B \setminus \{0\}) \simeq p_2^{-1}(B \setminus \{0\})$.*

Proof. If $f\colon X_1 \dashrightarrow X_2$ is a bimeromorphic map between $\mathbb{Q}-$factorial and terminal primitive symplectic varieties, then it is an isomorphism in codimension 1 (see

[MR25, Lemma 3.2,3.4], for instance), inducing an isomorphism on the rational Picard groups. Moreover, $f^*\colon \mathrm{H}^2(X_2,\mathbb{C}) \to \mathrm{H}^2(X_1,\mathbb{C})$ defines an isomorphism of Hodge structures, so that X_1 and X_2 have the same period $P_{X_1}(0) = P_{X_2}(0)$, and there exists a one dimensional disk B in D_Λ passing through it. Its lift via the local period maps defines two one dimensional disks $B_i \subseteq Def_{\mathrm{lt}}(X_i)$, centered at $0 = X_i$, for $i = 1, 2$. The latter define two locally trivial deformations $p_1\colon \mathcal{X}_1 \to B$ and $p_2\colon \mathcal{X}_2 \to B$ of X_1 and X_2, respectively, over the same one dimensional disk B, under the identification $B_1 \simeq B_2$ given by the local period maps, as they are local isomorphisms. Now the proof can be concluded as in [BL21, Theorem 4.9], replacing the symplectic resolutions Y, Y' with X_1, X_2 themselves. □

Corollary 2.3.7. *Let X_1 and X_2 be two primitive symplectic varieties satisfying the hypotheses of Lemma 2.3.5 or Lemma 2.3.6. Then, for every choice of a marking η_1 on X_1, there exists a marking η_2 on X_2 such that the pairs (X_1, η_1) and (X_2, η_2) are inseparable points in $\mathfrak{M}^0_\Lambda$.*

Proof. See [BL22, Corollary 6.17]. □

Surjectivity of the global period map $P^0_\Lambda\colon \mathfrak{M}^0_\Lambda \to D_\Lambda$ is a more delicate issue in the singular setting. In [BL21, Theorem 1.3], the authors prove it in the case in which $\mathfrak{M}_\Lambda$ parametrizes $\Lambda-$marked primitive symplectic varieties admitting a crepant resolution via an IHS manifold, adapting Verbitsky's and Huybrecths' proof, relying on the existence of hyperkähler metrics and twistor deformations. The more general statement below, obtained replacing resolutions with $\mathbb{Q}-$factorial terminalizations, follows instead from a generalization to the $\mathbb{Q}-$factorial terminal case of a work of Kollár-Laza-Saccà-Voisin ([KLSV18]) on limits of projective families, and the existence of a $\mathbb{Q}-$factorial terminalization in the projective case, by [BCHM10].

One of the key ingredients is the following projectivity criterion, due to Huybrechts in the smooth case of IHS manifolds ([Huy99, Theorem 3.11]).

Theorem 2.3.8. *Let X be a primitive symplectic variety and let $\alpha \in \mathrm{H}^2(X,\mathbb{Q})$ be a $(1,1)-$class with $q_X(\alpha) > 0$. Then X is projective.*

Proof. See [BL22, Theorem 6.9]. □

As a consequence, we get the following.

Corollary 2.3.9. *Every primitive symplectic variety is locally trivial deformation equivalent to a projective primitive symplectic variety.*

This is Corollary 1.3 of [BL22]. More precisely, the authors prove that, for any primitive symplectic variety X, the set of points of $Def_{\mathrm{lt}}(X)$ over which the fiber is projective is dense ([BL22, Corollary 6.10, Corollary 6.11]).

The outcome is the following formulation of Global Torelli Theorem for primitive symplectic varieties, which is the main Theorem of [BL22].

Theorem 2.3.10 (Global Torelli Theorem for primitive symplectic varieties)**.** *Let Λ be a lattice of signature $(3, b-3)$, with $b \geq 5$ and let $\mathfrak{M}_\Lambda^0$ be a connected component of the moduli space $\mathfrak{M}_\Lambda$ of $\Lambda-$marked pairs.*

(1) The restriction $P_\Lambda^0 \colon \mathfrak{M}_\Lambda^0 \to D_\Lambda$ of the global period map is generically injective. For any $x \in D$, the fiber $(P_\Lambda^0)^{-1}(x)$ consists of pairwise inseparable points, whose underlying primitive symplectic varieties are bimeromorphic.

(2) The image of P_Λ^0 is contained in the complement of countably many maximal Picard rank periods.

(3) If one point of $\mathfrak{M}_\Lambda^0$ corresponds to a $\mathbb{Q}-$factorial and terminal primitive symplectic variety, then the same is true for any point of $\mathfrak{M}_\Lambda^0$ and P_Λ^0 is surjective.

Proof. See [BL22, Theorem 8.2]. □

Remark 2.3.11 (Global Torelli Theorem for irreducible symplectic orbifolds)**.** A formulation for Global Torelli Theorem for irreducible symplectic orbifolds with second Betti number greater or equal than 3 already appeared in a previous work of Menet [Men20], with the same characterization of fibers of the period map and surjectivity of the latter. In particular, Menet's proof works regardless the additional hypothesis $b_2(X) \geq 5$, as it relies on the existence of twistor deformations, as in the smooth case and as in [BL21].

2.3.2 Locally trivial monodromy operators and monodromy group

We will now complete the description of the global geometry of the moduli space of $\Lambda-$marked primitive symplectic varieties and of the fibers of the period map by introducing the natural counter-part of the monodromy group (see Section 2.1.3) in the singular setting. Aim of this Section is to define the *locally trivial monodromy group* of a primitive symplectic variety, following [BL22] and review its fundamental properties (see [OPR24, Section 1.3]).

As already recalled in Remark 2.3.1, any locally trivial deformation $p \colon \mathcal{X} \to T$ of primitive symplectic varieties endows the base T with local systems carrying important information on the cohomology of the fibers and providing remarkable classes of isomorphism and isometries between those. Among these, we identify the following special classes of isometries between the respective Beauville-Bogomolov-Fujiki lattices (see part 2 of Section 1.2.2).

Definition 2.3.12. Let X, X_1 and X_2 be three primitive symplectic varieties.

(1) An isometry $g \in \mathrm{O}(\mathrm{H}^2(X_1, \mathbb{Z})_{\mathrm{tf}}, \mathrm{H}^2(X_2, \mathbb{Z})_{\mathrm{tf}})$ is a *locally trivial parallel transport operator from* X_1 *to* X_2 if there exist a locally trivial family of primitive symplectic varieties $p \colon \mathcal{X} \to T$, two points $t_1, t_2 \in T$ such that $\mathcal{X}_{t_i} \simeq X_i$, for $i = 1, 2$, and a continuous path γ in T from t_1 to t_2 such that g is the parallel transport $\mathsf{PT}_p(\gamma)$ along γ in the local system $R^2 p_* \mathbb{Z}$.

(2) An isometry $g \in \mathrm{O}(\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}})$ is a *locally trivial monodromy operator on* X if it is a locally trivial parallel transport operator from X to itself.

If X_1 and X_2 are two primitive symplectic varieties, we will denote by

$$\mathsf{PT}^2_{\mathrm{lt}}(X_1, X_2) \subseteq \mathrm{O}(\mathrm{H}^2(X_1,\mathbb{Z})_{\mathrm{tf}}, \mathrm{H}^2(X_2,\mathbb{Z})_{\mathrm{tf}})$$

the set of locally trivial parallel transport operators from X_1 to X_2. If $X_1 = X_2 =: X$, we will denote by

$$\mathrm{Mon}^2_{\mathrm{lt}}(X) := \mathsf{PT}^2_{\mathrm{lt}}(X, X) \subseteq \mathrm{O}(\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}}) \tag{2.9}$$

the set of locally trivial monodromy operators of X.

Remark 2.3.13. As already remarked in [OPR24, Lemma 1.12], arguing as in the smooth case in Remark 2.1.7 and by using locally trivial families, it can be easily shown that the set $\mathrm{Mon}^2_{\mathrm{lt}}(X)$ of locally trivial monodromy operators of a primitive symplectic variety X is actually a subgroup of $\mathrm{O}(\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}})$.

Definition 2.3.14. Let X be a primitive symplectic variety. The group $\mathrm{Mon}^2_{\mathrm{lt}}(X)$ defined in (2.9) is called the *locally trivial monodromy group of* X.

Remark 2.3.15. If X is smooth, then $\mathrm{Mon}^2_{\mathrm{lt}}(X) = \mathrm{Mon}^2(X)$, according to the classical definition (see Definition 2.1.8), as smooth deformations of X are locally trivial (see Remark 2.2.6).

As in the smooth case (see Remark 2.1.9), the locally trivial monodromy group carries important information concerning the geometry of the moduli space $\mathfrak{M}_\Lambda$ of $\Lambda-$marked primitive symplectic varieties (see (2.7)). For any marked primitive symplectic variety $(X, \eta\colon \mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}} \to \Lambda)$, let us set $\mathrm{Mon}^2_{\mathrm{lt}}(\Lambda) := \eta \circ \mathrm{Mon}^2_{\mathrm{lt}}(X) \circ \eta^{-1} \subseteq \mathrm{O}(\Lambda)$.

Proposition 2.3.16. *Let Λ be a lattice of signature $(3, b-3)$, with $b \geq 5$, let $\mathfrak{M}_\Lambda$ be the moduli space of $\Lambda-$marked primitive symplectic varieties and let $\mathfrak{M}^0_\Lambda$ be a connected component. Then $\mathrm{Mon}^2_{\mathrm{lt}}(\Lambda)$ is the stabilizer of $\mathfrak{M}^0_\Lambda$ under the $\mathrm{O}(\Lambda)-$action on $\mathfrak{M}_\Lambda$ given by composition on the markings. In particular, for any primitive symplectic variety X, the index $[\mathrm{O}(\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}})\colon \mathrm{Mon}^2_{\mathrm{lt}}(X)]$ is finite.*

Proof. This is the content of [BL22, Theorem 8.2 (1)]. See also [BL22, Section 8.6, Step 4] and Remark 2.1.9. □

Analogously as the smooth case (see Remark 2.1.15), in order to refine the inclusion of groups of Remark 2.3.13, we introduce an *orientation* on $\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}}$, when X is a primitive symplectic variety. For further details, general facts and notations, we refer to Appendix B.1.2, or to [Mar11, Section 4] and [Mar08, Section 4.1].

Remark 2.3.17 (Orientations II)**.** We recall that, if X is a primitive symplectic variety, the Beauville-Bogomolov-Fujiki lattice $(\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}}, q_X)$ has signature $(3, b_2(X)-3)$ (see part 2 of Section 1.2.2). Hence, in order to find an orientation for the big positive cone

$$\widetilde{\mathcal{C}_X} := \{\alpha \in \mathrm{H}^2(X,\mathbb{R}) \colon q_X(\alpha) > 0\}$$

of X, we may choose a basis of a $3-$dimensional positive space. As explained in the proof of [OPR24, Lemma 1.13], by the properties of the BBF form (see Proposition 1.2.12 and its proof), if $\omega \in \mathrm{H}^1(X,\Omega_X^{[1]})$ is a Kähler class and $\sigma \in \mathrm{H}^0(X,\Omega_X^{[2]})$ is a reflexive $2-$form, then the basis $\{\omega, \mathrm{Re}(\sigma), \mathrm{Im}(\sigma)\}$ determines an orientation of $\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}}$. Moreover, such an orientation does not depend on the choice of the Kähler class and of the symplectic form and it is preserved by locally trivial parallel transport operators in families of primitive symplectic varieties.

As a consequence of Proposition 2.3.16 and Remark 2.3.17 (see also [OPR24, Lemma 1.13]), we get the following.

Corollary 2.3.18. *Let X be a primitive symplectic variety with $b_2(X) \geq 5$. Then $\mathrm{Mon}^2_{\mathrm{lt}}(X)$ is a subgroup of finite index of $\mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}})$.*

For later use, we make few comments that will be helpful in order to give an efficient characterization of orientation preserving Hodge isometries.

Remark 2.3.19. If X and Y are two primitive symplectic varieties and $g\colon \mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}} \to \mathrm{H}^2(Y,\mathbb{Z})_{\mathrm{tf}}$ is a Hodge isometry, we may choose generators $\sigma_X \in \mathrm{H}^0(X,\Omega_X^{[2]})$ and $\sigma_Y \in \mathrm{H}^0(Y,\Omega_Y^{[2]})$ such that $g(\sigma_X) = \sigma_Y$. If ω_X is a Kähler class on X, as g is an isometry, its image $g(\omega_X)$ must be orthogonal to σ_Y, namely, $g(\omega_X) \in \mathrm{H}^1(X,\Omega_Y^{[1]})$. Hence, we can choose two natural orientations for X and Y, given, respectively, by $\{\omega_X, \mathrm{Re}(\sigma_X), \mathrm{Im}(\sigma_X)\}$ and $\{\omega_Y, \mathrm{Re}(\sigma_Y), \mathrm{Im}(\sigma_Y)\}$, where ω_Y is a Kähler class on Y, and deduce that g preserves the given orientations if and only if the orientation of the cone

$$\mathcal{C}'_X = \{\alpha \in \mathrm{H}^{1,1}(X,\mathbb{R}) \colon \alpha \cdot \alpha > 0\}$$

is preserved. According to Appendix B.1.2 (a) this happens if and only if g maps the connected component $\mathcal{C}_X$ of $\mathcal{C}'_X$ containing ω_X to the connected component $\mathcal{C}_Y$ of $\mathcal{C}'_Y$ containing ω_Y.

In parallel with Section 2.1.3, we conclude the discussion concerning the locally trivial monodromy group of a primitive symplectic variety and its geometric interpretation in terms of moduli with some considerations on Hodge Theoretic versions of Global Torelli Theorem (Theorem 2.3.10). In the smooth case of IHS manifolds, part (1) of Theorem 2.1.13 identified the monodromy group as a fundamental tool to detect bimeromorphism classes once a deformation class was fixed. In the singular setting, an analogous statement, with the same kind of generality, cannot be deduced from Theorem 2.3.10 (1), essentially due to technicalities needed to ensure that a bimeromorphism defines inseparable points in the moduli space of marked primitive symplectic varieties.

In particular, working verbatim as in the smooth case (see [Mar11, Section 3.1] or the proof of Theorem 2.1.13 (1)), from Lemma 2.3.5 and Corollary 2.3.7, we can deduce the following.

Proposition 2.3.20. *Let X_1 and X_2 be two projective primitive symplectic varieties, and let $f\colon X_1 \to X_2$ be a birational map. Suppose that f is an isomorphism in codimension 1 and that $f_* : \mathrm{Pic}(X_1)_{\mathbb{Q}} \to \mathrm{Pic}(X_2)_{\mathbb{Q}}$ is well-defined and an isomorphism. Then the pushforward*

$$f_*\colon \mathrm{H}^2(X_1,\mathbb{Z})_{\mathrm{tf}} \longrightarrow \mathrm{H}^2(X_2,\mathbb{Z})_{\mathrm{tf}}$$

is well defined and a locally trivial parallel transport operator.

Proof. By Lemma 2.3.5, the birational map $f\colon X_1 \to X_2$ induces two locally trivial deformations $p_1\colon \mathcal{X}_1 \to B$ and $p_2\colon \mathcal{X}_2 \to B$ of X_1 and X_2, respectively, over the same one dimensional disk B, with $0 \in B$ such that $\mathcal{X}_{i,0} \simeq X_i$ for $i = 1, 2$, such that $\mathcal{X}_1$ and $\mathcal{X}_2$ are birational over B and $p_1^{-1}(B \setminus \{0\}) \simeq p_2^{-1}(B \setminus \{0\})$. Let $b \in B \setminus \{0\}$ and let us consider two locally trivial parallel transport operators $g_1 \in \mathsf{PT}^2_{\mathrm{lt}}(X_1, \mathcal{X}_{1,b})$, $g_2 \in \mathsf{PT}^2_{\mathrm{lt}}(X_2, \mathcal{X}_{2,b})$ in the families p_1 and p_2, respectively. By gluing the families along b, under the isomorphism $\mathcal{X}_{1,b} \simeq \mathcal{X}_{2,b}$ (see Remark 2.1.7 and Remark 2.3.13), we get a locally trivial parallel transport operator $g := g_2^{-1} \circ g_1 \in \mathsf{PT}^2_{\mathrm{lt}}(X_1, X_2)$ that, as in [Mar11, Section 3.1], is a Hodge isometry induced by the correspondence defined by the graph Γ_f of f, coinciding with f_*. □

Remark 2.3.21. By Lemma 2.3.6, we get an analogous statement, in the analytic category, for any bimeromorphic irreducible symplectic varieties that are $\mathbb{Q}-$factorial and terminal.

On the other hand, one of the two implications of Theorem 2.1.13 (1) can be proved in full generality, as a consequence of Proposition 2.3.4.

Corollary 2.3.22. *Let X_1 and X_2 be two locally trivial deformation equivalent primitive symplectic varieties and assume that $b_2(X_1) \geq 5$. If $g\colon \mathrm{H}^2(X_1,\mathbb{Z})_{\mathrm{tf}} \to \mathrm{H}^2(X_2,\mathbb{Z})_{\mathrm{tf}}$ is locally trivial parallel transport operator which is an isomorphism of Hodge structures, then there exists a bimeromorphism $f\colon X_1 \to X_2$.*

Proof. The proof works verbatim as in Theorem 2.1.13 (1). If $g\colon \mathrm{H}^2(X_1,\mathbb{Z})_{\mathrm{tf}} \to \mathrm{H}^2(X_2,\mathbb{Z})_{\mathrm{tf}}$ is a locally trivial parallel transport as above, let $\eta_2\colon \mathrm{H}^2(X_2,\mathbb{Z}) \to \Lambda$ be a marking on X_2. Then $\eta_1 := \eta_2 \circ g$ is a marking on X_1 and, by Proposition 2.3.16, the marked pairs (X_1, η_1) and (X_2, η_2) belong to the same connected component $\mathfrak{M}^0_\Lambda$ of $\mathfrak{M}_\Lambda$ and, as g is an isomorphism of Hodge structures, they have the same period. Consequently, by part (1) of Theorem 2.3.10, they are inseparable points and, by Proposition 2.3.4, the underlying primitive symplectic varieties are bimeromorphic. □

Remark 2.3.23. From Corollary 2.3.22 we deduce that, in the singular setting, the monodromy index $[\mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}})\colon \mathrm{Mon}^2_{\mathrm{lt}}(X)]$ provides a lower bound on the number of bimeromorphism classes for any locally trivial deformation class of primitive symplectic varieties. More precisely, arguing as in Remark 2.1.17, we get that, for any primitive symplectic variety X with $b_2(X) \geq 5$, there are at least $[\mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}})\colon \mathrm{Mon}^2_{\mathrm{lt}}(X)]$ primitive symplectic varieties locally trivial deformation equivalent to X that are Hodge-isometric, but not bimeromorphic to X, for any generic weight 2 Hodge-isometry class of $\mathrm{H}^2(X,\mathbb{Z})_{\mathrm{tf}}$.

On the other hand, equality holds only in those cases in which a Hodge Theoretic formulation of Global Torelli Theorem can be stated.

For instance, if we restrict to deal with $\mathbb{Q}-$factorial and terminal primitive symplectic varieties - in which case we recall that all deformations are locally trivial, by Remark 2.2.6 - we can state the following formulation of the Hodge theoretic Global Torelli Theorem, as a consequence of Proposition 2.3.20, Remark 2.3.21 and Corollary 2.3.22.

Corollary 2.3.24 (Hodge Theoretic Global Torelli Theorem for $\mathbb{Q}-$factorial and terminal primitive symplectic varieties)**.** *Let X_1 be a $\mathbb{Q}-$factorial and terminal primitive symplectic variety, with $b_2(X_1) \geq 5$ and let X_2 be a deformation of X_1. Then X_1 and X_2 are bimeromorphic if and only if there exists a parallel transport operator $g\colon \mathrm{H}^2(X_1,\mathbb{Z})_{\mathrm{tf}} \to \mathrm{H}^2(X_2,\mathbb{Z})_{\mathrm{tf}}$ which is an isomorphism of Hodge structures.*

Remark 2.3.25. (1) The above formulation applies to $\mathbb{Q}-$factorial and terminal irreducible symplectic varieties as well. Indeed, by Proposition 2.2.4, any small locally trivial deformation of a terminal irreducible symplectic variety is again an irreducible symplectic variety, which is terminal. By [BL22, Lemma 5.20], also $\mathbb{Q}-$factoriality is preserved along locally trivial deformations. Also notice that, by Remark 2.2.6, all deformations of these kind of symplectic varieties are locally trivial. Therefore, we can analogously define a global period map on the moduli space $\mathfrak{M}_\Lambda$ of $\Lambda-$marked irreducible symplectic varieties and proceed as above.

(2) In the case of irreducible symplectic orbifolds (see Definition 1.2.22), the same formulation of Hodge Theoretic Global Torelli Theorem was already been achieved by [MR25, Theorem 1.1], together with the study of the action of parallel transport operators on the Kähler cone. Also in this case, by [Men20, Proposition 3.10], all deformations of irreducible symplectic orbifolds are locally trivial.

Hence, for primitive or irreducible symplectic varieties as above, the description of the locally trivial monodromy group becomes a particularly interesting deformation invariant, from a bimeromorphic classification point of view. As previously discussed in Section 1.2.3, there is a wealth of new and diverse examples of irreducible symplectic varieties and a classification of these is far from being obtained.

In particular, the description of the locally trivial monodromy group is, in general, an open problem. To the best of our knowledge, the latter has been solved only for those locally trivial deformation classes which have been studied and understood the most in the last years, namely *moduli spaces of sheaves on K3 and Abelian surfaces* (Section 3.1) and orbifolds of Nikulin type (Remark 1.2.24).

Example 2.3.26 (Monodromy of orbifolds of Nikulin type)**.** Among irreducible symplectic orbifolds, orbifolds of Nikulin type (Remark 1.2.24) define a distinguished deformation class. A description of their monodromy group has recently been achieved by [BMM24] (see also [Nan25]). In particular, for any irreducible symplectic orbifold X of Nikulin type, it holds

$$\mathrm{Mon}^2(X) = \mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z})).$$

This, together with Remark 2.3.25 (2), shows that for this deformation type the classic formulation of Bimeromorphic Global Torelli Theorem (Conjecture 2.0.3) holds.

The case of moduli spaces of sheaves will be treated in full detail in the next Chapters. In particular, the case of singular moduli spaces of sheaves on K3 surfaces will be discussed in Section 3.4, while the second Part of this work will be devoted to the computation of the remaining case of singular moduli spaces of sheaves on Abelian surfaces (Section 6.3).

Chapter 3

Moduli spaces of sheaves on K3 and Abelian surfaces

This Chapter is dedicated to the theory of moduli spaces of sheaves on K3 and Abelian surfaces, specifically focusing on their construction and their fundamental geometric and cohomological properties. These objects constitute the main focus of the present work and belong to a remarkable line of research in modern algebraic geometry, as they provide a systematic method for constructing and investigating higher-dimensional varieties. In particular, they play a central role in hyperkähler geometry, as they provide a rich source of examples of irreducible symplectic varieties, both smooth and singular, building on the general philosophy that moduli spaces of sheaves inherit crucial geometric properties from the underlying variety.

Section 3.1 is meant to outline the construction and describe the main properties of such moduli spaces, along with their structure of symplectic variety. Section 3.2 is devoted to the study of locally trivial deformations of moduli spaces of sheaves, while Section 3.3 addresses the study of their second integral cohomology groups and their lattice and Hodge structure, with a view towards the classification issues presented in Chapter 2. Finally, in Section 3.4, a description of the locally trivial monodromy group of moduli spaces of sheaves is provided in all cases where this computation is known in the literature, as a prelude to the next Part of this work, which is meant to complete this framework.

3.1 Definitions and main properties

In this Section we provide an overview of the construction and main properties of moduli spaces of sheaves on surfaces with trivial canonical bundle - hence, admitting a holomorphic symplectic form. In order to do so, we apply the theory developed in Appendix A to the specific case in which the underlying polarized variety is either a projective K3 or Abelian surface. In this case, the numerical invariants of the parametrized sheaves are fixed by the choice of a *Mukai vector*, which in turn determines the Hilbert polynomial of the latter. The machinery described in Appendix A.3 yields projective varieties that turn out to provide examples of irreducible symplectic varieties, both smooth and singular, mainly due to the work of Mukai, Yoshioka, O'Grady, Kaledin, M. Lehn, Sorger, Perego and Rapagnetta. For a deeper discussion on this theory, we refer to [HL97, Chapter 6], [PR23] and [PR24].

3.1.1 Fixing numerical invariants: Mukai vectors and Mukai lattice

Let us start by recalling that, given a complex projective variety X and a coherent sheaf F on X, by Hirzebruch-Riemann-Roch formula

$$\chi(F) = \int_X \mathrm{ch}(F).\mathrm{td}(X), \tag{3.1}$$

we deduce that, once a polarization H is fixed, the Hilbert polyonimal $P_H(F)$ (see (A.1) of F only depends on the total *Chern character* $\mathrm{ch}(F)$ - and hence, on the Chern classes - of F. Indeed, the *Todd class* $\mathrm{td}(X) := \mathrm{td}(\mathcal{T}_X)$ *of* X only depends on the underlying variety.

Remark 3.1.1. If (S, H) is a polarized surface with $\mathrm{c}_1(H) = h$, we can make this relation more explicit by computing the Hilbert polynomial of a coherent sheaf F of rank r and Chern classes $\mathrm{c}_1(F) = \mathrm{c}_1$, $\mathrm{c}_2(F) = \mathrm{c}_2$:

$$P_H(F)(m) = \frac{rh^2}{2}m^2 + \left(h\cdot \mathrm{c}_1 - \frac{r\mathrm{c}_1(K_S)\cdot h}{2}\right)m + \left(r\chi(\mathcal{O}_S) - \frac{\mathrm{c}_1\cdot \mathrm{c}_1(K_S) + \mathrm{c}_1^2}{2} - c_2\right).$$

If S is either a projective K3 surface - so that $\chi(\mathcal{O}_S) = 2$ - or an Abelian surface - so that $\chi(\mathcal{O}_S) = 0$ - we get, as $\mathrm{c}_1(K_S) = 0$, the following expression

$$P_H(F)(m) = \frac{rh^2}{2}m^2 + (h\cdot \mathrm{c}_1)\,m + \left(2\epsilon(S)r + \frac{\mathrm{c}_1^2}{2} - c_2\right), \tag{3.2}$$

where we set

$$\epsilon(S) := \begin{cases} 1 & \text{if } S \text{ is a K3 surface} \\ 0 & \text{if } S \text{ is an Abelian surface.} \end{cases} \tag{3.3}$$

Definition 3.1.2. Let X be a complex manifold. For any $F \in Coh(X)$, we define its *Mukai vector* as

$$v(F) := \mathrm{ch}(F).\sqrt{\mathrm{td}(X)} \in \mathrm{H}^{\mathrm{ev}}(X, \mathbb{Q}) := \bigoplus_{i=0}^{\dim X} \mathrm{H}^{2i}(X, \mathbb{Q}).$$

If $X = S$ is either a projective K3 surface, or an Abelian surface, set $\epsilon(S)$ as in (3.3), we get that, for any $F \in Coh(S)$,

$$v(F) = (\mathrm{rk}(F), \mathrm{c}_1(F), \mathrm{ch}_2(F) + \epsilon(S)\mathrm{rk}(F)) \in \mathrm{H}^{\mathrm{ev}}(S, \mathbb{Z}) := \bigoplus_{i=0}^{2} \mathrm{H}^{2i}(S, \mathbb{Z}). \tag{3.4}$$

Notice that, if S is an Abelian surface, then $\mathrm{td}(S) = (1, 0, 0)$, hence the Mukai vector of a coherent sheaf on S is nothing but its total Chern character.

The graded $\mathbb{Z}-$module $\mathrm{H}^{\mathrm{ev}}(S, \mathbb{Z})$ appearing in (3.4) is a free $\mathbb{Z}-$module of rank 24 or 8, respectively, but actually carries a richer structure.

Let us start by recalling that, for any pair (E, F) of coherent sheaves on a complex projective manifold X, the *Euler characteristic* of the pair (E, F) is defined as

$$\chi(E, F) := \sum_{i=0}^{\dim X} (-1)^i \mathrm{ext}^i(E, F) = \chi(E^\vee \otimes F).$$

The latter is bilinear in E and F and can be expressed in terms of their Mukai vectors by means of (3.1) as follows:

$$\chi(E, F) = \int_X \mathrm{ch}(E^\vee).\mathrm{ch}(F).\mathrm{td}(X). \tag{3.5}$$

If we set, for any $v = (v_i)_{i=0,\ldots,2\dim X} \in \mathrm{H}^{\mathrm{ev}}(X, \mathbb{Q})$,

$$v^\vee := ((-1)^i v_i)_{i=0,\ldots,2\dim X}, \tag{3.6}$$

we get a bilinear form on $\mathrm{H}^{\mathrm{ev}}(X, \mathbb{Q})$, called *Mukai pairing* (see [HL97, Definition 6.1.5]), defined, for any $v, w \in \mathrm{H}^{\mathrm{ev}}(X, \mathbb{Q})$, as

$$v \cdot w := -\int_X v^\vee . w, \tag{3.7}$$

such that $v(E) \cdot v(F) = -\chi(E, F)$ for any $E, F \in Coh(X)$, by (3.5).

If S is either a projective K3 or an Abelian surface, the Mukai pairing (3.7) defines a non-degenerate integral symmetric bilinear form on $\mathrm{H}^{\mathrm{ev}}(S, \mathbb{Z})$, expressed as

$$(r_1, \xi_1, a_1) \cdot (r_2, \xi_2, a_2) := \xi_1 \cdot \xi_2 - r_1 a_2 - r_2 a_1, \tag{3.8}$$

for any $(r_i, \xi_i, a_i) \in \mathrm{H}^{\mathrm{ev}}(S, \mathbb{Z})$, for $i = 1, 2$, where the operation $\cdot$ on the right hand side coincides with the intersection pairing on $\mathrm{H}^2(S, \mathbb{Z})$.

Definition 3.1.3. Let S be a projective K3 or an Abelian surface. We define the *Mukai lattice* of S as the lattice $\widetilde{\mathrm{H}}(S, \mathbb{Z})$ given by the pair $(\mathrm{H}^{\mathrm{ev}}(S, \mathbb{Z}), \cdot)$, where $\cdot$ is the Mukai pairing defined in (3.8).

Remark 3.1.4. It is immediate to notice that, if Λ_S is the lattice given by $\mathrm{H}^2(S, \mathbb{Z})$ equipped with the intersection pairing, then $\widetilde{\mathrm{H}}(S, \mathbb{Z}) \simeq \Lambda_S \oplus U$, where U is the unimodular hyperbolic lattice of rank 2.

* If S is a K3 surface, then $\widetilde{\mathrm{H}}(S, \mathbb{Z}) \simeq \Lambda_{K3} \oplus U \simeq U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$, as seen in (1.3), which is an even unimodular rank 24 lattice of signature $(4, 20)$.

∗ If S is an Abelian surface, as $\Lambda_S \simeq U^{\oplus 3}$, we get $\widetilde{\mathrm{H}}(S,\mathbb{Z}) \simeq U^{\oplus 4}$, which is an even unimodular rank 8 lattice of signature $(4,4)$.

Furthermore, the Mukai lattice of S inherits a pure weight two *Hodge structure* from the one of S, by setting

$$\begin{aligned} &\widetilde{\mathrm{H}}^{2,0}(S) := \mathrm{H}^{2,0}(S), \qquad \widetilde{\mathrm{H}}^{0,2}(S) := \mathrm{H}^{0,2}(S), \\ &\widetilde{\mathrm{H}}^{1,1}(S) := \mathrm{H}^{0}(S,\mathbb{C}) \oplus \mathrm{H}^{1,1}(S) \oplus \mathrm{H}^{4}(S,\mathbb{C}). \end{aligned} \tag{3.9}$$

Among elements of type $(1,1)$ with respect to the Hodge decomposition (3.9), we identify the following special class, generalizing Definition 3.1.2.

Definition 3.1.5. Let S be a projective K3 or an Abelian surface. A *Mukai vector* is an element $v = (r, \xi, a) \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$ such that $r \geq 0$ and $\xi \in \mathrm{NS}(S)$ and, if $r = 0$ one of the two following conditions holds:

(a) ξ is the first Chern class of a strictly effective divisor;

(b) $\xi = 0$ and $a > 0$.

Remark 3.1.6. The definition above is tailored to ensure that, for any Mukai vector $v \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$, there always exists a coherent sheaf F on S such that $v(F) = v$, thus making the choice of a Mukai vector a reasonable way to fix numerical invariants for sheaves on S to be parametrized. Indeed, if H is a polarization on S and F is a coherent sheaf on S, then its Hilbert polynomial $P_H(F)$ is determined by its Mukai vector $v(F)$ (see Remark 3.1.1). More explicitly, for any $m \in \mathbb{N}$, identity (3.5) yields

$$P_H(F)(m) = \chi(F \otimes \mathcal{O}_S(mH)) = v(F) \cdot v(\mathcal{O}_S(-mH)) =: P_{v,H}(m).$$

Therefore, in the following, once fixed a Mukai vector $v \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$, we will turn our attention to the - possibly empty - subset $M_v(S,H) \subseteq M_{P_{v,H}}(S,H)$ of the moduli space of $H-$semistable sheaves with Hilbert polynomial $P_{v,H}$ (see Definition A.2.1), parametrizing sheaves F with Mukai vector $v(F) = v$. In particular, by [HL97, Theorem 2.1.5], the moduli space $M_{P_{v,H}}(S,H)$ decomposes as a finite disjoint union of locally closed subschemes, each of which is of the form $M_{v'}(S,H)$, with v' a Mukai vector compatible with the Hilbert polynomial $P_{v,H}$, i.e. such that $P_{v',H} = P_{v,H}$.

Analogously, we can study the subset $M^s_v(S,H) \subseteq M^s_{P_{v,H}}(S,H)$ parametrizing $H-$stable sheaves with Mukai vector v.

3.1.2 The moduli spaces $M_v(S,H)$

Let S be a projective K3 or an Abelian surface. In the following, we will call a *polarization* on S any primitive ample divisor H on S. Let H be a polarization on S and let $v \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$ a Mukai vector.

Definition 3.1.7. (1) The subset $M_v(S,H) \subseteq M_P(S,H)$ parametrizing $H-$semistable sheaves with Mukai vector v is called *moduli space of $H-$semistable sheaves on S with Mukai vector v*.

(2) The subset $M_v^s(S,H) \subseteq M_P^s(S,H)$ parametrizing $H-$stable sheaves with Mukai vector v is called *moduli space of $H-$stable sheaves on S with Mukai vector v*.

Of course, the natural inclusion $M_v^s(S,H) \subseteq M_v(S,H)$ holds, but in order to address the problems of non-emptyness of the latter and of whether these admit the structure of a variety, we need to restrict the discussion to deal only with polarizations guaranteeing a good behavior of the moduli spaces, namely *$v-$generic polarizations*. This definition is not uniform across the literature (compare [HL97, Appendix 4.C] and [PR23, Section 2.1.2]), and we will postpone the comparison of different definitions to Section 3.1.4. In this work, we will adopt the notion of *$v-$genericity* introduced in [Yos09], following [PR23], where the notion of *$v-$generic* polarization appears under the name *$v-$general* polarization.

Definition 3.1.8 (Generic polarization)**.** Let S be a projective K3 or Abelian surface, and let $v \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$ be a Mukai vector. A polarization H on S is called *$v-$generic* if it satisfies one of the following conditions:

(1) If $r > 0$, then for every μ_H-semistable sheaf E such that $v(E) = v$ and for every $0 \neq F \subseteq E$ such that $\mu_H(E) = \mu_H(F)$, the equality $\mathrm{c}_1(F)/\mathrm{rk}(F) = \mathrm{c}_1(E)/\mathrm{rk}(E)$ holds.

(2) If $r = 0$, then for every $H-$semistable sheaf E such that $v(E) = v$ and for every $0 \neq F \subseteq E$ such that $\chi(E)/(\mathrm{c}_1(E) \cdot \mathrm{c}_1(H)) = \chi(F)/(\mathrm{c}_1(F) \cdot \mathrm{c}_1(H))$, it holds $v(F) \in \mathbb{Q}v$.

The definition above guarantees that, if v is a *primitive* Mukai vector, i.e. indivisible in $\widetilde{\mathrm{H}}(S,\mathbb{Z})$, then any $H-$semistable sheaf with Mukai vector v is actually $H-$stable, yielding the equality $M_v^s(S,H) = M_v(S,H)$ (see Remark 2.4 and Section 2.1.2 of [PR23]).

In the following, we will assume that S is a projective K3 or an Abelian surface, v is a Mukai vector and H is a $v-$generic polarization on S, and we will denote by $M_v^{(s)} := M_v^{(s)}(S,H)$. For the moment, we will also assume that v is a primitive Mukai vector, so that $M_v^s = M_v$. The starting point is the following result due to Mukai and Yoshioka.

Theorem 3.1.9. *Let S be a projective K3 or Abelian surface, v a primitive Mukai vector and H a $v-$generic polarization on S.*

(1) If S is a K3 surface, then $M_v \neq \emptyset$ if and only if $v^2 \geq -2$. If S is Abelian, then $M_v \neq \emptyset$ if and only if $v^2 \geq 0$.

(2) If $M_v \neq \emptyset$, then it is a smooth projective manifold of dimension $v^2 + 2$ that admits a holomorphic symplectic form.

Proof. The non-emptyness statement in part (1) is contained in [Yos99c, Theorem 0.1] and [Yos01a, Theorem 0.1], respectively. Part (2) is the content of [Mu84], to which we refer for the complete proof. Here we only sketch some aspects that can be deduced by the discussion in Section A.3.

Smoothness and expected dimension. By [Mu84] and Remark 3.1.6, the moduli space M^s_v is an irreducible component of $M^s_P(S,H)$, hence they have the same dimension and the same obstruction theory.

As $K_S = \mathcal{O}_S$, Serre duality yields $\mathrm{ext}^i(F,F)_0 = \mathrm{ext}^{2-i}(F,F)_0$ for any coherent sheaf F on S. Hence for any stable sheaf $F \in M^s_v$, it holds

$$\mathrm{ext}^2(F,F)_0 = \mathrm{ext}^0(F,F)_0 = 0,$$

as F is simple. The vanishing of the obstruction space $\mathrm{Ext}^2(F,F)_0$ implies the smoothness of F, by Theorem A.3.6 (2). Moreover, part (1) of Proposition A.3.5 implies

$$\dim_F M^s_v = \mathrm{ext}^1(F,F) = -\chi(F,F) + \mathrm{ext}^0(F,F) + \mathrm{ext}^2(F,F) = v(F)^2 + 2 = v^2 + 2,$$

by (3.5), Serre duality and the fact that F is simple.

Symplectic form. We will now briefly explain how a holomorphic symplectic form on S can induce a holomorphic symplectic form on M^s_v, referring to [HL97, Chapter 10] for further details. Let us set $X := S \times M^s_v$, let $\Delta \subset X \times X$ be the diagonal and let $\mathcal{I}$ be its ideal sheaf. If we set $\mathcal{O}_{2\Delta} := \mathcal{O}_{X\times X}/\mathcal{I}^2$, we get the following short exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_{2\Delta} \to \mathcal{O}_\Delta \to 0.$$

Let $p_1, p_2 \colon X \times X \to X$ denote the projections onto the two factors and let $E^\bullet$ be a bounded complex of locally free sheaves on X. Since $\mathcal{O}_\Delta$ is p_2-flat, p_1 is flat and $E^\bullet$ is locally free, the compositions of the functors $p_2^* E^\bullet \otimes (\cdot)$ and p_{1*} induce an exact sequence

$$0 \to F^\bullet \otimes \Omega_X \to p_{1*}(p_2^* F^\bullet \otimes \mathcal{O}_{2\Delta}) \to F^\bullet \to 0$$

defining a class $A(F^\bullet) \in \mathrm{Hom}_{D^b(X)}(F^\bullet, F^\bullet[1] \otimes \Omega_X) = \mathrm{Ext}^1(F^\bullet, F^\bullet \otimes \Omega_X)$, compatible with the quasi-isomorphism equivalence relation, called the *Atiyah class* of $F^\bullet$. If we consider a quasi-universal family $\mathcal{E}$ of $H-$stable sheaves on X, we can define its Atiyah class $A(\mathcal{E}) = A(F^\bullet)$ for any locally free resolution $F^\bullet$ of $\mathcal{E}$.

The composition $A(\mathcal{E})^{\otimes 2}$ of two copies of $A(\mathcal{E})$ induces a class

$$A^2(\mathcal{E}) \in \mathrm{Ext}^2(\mathcal{E}, \mathcal{E} \otimes \Omega^2_X)$$

under the morphism $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E} \otimes \Omega_X^{\otimes 2}) \to \mathrm{Ext}^2(\mathcal{E}, \mathcal{E} \otimes \Omega^2_X)$ induced by the canonical map $\Omega_X^{\otimes 2} \to \Omega^2_X$. Then the trace map (A.7) induces a morphism

$$\mathrm{tr}^2 \colon \mathrm{Ext}^2(\mathcal{E}, \mathcal{E} \otimes \Omega^2_X) \to \mathrm{H}^2(X, \Omega^2_X),$$

therefore we get a class

$$\mathrm{tr}^2(A^2(\mathcal{E})) \in \mathrm{H}^2(X, \Omega^2_X) = \bigoplus_{p+q=2, i+j=2} \mathrm{H}^i(S, \Omega^p_S) \otimes \mathrm{H}^j(M^s_v, \Omega^q_{M^s_v}),$$

by Künneth decomposition. The component of $\mathrm{tr}^2(A^2(\mathcal{E}))$ lying in $\mathrm{H}^2(S, \mathcal{O}_S) \otimes \mathrm{H}^0(M_v^s, \Omega^2_{M_v^s}) \simeq \mathrm{Hom}(\mathrm{H}^2(S, \mathcal{O}_S)^*, \mathrm{H}^0(M_v^s, \Omega^2_{M_v^s}))$ induces a morphism

$$\tau_{\mathcal{E}} \colon \mathrm{H}^0(S, \Omega^2_S) \simeq \mathrm{H}^2(S, \mathcal{O}_S)^* \to \mathrm{H}^0(M_v^s, \Omega^2_{M_v^s})$$

that maps a holomorphic symplectic form σ_S on S into a holomorphic symplectic form $\tau_{\mathcal{E}}(\sigma_S) =: \sigma_{M_v^s}$ on M_v^s (see [HL97, Corollary 10.4.2]). For any $F \in M_v^s$, under the canonical identification $T_F M_v^s \simeq \mathrm{Ext}^1(F, F)$, the symplectic form $\sigma_{M_v^s}$ acts pointwisely as follows:

$$\sigma_{M_v^s, F} \colon \mathrm{Ext}^1(F, F) \times \mathrm{Ext}^1(F, F) \xrightarrow{\circ} \mathrm{Ext}^2(F, F) \xrightarrow{\mathrm{tr}^2} \mathrm{H}^2(S, \mathcal{O}_S) \xrightarrow{\sigma_S} \mathrm{H}^2(S, K_S) \simeq \mathbb{C},$$

where $\circ \colon \mathrm{Ext}^1(F, F) \times \mathrm{Ext}^1(F, F) \to \mathrm{Ext}^2(F, F)$ is the *Yoneda pairing* defined under the canonical isomorphism $\mathrm{Ext}^i(E^\bullet, F^\bullet) \simeq \mathrm{Hom}_{D^b(Y)}(E^\bullet, F^\bullet[i])$, for any $E^\bullet, F^\bullet \in D^b(Y)$, for any smooth projective variety Y and for any $i = 0, \dots, \dim Y$. □

In particular, we get the following

Corollary 3.1.10. *If S is a projective K3 or Abelian surface, v is a primitive Mukai vector of square $v^2 \geq -2$ or $v^2 \geq 0$, respectively, and H is a $v-$generic polarization on S, then $M_v(S, H)$ is smooth.*

Moreover, for a primitive Mukai vector v as above, the moduli space $M_v(S, H)$ is a compact smooth projective manifold, which is even dimensional (see Remark 3.1.4) and carries a holomorphic symplectic form. Therefore, it is natural to ask whether this construction provides - possibly new - examples of irreducible holomorphic symplectic manifolds, as motivated by the following Example.

Example 3.1.11. Let S be a projective K3 surface, let us consider the Mukai vector $v = (1, 0, -n+1)$, with $n \geq 2$ and H any $v-$generic polarization. Then the moduli space $M_v(S, H)$ is isomorphic to the Hilbert scheme $Hilb^n(S)$ of n points on S (see part 2 of Section 1.1.2). Indeed, any $0-$dimensional closed subscheme Z of length n can be identified with its ideal sheaf $\mathcal{I}_Z$, which is a $H-$stable sheaf with Mukai vector v. Conversely, any sheaf $F \in M_v(S, H)$ is a rank 1 coherent sheaf on S fitting in the following short exact sequence

$$0 \to F \to F^{\vee\vee} \simeq \mathcal{O}_S \to Q_F \to 0,$$

where the quotient Q_F is a rank 0 coherent sheaf supported on a closed subscheme $Z \subseteq X$ of length n. We can therefore identify F with $supp(Q_F) = Z$. For further details, we refer to [HL97, Example 4.3.6, Example 4.5.10].

Example 3.1.11 generalizes to the following result.

Theorem 3.1.12. *Let S be a projective K3 or Abelian surface, v a primitive Mukai vector of square $v^2 = 2k > 0$ and H a $v-$generic polarization on S.*

(1) *If S is a K3 surface, then $M_v(S,H)$ is an irreducible holomorphic symplectic manifold, deformation equivalent to $Hilb^{k+1}(S)$.*

(2) *If S is an Abelian surface, then $M_v(S,H)$ is deformation equivalent to $\hat{S} \times Hilb^k(S)$, where $\hat{S} = \mathrm{Pic}^0(S)$ is the dual surface of S.*

Proof. See [Mu87], [Huy99, Corollary 4.8] for part (1) and [Yos01a, Theorem 0.1] for part (2). □

Thus, in the primitive case, moduli spaces of stable sheaves on projective K3 surfaces provide examples of IHS manifolds, all of $K3^{[n]}-$type. On the other hand, if the underlying surface is Abelian, then the moduli space obtained is not simply connected, by part (2) of Theorem 3.1.12. Therefore, it will involve a further construction, building on the same philosophy of generalized Kummer manifolds (see part 3 of Section 1.1.2), that will be addressed below.

In order to address the Abelian case with full generality, but also to complete the description of the moduli spaces $M_v(S,H)$, we provide the following generalization of Theorem 3.1.9 to the non-primitive case.

Theorem 3.1.13. *Let S be a projective K3 or Abelian surface, v a Mukai vector of square $v^2 > 0$ and H a $v-$generic polarization on S. Then, the moduli space $M_v(S,H)$ is a non-empty, irreducible, normal, projective variety of dimension v^2+2. Its smooth locus is $M_v^s(S,H)$ and the latter carries a holomorphic symplectic form.*

Proof. See [KLS06, Theorem 4.4]. □

3.1.3 The Abelian case - the moduli spaces $K_v(S,H)$

Let S be an Abelian surface. We recall that its *dual surface* $\hat{S}$, defined as

$$\hat{S} := \mathrm{Pic}^0(S) = \ker(\mathrm{c}_1 \colon \mathrm{Pic}(S) \to \mathrm{H}^2(S,\mathbb{Z})),$$

is again a $2-$dimensional complex torus, as it is isomorphic to $\mathrm{H}^1(S,\mathcal{O}_S)/\mathrm{H}^1(S,\mathbb{Z})$ via the exponential sequence. Moreover, it can easily be checked (See [Huy06, Section 9.1] that the latter coincides with the subgroup

$$\{L \in \mathrm{Pic}(S) \colon \tau_p^* L \simeq L \text{ for all } p \in S\} \subseteq \mathrm{Pic}(S)$$

of *translation invariant line bundles* on S. Furthermore, we can define the *Picard functor $\underline{\mathcal{P}ic}^0_S$ of S*, sending any variety X to the set

$$\underline{\mathcal{P}ic}^0_S(X) := \{M \in \mathrm{Pic}(S\times X) \colon M_x := M_{|S\times\{x\}} \in \mathrm{Pic}^0(S) \text{ for all } x \in X \text{ closed}\}_{\diagup\sim},$$

where $M \sim M'$ if and only if there exists $L \in \mathrm{Pic}(X)$ such that $M \otimes \pi_X^* L \simeq M'$, and any morphism $f \colon Y \longrightarrow X$ to its pull-back $(id_S \times f)^* \colon \underline{\mathcal{P}ic}^0_S(X) \longrightarrow \underline{\mathcal{P}ic}^0_S(Y)$.

By [Huy06, Theorem 9.14], the Picard functor is representable (see Definition A.2.1) by the dual surface $\hat{S}$, so that there exists an isomorphism of functors

$$\alpha\colon \mathrm{Hom}(\cdot, \hat{S}) \simeq \underline{\mathcal{P}ic}^0_S.$$

In particular, we get that $\hat{S}$ is projective, hence it is an Abelian surface, and that there exists a universal line bundle $\mathcal{P} \in \mathrm{Pic}(S \times \hat{S})$, corresponding to $\mathrm{id}_{\hat{S}}$ under the isomorphism $\alpha_{\hat{S}}$. The line bundle $\mathcal{P}$ is called *Poincaré line bundle* and it satisfies the following universal property: for any $L \in \hat{S}$, it holds $\mathcal{P}_{|S\times\{L\}} \simeq L$, and $\mathcal{P}_{|\{0\}\times\hat{S}} = 0$ in $\mathrm{Pic}^0(\hat{S}) \simeq S$.

Yoshioka's fibration

We will now review the construction of the moduli space which will be the main focus of this work, following [PR23, Section 2.2.1], [Yos99a] and [Yos01a].

Let us fix a Mukai vector $v \in \widetilde{\mathrm{H}}(S, \mathbb{Z})$ of positive square $v^2 > 0$ and let us fix a $v-$generic polarization and a coherent sheaf F_0 on S such that $v(F_0) = v$. Let us consider the *Fourier-Mukai transform* (see Section 4.2, or [Huy06, Chapter 5] for further details)

$$\mathrm{FM}_{\mathcal{P}}\colon D^b(S) \longrightarrow D^b(\hat{S}), \qquad F \longmapsto R\pi_{\hat{S}*}(\pi_S^*(F) \overset{L}{\otimes} \mathcal{P}),$$

with kernel the Poincaré line bundle, which is an equivalence by [Mu81] (see also [Huy06, Proposition 9.19]). The map

$$\begin{aligned} a_v = (a_v^1, a_v^2)\colon M_v(S,H) &\to S \times \hat{S} \\ F &\mapsto (\det(\mathrm{FM}_{\mathcal{P}}(F)) \otimes \det(\mathrm{FM}_{\mathcal{P}}(F_0))^\vee, \det(F) \otimes \det(F_0)^\vee), \end{aligned}$$

often called *Yoshioka's fibration*, is well defined under the canonical isomorphism $\hat{\hat{S}} \simeq S$. It was first defined by Yoshioka in [Yos99a] and [Yos01a] for a primitive Mukai vector v and then generalized to the non-primitive case in [PR23, Section 2.2.1]. Let us denote by

$$K_v(S,H) := a_v^{-1}(0_S, \mathcal{O}_S)$$

the fiber of a_v over the $0-$point of the group $S \times \hat{S}$. Then, the map a_v satisfies the following properties.

Theorem 3.1.14. *Let S be an Abelian surface, v a Mukai vector with $v^2 > 0$ and H a $v-$generic polarization.*

(1) The Yoshioka fibration $a_v\colon M_v(S,H) \to S \times \hat{S}$ is an isotrivial fibration. In particular, all fibers are isomorphic to $K_v(S,H)$.

(2) The Abelian variety $S \times \hat{S}$ is the Albanese variety of $M_v(S,H)$ and the Yoshioka fibration coincides with the Albanese morphism $alb\colon M_v(S,H) \to Alb(M_v(S,H))$.

Proof. (1) Isotriviality follows from the fact that the morphism

$$\begin{aligned} \tau_v\colon K_v(S,H)\times S\times \hat{S} &\longrightarrow M_v(S,H) \\ (F,p,L) &\longmapsto \tau_p^*(F)\otimes L \end{aligned} \tag{3.10}$$

is a finite étale cover, as shown in [Yos01a, Section 4.2] and [PR23, Section 2.2.1].

(2) This is the content of [Yos01a, Theorem 0.1 (1)] in the primitive case. The generalization to the non-primitive case is addressed in [PR23, Corollary 3.7], where the authors use the fact that, for a normal projective variety X with at most rational singularities, one can set $Alb(X) := Alb(\widetilde{X})$ for any desingularization $\widetilde{X}$ of X and construct the Albanese morphism $alb\colon X \to Alb(X)$ descending the usual Albanese morphism of $\widetilde{X}$ (see [Rei83, Proposition 2.3], [Kaw85, Lemma 8.1]). □

Hence, if non-empty, we expect the fiber $K_v(S,H)$ to be a variety of dimension

$$\dim(M_v(S,H)) - \dim(S\times\hat{S}) = v^2+2-4 = v^2-2.$$

As a consequence of isotriviality, the isomorphism class of $K_v(S,H)$ does not depend on the choice of the coherent sheaf F_0. With a slight abuse of terminology, we will refer to $K_v(S,H)$ as *the moduli space of $H-$semistable sheaves on the Abelian surface S* and to $K_v^s(S,H) := M_v^s(S,H)\cap K_v(S,H)$ as *the moduli space of $H-$stable sheaves on the Abelian surface S*.

We can now complete the picture stating Theorem 3.1.13 and Theorem 3.1.12 for the moduli spaces of sheaves above-introduced in the Abelian case. As it will be necessary to highlight some numerical invariants of the Mukai vector v, to keep notation concise, we introduce the following definition.

Definition 3.1.15 ((m,k)$-$triple)**.** Let $m,k\in\mathbb{N}\setminus\{0\}$ be two strictly positive integers. A triple (S,v,H) will be called an *$(m,k)-$triple* if S is a projective K3 or an Abelian surface[1], v is a Mukai vector on S of the form $v=mw$, where w is a primitive Mukai vector such that $w^2=2k$ and H is a $v-$generic polarization on S. In this case, we will also refer to v as a *Mukai vector of type (m,k)*.

Remark 3.1.16. Exactly as in [OPR24, Remark 1.22], it can easily be checked that, if (S,v,H) is an $(m,k)-$triple with $v=mw$, then (S,w,H) is a $(1,k)-$triple.

Theorem 3.1.17. *Let (S,v,H) be an $(m,k)-$triple, with S an Abelian surface.*

[1] In this Chapter, we will deal both with K3 and Abelian surfaces. Throughout Part II we will work only with Abelian surfaces, hence this Definition will be used only in the Abelian case (see Assumption 1)

(1) *If $(m,k) \neq (1,1)$, then $K_v(S,H)$ is a non-empty, irreducible, normal projective variety of dimension $v^2 - 2 = 2m^2k - 2$. Its smooth locus is $K_v^s(S,H)$, which carries a holomorphic symplectic form.*

(2) *If $m = 1$ and $k > 1$, then $K_v(S,H) = K_v^s(S,H)$ is smooth and it is an irreducible holomorphic symplectic manifold, deformation equivalent to $Kum^{k-1}(S)$.*

Proof. Both statements follow from [Yos01a, Theorem 0.1, Theorem 0.2] in the primitive case, and from [KLS06, Theorem 4.4] in the non-primitive case. □

By part (2) of Theorem 3.1.17, we get that also moduli spaces of stable sheaves on Abelian surfaces provide examples of irreducible holomorphic symplectic manifolds. Again, those examples fall under a deformation type that we encountered before - indeed, they provide a modular construction of generalized Kummer manifolds. An explicit example of construction is sketched by the following.

Example 3.1.18. Let S be an Abelian surface, let us consider the Mukai vector $v = (1, 0, -n)$, with $n \geq 3$ and let H be a $v-$generic polarization. Then $M_v(S,H)$ is isomorphic to $Hilb^n(S) \times \hat{S}$, where the identification on the first component is given, as in Example 3.1.11, by the morphism h associating to each subsheaf $F \in M_v(S,H)$ the support of its quotient $\mathcal{O}_S/F$. Yoshioka's fibration factorizes through this isomorphism

$$\begin{tikzcd} M_v(S,H) \arrow[rr, "a_v=(a_v^1, a_v^2)"] \arrow[dr, "(h, a_v^2)"', "\simeq"] & & S \times \hat{S} \\ & Hilb^n(S) \times \hat{S} \arrow[ur, "(a_{n-1}, \mathrm{id}_{\hat{S}})"'] & \end{tikzcd}$$

where $a_{n-1} = \Sigma_n \circ \mathrm{HC}^n$ is the composition of the summation map with the Hilbert-Chow morphism, defined in (1.4). Under this identification, we get an isomorphism $K_v(S,H) \simeq Kum^{n-1}(S)$.

If $n = 2$, the above construction collapses to the construction of the Kummer surface $Kum(S)$, which is a K3 surface.

Remark 3.1.19 ("Low dimensional" cases)**.** We point out that all the previous statements, including the very definition of $(m,k)-$triple, were given under the assumption that the Mukai vectors v taken under consideration had strictly positive square, i.e. if and only if $k > 0$. This hypothesis has an underlying geometric meaning, not only related to non-emptyness of the moduli spaces, which we will clarify in the following. Let us consider an $(m,k)-$triple (S, v, H), so that $v = mw$, with w primitive of square $w^2 = 2k$.

$v^2 < 0$: equivalently, $k < 0$.

* If S is K3, then $M_v(S,H)$ either empty (if $k < -1$) or a point (if $k = -1$, see [Mu87]).

* If S is Abelian, then $M_v(S,H)$ is empty (see [Yos01a]), so it is $K_v(S,H)$.

$v^2 = 0$: equivalently, $k = 0$.

$m = 1$: $M_v(S,H)$ is deformation equivalent to S ([Mu87]). In particular,

* If S is K3, then $M_v(S,H)$ is a K3 surface.
* If S is Abelian, then $M_v(S,H)$ is an Abelian surface and the analogue of $K_v(S,H)$ is a point.

$m > 1$: $M_v(S,H)$ is isomorphic to the symmetric product $M_w(S,H)^{(m)}$ (see [KLS06, Section 1]).

* If S is K3, then $M_v(S,H)$ is the $m-$th symmetric product of a K3 surface, which is a primitive symplectic variety admitting a symplectic resolution via $Hilb^m(M_w(S,H))$ (see Example 1.2.17).
* If S is Abelian, then $M_v(S,H)$ is the $m-$th symmetric product of an Abelian surface. The analogue of $K_v(S,H)$ is the fiber of the summation map Σ^m, which is again a primitive symplectic variety - admitting $Kum^{m-1}(M_w(S,H))$ as symplectic resolution - which is not irreducible symplectic, analogously as in Example 1.2.17.

For this reasons, we will be only interested in $(m,k)-$triples with $k > 0$. We only point out that, if $v = w$ is primitive, then we have to distinguish the following situations:

* If S is K3, then $M_w(S,H)$ is of $K3^{[k+1]}-$type for any $k > 0$.
* If S is Abelian, then, according to k, the following cases can occur for $K_w(S,H)$:

$k = 1$: $K_w(S,H)$ is a point.
$k = 2$: $K_w(S,H)$ is the Kummer surface of S, which is a K3 surface.
$k > 2$: $K_w(S,H)$ is of $Kum^{k-1}-$type.

This low-square jump phenomenon, occurring only in the Abelian case, will turn crucial in the second part of this work (see Section 6.4).

We observed that, given a $(1,k)-$triple (S,w,H), with S a projective K3 or Abelian surface, the smooth moduli spaces $M_w(S,H)$ and $K_w(S,H)$, respectively, provide examples of irreducible holomorphic symplectic manifolds. On the other hand, if (S,v,H) is an $(m,k)-$triple, with v non-primitive, the moduli spaces $M_v(S,H)$ and $K_v(S,H)$, respectively, are normal projective varieties whose smooth locus carries a holomorphic symplectic form. Therefore, it is natural to ask whether the latter yield examples of possibly singular symplectic varieties, or can be used to provide, via symplectic desingularizations, other examples of IHS manifolds. Indeed, moduli spaces of semistable sheaves as above fall precisely under this two cases.

Theorem 3.1.20. *Let (S, v, H) be a $(2,1)-$triple, with S either a projective K3 or Abelian surface.*

(1) If S is K3, then $M_v(S,H)$ admits a symplectic resolution $\widetilde{M_v(S,H)}$ that is an irreducible holomorphic symplectic manifold of deformation type OG_{10}.

(2) If S is Abelian, then $K_v(S,H)$ admits a symplectic resolution $\widetilde{K_v(S,H)}$ that is an irreducible holomorphic symplectic manifold of deformation type OG_6.

Proof. The statement was first proved by O'Grady in [OG99] and [OG03], respectively, for a specific Mukai vector $v = (2, 0, -2)$, giving the very first construction of these kind of examples (see also part 4 of Section 1.1.2). In [LS06], the existence of a symplectic resolution, obtained by blowing up the strictly semistable locus - $M_v(S,H) \setminus M_v^s(S,H)$ and $K_v(S,H) \setminus K_v^s(S,H)$ respectively - with reduced structure, is shown for any $(2,1)-$triple. Finally, in [PR13], it is shown that such symplectic resolutions are irreducible holomorphic symplectic manifolds deformation equivalent to OG_{10} and OG_6, respectively. □

Before examining the other case, given by $(m,k)-$triples different from $(2,1)$, we discuss factoriality properties of singular moduli spaces as above.

Theorem 3.1.21. *Let (S, v, H) be an $(m,k)-$triple, with $m > 1$, and S either a projective K3 or Abelian surface.*

(1) If $(m,k) = (2,1)$, then $M_v(S,H)$ and $K_v(S,H)$, respectively, are $2-$factorial.

(2) If $(m,k) \neq (2,1)$, then $M_v(S,H)$ and $K_v(S,H)$, respectively, are locally factorial.

Proof. Part (1) is Theorem 1.1 of [Per09], while part (2) is Theorem A of [KLS06]. □

Remark 3.1.22. In particular, we get that, for any $(m,k)-$triple (S, v, H), with v a non-primitive Mukai vector and S either a projective K3 or Abelian surface, the singular moduli spaces $M_v(S,H)$ and $K_v(S,H)$ are all $\mathbb{Q}-$factorial. Moreover, these are terminal if and only if $(m,k) \neq (2,1)$.

Indeed, both $M_v(S,H)$ and $K_v(S,H)$ are normal, irreducible, projective varieties admitting a holomorphic symplectic form on their smooth loci, and their singular loci have codimension equal to 2 if $(m,k) = (2,1)$, and at least equal to 4 otherwise (see [KLS06, Proposition 6.1]). By Flenner's Theorem ([Fle88]), the symplectic form extends for any resolution of singularities, hence these are symplectic varieties. By [Nam01c, Corollary 1] (see also Remark 1.2.13), a symplectic variety is terminal if and only if the codimension of the singular locus is at least equal to 4, which is the case for $(m,k) \neq (2,1)$.

From the previous Remark, we deduce the following (see [KLS06, Theorem B]):

Corollary 3.1.23. *Let (S, v, H) be an $(m,k)-$triple, with $m > 1$, and S either a projective K3 or Abelian surface. If $(m,k) \neq (2,1)$, then $M_v(S,H)$ and $K_v(S,H)$, respectively, do not admit any symplectic resolution.*

Although no further constructions of irreducible holomorphic symplectic manifolds can be obtained from singular moduli spaces as above, we have the following foundational result, which is the main Theorem of [PR23].

Theorem 3.1.24. *Let $m, k \in \mathbb{N} \setminus \{0\}$ and let (S, v, H) be an $(m,k)-$triple, with S either a projective K3 or Abelian surface.*

(1) If S is K3, then $M_v(S, H)$ is an irreducible symplectic variety.

(2) If S is Abelian and $(m, k) \neq (1, 1)$, then $K_v(S, H)$ is an irreducible symplectic variety.

Proof. See [PR23, Theorem 1.10], where the statement is proved for any $(m,k)-$triple as above, with $m > 1$. For the reader's convenience, we also include the previously addressed primitive cases (see Theorem 3.1.12 and Theorem 3.1.17 (2)). The pair $(1, 1)$ has been excluded in the Abelian case, as the associated moduli space of sheaves collapses to a point, by Remark 3.1.19. □

Therefore, all moduli spaces of semistable sheaves as above define examples of irreducible symplectic varieties, all $\mathbb{Q}-$factorial, and all terminal with only one exception. In the next Section (3.2) their locally trivial deformation classes will be characterized in terms of the chosen Mukai vector, showing that this construction provides infinitely many new and different examples of irreducible symplectic varieties.

Before proceeding, we address the comparison of the different definitions of genericity for a polarization, that was postponed at the beginning of the Section.

3.1.4 Generic polarizations: comparison of definitions and congruence relations

In Section 3.1.2 we introduced the notion of *generic polarization* (Definition 3.1.8) with respect to a Mukai vector. Subsequently, we discussed the theory of moduli spaces of semistable sheaves, constructed starting from an $(m,k)-$triple (S, v, H) defined by a projective K3 or Abelian surface S, a Mukai vector v and a $v-$generic polarization, according to Definition 3.1.8. On the other hand, most of the abovementioned theory was developed for triples using polarizations that are $v-$generic according to a different definition (see, for instance, [HL97, Appendix 4.C]), that we will state and discuss below.

Let S be a projective K3 or Abelian surface, and let $v = (v_0, v_1, v_2) \in \widetilde{\mathrm{H}}(S, \mathbb{Z})$ be a Mukai vector. Then the latter induces a *wall and chamber decomposition* of the ample cone $Amp(S)$ of S as follows.

Case 1: $v_0 > 0$. Set

$$|v| := \frac{v_0^2}{4} v \cdot v + \frac{v_0^{2\epsilon(S)+2}}{2},$$

where $\cdot$ is the Mukai pairing, and we recall that $\epsilon(S) = 1$ if S is K3 and 0 if S is Abelian. The rational number $|v|$ is always strictly positive, if v is a Mukai vector of type (m,k), with $m,k > 0$. In that case, we set

$$W_v := \{D \in \mathrm{NS}(S)\colon\ -|v| \leq D^2 < 0\},$$

while we set $W_v := \emptyset$ if $|v| = 0$.

Case 2: $v_0 = 0$**.** For every pure sheaf E of Mukai vector v, and for every subsheaf $0 \neq F \subseteq E$ of Mukai vector $v(F) = (0, u_1, u_2)$, we define the *divisor associated to the pair* (E,F) as

$$D_{E,F} := u_2 v_1 - v_2 u_1. \tag{3.11}$$

We therefore define W_v as the set of non-numerically trivial divisors $D_{E,F}$ as in (3.11), for any pair (E,F) as above.

For any Mukai vector v either in case 1 or case 2, we give the following definition.

Definition 3.1.25 (Generic polarization - alternative definition)**.** Let S be a projective K3 or Abelian surface, and let $v \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$ be a Mukai vector. A polarization H on S is $v-$*generic* if $\mathrm{c}_1(H) \cdot D \neq 0$ for every $D \in W_v$.

Remark 3.1.26. If S has Picard rank $\rho(S) = 1$, then the ample generator of $\mathrm{Pic}(S)$ is $v-$generic with respect to Definition 3.1.25 for every Mukai vector v.

If $\rho(S) > 1$, we characterize this notion of genericity as follows.

Definition 3.1.27. If $D \in W_v$, we define the *$v-$wall associated to D* as the orthogonal complement $D^\perp$ of D in $Amp(S)$, with respect to the intersection pairing.

For any $D \in W_v$, the associated $v-$wall $D^\perp$ is an hyperplane in $Amp(S)$ and the set of such $v-$walls is locally finite if $v_0 > 0$ (see [HL97, Theorem 4.C.2]), or even finite if $v_0 = 0$ (see [Yos01a, Section 1.4]).

Definition 3.1.28. Suppose that $\rho(S) > 1$. Any connected component of $Amp(S) \setminus \bigcup_{D \in W_v} D^\perp$ is called *$v-$chamber*.

Remark 3.1.29. Hence, if $\rho(S) > 1$, a polarization H is $v-$generic according to Definition 3.1.25 if and only if $\mathrm{c}_1(H)$ lies in a $v-$chamber. We moreover point out the following important properties, assuming $\rho(S) > 1$:

(1) As the set of $v-$walls is locally finite in $Amp(S)$, for any Mukai vector v as above, a $v-$generic polarization (Definition 3.1.25) always exists.

(2) Changing polarization inside a $v-$chamber does not affect the moduli space $M_v(S,H)$ (and neither $K_v(S,H)$, if S is Abelian). More precisely, if $\mathcal{C}$ is a $v-$chamber and $H_1, H_2 \in \mathcal{C}$, then, by [PR23, Proposition 2.5], any sheaf of Mukai vector v is H_1-semistable if and only if it is H_2 semistable, yielding identifications

$$M_v(S,H_1) = M_v(S,H_2) \text{ and } M_v^s(S,H_1) = M_v^s(S,H_2).$$

(3) If $v \neq (0, v_1, 0)$, H is a $v-$generic polarization and E a $H-$semistable sheaf with Mukai vector v, then any $H-$destabilizing subsheaf F of E satisfies $v(F) \in \mathbb{Q}v$. We therefore deduce that, if v is primitive and H is $v-$generic (Definition 3.1.25), then any $H-$semistable sheaf of Mukai vector v is $H-$stable.

Remark 3.1.30 (The case $v = (0, v_1, 0)$ and $\rho(S) > 1$)**.** If $v = (0, v_1, 0)$ and $\rho(S) > 1$, two problems may arise. In that case, any divisor $D \in W_v$ must be of the form bv_1, for some $b \neq 0$, yielding $H \cdot D \neq 0$ for any $v-$generic polarization, as v_1 is the class of an effective divisor. Hence, any polarization is $v-$generic according to Definition 3.1.25.

Moreover, the statement in part (3) of Remark 3.1.29 is no longer true, as shown in [PR23, Example 2.7]. In particular, if $\rho(S) > 1$ and v_1 is primitive, for any polarization H we get

$$M^s_{(0,v_1,0)}(S, H) \subsetneq M_{(0,v_1,0)}(S, H),$$

losing the main motivation for introducing the notion of genericity.

As Definition 3.1.25 is not well adapted for Mukai vectors of the form $v = (0, v_1, 0)$, when $\rho(S) > 1$, the notion of $v-$genericity appearing in Definition 3.1.8 was introduced in [Yos01a], and in [PR23, Section 2.1.2] it is explained how the latter can replace the first one, keeping consistent the theory of moduli spaces of sheaves discussed in the previous Sections. More precisely, this means that it is always possible to find isomorphisms, or even identifications, of moduli spaces

$$M_v(S, H_1) \simeq M_v(S, H_2) \text{ and } M^s_v(S, H_1) \simeq M^s_v(S, H_2),$$

where H_1 is $v-$generic according to Definition 3.1.8 and H_2 is $v-$generic according to Definition 3.1.25, as explained below.

Remark 3.1.31. Let S be a projective K3 or Abelian surface and let $v = (v_0, v_1, v_2)$ be a Mukai vector.

(1) If $\rho(S) = 1$, then the ample generator of $\mathrm{Pic}(S)$ is $v-$generic with respect to both Definitions 3.1.8 and 3.1.25, by [PR23, Lemma 2.9 (1)].

(2) If $\rho(S) > 1$, assume that, if $v = (0, v_1, v_2)$, then $v_2 \neq 0$. Then, the following hold:

  (a) Any polarization that is $v-$generic according to Definition 3.1.25 is $v-$generic according to Definition 3.1.8, [PR23, Lemma 2.9 (2)].

  (b) If H is a $v-$generic polarization according to Definition 3.1.8, it is not necessarily $v-$generic according to Definition 3.1.25, so it may lie on a $v-$wall. In that case, let $\mathcal{C}$ be the $v-$chamber such that $c_1(H) \in \overline{\mathcal{C}}$ (the closure of $\mathcal{C}$ in $Amp(S)$) and let $H' \in \mathcal{C}$. Then, by [PR23, Lemma 2.10], there are identifications of moduli spaces $M^{(s)}_v(S, H) = M^{(s)}_v(S, H')$, and $K^{(s)}_v(S, H) = K^{(s)}_v(S, H')$, if S is Abelian.

(c) If $v = (0, v_1, 0)$, then it may happen that no $v-$generic polarizations (according to Definition 3.1.8) exist at all (see [PR23, Example 2.11]).
Nonetheless, tensorization with H induces isomorphisms (see Section 4.2.1 and [PR23, Lemma 2.24]) $M_v^{(s)}(S,H) \simeq M_{v_H}^{(s)}(S,H)$, and $K_v^{(s)}(S,H) \simeq K_{v_H}^{(s)}(S,H)$ if S is Abelian, where $v_H := (0, v_1, v_1 \cdot H)$. Moreover, H is $v-$generic if and only if it is v_H-generic (according to Definition 3.1.8). Thus, case (b) can be applied to v_H, providing a v_H-generic polarization H' (according to Definition 3.1.25) such that $M_v^{(s)}(S,H) \simeq M_{v_H}^{(s)}(S,H')$.

Remark 3.1.31 implies that Definition 3.1.8 is well suited for dealing with the theory of moduli spaces of sheaves for any choice of a Mukai vector as in Definition 3.1.2, thus from now on we will adopt Definition 3.1.8 as the standard one. On the other hand, Remark 3.1.31 will allow us to restrict to considering polarizations that are $v-$generic according to Definition 3.1.25 when proving results for moduli spaces defined by triples (S, v, H), with H $v-$generic according to the standard definition.

For later use, we introduce an equivalence relation designed to formalize the identifications of moduli spaces mentioned in Remark 3.1.31.

Definition 3.1.32. Two $(m,k)-$triples (S_1, v_1, H_1) and (S_2, v_2, H_2) are said to be *congruent* if $S_1 = S_2 =: S$, $v_1 = v_2 =: v$ and, for every coherent sheaf F on S such that $v(F) = v$, we have that F is H_1-semistable if and only if F is H_2-semistable.

The last property yields identifications $M_v(S,H_1) = M_v(S,H_2)$ and $K_v(S,H_1) = K_v(S,H_2)$ (see [PR23, Lemma 2.16]) and we denote by

$$\chi_{H_1,H_2} \colon K_v(S,H_1) \longrightarrow K_v(S,H_2) \tag{3.12}$$

the identity morphism.

Remark 3.1.33. It is straightforward from Remark 3.1.31 (2.b) that, if H_1 and H_2 are two $v-$generic polarizations lying in the closure of the same $v-$chamber, then the triples (S, v, H_1) and (S, v, H_2) are congruent.

3.2 Deformations of moduli spaces of sheaves

In Section 3.1 we discussed the construction of moduli spaces of semistable sheaves on K3 and Abelian surfaces and how the latter yield examples of irreducible symplectic varieties, both smooth and singular. More precisely, Theorem 3.1.24 establishes that, for any choice of an $(m,k)-$triple (S, v, H), with S either a projective K3 or Abelian surface, the moduli spaces $M_v(S,H)$ and $K_v(S,H)$, respectively, are irreducible symplectic varieties, with the only exception of points and symmetric products. The following result in [PR23] shows that the locally trivial deformation class of moduli spaces as above is determined by the choice of the pair (m,k).

Theorem 3.2.1. *Let $m, k \in \mathbb{N} \setminus \{0\}$ and let (S_1, v_1, H_1) and (S_2, v_2, H_2) be two $(m,k)-$triples.*

(1) If S_1 and S_2 are both K3 or both Abelian surfaces, then $M_{v_1}(S_1, H_1)$ and $M_{v_2}(S_2, H_2)$ are locally trivial deformation equivalent.

(2) If S_1 and S_2 are both Abelian surfaces, then $K_{v_1}(S_1, H_1)$ and $K_{v_2}(S_2, H_2)$ are locally trivial deformation equivalent.

Proof. In the primitive case - i.e. for $m = 1$ - this result is due to [Mu84], [Bea83], [OG97], [Yos99c], [Yos01a] while the case $(m,k) = (2,1)$ is dealt in [PR13]. For $(m,k) \neq (2,1)$ and $m > 1$, it is due to [Yos09] and [PR23]. For a complete proof, see [PR23, Theorem 1.7]. □

Remark 3.2.2. A remarkable feature of this result and its proof is that the locally trivial families defining the above-stated equivalence relation are constructed from deformations of the underlying surface, together with the Mukai vector and the polarization, inducing a locally trivial deformation by means of relative moduli spaces of sheaves (see Appendix A.3.3). This construction will play a fundamental role in the second Part of this work, and will be discussed in more detail in Section 4.1.

Theorem 3.2.1 states that the locally trivial deformation class of a moduli space of sheaves constructed from an $(m,k)-$triple is determined by the pair (m,k). In the next Section we will see that the correspondence is one to one, i.e. that every pair determines a different locally trivial deformation class of irreducible symplectic varieties.

In fact, as a consequence of Proposition 2.2.4, we deduce that any locally trivial deformation of moduli spaces of sheaves built from $(m,k)-$triples with $(m,k) \neq (2,1)$ are again $\mathbb{Q}-$factorial and terminal irreducible symplectic varieties.

Corollary 3.2.3. *Let m and k be two strictly positive integers and let (S, v, H) be an $(m,k)-$triple.*

(1) If $(m,k) \neq (2,1)$, any small locally trivial deformation of $M_v(S,H)$, and of $K_v(S,H)$, if S is Abelian, is a $\mathbb{Q}-$factorial and terminal irreducible symplectic variety.

(2) If $(m,k) = (2,1)$ and S is K3, then any small locally trivial deformation of $M_v(S,H)$ is a $\mathbb{Q}-$factorial irreducible symplectic variety.

(3) If $(m,k) = (2,1)$ and S is Abelian, any small locally trivial deformation of $K_v(S,H)$ is a $\mathbb{Q}-$factorial irreducible symplectic variety, provided that the family $f\colon \mathcal{X} \to T$ is projective and T is quasi-projective.

Proof. (1) If $(m,k) \neq (2,1)$, then the moduli spaces $M_v(S,H)$, and $K_v(S,H)$, if S is Abelian, are $\mathbb{Q}-$factorial and terminal, by Theorem 3.1.21 and Remark 3.1.22, hence, by Proposition 2.2.4 (2) any small locally trivial deformation of the latter is again a terminal irreducible symplectic variety, and is $\mathbb{Q}-$factorial by Proposition 2.2.5.

(2) In that case, the moduli spaces $M_v(S,H)$ have all simply connected smooth locus, so the claim follows from Proposition 2.2.4 (1), together with Proposition 2.2.5.

(3) If S is Abelian and $(m,k)=(2,1)$, then $K_v(S,H)$ is a $\mathbb{Q}-$factorial irreducible symplectic variety with canonical singularities that are not terminal (see Remark 3.1.22), and, by [PR23, Theorem 3.6], it holds $\pi_1(K_v^s(S,H))=\mathbb{Z}/2\mathbb{Z}$. Hence, the only criterion that can be applied in this case is Proposition 2.2.4 (3), for which projectivity is a requirement. □

3.3 Second integral cohomology of moduli spaces of sheaves

In this Section we will address the study of the second integral cohomology groups of moduli spaces of sheaves, describing both its lattice and weight two Hodge structure, following [PR24].

We recall that, by Proposition 1.2.12, to any irreducible symplectic variety X can be associated its Beauville-Bogomolov-Fujiki lattice $(\mathrm{H}^2(X,\mathbb{Z}),q_X)$, of signature $(3,b_2(X)-3)$, which is a locally trivial deformation invariant, by [BL22, Lemma 5.7] (see also Section 2.3.1). Its description in the case in which $X=M_v(S,H)$ or $X=K_v(S,H)$, for any $(m,k)-$triple (S,v,H), with S either a projective K3 or Abelian surface, respectively, will allow us to deduce that every pair (m,k) defines a unique locally trivial deformation class.

Moreover, by Global Torelli Theorem (Theorem 2.3.10), the BBF lattice structure and the weight two Hodge structure of $\mathrm{H}^2(X,\mathbb{Z})$ play a fundamental role for the bimeromorphic classification of such varieties, together with the locally trivial monodromy group, that will be discussed in the next Section (3.4) and in the second Part of this work.

The first aspect we discuss is the relation between the second integral cohomology group of a moduli space of sheaves and that of its smooth locus. Let (S,v,H) be an $(m,k)-$triple, with S either a projective K3 or Abelian surface and let

$$i_v\colon M_v^s(S,H)\to M_v(S,H)$$

be the open embedding of the smooth locus. If S is Abelian, let us denote by

$$i_v^0\colon K_v^s(S,H)\to K_v(S,H)$$

its restriction to the Yoshioka fibers.

Proposition 3.3.1. *Let (S,v,H) be an $(m,k)-$triple, with $m>1$ and S either a projective K3 or Abelian surface.*

(1) *If* $(m,k) \neq (2,1)$, *then the morphism* $i_v^*\colon \mathrm{H}^2(M_v(S,H),\mathbb{Z}) \to \mathrm{H}^2(M_v^s(S,H),\mathbb{Z})$ *is an isomorphism of* $\mathbb{Z}-$*modules and of weight* 2 *Hodge structures.*

If S *is Abelian, then also* $(i_v^0)^*\colon \mathrm{H}^2(K_v(S,H),\mathbb{Z}) \to \mathrm{H}^2(K_v^s(S,H),\mathbb{Z})$ *is an isomorphism of* $\mathbb{Z}-$*modules and of weight* 2 *Hodge structures.*

(2) *If* $(m,k) = (2,1)$, *then the morphism* $i_v^*\colon \mathrm{H}^2(M_v(S,H),\mathbb{Z}) \to \mathrm{H}^2(M_v^s(S,H),\mathbb{Z})$ *is an injective morphism of* $\mathbb{Z}-$*modules and of weight* 2 *Hodge structures.*

If S *is Abelian, then also* $(i_v^0)^*\colon \mathrm{H}^2(K_v(S,H),\mathbb{Z}) \to \mathrm{H}^2(K_v^s(S,H),\mathbb{Z})$ *is an injective morphism of* $\mathbb{Z}-$*modules and of weight* 2 *Hodge structures.*

Proof. See [PR24, Proposition 3.5] and [PR13, Lemma 3.6], respectively, and [PR24, Corollary 3.3]. □

Now we will relate the integral cohomology of a moduli space to that of the underlying surface, by means of a quasi-universal family (see Appendix A.3.1) on its smooth locus and exploiting the isomorphism of Proposition 3.3.1. The following construction is originally due to Donaldson and Mukai ([Mu87]) for moduli spaces of rank 2 vector bundles, then generalized by O'Grady ([OG97]) and Yoshioka ([Yos99c]) to moduli spaces of sheaves with primitive Mukai vector, and by Perego and Rapagnetta ([PR13], [PR24]) in the non-primitive case. For further details, we refer to [PR24, Section 4.1].

Let S be a projective K3 or Abelian surface and let $K^0_{top}(S)$ be its topological Grothendieck group. Let us recall that the Mukai vector induces an isomorphism $v\colon K^0_{top}(S) \to \mathrm{H}^{\mathrm{ev}}(S,\mathbb{Z})$, so that, for any class $\alpha \in \mathrm{H}^{\mathrm{ev}}(S,\mathbb{Z})$ there exists a class $E_\alpha \in K^0_{top}(S)$ such that $v(E_\alpha) = \alpha$. If Y is a complex quasi-projective variety, and $\mathcal{F}$ is a $Y-$flat family of coherent sheaves on $S \times Y$, we define a group morphism

$$\begin{aligned} \mu_{\mathcal{F}}\colon \mathrm{H}^{\mathrm{ev}}(S,\mathbb{Z}) &\to \mathrm{H}^2(Y,\mathbb{Z}) \\ \alpha &\mapsto \mathrm{c}_1(p_{Y!,top}(p^*_{S,top}(E^\vee_\alpha) \otimes_{top} [\mathcal{F}])), \end{aligned} \tag{3.13}$$

whose fundamental properties are described in [PR24, Lemma 4.3] and [HL97, Lemma 8.1.2]. We will use the above-defined morphism in the following two situations:

(1) If $Y = M_v^s(S,H)$ and $\mathcal{F}$ is a quasi-universal family on $S \times M_v^s(S,H)$ (see Appendix A.3.1), where (S,v,H) is an $(m,k)-$triple.

(2) Let $\mathcal{Q}_v$ be the tautological quotient on $S \times Quot_S(\mathcal{H}_v, P_{v,H})$ defined in (A.5), where $Quot_S(\mathcal{H}_v, P_{v,H})$ is the Quot scheme associated to this moduli problem (see Step 2 of Appendix A.3 and Remark 3.1.6), let $R_v^s \subseteq Quot_S(\mathcal{H}_v, P_{v,H})$ be the open subset parametrizing stable sheaves and let $\mathcal{Q}_v^s$ be the restriction of $\mathcal{Q}_v$ to R_v^s. Moreover, let $q_v^s\colon R_v^s \to M_v^s(S,H)$ be the classifying morphism induced by $\mathcal{Q}_v^s$ (see Remark A.2.2 (2)). Then, we will set $Y = M_v^s(S,H)$ and $\mathcal{F} = \mathcal{Q}_v^s$.

Recall that $\mathrm{H}^{\mathrm{ev}}(S,\mathbb{Z})$ can be equipped with the Mukai pairing $\cdot$, and for any vector $v \in \mathrm{H}^{\mathrm{ev}}(S,\mathbb{Z})$ we can consider its orthogonal complement $v^{\perp}$ in the Mukai lattice $\widetilde{\mathrm{H}}(S,\mathbb{Z}) = (\mathrm{H}^{\mathrm{ev}}(S,\mathbb{Z}),\cdot)$ (see Definition 3.1.3). With the notation above introduced, we state the following result.

Proposition 3.3.2. *Let (S,v,H) be an $(m,k)-$triple.*

(1) There is a unique morphism

$$\lambda_v^s\colon v^{\perp} \to \mathrm{H}^2(M_v^s(S,H),\mathbb{Z}) \tag{3.14}$$

such that $q_v^{s*} \circ \lambda_v^s = \mu_{Q_v^s}$.

(2) If $\mathcal{F}$ is a quasi-universal family on $S \times M_v^s(S,H)$ of similitude ρ, then, for any $\alpha \in v^{\perp}$, it holds

$$\rho\lambda_v^s(\alpha) = \mu_{\mathcal{F}}(\alpha).$$

(3) The $\mathbb{Q}-$linear extension $\lambda_{v,\mathbb{Q}}^s$ of λ_v^s can be characterized as

$$\lambda_{v,\mathbb{Q}}^s(\alpha) = \frac{1}{\rho}[p_{M_v^s*}(p_S^*(\alpha^{\vee}.\sqrt{td(S)}).ch(\mathcal{F}))]_2.$$

Proof. See [PR24, Proposition 4.4 (1), (2)] and [PR24, Remark 4.5]. □

Proposition 3.3.1 and Proposition 3.3.2 allows us to give the following definition.

Definition 3.3.3. Let (S,v,H) be an $(m,k)-$triple.

(1) We define $\lambda_v\colon v^{\perp} \to \mathrm{H}^2(M_v(S,H),\mathbb{Z})$ as the unique group morphism such that $i_v^* \circ \lambda_v = \lambda_v^s$.

(2) If S is Abelian, set $\lambda_v^{s,0} := (i_v^0)^* \circ \lambda_v^s\colon v^{\perp} \to \mathrm{H}^2(K_v^s(S,H),\mathbb{Z})$. We define $\lambda_v^0\colon v^{\perp} \to \mathrm{H}^2(K_v(S,H),\mathbb{Z})$ as the unique group morphism such that $(i_v^0)^* \circ \lambda_v^0 = \lambda_v^{s,0}$.

The morphisms λ_v and λ_v^0 above-defined relate the second integral cohomology of moduli spaces of sheaves with that of the underlying surface. Such relation is deeper than just a group morphism, as it turns out to preserve both the lattice and Hodge structures involved. Recall that, if $v^2 \neq 0$, then $v^{\perp} \subseteq \widetilde{\mathrm{H}}(S,\mathbb{Z})$ embeds as a sublattice and, moreover, it inherits a pure weight two Hodge structure from the one of $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ (see (3.9)).

Theorem 3.3.4. *Let $m,k \in \mathbb{N} \setminus \{0\}$ and let (S,v,H) be an $(m,k)-$triple.*

(1) If S is K3, then the morphism

$$\lambda_v\colon v^{\perp} \to \mathrm{H}^2(M_v(S,H),\mathbb{Z})$$

is an isomorphism of $\mathbb{Z}-$modules and a Hodge isometry.

(2) If S is Abelian, assume that $(m,k) \neq (1,1)$. Then the morphism

$$\lambda_v^0 \colon v^\perp \to \mathrm{H}^2(K_v(S,H),\mathbb{Z})$$

is an injective Hodge isometry. If, moreover $(m,k) \neq (1,2)$, then it is an isomorphism of $\mathbb{Z}-$modules. In that case, there exists an isomorphism of $\mathbb{Z}-$modules and Hodge isometry

$$(\lambda_v, a_v^*) \colon v^\perp \oplus \mathrm{H}^2(S \times \hat{S},\mathbb{Z}) \to \mathrm{H}^2(M_v(S,H),\mathbb{Z}).$$

Proof. See [PR24, Theorem 1.6, Lemma 5.1 and Proposition 6.2, Corollary 5.13]. □

Notation 3.3.5. As we will be primarily interested in the morphism λ_v^0 in the Abelian case, if no confusion arises, in the following we will denote $\lambda_v := \lambda_v^0$ if (S,v,H) is an $(m,k)-$triple with S Abelian.

Remark 3.3.6. Theorem 3.3.4 provides a lattice theoretic description of $\mathrm{H}^2(M_v(S,H),\mathbb{Z})$ and $\mathrm{H}^2(K_v(S,H),\mathbb{Z})$ only in terms of the Mukai vector v. More precisely, if v is of type (m,k) then their isometry class is completely determined by k. In fact, if $v = mw$ is a Mukai vector of positive square $v^2 = 2m^2k$, with w primitive and $m \geq 1$, then $v^\perp = w^\perp$ and its discriminant $A_{v^\perp}$ (see Appendix B.1.3) is a cyclic group of order $2k$. In particular,

(1) if S is K3, then $v^\perp$ is abstractly isometric to the even lattice $\Lambda_{K3} \oplus \langle -2k \rangle = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2k \rangle$ of signature $(3,20)$,

(2) if S is Abelian, then $v^\perp$ is abstractly isometric to the even lattice $\Lambda_{Ab} \oplus \langle -2k \rangle = U^{\oplus 3} \oplus \langle -2k \rangle$ of signature $(3,4)$.

As a consequence we deduce the following.

Corollary 3.3.7. *For $i = 1,2$, let (S_i, v_i, H_i) be two $(m_i,k_i)-$triples.*

(1) If S_1 and S_2 are both K3 surfaces, then $M_{v_1}(S_1,H_1)$ and $M_{v_2}(S_2,H_2)$ are locally trivial deformation equivalent if and only if $(m_1,k_1) = (m_2,k_2)$.

(2) If S_1 and S_2 are both Abelian surfaces, then $K_{v_1}(S_1,H_1)$ and $K_{v_2}(S_2,H_2)$ are locally trivial deformation equivalent if and only if $(m_1,k_1) = (m_2,k_2)$.

(3) If S_1 is K3 and S_2 is Abelian, then $M_{v_1}(S_1,H_1)$ and $K_{v_2}(S_2,H_2)$ are not locally trivial deformation equivalent.

Proof. This is the content of [PR24, Corollary 6.4]. The statement easily follows from Remark 3.3.6, which uniquely determines k, and from the fact that locally trivial deformation equivalent moduli spaces must have the same dimension $2m^2k \pm 2$, from which we uniquely determine m. □

Remark 3.3.8 (Fujiki constants)**.** Another fundamental locally trivial deformation invariant, intrinsecally related to the BBF lattice structure, is the Fujiki constant (see equation (1.2)) of an irreducible symplectic variety. Consistently with Corollary 3.3.7, we point out that the Fujiki constants of moduli spaces of sheaves only depend on the pair (m,k). Let (S,v,H) be an $(m,k)-$triple.

(1) If S is K3, then $c_{M_v} = \frac{(2m^k+2)!}{(m^2k+1)!2^{m^2k+1}}$.

The computation is due to [Bea83] in the primitive case, to [Rap08] and [PR13] for $(2,1)-$triples and to [PR24] in the general non-primitive case.

(2) If S is Abelian and $(m,k) \neq (1,1),(1,2)$, then $c_{K_v} = \frac{(2m^k-2)!m^2k}{(m^2k-1)!2^{m^2k-1}}$.

The computation is due to [Bea83] in the primitive case, to [Rap07] and [PR13] for $(2,1)-$triples and to [PR24] in the general non-primitive case.

Notice that, if $m = 1$ - and $k \neq 1$ in the Abelian case - then the latter are consistent with those of Hilbert schemes of points and generalized Kummer manifolds introduced in Section 1.1.2.

We conclude this Section with a quick overview of the classification - up to locally trivial deformation - of the moduli spaces $M_v(S,H)$ and $K_v(S,H)$, according to the values of the pair (m,k), based on the discussion in Section 3.1.

m	k	$M_v(S,H)$, with S K3
≥ 1	≤ -4	empty
≥ 1	-2	point
1	0	S, K3 surface
≥ 2	0	PSV not ISV
1	≥ 1	IHS manifold, $Hilb^{k+1}(S)$
2	1	ISV with symplectic resolution OG_{10}, deformation type $(2,1)$
≥ 2	≥ 1; > 1 if $m=2$	ISV without symplectic resolution, deformation type (m,k)

Table 3.1: Classification of $M_v(S,H)$ up to (l.t.) deformation, for S K3.

m	k	$K_v(S,H)$, with S Abelian
≥ 1	≤ -2	empty
1	0	point
≥ 2	0	PSV not ISV
1	1	point
1	2	$Kum(S)$, K3 surface
1	≥ 3	IHS manifold, $Kum^{k-1}(S)$
2	1	ISV with symplectic resolution OG_6, deformation type $(2,1)$
≥ 2	≥ 1; > 1 if $m=2$	ISV without symplectic resolution, deformation type (m,k)

Table 3.2: Classification of $K_v(S,H)$ up to (l.t.) deformation, for S Abelian.

3.4 Monodromy of moduli spaces of sheaves: the state of the art

We conclude the Chapter - and the first Part of this work - by discussing the issue of the computation of the locally trivial monodromy group of moduli spaces of sheaves. We recall that, by Global Torelli Theorem (Theorem 2.1.12 and Theorem 2.3.10), the description of the (locally trivial) monodromy group is a crucial step in order to address the bimeromorphic classification of primitive and irreducible symplectic varieties, both smooth and singular. In the smooth and in the $\mathbb{Q}-$factorial and terminal case - which is mainly the case of moduli spaces of sheaves - this motivation is made even more explicit by the Hodge-Theoretic formulation of Global Torelli Theorem (see Theorem 2.1.13 and Corollary 2.3.24). Moreover, in that case, all deformations turn out to be locally trivial, by Remark 2.2.6.

In the following, we will provide an overview of the computations of locally trivial monodromy groups of moduli spaces of sheaves achieved so far.

The primitive case

Let (S, v, H) be an $(m, k)-$triple, with S either a projective K3 or Abelian surface. If $m = 1$, or, equivalently $v = w$ is a primitive Mukai vector, then the associated moduli spaces of sheaves are smooth and yield examples of irreducible holomorphic symplectic manifolds of deformation type $K3^{[k+1]}$ or Kum^{k-1}, respectively. Hence, their monodromy groups fall under the cases described in Remark 2.1.19, but we include them for the sake of completeness.

Theorem 3.4.1. *Let (S, w, H) be a $(1, k)-$triple, with S a projective K3 surface, and let X an irreducible holomorphic symplectic manifold deformation equivalent to $M_w(S, H)$. Then*

$$\mathrm{Mon}^2(X) = \mathsf{W}(X).$$

Proof. See [Mar08, Corollary 1.8] and [Mar10, Theorem 2.2]. □

We recall that, for any irreducible symplectic variety X, the group $\mathsf{W}(X)$ is the subgroup of $\mathrm{O}(\mathrm{H}^2(X, \mathbb{Z}), q_X)$ of orientation preserving isometries acting as $\pm\mathrm{id}$ on the discriminant group of $\mathrm{H}^2(X, \mathbb{Z})$ (see Appendix B.4), often called *Weyl group of reflections*.

Remark 3.4.2. If X is as above, namely, of $K3^{[k+1]}-$type, by [Mar10, Lemma 4.1], the index of $\mathsf{W}(X)$ inside $\mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z}))$ is $2^{\rho(k)-1}$, where $\rho(k)$ is the Euler number of k, counting the number of distinct primes occurring in its factorization. In particular, we deduce that, for a moduli space $M_w(S, H)$ as above, its monodoromy group is maximal if and only if $k = p^r$ for some prime number p and some positive integer r.

On the other hand, in the Abelian case, the following description holds.

Theorem 3.4.3. *Let (S, w, H) be a $(1, k)-$triple, with $k > 2$ and S an Abelian surface, and let X be an irreducible holomorphic symplectic manifold deformation equivalent to a moduli space $K_w(S, H)$. Then*

$$\mathrm{Mon}^2(X) = \mathsf{N}(X).$$

Proof. See [Mar22, Theorem 1.4], [Mon16, Theorem 2.3]. □

We recall that, for any irreducible symplectic variety X, the group $\mathsf{N}(X)$ is the index 2 subgroup of $\mathsf{W}(X)$ of isometries in the kernel of the character $\det \cdot \mathrm{disc}$ (see (B.4)).

Remark 3.4.4. Again by [Mar10, Lemma 4.1], the index of $\mathsf{N}(X)$ inside $\mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z}))$ is $2^{\rho(k)}$. We deduce that, for moduli spaces as above, or more generally, for irreducible holomorphic symplectic manifolds *of Kummer type*, the monodromy group is never maximal.

The non-primitive case

If (S, v, H) is an $(m, k)-$triple, with S either a projective K3 or Abelian surface, and $m > 1$, then the associated moduli spaces of sheaves are singular irreducible symplectic varieties, admitting a symplectic resolution if and only if $(m, k) = (2, 1)$.

The first computation of locally trivial monodromy of singular moduli spaces of sheaves was achieved precisely in the case of $(2, 1)-$triples, and their description is fundamentally related to that of their symplectic resolutions - namely, O'Grady's examples.

Theorem 3.4.5. *Let (S, v, H) be a $(2, 1)-$triple.*

(1) If S is K3 and X is an irreducible symplectic variety locally trivial deformation equivalent to $M_v(S, H)$, then

$$\mathrm{Mon}^2_{\mathrm{lt}}(X) = \widetilde{\mathrm{O}}^+(\mathrm{H}^2(X, \mathbb{Z})),$$

where the latter is the group of orientation preserving isometries of $\mathrm{H}^2(X, \mathbb{Z})$ acting trivially on its discriminant group (see Appendix B.1.3).

(2) If S is Abelian and X is an irreducible symplectic variety locally trivial deformation equivalent to $K_v(S, H)$, then

$$\mathrm{Mon}^2_{\mathrm{lt}}(X) = \mathrm{O}^+(\mathrm{H}^2(X, \mathbb{Z})),$$

Proof. Part (1) is [Ono22, Theorem 6.1], while part (2) is [MR21, Proposition 4.2]. □

We point out that, in part (1) of Theorem 3.4.5, the group $\widetilde{\mathrm{O}}^+(\mathrm{H}^2(X, \mathbb{Z}))$ coincides with $\mathsf{W}(X)$, as the discriminant group of $\mathrm{H}^2(X, \mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Hence, the monodromy description is the same as in Theorem 3.4.1.

After these two sporadic examples, the locally trivial monodromy group of a complete distinguished class of singular moduli spaces of sheaves has been described by [OPR24], namely for $(m,k)-$triples (S,v,H), with S a K3 surface and $m > 1$.

Theorem 3.4.6. *Let (S,v,H) be an $(m,k)-$triple, with $m > 1$ and S a projective K3 surface, and let X be an irreducible symplectic variety locally trivial deformation equivalent to $M_v(S,H)$.*

(1) The locally trivial monodromy group of X is

$$\mathrm{Mon}^2_{\mathrm{lt}}(X) = \mathsf{W}(X).$$

(2) If $Y \subseteq X$ is the most singular locus of X, then Y is an irreducible holomorphic symplectic manifold deformation equivalent to $M_w(S,H)$ and its closed embedding $i_{Y,X}\colon Y \hookrightarrow X$ induces a group isomorphism

$$i^{\sharp}_{Y,X}\colon \mathrm{Mon}^2_{\mathrm{lt}}(X) \to \mathrm{Mon}^2(Y).$$

Proof. See [OPR24, Theorem A.1] for part (1) and [OPR24, Theorem B.1] for part (2). □

The analogous case of singular moduli spaces of sheaves on Abelian surfaces, built from $(m,k)-$triples (S,v,H) with $m > 1$ and $k > 2$, will be treated in the next Part of this work. Indeed, Chapters 6, 4 and 5 are devoted to computing the locally trivial monodromy group of moduli spaces $K_v(S,H)$, providing a lattice-theoretic description as in part (1) of Theorem 3.4.6, along with - and in relation to - a geometric description analogous to that of part (2).

Part II

Locally trivial monodromy of moduli spaces of sheaves on Abelian surfaces

Introduction to Part II

Having established the necessary background in Part I, we now turn to the original research contributions of this work, which consists of the computation of the locally trivial monodromy group of singular moduli spaces of sheaves on Abelian surfaces.

Indeed, as explained in Section 3.1 moduli spaces of sheaves on projective K3 and Abelian surfaces provide two distinguished and rich classes of irreducible symplectic varieties. By Global Torelli Theorem and the discussion in Chapter 2, the general approach to their classification problem strongly relies on the study of their locally trivial deformation theory, the lattice and Hodge structure of their second integral cohomology groups and, in particular of their locally trivial monodromy groups. The first two topics have been addressed, respectively, in Sections 3.2 and 3.3. The state of the art of the computation of the locally trivial monodromy group of moduli spaces of sheaves has been presented in Section 3.4, concluding Part I.

The picture emerging from the previous discussion is the following. The locally trivial monodromy group of moduli spaces of sheaves has been computed:

(1) in the primitive - hence smooth - case, by Markman and Mongardi (Theorems 3.4.1 and 3.4.3);

(2) for singular O'Grady's moduli spaces, by Mongardi-Rapagnetta and Onorati (Theorem 3.4.5);

(3) for all singular moduli spaces of sheaves on projective K3 surfaces, by Onorati-Perego-Rapagnetta (Theorem 3.4.6).

The aim of this Part is to complete the framework by computing the remaining case of singular moduli spaces of sheaves on Abelian surfaces. Case (3) will serve as methodological model: we aim to establish a result analogous to Theorem 3.4.6, providing both a lattice-theoretic description and a clear geometric interpretation of the latter. Indeed, we will show that the techniques developed for the K3 setting can be adapted to the Abelian case.

For this reason, from now on, we will focus only on the theory of this distinguished class of moduli spaces, and we will adopt the following notation throughout Part II.

Assumption 1. From now on, unless otherwise specified, we will use Definition 3.1.15 only in the Abelian case. Hence, in the following, by $(m,k)-$*triple* we will mean a triple (S, v, H), with S an Abelian surface, $v = mw \in \widetilde{\mathrm{H}}(S, \mathbb{Z})$ a Mukai vector, $m \geq 1$, w primitive such that $w^2 = 2k > 0$, and H a $v-$generic polarization on S.

With the above-introduced notation, we state the first main result of this part, which provides a lattice-theoretic description of the locally trivial monodromy group of a moduli space as above. The case that will be the most relevant to us is that of $(m, k)-$triples with $k > 2$, in which all deformations turn out to be locally trivial (see Remarks 2.2.6, 3.1.22 and 6.3.1).

Theorem A.1 (Theorem 5.2.1, Theorem 6.3.4, Corollary 6.3.6)**.** *Let (S, v, H) be an $(m, k)-$triple, and let X be an irreducible symplectic variety that is deformation equivalent to a moduli space $K_v(S, H)$.*

(1) If $(m, k) \neq (1, 1), (1, 2)$, then

$$\mathrm{Mon}^2_{\mathrm{lt}}(X) \supseteq \mathsf{N}(X).$$

(2) For any $m \geq 1$ and $k > 2$, it holds

$$\mathrm{Mon}^2(X) = \mathrm{Mon}^2_{\mathrm{lt}}(X) = \mathsf{N}(X).$$

The group $\mathsf{N}(X) := \mathsf{N}(\mathrm{H}^2(X, \mathbb{Z}))$ is defined in Appendix B.4 and has been discussed in Section 3.4. It is characterized as the group of orientation preserving isometries of $\mathrm{H}^2(X, \mathbb{Z})$ acting as $\pm\mathrm{id}$ on its discriminant group and belonging to the kernel of the character det · disc.

For moduli spaces built from $(m, k)-$triples with $k > 2$, the description in Theorem A.1 coincides with that achieved by Markman and Mongardi in the smooth case of generalized Kummer manifolds (Theorem 3.4.3). This analogy is not a coincidence and has a clear geometric origin, as explained in the following equivalent formulation of Theorem A.1 (2).

Theorem B.1 (Corollary 6.3.3, Corollary 6.3.6)**.** *Let (S, v, H) be an $(m, k)-$triple, with $m \geq 1$ and $k > 2$, let X be an irreducible symplectic variety that is deformation equivalent to a moduli space $K_v(S, H)$ and let Z be a connected component of the most singular locus of X. Then Z is an irreducible holomorphic symplectic manifold deformation equivalent to $K_w(S, H)$ and its closed embedding $i_{Z,X} \colon Z \to X$ induces an isomorphism of groups*

$$i^{\sharp}_{Z,X} \colon \mathrm{Mon}^2(X) \longrightarrow \mathrm{Mon}^2(Z).$$

In conclusion, we use the monodromy description given by Theorem B.1 to provide a proof of the SYZ conjecture for any irreducible symplectic variety X deformation equivalent to a moduli space $K_v(S, H)$ as above. According to this conjecture, nef isotropic line bundles on symplectic varieties are expected to define Lagrangian fibrations on the latter. This is precisely the content of the following Theorem.

Theorem A.2 (Theorem 7.2.4, Corollary 7.2.14)**.** *Let X be an irreducible symplectic variety deformation equivalent to a moduli space $K_v(S,H)$ as above. If L is a nef line bundle on X such that $q_X(L) = 0$, then there exists a Lagrangian fibration $f\colon X \to B$ such that $L = f^*\mathcal{O}_B(1)$ and of polarization type $\underline{\mathrm{d}}(f) = (1,\ldots,1,\mathrm{div}(L),\frac{m^2k}{\mathrm{div}(L)})$.*

Also in this case, the outcome is analogous to that of singular moduli spaces of sheaves on K3 surfaces, which has been recently achieved by Onorati-Ortiz ([OO25]).

Outline of the proof of Theorems A.1 and B.1

As the (locally trivial) monodromy group is a (locally trivial) deformation invariant, we start by reducing to deal with $X = K_v(S,H)$, with (S,v,H) is an $(m,k)-$triple, with $m \geq 1$ and $k > 2$.

As a first step, we construct a rich and important class of monodromy operators, by showing the inclusion of groups

$$\mathsf{N}(K_v(S,H)) \subseteq \mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H)) \tag{I}$$

for any $m,k \in \mathbb{N}\setminus\{0\}$, with $(m,k) \neq (1,1),(1,2)$ (Theorem 5.2.1). Paralleling [Mar22], we prove the inclusion by describing the generators of $\mathsf{N}(K_v(S,H))$ as locally trivial monodromy operators induced by

- $*$ monodromy operators of the base surface S (Section 4.1 and Section 5.1)
- $*$ pushforwards of isomorphisms of moduli spaces induced by some Fourier-Mukai equivalences on $D^b(S)$ (Section 4.2 and Section 5.2).

This procedure applies to the primitive case as well, and differs from the one in [Mar22] only by a slight different choice of generators (Definition 4.2.7) and the avoidance of non-projective families (Section 5.1).

The second step consists of the construction of an injective morphism

$$i^\sharp_{w,m}\colon \mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H)) = \mathrm{Mon}^2(K_v(S,H)) \longrightarrow \mathrm{Mon}^2(K_w(S,H)), \tag{II}$$

where the first equality follows from the assumption $k > 2$ (Remark 6.3.1). The morphism $i^\sharp_{w,m}$ is induced - via conjugation on isometries - by the pull-back action on the respective second integral cohomology groups of the closed embedding

$$i_{w,m}\colon K_w(S,H) \to K_v(S,H)$$

as one of the finite connected components of the most singular locus of $K_v(S,H)$. This is the content of Corollary 6.3.3 (see also Theorem 6.3.2). Proposition 6.2.4 and Lemma 6.2.7 guarantee the compatibility of the action of $i^\sharp_{w,m}$ with the lattice-theoretic descriptions of the second integral cohomology groups of the moduli spaces involved and of their respective orthogonal groups, provided by Theorem

3.3.4. According to the latter, there exists a chain of isometries - and obvious identifications -

$$\mathrm{H}^2(K_v(S,H),\mathbb{Z}) \xrightarrow{\lambda_v^{-1}} v^{\perp} = w^{\perp} \xrightarrow{\lambda_w} \mathrm{H}^2(K_w(S,H),\mathbb{Z}). \qquad \text{(III)}$$

Lemma 6.2.7 asserts that the conjugation action of the composition (III) coincides with $i^{\sharp}_{w,m}$.

For $k > 2$, the inclusion (I) and the compatibility law proven in Lemma 6.2.7 imply the equality

$$i^{\sharp}_{w,m}(\mathsf{N}(K_v(S,H))) = \mathsf{N}(K_w(S,H)),$$

which, combined with Theorem 3.4.3, forces surjectivity of $i^{\sharp}_{w,m}$ and simultaneously concludes the proof of both Theorem A.1 and Theorem B.1, as summarized in the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Mon}^2(K_v(S,H)) \arrow[rrr, hook, dashed, "i^{\sharp}_{w,m}", "\text{Section 6.3}"'] & & & \mathrm{Mon}^2(K_w(S,H)) \\
\mathsf{N}(K_v(S,H)) \arrow[u, hook, dashed, "\cup\mid"', "\text{Section 5.2}"] \arrow[r, "(\lambda_v^{\sharp})^{-1}", "\sim"'] \arrow[rrr, bend right, "i^{\sharp}_{w,m}"'] & \mathsf{N}(v^{\perp}) \arrow[r, equal] & \mathsf{N}(w^{\perp}) \arrow[r, "\lambda_w^{\sharp}", "\sim"'] & \mathsf{N}(K_w(S,H)) \arrow[u, equal, "\text{Theorem 3.4.3}", "\text{([Mar22], [Mon16])}"']
\end{tikzcd}$$

Structure of Part II

* In Chapter 4 we introduce a groupoid representation designed to encode the monodromy information of a moduli space in a natural way, extending [Mar22, Section 9]. This is the technical tool that will allow us to construct locally trivial monodromy operators in the next Chapter.
* In Chapter 5, we exhibit an important subgroup of the locally trivial monodromy group under study, which will play a fundamental role in its lattice-theoretic description.
* In Chapter 6 we relate the monodromy group of a singular moduli space with the monodromy group of a smooth moduli space by means of an isomorphism which arises from an explicit geometric construction.
* In Chapter 7, we discuss some applications of the monodromy description achieved in Chapter 6. In particular, the SYZ conjecture for singular moduli spaces of sheaves on Abelian surfaces is proven.

Chapter 4

A Groupoid representation

This Chapter is devoted to the development of the technical tools that will provide an embedding of the group $\mathsf{N}(K_v(S,H))$ - defined in Appendix B.4 and discussed in Remark 2.1.19 and Section 3.4 - in the locally trivial monodromy group $\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H))$ in a natural way, regardless of the choice of the $(m,k)-$triple (S,v,H). In Sections 4.1 and 4.2 we will define a groupoid $\mathcal{G}^{m,k}$ of $(m,k)-$triples, whose morphisms arise from deformations of the triple itself, or from Fourier-Mukai equivalences. In Section 4.3 we will construct two different representations of $\mathcal{G}^{m,k}$ with values in groupoids of free $\mathbb{Z}-$modules, designed to encode the monodromy information of the moduli spaces.

We wish to point out that this Chapter parallels Section 2 of [OPR24], to which we refer for further details. The aim of this part of the work is, indeed, to review the main results and constructions, check their compatibility with the Abelian case and replace them with a suitable analogue if any difference occurs.

Let us start with a quick overview of the main definitions and constructions concerning groupoids that we will use in the following, referring the reader to the lecture notes [Hig71] for a deeper discussion of this theory.

Definition 4.0.1. A *groupoid* $\mathcal{G}$ is a small category whose morphisms are all isomorphisms.

(1) If $\mathcal{G}$ is a groupoid and x is an object of $\mathcal{G}$, we define its *isotropy group* as

$$\mathrm{Aut}_{\mathcal{G}}(x) := \mathrm{Hom}_{\mathcal{G}}(x,x).$$

If $F\colon \mathcal{G} \to \mathcal{H}$ is a functor between two groupoids, for every $x \in Ob(\mathcal{G})$ we will denote its action on the respective isotropy groups as

$$F_x\colon \mathrm{Aut}_{\mathcal{G}}(x) \to \mathrm{Aut}_{\mathcal{H}}(F(x)).$$

(2) If $\mathcal{G}$ is a groupoid, a *representation of* $\mathcal{G}$ is a functor $F\colon \mathcal{G} \to \mathcal{A}$, where $\mathcal{A}$ is a suitable groupoid of $\mathbb{Z}-$modules.

(3) If $\mathcal{G}$ and $\mathcal{H}$ are two groupoids having the same objects, we define the *free product of $\mathcal{G}$ and $\mathcal{H}$* as the groupoid $\mathcal{G} * \mathcal{H}$ whose objects are the objects of $\mathcal{G}$ (and $\mathcal{H}$) and whose morphisms are defined in the following way: if $x, y \in \mathcal{G} * \mathcal{H}$ are two objects, a morphism $f \in \mathrm{Hom}_{\mathcal{G}*\mathcal{H}}(x, y)$ is a formal combination (with usual cancellation properties)

$$f = f_1 * \cdots * f_l,$$

where f_i is a morphism from $x_i \in Ob(\mathcal{G} * \mathcal{H})$ to $x_{i+1} \in Ob(\mathcal{G} * \mathcal{H})$ either in $\mathcal{G}$ or $\mathcal{H}$ for every $i = 1, \ldots, l$ and such that $x_1 = x$ and $x_{l+1} = y$.

4.1 Deformations of $(m, k)-$triples and their groupoid

We start by recalling that, by the discussion in Chapter 3, the choice of an $(m, k)-$triple (S, v, H), as in Definition 3.1.15 and Assumption 1, determines a moduli space of sheaves $K_v(S, H)$, which is an irreducible symplectic variety, by Theorem 3.1.24, and whose locally trivial deformation class is uniquely determined by the pair (m, k), by Corollary 3.3.7. The content of this Section is meant to be a categorical translation of the main ideas and tools that are involved in Theorem 3.2.1. Indeed, as already pointed out in Remark 3.2.2, one of the key points of its proof is the construction of a deformation of a moduli space of sheaves induced by a deformation of the base surface, together with the Mukai vector and the generic polarization, which we will recall in the following.

Let (S, v, H) be an $(m, k)-$triple, with $v = (r, \xi, a)$, and let $L \in \mathrm{Pic}(S)$ be a line bundle such that $\mathrm{c}_1(L) = \xi$. If $f \colon \mathcal{X} \to T$ is a morphism and $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$, we will denote $\mathcal{L}_t := \mathcal{L}_{|\mathcal{X}_t}$.

Definition 4.1.1. Let (S, v, H) be an $(m, k)-$triple as above and T a smooth and connected algebraic variety. A *deformation of (S, v, H) along T* is a triple $(f \colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H})$, where

(1) $f \colon \mathcal{S} \to T$ is a smooth, projective deformation of S, with $0 \in T$ the point such that $\mathcal{S}_0 \simeq S$;

(2) $\mathcal{L}$ is a line bundle on $\mathcal{S}$ such that $\mathcal{L}_0 \simeq L$;

(3) $\mathcal{H}$ is a line bundle on $\mathcal{S}$ such that $\mathcal{H}_t$ is a v_t-generic polarization on $\mathcal{S}_t$ for every $t \in T$ and such that $\mathcal{H}_0 \simeq H$,

where, for every $t \in T$, we set $v_t := (r, \mathrm{c}_1(\mathcal{L}_t), a)$.

Remark 4.1.2. For later use, we remark that $v-$genericity is an open property in the Zariski topology, as shown in Proposition 2.14 of [PR23]. In fact, up to replacing the base T with the complement of the Zariski closed subset

$$Z := \{t \in T \text{ such that } \mathcal{H}_t \text{ is not } v_t - \text{generic}\},$$

we can find a deformation as in Definition 4.1.1 such that property (3) is satisfied. Notice that non-emptiness of $T \setminus Z$ is a consequence of the assumption that (S, v, H) is an $(m, k)-$triple and $\mathcal{H}_0 \simeq H$.

Definition 4.1.3. Let (S_1, v_1, H_1) and (S_2, v_2, H_2) be two $(m, k)-$triples. A *deformation path from* (S_1, v_1, H_1) *to* (S_2, v_2, H_2) is a $6-$tuple

$$\alpha := (f\colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H}, t_1, t_2, \gamma),$$

where

(1) the triple $(f\colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H})$ is a deformation of both (S_1, v_1, H_1) and (S_2, v_2, H_2);

(2) for $i = 1, 2$, the point $t_i \in T$ is such that $(\mathcal{S}_{t_i}, v_{t_i}, \mathcal{H}_{t_i}) = (S_i, v_i, H_i)$;

(3) γ is a continuous path in T from t_1 to t_2.

We will now explain, with a few remarks, how deformation paths provide a tool to produce locally trivial monodromy operators in a natural way. We start by recalling the following

Definition 4.1.4. Let S_1 and S_2 be two Abelian surfaces. An isometry $g\colon \widetilde{\mathrm{H}}(\mathcal{S}_{t_1}, \mathbb{Z}) \to \widetilde{\mathrm{H}}(\mathcal{S}_{t_2}, \mathbb{Z})$ is a *parallel trasport operator* if there exists a smooth family $f\colon \mathcal{S} \to T$ of Abelian surfaces, $t_1, t_2 \in T$ such that $\mathcal{S}_{t_i} \simeq S_i$ for $i = 1, 2$ and a continuous path γ in T from t_1 to t_2 such that g is the parallel transport along γ inside the local system $R^{\mathrm{ev}} f_* \mathbb{Z} = \bigoplus_{i=0}^{2} R^{2i} f_* \mathbb{Z}$ (see Definition 2.1.1 (2)), which we will denote by $\mathsf{PT}_f(\gamma)$.

Remark 4.1.5. If (S, v, H) is an $(m, k)-$triple and $(f\colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H})$ is a deformation of (S, v, H) along T, as explained in Section 2.3 of [PR23], it induces a locally trivial deformation of the moduli space $K_v(S, H)$ as follows: we consider the relative moduli space of semistable sheaves $\phi\colon M_v(\mathcal{S}/T, \mathcal{H}) \to T$, so that for every $t \in T$ we have $M_v(\mathcal{S}/T, \mathcal{H})_t = M_{v_t}(\mathcal{S}_t, \mathcal{H}_t)$ (see Appendix A.3.3). Let $\widehat{\mathcal{S}} \to T$ be the dual family of $\mathcal{S} \to T$, so that $\widehat{\mathcal{S}}_t$ is the dual surface of $\mathcal{S}_t$ for every $t \in T$, whose existence and characterization is guaranteed by representability of the relative Picard functor (see Section 3.1.3 and again Appendix A.3.3). By [PR23, Remark 2.19] (see also condition $(\star)$ therein), up to shrinking the base T or taking a finite étale cover of the latter, we can assume that both the families $\phi\colon M_v(\mathcal{S}/T, \mathcal{H}) \to T$ and $f\colon \mathcal{S} \to T$ admit a $0-$section. Consequently, a relative Yoshioka fibration

$$a\colon M_v(\mathcal{S}/T, \mathcal{H}) \to \mathcal{S} \times_T \widehat{\mathcal{S}}$$

is defined and its restriction on each fiber $M_v(\mathcal{S}/T, \mathcal{H})_t$ of ϕ coincides with the Yoshioka fibration $a_{v_t}\colon M_{v_t}(\mathcal{S}_t, \mathcal{H}_t) \to \mathcal{S}_t \times \widehat{\mathcal{S}}_t$ defined in Section 3.1.3. If we set

$$Z := \{(0_{\mathcal{S}_t}, \mathcal{O}_{\mathcal{S}_t}) \in \mathcal{S}_t \times_T \widehat{\mathcal{S}}_t |\ t \in T\} \subseteq \mathcal{S}_t \times \widehat{\mathcal{S}}_t,$$

the restriction of ϕ to $K_v(\mathcal{S}/T, \mathcal{H}) := a^{-1}(Z)$ provides us a morphism

$$p\colon K_v(\mathcal{S}/T, \mathcal{H}) \to T,$$

whose fiber over $t \in T$ is $K_v(\mathcal{S}/T, \mathcal{H})_t = K_{v_t}(\mathcal{S}_t, \mathcal{H}_t)$. By Lemma 2.21 of [PR23], the family $p\colon K_v(\mathcal{S}/T, \mathcal{H}) \to T$ is a locally trivial deformation of $K_v(S, H)$ along T.

In light of Remark 4.1.5, if $\alpha = (f\colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H}, t_1, t_2, \gamma)$ is a deformation path between two $(m,k)-$triples (S_1, v_1, H_1) and (S_2, v_2, H_2) and $p\colon K_v(\mathcal{S}/T, \mathcal{H}) \to T$ is the relative moduli space associated to the deformation $(f\colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H})$, then α induces two natural locally trivial parallel transport operators:

(a) the parallel transport operator

$$p_\alpha := \mathsf{PT}_f(\gamma)\colon \widetilde{\mathrm{H}}(S_1, \mathbb{Z}) \longrightarrow \widetilde{\mathrm{H}}(S_2, \mathbb{Z}) \tag{4.1}$$

along γ in the local system $R^{\mathrm{ev}} f_* \mathbb{Z}$;

(b) the locally trivial parallel transport operator

$$g_\alpha := \mathsf{PT}_p(\gamma)\colon \mathrm{H}^2(K_{v_1}(S_1, H_1), \mathbb{Z}) \longrightarrow \mathrm{H}^2(K_{v_2}(S_2, H_2), \mathbb{Z}) \tag{4.2}$$

along γ in the local system $R^2 p_* \mathbb{Z}$.

Remark 4.1.6. In order to clarify the relation between the two locally trivial parallel transport operators p_α and g_α, we compare the two local systems involved.

(1) Due to condition (2) in Definition 4.1.1, the local system $R^{\mathrm{ev}} f_* \mathbb{Z}$ admits a flat section v such that, for every $t \in T$, we have $\mathsf{v}_t = v_t = (r, \mathrm{c}_1(\mathcal{L}_t), a) \in \widetilde{\mathrm{H}}(\mathcal{S}_t, \mathbb{Z})$. This allows us to consider the sub-local system

$$\mathsf{v}^\perp \subseteq R^{\mathrm{ev}} f_* \mathbb{Z}.$$

(2) If (S, v, H) is an $(m,k)-$triple with $(m,k) \neq (1,1), (1,2)$, then the isometry

$$\lambda_v\colon v^\perp \to \mathrm{H}^2(K_v(S, H), \mathbb{Z})$$

in Theorem 3.3.4 (2) extends to an isometry of local systems

$$\lambda\colon \mathsf{v}^\perp \longrightarrow R^2 p_* \mathbb{Z}.$$

Indeed, the relative moduli space of stable sheaves $p^s\colon K^s_v(\mathcal{S}/T, \mathcal{H}) \to T$ induced by $(f\colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H})$ naturally includes in $p\colon K_v(\mathcal{S}/T, \mathcal{H}) \to T$ via a relative open embedding, inducing an isomorphism $\iota\colon R^2 p^s_* \mathbb{Z} \to R^2 p_* \mathbb{Z}$ of local systems, by Proposition 3.3.1 (1). By Proposition 3.3.2 and Definition 3.3.3, a quasi-universal family of $M^s_v(\mathcal{S}/T, \mathcal{H}) \to T$ induces an isomorphism $\lambda^s\colon v^\perp \to R^2 p^s_* \mathbb{Z}$ of local systems such that the composition

$$\lambda := \iota \circ \lambda^s\colon v^\perp \longrightarrow R^2 p_* \mathbb{Z}$$

coincides, over every $t \in T$, with λ_{v_t}. In other words, the isometry λ_v behaves well in deformations of $(m,k)-$triples.

(3) By definition, the parallel transport operator p_α is constant along v, i.e. $p_\alpha(v_{t_1}) = v_{t_2}$ for every $t_1, t_2 \in T$, hence its restriction $p_{\alpha|v_{t_1}^\perp}$ defines a parallel transport operator in the local system $\mathsf{v}^\perp$, which is isomorphic, due to point (2), to $R^2p_*\mathbb{Z}$. Since g_α and $p_{\alpha|\mathsf{v}^\perp}$ are parallel transport operators over the same path in isomorphic local systems, we deduce that p_α uniquely determines g_α.

In light of the last Remark and in order to give a well defined composition law for deformation paths, we introduce the following

Definition 4.1.7. Let (S_1, v_1, H_1) and (S_2, v_2, H_2) be two $(m,k)-$triples. Two deformation paths α and α' from (S_1, v_1, H_1) to (S_2, v_2, H_2) are *equivalent* if $p_\alpha = p'_\alpha$. We will denote the equivalence class of α by $\overline{\alpha}$.

We immediately notice that, by point (3) of Remark 4.1.6, if two deformation paths α and α' are equivalent, then also $g_\alpha = g_{\alpha'}$.

Let (S_1, v_1, H_1), (S_2, v_2, H_2) and (S_3, v_3, H_3) be three $(m,k)-$triples and let $\alpha = (f\colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H}, t_1, t_2, \gamma)$ be a deformation path from (S_1, v_1, H_1) to (S_2, v_2, H_2) and $\alpha' = (f'\colon \mathcal{S}' \to T', \mathcal{L}', \mathcal{H}', t'_1, t'_2, \gamma')$ a deformation path from (S_2, v_2, H_2) to (S_3, v_3, H_3).

Definition 4.1.8. The *concatenation of α with α'* is the $6-$tuple

$$\alpha \star \alpha' := (f''\colon \mathcal{S}'' \to T'', \mathcal{L}'', \mathcal{H}'', t''_1, t''_2, \gamma''),$$

where

* T'' is obtained by gluing T and T' along t_2 and t'_1;
* $\mathcal{S}''$ is obtained by gluing $\mathcal{S}$ and $\mathcal{S}'$ along $\mathcal{S}_{t_2}$ and $\mathcal{S}'_{t'_1}$;
* f'' is obtained by gluing f and f' along $\mathcal{S}_{t_2}$ and $\mathcal{S}'_{t'_1}$;
* $\mathcal{L}''$ is obtained by gluing $\mathcal{L}$ and $\mathcal{L}'$ along $\mathcal{S}_{t_2}$ and $\mathcal{S}'_{t'_1}$;
* $\mathcal{H}''$ is obtained by gluing $\mathcal{H}$ and $\mathcal{H}'$ along $\mathcal{S}_{t_2}$ and $\mathcal{S}'_{t'_1}$;
* t''_1 is the image of t_1 in T'';
* t''_2 is the image of t_3 in T'';
* γ'' is the concatenation of the image of the path γ in T'' with the image of the path γ' in T''.

Remark 4.1.9. We remark that, in general, the concatenation $\alpha \star \alpha'$ does not define a deformation path in the sense of Definition 4.1.3, since the base T'' obtained by gluing might not be smooth. Nonetheless, if we set

$$p_{\alpha\star\alpha'} := p_{\alpha'} \circ p_\alpha \text{ and } g_{\alpha\star\alpha'} := g_{\alpha'} \circ g_\alpha,$$

we get two well defined locally trivial parallel transport operators, by Remark 2.3.13 and 2.1.7. We can then extend the equivalence relation in Definition 4.1.7 to concatenations of $(m,k)-$triples and notice that, if α is equivalent to β and α' is equivalent to β', then $\alpha \star \alpha'$ is equivalent to $\beta \star \beta'$. As before, we will denote the equivalence class of $\alpha \star \alpha'$ with $\overline{\alpha \star \alpha'}$.

Thanks to the previous definitions and remarks, we are now able to introduce the first of the two groupoids needed to define the groupoid $\mathcal{G}^{m,k}_{\text{def}}$ of deformations of $(m,k)-$triples.

Definition 4.1.10. Let $m,k \in \mathbb{N} \setminus \{0\}$. The groupoid $\widetilde{\mathcal{G}}^{m,k}_{def}$ is defined as follows:

* the objects of $\widetilde{\mathcal{G}}^{m,k}_{def}$ are the $(m,k)-$triples;
* if (S_1, v_1, H_1) and (S_2, v_2, H_2) are two $(m,k)-$triples, an *elementary morphism* between them is an equivalence class of deformation paths from (S_1, v_1, H_1) to (S_2, v_2, H_2);
* if (S_1, v_1, H_1) and (S_2, v_2, H_2) are two $(m,k)-$triples, a morphism in $\widetilde{\mathcal{G}}^{m,k}_{def}$ from (S_1, v_1, H_1) to (S_2, v_2, H_2) is a formal concatenation

$$\overline{\alpha_1} * \cdots * \overline{\alpha_l}$$

of elementary morphisms and their formal inverses, subject to usual cancellation rules, prescribed by

$$\overline{\alpha_i} * \overline{\alpha_{i+1}} := \overline{\alpha_i \star \alpha_{i+1}}, \tag{4.3}$$

for every $i = 1, \ldots, l-1$.

Remark 4.1.11. By Remark 4.1.9, we get the good definition of the composition law (4.3) and an explicit description of

* the *identity morphism*

$$\mathrm{id}_{(S,v,H)} := \overline{(S \to \{p\}, L, H, p, p, k_p)},$$

where k_p is the constant path in p, for any object (S, v, H) in $\widetilde{\mathcal{G}}^{m,k}_{def}$;
* the *formal inverse* of any elementary morphism $\overline{\alpha} = \overline{(f\colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H}, t_1, t_2, \gamma)}$, defined by $\overline{\alpha}^{-1} := \overline{\alpha^{-1}}$, where

$$\alpha^{-1} := (f\colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H}, t_2, t_1, \gamma^{-1}).$$

We now complete the setting with the groupoid $\mathcal{P}^{m,k}$ of congruent $(m,k)-$triples, according to Definition 3.1.32.

Definition 4.1.12. Let $m,k \in \mathbb{N} \setminus \{0\}$. The groupoid $\mathcal{P}^{m,k}$ is defined as follows:

* the objects of $\mathcal{P}^{m,k}$ are the $(m,k)-$triples;
* if (S_1, v_1, H_1) and (S_2, v_2, H_2) are two $(m,k)-$triples, we set

$$\mathrm{Hom}_{\mathcal{P}^{m,k}}((S_1, v_1, H_1), (S_2, v_2, H_2)) := \{\chi_{H_1,H_2}\}$$

as in (3.12), if (S_1, v_1, H_1) and (S_2, v_2, H_2) are congruent, and otherwise

$$\mathrm{Hom}_{\mathcal{P}^{m,k}}((S_1, v_1, H_1), (S_2, v_2, H_2)) := \emptyset.$$

We have now collected all the material needed to define the groupoid $\mathcal{G}^{m,k}_{\mathrm{def}}$.

Definition 4.1.13. Let $m, k \in \mathbb{N} \setminus \{0\}$. We define the groupoid

$$\mathcal{G}^{m,k}_{\mathrm{def}} := \widetilde{\mathcal{G}}^{m,k}_{def} * \mathcal{P}^{m,k}$$

as the free product of $\widetilde{\mathcal{G}}^{m,k}_{def}$ and $\mathcal{P}^{m,k}$, according to Definition 4.0.1 (3).

4.2 Fourier-Mukai equivalences and their groupoid

In this Section we will introduce the groupoid $\mathcal{G}^{m,k}_{\mathrm{FM}}$, whose morphisms are defined by some Fourier-Mukai equivalences on $D^b(S)$ inducing isomorphisms on the moduli spaces $K_v(S, H)$, building on foundational results due to [Yos01a] and [PR23].

Let us recall that, given two Abelian surfaces S_1 and S_2 and an object $\mathcal{K} \in D^b(S_1 \times S_2)$, the *Fourier-Mukai transform with kernel* $\mathcal{K}$ is the derived functor defined as

$$\mathrm{FM}_{\mathcal{K}} \colon D^b(S_1) \longrightarrow D^b(S_2), \qquad F \longmapsto R\pi_{S_2 *}(\pi^*_{S_1}(F) \overset{L}{\otimes} \mathcal{K}),$$

where $\pi_{S_i} \colon S_1 \times S_2 \to S_i$ are the two natural projections, for $i = 1, 2$. For further details concerning this class of derived functors, we refer to [Huy06, Chapters 5 and 9].

For a Fourier-Mukai transform $\mathrm{FM}_{\mathcal{K}} \colon D^b(S_1) \to D^b(S_2)$ to induce a morphism between the corresponding moduli spaces $K_{v_1}(S_1, H_1)$ and $K_{v_2}(S_2, H_2)$, the following properties must reasonably be satisfied: $\mathrm{FM}_{\mathcal{K}}$ must send sheaves to sheaves, and it must preserve semistability with respect to the chosen polarizations H_1 and H_2. Accordingly, we introduce the following notion (see [Yos01a, Definition 2.1], see also [PR23, Definition 2.25]), which will be useful in the next discussion.

Definition 4.2.1. Let S_1 and S_2 be two Abelian surfaces, let $\phi \colon D^b(S_1) \to D^b(S_2)$ a Fourier-Mukai transform and let $i \in \{0, 1, 2\}$. A coherent sheaf $E \in D^b(S_1)$ is said to satisfy the WIT_i property with respect to ϕ if $\phi(E) = \phi^i(E)[-i]$.

4.2.1 Tensorization with line bundles

Let S be an Abelian surface, let $L \in \mathrm{Pic}(S)$ be a line bundle and let us consider the derived equivalence

$$\mathsf{L}\colon D^b(S) \longrightarrow D^b(S), \qquad F \longmapsto F \otimes L, \tag{4.4}$$

which is isomorphic to the Fourier-Mukai transform $\mathrm{FM}_{\Delta_* L}$, where $\Delta\colon S \hookrightarrow S \times S$ is the diagonal embedding. Any coherent sheaf F on S naturally satisfies the WIT_0 property with respect to L. Moreover, if F has Mukai vector $v(F) = v$, then

$$v(\mathsf{L}(F)) = v(F).\mathrm{ch}(L) =: v_L \tag{4.5}$$

and the equivalence (4.4) induces a morphism

$$\mathsf{L}\colon K_v(S, H) \longrightarrow K_{v_L}(S, H).$$

The next result provides a sufficient criterion for the morphism L to be an isomorphism.

Lemma 4.2.2. *Let S be an Abelian surface, $v = (r, \xi, a)$ a Mukai vector and H an ample line bundle on S.*

(1) For any $d \in \mathbb{Z}$, the morphism $\mathsf{dH}\colon K_v(S, H) \to K_{v_{dH}}(S, H)$ is an isomorphism.

(2) If $r > 0$ and H is $v-$generic, the morphism $\mathsf{L}\colon K_v(S, H) \to K_{v_L}(S, H)$ is an isomorphism.

Proof. See [PR23, Lemma 2.24]. □

4.2.2 The Poincaré line bundle as kernel

Let S be an Abelian surface, let $\mathcal{P} \in D^b(S \times \hat{S})$ be the Poincaré line bundle on S (see Section 3.1.3). The Fourier-Mukai transform

$$\mathrm{FM}_{\mathcal{P}}\colon D^b(S) \longrightarrow D^b(\hat{S}), \qquad F \longmapsto R\pi_{\hat{S}*}(\pi_S^*(F) \overset{L}{\otimes} \mathcal{P}),$$

where $\pi_S\colon S \times \hat{S} \to S$ and $\pi_{\hat{S}}\colon S \times \hat{S} \to \hat{S}$ are the two natural projections, is known to be an equivalence ([Mu81], see also [Huy06, Proposition 9.19]). For any $L \in \mathrm{Pic}(S)$, we set $\hat{L} := \det(\mathrm{FM}_{\mathcal{P}}(L))^{-1} \in \mathrm{Pic}(\hat{S})$ and, if $\mathrm{c}_1(L) = \xi$, we set $\hat{\xi} := \mathrm{c}_1(\hat{L})$. If (r, ξ, a) is a Mukai vector on S, then the cohomological action $\mathrm{FM}^{\mathrm{H}}_{\mathcal{P}}$ of $\mathrm{FM}_{\mathcal{P}}$ (see [Huy06, Lemma 9.23]) provides the equality

$$\mathrm{FM}^{\mathrm{H}}_{\mathcal{P}}(r, \xi, a) = (a, -\hat{\xi}, r) =: \widetilde{v}, \tag{4.6}$$

where the latter is a Mukai vector on $\hat{S}$.

We now consider the derived duality operation

$$D_S\colon D^b(S) \longrightarrow D^b(S), \qquad F \longmapsto F^\vee,$$

which is a contravariant autoequivalence, as S is smooth. Its cohomological action, restricted to $\widetilde{\mathrm{H}}(S,\mathbb{Z})$, provides an involution and isometry satisfying

$$D_S^{\mathrm{H}}(r,\xi,a) = (r,-\xi,a) = (r,\xi,a)^\vee, \tag{4.7}$$

for any Mukai vector $v = (r,\xi,a)$ on S, where the last equality is written according to notation (3.6). Combining the two previous derived equivalences we can define

$$\mathrm{FM}_{\mathcal{P}}^\vee := D_{\hat{S}} \circ \mathrm{FM}_{\mathcal{P}}\colon D^b(S) \longrightarrow D^b(\hat{S}), \qquad F \longmapsto (R\pi_{\hat{S}*}(\pi_S^*(F) \overset{L}{\otimes} \mathcal{P}))^\vee$$

and get, by (4.6) and (4.7), for any Mukai vector $v = (r,\xi,a)$,

$$(\mathrm{FM}_{\mathcal{P}}^\vee)^{\mathrm{H}}(r,\xi,a) = (a,\hat{\xi},r) =: \hat{v}. \tag{4.8}$$

Again, we turn our attention to the morphisms induced on the corresponding moduli spaces by the previous derived functors. For this purpose, we recall that, if H is an ample line bundle on S, then $\hat{H}$ is an ample line bundle on $\hat{S}$ (see [Mu81, Proposition 3.11]).

Lemma 4.2.3. *Let S be an Abelian surface, H an ample line bundle on S with $\mathrm{c}_1(H) =: h$ and $n, a \in \mathbb{Z}$.*

(1) Suppose that $\mathrm{NS}(S) = \mathbb{Z}h$ and $v = (r, nh, a)$, with $r > 0$. Then there exists an integer $n_0 \gg 0$ such that for every $n > n_0$ the Fourier-Mukai equivalence $\mathrm{FM}_{\mathcal{P}}$ induces an isomorphism

$$\mathrm{FM}_{\mathcal{P},v}\colon K_v(S,H) \longrightarrow K_{\widetilde{v}}(\hat{S},\hat{H}).$$

(2) Set $v = (0,\xi,a)$ and suppose that H is $v-$generic and $\hat{H}$ is $\widetilde{v}-$generic. Then there exists an integer $a_0 \gg 0$ such that for every $a > a_0$ the Fourier-Mukai equivalence $\mathrm{FM}_{\mathcal{P}}$ induces an isomorphism

$$\mathrm{FM}_{\mathcal{P},v}\colon K_v(S,H) \longrightarrow K_{\widetilde{v}}(\hat{S},\hat{H}).$$

(3) In any of the two cases above, the Fourier-Mukai equivalence $\mathrm{FM}_{\mathcal{P}}^\vee$ induces an isomorphism

$$\mathrm{FM}_{\mathcal{P},v}^\vee\colon K_v(S,H) \longrightarrow K_{\hat{v}}(\hat{S},\hat{H}).$$

Proof. Exactly as in [OPR24, Lemma 2.15], the claim is a straightforward application of Proposition 2.29, Proposition 2.33 and Lemma 2.28 of [PR23]. See [Yos01b], Theorem 3.18 for a more general statement. □

Remark 4.2.4. For later use, we furthermore point out that, under the hypotheses of Lemma 4.2.3, any $H-$semistable sheaf E with Mukai vector v satisfies the WIT_0 property with respect to $\mathrm{FM}_{\mathcal{P}}$ and $\mathrm{FM}_{\mathcal{P}}(E) = \mathrm{FM}^0_{\mathcal{P}}(E)$ is a locally free sheaf of Mukai vector $\widetilde{v}$, by [Yos01b, Theorem 3.18] (see also [PR23, Lemma 2.28]). Consequently, the same holds with respect to $\mathrm{FM}^{\vee}_{\mathcal{P}}$, in which case $\mathrm{FM}_{\mathcal{P}}(E) = \mathrm{FM}^0_{\mathcal{P}}(E)^{\vee}$ is a locally free sheaf of Mukai vector $\hat{v}$.

4.2.3 A relative Poincaré line bundle as kernel in the elliptic case

In the following, we will let S be isomorphic to the product $E \times F$ of two elliptic curves E, F. It will be sometimes convenient to look at the latter as an elliptic Abelian surface $p := \pi_E \colon S \to E$ with fiber F and $s \colon E \to S$ a $0-$section, which is actually equivalent to the first condition. We will denote by $f := \mathrm{c}_1(p^*(\mathcal{O}_E(1))) \in \mathrm{NS}(S)$ the class of a fiber of p and by $e := \mathrm{c}_1(s(E)) \in \mathrm{NS}(S)$ the class of the section s - or, equivalently, $e := \mathrm{c}_1([E])$ and $f := \mathrm{c}_1([F])$. Then, one can easily check that $e^2 = 0 = f^2$ and $e \cdot f = 1$, so that, if we assume that $\mathrm{NS}(S) = \langle e, f \rangle$, then $\mathrm{NS}(S)$ is isometric to the unimodular rank 2 hyperbolic lattice U.

Let $H \in \mathrm{Pic}(S)$ such that $h := \mathrm{c}_1(H) = e + tf$ and we assume $t \gg 0$, so that, by Nakai-Moishezon criterion, the class h is ample. Moreover, it can be checked (see Lemma 2.12 of [PR23]) that the polarization H is generic with respect to the Mukai vector $(0, f, 0)$. As explained in [Bri98, Section 4], the moduli space $M_{(0,f,0)}(S, H)$ is fine and parametrizes stable sheaves F of pure dimension 1, supported on the fibers of p, where their restriction has rank 1 and degree $\mathrm{c}_1(F) \cdot f = 0$. Furthermore, it is equipped by a fibration $M_{(0,f,0)}(S, H) \to E$ induced by p, whose fibers are isomorphic to the fibers of p by [At57, Theorem 7] and, indeed,

$$M_{(0,f,0)}(S, H) \simeq S \tag{4.9}$$

as elliptic surfaces (see [Bri98, Section 4.2]). By Theorem 1.2 of [Bri98], there exist tautological sheaves $\mathcal{P}$ on $M_{(0,f,0)}(S, H) \times S$ - regarded as coherent sheaves on $S \times S$ via the isomorphism (4.9) - such that the Fourier-Mukai transform $\mathrm{FM}_{\mathcal{P}} \colon D^b(S) \longrightarrow D^b(S)$ with kernel $\mathcal{P}$ is an equivalence. We choose one of such tautological sheaves - often called *relative Poincaré line bundles* - and we denote it by $\mathcal{E}$, to avoid confusion with the actual Poincaré line bundle $\mathcal{P}$ used in Section 4.2.2. We consider the Fourier-Mukai equivalence

$$\mathrm{FM}_{\mathcal{E}} \colon D^b(S) \longrightarrow D^b(S), \qquad F \longmapsto (R\pi_{2*}(\pi_1^*(F) \overset{L}{\otimes} \mathcal{E}))[1], \tag{4.10}$$

where $\pi_i \colon S \times S \to S$ is the projection on the $i-$th factor for $i = 1, 2$.

Lemma 4.2.5. *Let S be an elliptic Abelian surface as above and let $H \in \mathrm{Pic}(S)$ such that $h := \mathrm{c}_1(H) = e + tf$, with $t \gg 0$. Then $\mathrm{FM}_{\mathcal{E}}$ induces an isomorphism*

$$\mathrm{FM}_{\mathcal{E}} \colon K_{(m,0,-mk)}(S, H) \longrightarrow K_{(0,m(e+kf),m)}(S, H) \tag{4.11}$$

for every $m, k > 0$.

Proof. See [Yos01a, Theorem 3.15, Proposition 4.9]. □

Remark 4.2.6. The shift [1] in the definition (4.10) of $\mathrm{FM}_{\mathcal{E}}$ is motivated by the fact that, by [Yos01a, Lemma 3.9], if H is a polarization as in Lemma 4.2.5 and E is a $H-$semistable sheaf on S with Mukai vector $(m, 0, -mk)$, then it satisfies the WIT_1 property with respect to the Fourier-Mukai transform ϕ with kernel $\mathcal{E}$, in the classical sense, hence it satisfies the WIT_0 property with respect to $\mathrm{FM}_{\mathcal{E}} = \phi[1]$.

We can now introduce the groupoid $\mathcal{G}^{m,k}_{\mathrm{FM}}$.

Definition 4.2.7. Let $m, k \in \mathbb{N} \setminus \{0\}$. We define the groupoid $\mathcal{G}^{m,k}_{\mathrm{FM}}$ as follows:

* the objects of $\mathcal{G}^{m,k}_{\mathrm{FM}}$ are the $(m,k)-$triples;
* if (S_1, v_1, H_1) and (S_2, v_2, H_2) are two $(m,k)-$triples, an *elementary morphism* between them is one of the following:
  - the equivalence L if $(S_1, v_1, H_1) = (S, v, H)$ and $(S_2, v_2, H_2) = (S, v_L, H)$ are as in Lemma 4.2.2;
  - the Fourier-Mukai equivalence $\mathrm{FM}_{\mathcal{P}}$ if $(S_1, v_1, H_1) = (S, v, H)$ and $(S_2, v_2, H_2) = (\hat{S}, \tilde{v}, \hat{H})$ verify the hypotheses of Lemma 4.2.3 (1) or (2);
  - the Fourier-Mukai equivalence $\mathrm{FM}^{\vee}_{\mathcal{P}}$ if $(S_1, v_1, H_1) = (S, v, H)$ and $(S_2, v_2, H_2) = (\hat{S}, \hat{v}, \hat{H})$ verify the hypotheses of Lemma 4.2.3 (3) ;
  - the Fourier-Mukai equivalence $\mathrm{FM}_{\mathcal{E}}$ if $(S_1, v_1, H_1) = (S, (m, 0, -mk), H)$ and $(S_2, v_2, H_2) = (S, (0, m(e + kf), m), H)$ verify the hypotheses of Lemma 4.2.5;
* if (S_1, v_1, H_1) and (S_2, v_2, H_2) are two $(m,k)-$triples, a morphism in $\mathcal{G}^{m,k}_{\mathrm{FM}}$ from (S_1, v_1, H_1) to (S_2, v_2, H_2) is a formal concatenation of elementary morphisms and their formal inverses, subject to usual cancellation rules.

4.3 The groupoid $\mathcal{G}^{m,k}$ and its representations

Thanks to the notions introduced in Sections 4.1 and 4.2, we are finally in the position to define the groupoid $\mathcal{G}^{m,k}$ of $(m,k)-$triples.

Definition 4.3.1. Let $m, k \in \mathbb{N} \setminus \{0\}$. We define the groupoid

$$\mathcal{G}^{m,k} := \mathcal{G}^{m,k}_{\mathrm{def}} * \mathcal{G}^{m,k}_{\mathrm{FM}}$$

as the free product of the groupoids $\mathcal{G}^{m,k}_{\mathrm{def}}$ and $\mathcal{G}^{m,k}_{\mathrm{FM}}$.

Remark 4.3.2. In the proof of Theorem 3.2.1, it is shown that, if (S_1, v_1, H_1) and (S_2, v_2, H_2) are two $(m,k)-$triples, then one can always construct a path from (S_1, v_1, H_1) to (S_2, v_2, H_2) only using the following elementary morphisms in $\mathcal{G}^{m,k}$:

* equivalence classes of deformation paths of $(m,k)-$triples, inducing locally trivial deformations of moduli spaces;
* the identity morphism χ_{H_1,H_2} of $\mathcal{P}^{m,k}$, inducing identifications of moduli spaces;
* derived equivalences of the form L, where L is a suitable line bundle;
* the Fourier-Mukai equivalence $\mathrm{FM}_{\mathcal{P}}$,

where the last two morphisms induce isomorphisms of moduli spaces. Consequently, for any pair of $(m,k)-$triples as above, we get

$$\mathrm{Hom}_{\mathcal{G}^{m,k}}((S_1,v_1,H_1),(S_2,v_2,H_2)) \neq \emptyset.$$

4.3.1 The $\widetilde{\mathcal{H}}-$representation $\widetilde{\Phi}^{m,k}$ of $\mathcal{G}^{m,k}$

We start by recalling that, if S is an Abelian surface, then its Mukai lattice $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ is isometric to $U^{\oplus 4}$ (see Remark 3.1.4), where U is the unimodular rank 2 hyperbolic lattice. Motivated by this, in the following we will let $\widetilde{\Lambda}$ be an even unimodular lattice of signature $(4,4)$, which is isometric, by Milnor's Theorem (Theorem B.3.1), to the lattice $U^{\oplus 4}$, and we will endow it with an orientation ϵ, as defined in Appendix B.1.2 (b). We will use this data to define a new groupoid of $\mathbb{Z}-$modules, as follows:

Definition 4.3.3. Let $m,k \in \mathbb{N} \setminus \{0\}$. We define the groupoid $\widetilde{\mathcal{H}}^{m,k}$ as follows:

* the objects are triples $(\widetilde{\Lambda},v,\epsilon)$, where $\widetilde{\Lambda}$ is an even unimodular lattice of signature $(4,4)$, $v \in \widetilde{\Lambda}$ is of the form $v = mw$, where $w \in \widetilde{\Lambda}$ is primitive and $w^2 = 2k$, and ϵ is an orientation on $\widetilde{\Lambda}$;
* if $(\widetilde{\Lambda}_1,v_1,\epsilon_1)$ and $(\widetilde{\Lambda}_2,v_2,\epsilon_2)$ are two objects in $\widetilde{\mathcal{H}}^{m,k}$, then we set

$$\mathrm{Hom}_{\widetilde{\mathcal{H}}^{m,k}}((\widetilde{\Lambda}_1,v_1,\epsilon_1),(\widetilde{\Lambda}_2,v_2,\epsilon_2)) := \{g \in \mathrm{O}(\widetilde{\Lambda}_1,\widetilde{\Lambda}_2) \colon g(v_1) = v_2\}.$$

We notice that morphisms in $\widetilde{\mathcal{H}}^{m,k}$ are not necessarily orientation preserving, in the sense of Appendix B.1.2.

Example 4.3.4. As previously remarked, if S is an Abelian surface, then its Mukai lattice $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ is an even unimodular lattice of signature $(4,4)$. We now define an orientation on $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ starting from an orientation on $\mathrm{H}^2(S,\mathbb{Z})$, which is an even unimodular lattice of signature $(3,3)$.

(1) We point out that, from the description of the period domain and of the period map of Abelian surfaces ([Shi78, Theorem II]) we get that any holomorphic $2-$form $\sigma \in \mathrm{H}^0(S,\Omega^2_S)$ satisfies $\sigma^2 = 0$, $\sigma \cdot \overline{\sigma} > 0$ and $\sigma \perp \mathrm{H}^{1,1}(S)$, from which we deduce that $\mathrm{Re}(\sigma)$ and $\mathrm{Im}(\sigma)$ span a positive definite real subspace of $\mathrm{H}^2(S,\mathbb{R})$, orthogonal to $\mathbb{R}\omega$, where ω is any Kähler form, which is again positive definite by Hodge Index Theorem. We conclude that the basis

$\{\omega, \mathrm{Re}(\sigma), \mathrm{Im}(\sigma)\}$ defines an orientation on $\mathrm{H}^2(S,\mathbb{Z})$, which, as in Remark 2.3.17, does not depend on the choice of the symplectic form and of the Kähler form.

(2) Finally, we recall that $\mathrm{H}^2(S,\mathbb{Z})$ naturally embeds in $\widetilde{\mathrm{H}}(S,\mathbb{Z})$ and its orthogonal complement, with respect to the Mukai pairing, coincides with $\mathrm{H}^0(S,\mathbb{Z}) \oplus \mathrm{H}^4(S,\mathbb{Z})$. Hence, we can naturally identify $\mathrm{O}(\mathrm{H}^2(S,\mathbb{Z}))$ with the subgroup of $\mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Z}))$ of isometries acting as the identity on $\mathrm{H}^0(S,\mathbb{Z}) \oplus \mathrm{H}^4(S,\mathbb{Z})$, and use this to extend any orientation of $\mathrm{H}^2(S,\mathbb{Z})$ to an orientation of $\widetilde{\mathrm{H}}(S,\mathbb{Z})$. Indeed, by adding the 2−vector $(1,0,-1) \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$, orthogonal to $\mathrm{H}^2(S,\mathbb{Z})$, to the previous orientation, we get an orientation

$$\{\omega, \mathrm{Re}(\sigma), \mathrm{Im}(\sigma), (1,0,-1)\}$$

of $\widetilde{\mathrm{H}}(S,\mathbb{Z})$, which we will denote by ϵ_S.

The last Example allows us to define the representation $\widetilde{\Phi}^{m,k}_{\mathrm{def}} \colon \mathcal{G}^{m,k}_{\mathrm{def}} \to \widetilde{\mathcal{H}}^{m,k}$.

Definition 4.3.5. Let $m,k \in \mathbb{N} \setminus \{0\}$. We define the representation

$$\widetilde{\Phi}^{m,k}_{\mathrm{def}} \colon \mathcal{G}^{m,k}_{\mathrm{def}} \longrightarrow \widetilde{\mathcal{H}}^{m,k}$$

is defined as follows:

* if (S,v,H) is an object in $\mathcal{G}^{m,k}_{\mathrm{def}}$, then we set $\widetilde{\Phi}^{m,k}_{\mathrm{def}}((S,v,H)) := (\widetilde{\mathrm{H}}(S,\mathbb{Z}), v, \epsilon_S)$, where ϵ_S is the orientation defined in Example 4.3.4;
* if (S_1,v_1,H_1) and (S_2,v_2,H_2) are two objects in $\mathcal{G}^{m,k}_{\mathrm{def}}$ and $\alpha = (f \colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H}, t_1, t_2, \gamma)$ is a deformation path from (S_1,v_1,H_1) to (S_2,v_2,H_2), then

  $$\widetilde{\Phi}^{m,k}_{\mathrm{def}}(\overline{\alpha}) := p_\alpha$$

  is the parallel transport operator in the local system $R^{\mathrm{ev}} f_* \mathbb{Z}$ along the path γ (as in (4.1));
* if (S_1,v_1,H_1) and (S_2,v_2,H_2) are congruent, then we set

  $$\widetilde{\Phi}^{m,k}_{\mathrm{def}}(\chi_{H_1,H_2}) := \mathrm{id}_{\widetilde{\mathrm{H}}(S,\mathbb{Z})}.$$

Remark 4.3.6. Let $\alpha = (f \colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H}, t_1, t_2, \gamma)$ be a deformation path from (S_1,v_1,H_1) to (S_2,v_2,H_2).

(1) As already noticed in Remark 4.1.6 (1), the Mukai vectors v_1 and v_2 belong to the same flat section of the local system $R^{\mathrm{ev}} f_* \mathbb{Z}$. Hence, the parallel transport p_α maps v_1 to v_2 and the representation $\widetilde{\Phi}^{m,k}_{\mathrm{def}}$ is well defined.

(2) Moreover if $\epsilon_{S_i} = \{\omega_i, \mathrm{Re}(\sigma_i), \mathrm{Im}(\sigma_i), (1,0,-1)\}$ are the orientations for $\widetilde{\mathrm{H}}(S_i, \mathbb{Z})$ defined in Example 4.3.4, for $i = 1, 2$, then p_α maps ϵ_{S_1} to ϵ_{S_2}. Indeed, the classes ω_1 and ω_2 extend to flat sections of the same local system and the same holds for σ_1 and σ_2. As we will discuss in more detail in Section 5.1.1,

$$\mathrm{Mon}^2(S) \subseteq \mathrm{O}^+(\mathrm{H}^2(S, \mathbb{Z}))$$

(by [Shi78], see also [MR21]), from which we get that p_α is constant along such local systems. This implies, together with the fact that the vector $(1,0,-1)$ is constant along any locally trivial deformation, that morphisms in the image of $\widetilde{\Phi}^{m,k}_{\mathrm{def}}$ are always orientation preserving.

We shall now proceed to define the representation $\widetilde{\Phi}^{m,k}_{\mathrm{FM}} \colon \mathcal{G}^{m,k}_{\mathrm{FM}} \to \widetilde{\mathcal{H}}^{m,k}$.

Definition 4.3.7. Let $m, k \in \mathbb{N} \setminus \{0\}$. We define the representation

$$\widetilde{\Phi}^{m,k}_{\mathrm{FM}} \colon \mathcal{G}^{m,k}_{\mathrm{FM}} \longrightarrow \widetilde{\mathcal{H}}^{m,k}$$

is defined as follows:

* if (S, v, H) is an object in $\mathcal{G}^{m,k}_{\mathrm{FM}}$, then we set $\widetilde{\Phi}^{m,k}_{\mathrm{FM}}((S, v, H)) := (\widetilde{\mathrm{H}}(S, \mathbb{Z}), v, \epsilon_S)$;
* if (S_1, v_1, H_1) and (S_2, v_2, H_2) are two objects in $\mathcal{G}^{m,k}_{\mathrm{FM}}$ and $\phi \colon D^b(S_1) \to D^b(S_2)$ is a morphism in $\mathcal{G}^{m,k}_{\mathrm{FM}}$ from (S_1, v_1, H_1) to (S_2, v_2, H_2), then

  $$\widetilde{\Phi}^{m,k}_{\mathrm{FM}}(\phi) := \phi^{\mathrm{H}}$$

  is the isometry induced by ϕ on the respective Mukai lattices.

Remark 4.3.8. As recalled in Proposition 5.44 and Corollary 9.43 of [Huy06], any Fourier-Mukai equivalence between Abelian surfaces induces an isomorphism on the respective integral cohomologies, isometric with respect to the Mukai pairing, which is also parity preserving. Hence, if $\phi = \mathsf{L}$, $\mathrm{FM}_{\mathcal{P}}$ or $\mathrm{FM}_{\mathcal{E}}$, then it induces an isometry ϕ^{H} on the respective Mukai lattices, preserving the Mukai vectors by (4.5), (4.6) and (4.11). The claim follows for $\phi = \mathrm{FM}^{\vee}_{\mathcal{P}}$, as derived duality induces, via the Chern character, an isomorphism on the even cohomology prescribed by (4.7), and combining it with (4.8). Hence, the representation $\widetilde{\Phi}^{m,k}_{\mathrm{FM}}$ is again well defined.

Remark 4.3.9. As we will explain in more detail in Section 4.3.3, we remark that morphisms in the image of $\widetilde{\Phi}^{m,k}_{\mathrm{FM}}$ are not necessarily orientation preserving.

Definition 4.3.10. Let $m, k \in \mathbb{N} \setminus \{0\}$. We define the representation

$$\widetilde{\Phi}^{m,k} \colon \mathcal{G}^{m,k} \longrightarrow \widetilde{\mathcal{H}}^{m,k}$$

as the unique[1] representation restricting to $\widetilde{\Phi}^{m,k}_{\mathrm{def}}$ on $\mathcal{G}^{m,k}_{\mathrm{def}}$ and to $\widetilde{\Phi}^{m,k}_{\mathrm{FM}}$ on $\mathcal{G}^{m,k}_{\mathrm{FM}}$.

[1] See [OPR24, Remark 2.31].

4.3.2 The $\mathcal{A}_k-$representation $\mathsf{pt}^{m,k}$ of $\mathcal{G}^{m,k}$

We now turn our attention to lattices of the same isometry class of $\mathrm{H}^2(K_v(S,H),\mathbb{Z})$, where (S,v,H) is an $(m,k)-$triple, with $(m,k)\neq(1,1),(1,2)$. We recall that, by Theorem 3.3.4 (2), there is an isometry $\mathrm{H}^2(K_v(S,H),\mathbb{Z})\simeq v^{\perp}$. The latter, being the orthogonal complement of the lattice generated by $v=mw$, with $w^2=2k>0$, in $\widetilde{\mathrm{H}}(S,\mathbb{Z})\simeq U^{\oplus 4}$, is isometric to the even lattice $U^{\oplus 3}\oplus\langle -2k\rangle$ of rank 7 and signature $(3,4)$.

Definition 4.3.11. For any $k\in\mathbb{N}\setminus\{0\}$, we define the groupoid $\mathcal{A}_k$ as follows:

* the objects of $\mathcal{A}_k$ are even lattices Λ of signature $(3,4)$, isometric to the lattice $U^{\oplus 3}\oplus\langle -2k\rangle$.
* if Λ_1 and Λ_2 are two objects, then we set

$$\mathrm{Hom}_{\mathcal{A}_k}(\Lambda_1,\Lambda_2):=\mathrm{O}(\Lambda_1,\Lambda_2).$$

As in the previous Section, we start by defining the $\mathcal{A}_k-$representation for $\mathcal{G}^{m,k}_{\mathrm{def}}$.

Definition 4.3.12. Let $m,k\in\mathbb{N}\setminus\{0\}$, with $(m,k)\neq(1,1),(1,2)$. We define the representation

$$\mathsf{pt}^{m,k}_{\mathrm{def}}\colon \mathcal{G}^{m,k}_{\mathrm{def}}\longrightarrow\mathcal{A}_k$$

as follows:

* if (S,v,H) is an object in $\mathcal{G}^{m,k}_{\mathrm{def}}$, then we set

$$\mathsf{pt}^{m,k}_{\mathrm{def}}((S,v,H)):=\mathrm{H}^2(K_v(S,H),\mathbb{Z});$$

* if (S_1,v_1,H_1) and (S_2,v_2,H_2) are two objects in $\mathcal{G}^{m,k}_{\mathrm{def}}$ and $\alpha=(f\colon\mathcal{S}\to T,\mathcal{L},\mathcal{H},t_1,t_2,\gamma)$ is a deformation path from (S_1,v_1,H_1) to (S_2,v_2,H_2), then

$$\mathsf{pt}^{m,k}_{\mathrm{def}}(\overline{\alpha}):=g_\alpha$$

  is the parallel transport in the local system $R^2p_*\mathbb{Z}$ along the path γ (as in (4.2));
* if (S_1,v_1,H_1) and (S_2,v_2,H_2) are congruent, then we set

$$\mathsf{pt}^{m,k}_{\mathrm{def}}(\chi_{H_1,H_2}):=\mathrm{id}_{\mathrm{H}^2(K_v(S,H),\mathbb{Z})}.$$

We now define the $\mathcal{A}_k-$representation for $\mathcal{G}^{m,k}_{\mathrm{FM}}$.

Definition 4.3.13. Let $m,k\in\mathbb{N}\setminus\{0\}$, with $(m,k)\neq(1,1),(1,2)$. We define the representation

$$\mathsf{pt}^{m,k}_{\mathrm{FM}}\colon \mathcal{G}^{m,k}_{\mathrm{FM}}\longrightarrow\mathcal{A}_k$$

as follows:

* if (S, v, H) is an object in $\mathcal{G}^{m,k}_{\mathrm{FM}}$, then we set
$$\mathsf{pt}^{m,k}_{\mathrm{FM}}((S, v, H)) := \mathrm{H}^2(K_v(S, H), \mathbb{Z});$$
* if (S_1, v_1, H_1) and (S_2, v_2, H_2) are two objects in $\mathcal{G}^{m,k}_{\mathrm{FM}}$ and $\phi\colon D^b(S_1) \to D^b(S_2)$ is an elementary morphism from (S_1, v_1, H_1) to (S_2, v_2, H_2), then, by definition, it induces an isomorphism
$$\phi_{v_1}\colon K_{v_1}(S_1, H_1) \longrightarrow K_{v_2}(S_2, H_2)$$
of moduli spaces. We then set
$$\mathsf{pt}^{m,k}_{\mathrm{FM}}(\phi) := \phi_{v_1 *}\colon \mathrm{H}^2(K_{v_1}(S_1, H_1), \mathbb{Z}) \longrightarrow \mathrm{H}^2(K_{v_2}(S_2, H_2), \mathbb{Z});$$
* if $\phi = \phi_1 * \cdots * \phi_l \in \mathrm{Hom}_{\mathcal{G}^{m,k}_{\mathrm{FM}}}((S_1, v_1, H_1), (S_2, v_2, H_2))$ is a composition of elementary morphisms, we define $\mathsf{pt}^{m,k}_{\mathrm{FM}}(\phi)$ as the composition
$$\mathsf{pt}^{m,k}_{\mathrm{FM}}(\phi_l) \circ \cdots \circ \mathsf{pt}^{m,k}_{\mathrm{FM}}(\phi_1)$$
of the corresponding isometries.

Combining the two previous Defintions, as in the case of $\widetilde{\Phi}^{m,k}$, we get the following

Definition 4.3.14. Let $m, k \in \mathbb{N} \setminus \{0\}$, with $(m, k) \neq (1, 1), (1, 2)$. We define the representation
$$\mathsf{pt}^{m,k}\colon \mathcal{G}^{m,k} \longrightarrow \mathcal{A}_k$$
as the unique representation restricting to $\mathsf{pt}^{m,k}_{\mathrm{def}}$ on $\mathcal{G}^{m,k}_{\mathrm{def}}$ and to $\mathsf{pt}^{m,k}_{\mathrm{FM}}$ on $\mathcal{G}^{m,k}_{\mathrm{FM}}$.

The first straightforward, but nonetheless crucial, property satisfied by the representation just defined is stated in the following

Proposition 4.3.15. *Let $m, k \in \mathbb{N} \setminus \{0\}$, with $(m, k) \neq (1, 1), (1, 2)$, and let $A_1 := (S_1, v_1, H_1)$ and $A_2 := (S_2, v_2, H_2)$ be two objects in $\mathcal{G}^{m,k}$. Then there is an inclusion of sets*
$$\mathsf{pt}^{m,k}(\mathrm{Hom}_{\mathcal{G}^{m,k}}(A_1, A_2)) \subseteq \mathsf{PT}^2_{\mathrm{lt}}(K_{v_1}(S_1, H_1), K_{v_2}(S_2, H_2)).$$

Proof. First of all, the inclusion
$$\mathsf{pt}^{m,k}_{\mathrm{def}}(\mathrm{Hom}_{\mathcal{G}^{m,k}_{\mathrm{def}}}(A_1, A_2)) \subseteq \mathsf{PT}^2_{\mathrm{lt}}(K_{v_1}(S_1, H_1), K_{v_2}(S_2, H_2))$$
holds by definition. Moreover, since any elementary morphism in $\mathcal{G}^{m,k}_{\mathrm{FM}}$ induces isomorphisms of moduli spaces, by Proposition 2.3.20,
$$\mathsf{pt}^{m,k}_{\mathrm{FM}}(\mathrm{Hom}_{\mathcal{G}^{m,k}_{\mathrm{FM}}}(A_1, A_2)) \subseteq \mathsf{PT}^2_{\mathrm{lt}}(K_{v_1}(S_1, H_1), K_{v_2}(S_2, H_2))$$
holds, concluding the proof. □

According to the notation introduced in Definition 4.0.1 (1), we can therefore relate, via the representation $\mathsf{pt}^{m,k}$, the isotropy group of an $(m,k)-$triple with the monodromy group of the moduli space associated to the same triple:

Corollary 4.3.16. *Let $m,k \in \mathbb{N} \setminus \{0\}$, with $(m,k) \neq (1,1),(1,2)$, and let (S,v,H) be an object in $\mathcal{G}^{m,k}$. Then*

$$\mathrm{Im}(\mathsf{pt}^{m,k}_{(S,v,H)} \colon \mathrm{Aut}_{\mathcal{G}^{m,k}}((S,v,H)) \longrightarrow \mathrm{Aut}_{\mathcal{A}_k}(\mathrm{H}^2(K_v(S,H),\mathbb{Z}))) \subseteq \mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H)).$$

4.3.3 Relation between the two representations $\widetilde{\Phi}^{m,k}$ and $\mathsf{pt}^{m,k}$.

In order to compare the two representations $\widetilde{\Phi}^{m,k} \colon \mathcal{G}^{m,k} \to \widetilde{\mathcal{H}}^{m,k}$ and $\mathsf{pt}^{m,k} \colon \mathcal{G}^{m,k} \to \mathcal{A}_k$, we shall now connect them by means of a functor $\Psi^{m,k} \colon \widetilde{\mathcal{H}}^{m,k} \to \mathcal{A}_k$.

Definition 4.3.17. For any $m,k \in \mathbb{N} \setminus \{0\}$, with $(m,k) \neq (1,1),(1,2)$, we define the functor

$$\Psi^{m,k} \colon \widetilde{\mathcal{H}}^{m,k} \to \mathcal{A}_k$$

as follows:

∗ if $(\widetilde{\Lambda}, v, \epsilon)$ is an object in $\widetilde{\mathcal{H}}^{m,k}$, we set

$$\Psi^{m,k}(\widetilde{\Lambda}, v, \epsilon) := v^{\perp};$$

∗ if $(\widetilde{\Lambda}_1, v_1, \epsilon_1)$ and $(\widetilde{\Lambda}_2, v_2, \epsilon_2)$ are two objects in $\widetilde{\mathcal{H}}^{m,k}$ and $g \colon \widetilde{\Lambda}_1 \to \widetilde{\Lambda}_2$ is an isometry such that $g(v_1) = v_2$, then we set

$$\Psi^{m,k}(g) := (-1)^{\mathrm{or}(g)} g_{|v_1^{\perp}} \colon v_1^{\perp} \longrightarrow v_2^{\perp},$$

where $\mathrm{or} \colon \mathrm{O}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2) \to \mathbb{Z}/2\mathbb{Z}$ is the orientation character (B.1).

We are finally in the position to compare the two $\mathcal{A}_k-$representations

$$\Phi^{m,k} := \Psi^{m,k} \circ \widetilde{\Phi}^{m,k} \colon \mathcal{G}^{m,k} \longrightarrow \mathcal{A}_k \tag{4.12}$$

and $\mathsf{pt}^{m,k}$. For this purpose, we will translate naturality of the isometries $\lambda_v \colon v^{\perp} \longrightarrow \mathrm{H}^2(K_v(S,H),\mathbb{Z})$ of Theorem 3.3.4 (2) into the existence of an isomorphism of functors

$$\lambda \colon \Phi^{m,k} \longrightarrow \mathsf{pt}^{m,k}.$$

The proof of this result, which is the main of the Chapter, will be addressed in two parts, involving, respectively, the $\mathcal{A}_k-$representations of $\mathcal{G}^{m,k}_{\mathrm{def}}$ and of $\mathcal{G}^{m,k}_{\mathrm{FM}}$.

Let us start by setting

$$\Phi^{m,k}_{\mathrm{def}} := \Psi^{m,k} \circ \widetilde{\Phi}^{m,k}_{\mathrm{def}} \colon \mathcal{G}^{m,k}_{\mathrm{def}} \longrightarrow \mathcal{A}_k.$$

Proposition 4.3.18. *For any $m,k \in \mathbb{N} \setminus \{0\}$, with $(m,k) \neq (1,1),(1,2)$, there exists an isomorphism of functors*

$$\lambda_{\mathrm{def}} \colon \Phi^{m,k}_{\mathrm{def}} \longrightarrow \mathsf{pt}^{m,k}_{\mathrm{def}}.$$

Proof. For $i=1,2$, let $A_i := (S_i, v_i, H_i)$ be two objects in $\mathcal{G}^{m,k}_{\mathrm{def}}$. By Theorem 3.3.4 (2), for $i=1,2$, there exists an isometry

$$\lambda_{v_i} \colon v_i^{\perp} \longrightarrow \mathrm{H}^2(K_{v_i}(S_i,H_i),\mathbb{Z}).$$

In order to show that the assignment $\lambda(A_i) := \lambda_{v_i}$ defines an isomorphism of functors $\Phi^{m,k}_{\mathrm{def}} \to \mathsf{pt}^{m,k}_{\mathrm{def}}$, we need to show that the following diagram

$$\begin{tikzcd} v_1^{\perp} \arrow[r, "\lambda_{v_1}"] \arrow[d, "\Phi^{m,k}_{\mathrm{def}}(h)"'] & \mathrm{H}^2(K_{v_1}(S_1,H_1),\mathbb{Z}) \arrow[d, "\mathsf{pt}^{m,k}_{\mathrm{def}}(h)"] \\ v_2^{\perp} \arrow[r, "\lambda_{v_2}"] & \mathrm{H}^2(K_{v_1}(S_2,H_2),\mathbb{Z}). \end{tikzcd} \tag{4.13}$$

is commutative, for any morphism $h \in \mathrm{Hom}_{\mathcal{G}^{m,k}_{\mathrm{def}}}(A_1,A_2)$.

Case 1. Let $h = \overline{\alpha}$ be the elementary morphism given by the class of a deformation path $\alpha = (f \colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H}, t_1, t_2, \gamma)$ from A_1 to A_2. In this case,

$$\Phi^{m,k}_{def}(\alpha) = \Psi^{m,k}(\widetilde{\Phi}^{m,k}_{def}(\alpha)) = (-1)^{\mathrm{or}(p_\alpha)} p_\alpha|_{v_1^{\perp}} = p_\alpha|_{v_1^{\perp}},$$

as the parallel transport operator p_α in the local system $R^{\mathrm{ev}} f_* \mathbb{Z}$ (see (4.1) is orientation preserving by Remark 4.3.6 (b). On the other hand, if $p \colon K_v(\mathcal{S}/T,\mathcal{H}) \to T$ is the relative moduli space induced by the deformation $(f \colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H})$, then

$$\mathsf{pt}^{m,k}_{\mathrm{def}}(h) = g_\alpha$$

is the parallel transport operator in the local system $R^2 p_* \mathbb{Z}$ (see (4.2)). By Remark 4.1.6 (1), there exists a flat section v of $R^{\mathrm{ev}} f_* \mathbb{Z}$ such that $\mathsf{v}_{t_i} = v_i$ for $i=1,2$ and, by Remark 4.1.6 (2), the isometries λ_{v_i}, for $i=1,2$, fit in an isomorphism of local systems

$$\lambda_{\mathsf{v}} \colon \mathsf{v}^{\perp} \longrightarrow R^2 p_* \mathbb{Z}.$$

We deduce then that g_α and $p_{\alpha|v_1^{\perp}}$ are parallel transport operators along the same path inside local systems that are isomorphic via λ_{v}, by Remark 4.1.6 (3), from which commutativity of diagram (4.13) follows.

Case 2. Suppose that A_1 and A_2 are congruent and that $h = \chi_{H_1,H_2}$. In this case, commutativity of diagram (4.13) is automatic, as

$$\Phi^{m,k}_{def}(\chi_{H_1,H_2}) = \mathrm{id}_{v^{\perp}} \text{ and } \mathsf{pt}^{m,k}_{def}(\chi_{H_1,H_2}) = \mathrm{id}_{\mathrm{H}^2(K_v(S,H_1),\mathbb{Z})}$$

and we have an identification $\mathrm{H}^2(K_v(S,H_1),\mathbb{Z}) = \mathrm{H}^2(K_v(S,H_2),\mathbb{Z})$ and an identification $\lambda_{v_1} = \lambda_{v_2} = \lambda_v$. $\square$

We now set
$$\Phi^{m,k}_{\mathrm{FM}} := \Psi^{m,k} \circ \widetilde{\Phi}^{m,k}_{\mathrm{FM}} \colon \mathcal{G}^{m,k}_{\mathrm{FM}} \longrightarrow \mathcal{A}_k.$$

Proposition 4.3.19. *For any $m, k \in \mathbb{N} \setminus \{0\}$, with $(m,k) \neq (1,1), (1,2)$, there exists an isomorphism of functors*
$$\lambda_{\mathrm{FM}} \colon \Phi^{m,k}_{\mathrm{FM}} \longrightarrow \mathsf{pt}^{m,k}_{\mathrm{FM}}.$$

In order to prove Proposition 4.3.19, we need to show that there exists a commutative diagram as in (4.13), but defined by morphisms in $\mathcal{G}^{m,k}_{\mathrm{FM}}$. The first step is given by the following result due to Yoshioka.

Lemma 4.3.20. *Let (S_1, v_1, H_1) and (S_2, v_2, H_2) be two $(m,k)-$triples and let $\mathcal{K} \in D^b(S_1 \times S_2)$ be an object which is flat over each factor and is strongly simple over each factor, i.e. such that*

(1) $\mathrm{Hom}(\mathcal{K}_{|\{p_1\}\times S_2}, \mathcal{K}_{|\{p_1\}\times S_2}) = \mathbb{C}_{p_1}$ *for any $p_1 \in S_1$;*

(2) $\mathrm{Ext}^i(\mathcal{K}_{|\{p_1\}\times S_2}, \mathcal{K}_{|\{q_1\}\times S_2}) = 0$ *for any $p_1, q_1 \in S_1$, $p_1 \neq q_1$ and $i = 0, 1, 2$*

and the analogous condition holds for any $p_2, q_2 \in S_2$ as above. Then,

(a) If every $E \in K_{v_1}(S_1, H_1)$ satisfies the WIT_i property with respect to $\phi := \mathrm{FM}_{\mathcal{K}}$ and $\phi^i(E) \in K_{v_2}(S_2, H_2)$, then there exists the following commutative diagram

$$\begin{array}{ccc} v_1^{\perp} & \xrightarrow{\lambda_{v_1}} & \mathrm{H}^2(K_{v_1}(S_1, H_1), \mathbb{Z}) \\ {\scriptstyle (-1)^i \phi^{\mathrm{H}}}\downarrow & & \downarrow{\scriptstyle \phi_{v_1 *}} \\ v_2^{\perp} & \xrightarrow{\lambda_{v_2}} & \mathrm{H}^2(K_{v_1}(S_2, H_2), \mathbb{Z}). \end{array} \tag{4.14}$$

(b) If every $E \in K_{v_1}(S_1, H_1)$ satisfies the WIT_i property with respect to $\phi := \mathrm{FM}^{\vee}_{\mathcal{K}}$ and $\phi^i(E) \in K_{v_2^{\vee}}(S_2, H_2)$, then there exists the following commutative diagram

$$\begin{array}{ccc} v_1^{\perp} & \xrightarrow{\lambda_{v_1}} & \mathrm{H}^2(K_{v_1}(S_1, H_1), \mathbb{Z}) \\ {\scriptstyle (-1)^{i+1} \phi^{\mathrm{H}}}\downarrow & & \downarrow{\scriptstyle \phi_{v_1 *}} \\ (v_2^{\vee})^{\perp} & \xrightarrow{\lambda_{v_2^{\vee}}} & \mathrm{H}^2(K_{v_1}(S_2, H_2), \mathbb{Z}). \end{array} \tag{4.15}$$

Proof. See [Yos01a, Proposition 2.4, Proposition 2.5]. □

Corollary 4.3.21. *Let (S_1, v_1, H_1) and (S_2, v_2, H_2) be two objects in $\mathcal{G}^{m,k}_{\mathrm{FM}}$ and let $\phi \colon D^b(S_1) \to D^b(S_2)$ be an elementary morphism.*

(1) If $\phi = \mathsf{L}, \mathrm{FM}_{\mathcal{P}}$ or $\mathrm{FM}_{\mathcal{E}}$, then ϕ induces a commutative diagram as (4.14) with $i = 0$;

(2) If $\phi = \mathrm{FM}^{\vee}_{\mathcal{P}}$, then ϕ induces a commutative diagram as (4.15) with $i = 0$.

Proof. A straightforward computation shows that the respective kernels $\Delta_* L$, $\mathcal{P}$ and $\mathcal{E}$ of ϕ are all flat and strongly simple on both factors. Moreover, condition (a) of Lemma 4.3.20 holds with respect to $\phi = \mathsf{L}, \mathrm{FM}_{\mathcal{P}}$ or $\mathrm{FM}_{\mathcal{E}}$ by Remark 4.2.4 and Remark 4.2.6 and, analogously, condition (b) holds with respect to $\phi = \mathrm{FM}_{\mathcal{P}}^{\vee}$. The claim then follows from Lemma 4.3.20. □

The next step consists of showing that the alternating signs in diagrams (4.14) and (4.15) are compatible with that of Definition 4.3.17, according to whether the morphisms under study are orientation preserving or reversing.

As already noticed in Remark 4.3.9, morphisms in the image of $\widetilde{\Phi}^{m,k}_{\mathrm{FM}}$ are not necessarily orientation preserving. In order to prove the commutativity of a diagram as (4.13) for elementary morphisms in $\mathcal{G}^{m,k}_{\mathrm{FM}}$, we need to introduce some criteria to decide whether an isometry induced by a Fourier-Mukai equivalence is orientation preserving. The following is an adaptation of [HS05, Proposition 5.3] to the case in which the surface S is Abelian.

Lemma 4.3.22. *Let S_1 and S_2 be two Abelian surfaces and let $\phi\colon \widetilde{\mathrm{H}}(S_1, \mathbb{Z}) \to \widetilde{\mathrm{H}}(S_2, \mathbb{Z})$ be a Hodge isometry. Let $\omega \in \mathrm{H}^{1,1}(S_1, \mathbb{R})$ be a Kähler class on S_1 and let $\mathcal{C}_{S_i} \subseteq \mathrm{H}^{1,1}(S_i, \mathbb{R})$ be the connected component of the cone of positive forms containing the Kähler cone of S_i, for $i = 1, 2$. Set*

$$r := \phi(0,0,1)_0,\ \chi := \phi(1,0,0)_0,\ \chi_\omega := \phi(0,\omega,-\omega^2/2)_0 \in \mathrm{H}^0(S_2,\mathbb{Z})$$

and set $u_0 := (-r, 0, \chi)$, $u_1 := (0, -r\omega, r\omega^2/2 + \chi_\omega) \in \widetilde{\mathrm{H}}(S_1, \mathbb{Z})$.

(a) If $r \neq 0$, then ϕ is orientation preserving if and only if

$$\Big(\frac{\chi_\omega}{r} + \frac{\omega^2}{2}\Big)\phi(u_0)_2 - \Big(\frac{\chi}{r} - \frac{\omega^2}{2}\Big)\phi(u_1)_2 \in \mathcal{C}_{S_2};$$

(b) if $r = 0$, then ϕ is orientation preserving if and only if

$$\chi\phi(0,\omega,-\omega^2/2)_2 - \chi_\omega\phi(1,0,0)_2 - \frac{\omega^2}{2}(\phi(u_0)_2 - \phi(u_1)_2) \in \mathcal{C}_{S_2}.$$

Proof. We refer to [HS05, Section 5] for a detailed proof in the original case in which the surfaces involved are K3 surfaces and we sketch here the idea of the proof in the Abelian case, omitting computations. As ϕ is a Hodge isometry, arguing as in Remark 2.3.19, we get that the orientation of $\widetilde{\mathrm{H}}(S_1, \mathbb{Z})$ defined by $\{\omega, \mathrm{Re}(\sigma), \mathrm{Im}(\sigma), (1, 0, -1)\}$ - where σ is a holomorphic symplectic form on S_1 - is preserved by ϕ if and only if the orientation of the positive real plane $\langle \omega, (1, 0, -1)\rangle = \langle (1, \omega, -\omega^2/2)\rangle$ is preserved. The latter is completely encoded in the complex line spanned by $\exp(i\omega)$, whose image via the $\mathbb{C}-$linear extension $\phi_{\mathbb{C}}$ of ϕ must be of the form

$$\phi_{\mathbb{C}}(\exp(i\omega)) = \lambda(\exp(b + ia)),$$

with $\lambda \in \mathbb{C}^*$ and $a, b \in \mathrm{H}^{1,1}(S_2, \mathbb{R})$. We write $\phi_{\mathbb{C}}(\exp(i\omega))$ as a $\mathbb{Z}[i]-$linear combination of $\phi(1,0,0)$, $\phi(0,0,1)$ and $\phi(0,\omega,-\omega^2/2)$ and we use it to compute the scalar

$$\lambda = \phi_{\mathbb{C}}(\exp(i\omega))_0 = \chi - \frac{\omega^2}{2}r + i(\chi_\omega + \frac{\omega^2}{2}r)$$

and the real $(1,1)-$form

$$\begin{aligned} |\lambda|^2 a &= \mathrm{Im}(\overline{\lambda}(\exp(b+ia)))_2 = \mathrm{Im}(\overline{\lambda}\phi_{\mathbb{C}}(\exp(i\omega)))_2 = \\ &= \begin{cases} \left(\frac{\chi_\omega}{r} + \frac{\omega^2}{2}\right)\phi(u_0)_2 - \left(\frac{\chi}{r} - \frac{\omega^2}{2}\right)\phi(u_1)_2 & \text{if } r \neq 0 \\ \chi\phi(0,\omega,-\omega^2/2)_2 - \chi_\omega\phi(1,0,0)_2 - \frac{\omega^2}{2}(\phi(u_0)_2 - \phi(u_1)_2) & \text{if } r = 0. \end{cases} \end{aligned}$$

The latter is, up to multiplication with the positive scalar $|\lambda|^2$, the class $a \in \mathrm{H}^{1,1}(S_2, \mathbb{R})$, which determines the orientation given by $\{\mathrm{Re}(\lambda \exp(b+ia)), \mathrm{Im}(\lambda \exp(b+ia))\}$, or, equivalently the orientation defined by the real and imaginary part of $\exp(ia)$. If ω' is a Kähler class on S_2, then the real and imaginary part of $\exp(ia)$ induce the same orientation as the natural one given by $\exp(i\omega')$ if and only if a and ω' belong to the same connected component of the cone of positive classes in $\mathrm{H}^{1,1}(S_2, \mathbb{R})$, from which the claim follows. □

Corollary 4.3.23. *Let (S_1, v_1, H_1) and (S_2, v_2, H_2) two objects in $\mathcal{G}^{m,k}_{\mathrm{FM}}$, let $\phi\colon D^b(S_1) \to D^b(S_2)$ an elementary morphism and let $\phi^{\mathrm{H}}\colon \widetilde{\mathrm{H}}(S_1, \mathbb{Z}) \to \widetilde{\mathrm{H}}(S_2, \mathbb{Z})$ be the induced isometry on the respective Mukai lattices.*

(1) If $\phi = \mathsf{L}, \mathrm{FM}_{\mathcal{P}}$ or $\mathrm{FM}_{\mathcal{E}}$, then $\mathrm{or}(\phi^{\mathrm{H}}) = 0$;

(2) If $\phi = \mathrm{FM}^{\vee}_{\mathcal{P}}$, then $\mathrm{or}(\phi^{\mathrm{H}}) = 1$.

Proof. We recall that any Fourier-Mukai equivalence $\phi\colon D^b(S_1) \to D^b(S_2)$ induces an isomorphism of weight two Hodge structures as defined in (3.9) (see [Huy06, Proposition 5.39]), so we can apply Lemma 4.3.22.

If $\phi = \mathsf{L}$, $(S_1, v_1, H_1) = (S, v, H)$ and $(S_2, v_2, H_2) = (S, v_L, H)$ then, for any $(r, \xi, a) \in \widetilde{\mathrm{H}}(S, \mathbb{Z})$,

$$\mathsf{L}^{\mathrm{H}}(r, \xi, a) = (r, \xi + \mathrm{c}_1(L), a + \xi\mathrm{c}_1(L) + r\mathrm{c}_1(L)^2/2).$$

We get $\mathsf{L}^{\mathrm{H}}(0,0,1)_0 = 0$, hence we need to check the condition of case (b) of Lemma 4.3.22, which is shown equivalent to $\omega \in \mathcal{C}_S$, where ω is a Kähler class.

If $\phi = \mathrm{FM}_{\mathcal{P}}$, $(S_1, v_1, H_1) = (S, v, H)$ and $(S_2, v_2, H_2) = (\hat{S}, \widetilde{v}, \hat{H})$, then by (4.6) we get $\mathrm{FM}^{\mathrm{H}}_{\mathcal{P}}(0,0,1)_0 = 1$. Condition of case (a) turns out to be $(\omega^2/2)\hat{\omega} \in \mathcal{C}_{\hat{S}}$, which is satisfied, as $\omega \in \mathcal{C}_S$. Analogously, the Fourier-Mukai equivalence $\phi = \mathrm{FM}^{\vee}_{\mathcal{P}}$ falls into case (a) and produces the opposite class $-(\omega^2/2)\hat{\omega} \notin \mathcal{C}_{\hat{S}}$.

Lastly, if $\phi = \mathrm{FM}_{\mathcal{E}}$, $(S_1, v_1, H_1) = (S, v, H)$ and $(S_2, v_2, H_2) = (S, \mathrm{FM}^{\mathrm{H}}_{\mathcal{E}}(v), H)$, with $\mathrm{NS}(S) = \langle e, f \rangle$ as in Lemma 4.2.5, by [Bri98, Theorem 5.3] we get

$\phi^{\mathrm{H}}(0,0,1)_0 = 0$. We therefore proceed to check condition (b), by using [Bri98, Theorem 5.3] again and [Yos01a, Section 3.2, (3.14)] for Mukai vectors of positive rank, which equals to

$$(\omega \cdot f)(e + l\frac{\omega^2}{2}f) \in \mathcal{C}_S,$$

for a suitable $l > 0$, which is satisfied, as $\omega \in \mathcal{C}_S$. $\square$

Proof of Proposition 4.3.19. As in the previous case, for $i = 1,2$, let $A_i := (S_i, v_i, H_i)$ be two objects in $\mathcal{G}^{m,k}_{\mathrm{def}}$ and let us show that the following diagram

$$\begin{tikzcd} v_1^\perp \arrow[r, "\lambda_{v_1}"] \arrow[d, "\Phi^{m,k}_{\mathrm{FM}}(\phi)"'] & \mathrm{H}^2(K_{v_1}(S_1,H_1),\mathbb{Z}) \arrow[d, "\mathsf{pt}^{m,k}_{\mathrm{FM}}(\phi)"] \\ v_2^\perp \arrow[r, "\lambda_{v_2}"] & \mathrm{H}^2(K_{v_1}(S_2,H_2),\mathbb{Z}). \end{tikzcd} \tag{4.16}$$

is commutative, for any elementary morphism $\phi \in \mathrm{Hom}_{\mathcal{G}^{m,k}_{\mathrm{def}}}(A_1, A_2)$, i.e. for $\phi = \mathsf{L}, \mathrm{FM}_{\mathcal{P}}, \mathrm{FM}^\vee_{\mathcal{P}}$ and $\mathrm{FM}_{\mathcal{E}}$.

If $\phi = \mathsf{L}, \mathrm{FM}_{\mathcal{P}}$ or $\mathrm{FM}_{\mathcal{E}}$, Corollary 4.3.21 (1) produces the equality

$$\lambda_{v_2}^{-1} \circ \phi_{v_1,*} \circ \lambda_{v_1} = \phi^{\mathrm{H}}_{|v_1^\perp},$$

which is exactly the commutativity condition for diagram (4.16), by Corollary 4.3.23 (1). If $\phi = \mathrm{FM}^\vee_{\mathcal{P}}$, Corollary 4.3.21 (2) provides the identity

$$\lambda_{v_2}^{-1} \circ \phi_{v_1,*} \circ \lambda_{v_1} = -\phi^{\mathrm{H}}_{|v_1^\perp},$$

which concludes the proof, as ϕ^{H} is orientation reversing by Corollary 4.3.23 (2). $\square$

By defining

$$\lambda\colon \Phi^{m,k} \longrightarrow \mathsf{pt}^{m,k}$$

as the unique natural transformation of functors restricting to λ_{def} on $\mathcal{G}^{m,k}_{\mathrm{def}}$ and to λ_{FM} on $\mathcal{G}^{m,k}_{\mathrm{FM}}$, Proposition 4.3.18 and Proposition 4.3.19 yield the following

Corollary 4.3.24. *For any $m,k \in \mathbb{N} \setminus \{0\}$, with $(m,k) \neq (1,1),(1,2)$, there exists an isomorphism of functors*

$$\lambda\colon \Phi^{m,k} \longrightarrow \mathsf{pt}^{m,k}.$$

Chapter 5

Towards a lattice-theoretic description of $\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H))$

In this Chapter we apply the machinery developed in the previous Chapter to approach a first lattice theoretic description of the locally trivial monodormy group of moduli spaces of the form $K_v(S,H)$, where (S,v,H) is an $(m,k)-$triple, with $(m,k) \neq (1,1),(1,2)$. In particular, we will show that the group $\mathsf{N}(v^\perp)$ introduced in Appendix B.4 embeds in $\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H))$, by showing that is generators can be described as morphisms in the image of the representations defined in Section 4.3. This inclusion will culminate in an equality in the next Chapter, as soon as $k>2$, providing an explicit lattice theoretic description of the monodromy group $\mathrm{Mon}^2(K_v(S,H))$.

In Section 5.1 we describe locally trivial monodromy operators induced by monodromy operators of the Abelian surface S as morphisms in the image of the representation $\widetilde{\Phi}^{m,k}_{\mathrm{def}}$ defined in Section 4.3.1. In Section 5.2 we include the group $\mathsf{N}(v^\perp)$, already appearing in Theorem 3.4.3, in $\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H))$ by showing that its generators belong to the image of the representation $\Phi^{m,k}$ and using the isomorphism of functors λ defined in Section 4.3.3, for any $m,k \in \mathbb{N}\setminus\{0\}$, with $(m,k) \neq (1,1),(1,2)$.

5.1 Locally trivial monodromy operators of surface type

In this Section we show that monodromy operators in projective families of polarized Abelian surfaces can be lifted to some moduli spaces of sheaves $K_v(S,H)$ on the same surface. We will translate the lifting process in terms of action of the representation $\widetilde{\Phi}^{m,k}_{\mathrm{def}}$ defined in Section 4.3.1, providing an injective morphism

$$\mathrm{Mon}^2(S) \longrightarrow \mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H))$$

of groups, under the isomorphism of functors λ. The locally trivial monodromy operators in the image of this map will be called *of surface type*.

5.1.1 The monodromy group of Abelian surfaces

We start by recalling that, as a consequence of [Shi78, Theorem 1 and 2], if S is an Abelian surface, then

$$\mathrm{Mon}^2(S) = \mathrm{SO}^+(\mathrm{H}^2(S,\mathbb{Z})), \tag{5.1}$$

where the latter is the group of orientation preserving isometries of $\mathrm{H}^2(S,\mathbb{Z})$ of determinant 1 (see Appendix B.1.1 and B.1.2). In [MR21, Section 3], a finer result is proven, showing that the monodromy group of an Abelian surface can be generated by monodromy operators in a finite number of projective families:

Theorem 5.1.1. *Let S be an Abelian surface, with $\mathrm{NS}(S) \simeq U$. There exist four ample classes $h_1, \ldots, h_4 \in \mathrm{NS}(S)$ such that*

$$\mathrm{Mon}^2(S) = \langle \mathrm{SO}^+(\mathrm{H}^2(S,\mathbb{Z}))_{h_i} : i = 0, \ldots, 4 \rangle. \tag{5.2}$$

Moreover, for any $i = 0, \ldots, 4$, there exists a projective family $f_i \colon : \mathcal{S}_i \to T_i$ of polarized Abelian surfaces such that

$$\mathrm{SO}^+(\mathrm{H}^2(S,\mathbb{Z}))_{h_i} = \mathrm{Mon}^2_{f_i}(S), \tag{5.3}$$

where the latter is the group generated by monodromy operators in the family f_i.

Proof. See [MR21, Corollary 3.4, Corollary 3.6]. □

Theorem 5.1.1 allows us to consider only monodromy operators in projective families, which is a requirement in order to induce deformations of moduli spaces of the form $K_v(S,H)$, using the theory introduced in the previous Chapters.

For later use, we give some more details concerning the projective families $f_i \colon \mathcal{S}_i \to T_i$ and the ample classes h_i, for $i = 0, \ldots, 4$, involved in Theorem 5.1.1. For a more precise discussion, we refer to [MR21, Section 3].

Remark 5.1.2. Let (S,h) be a *polarized Abelian surface of degree* $2d \in \mathbb{N} \setminus \{0\}$, i.e. a pair made of an Abelian surface S and an indivisible class $h \in \mathrm{H}^2(S,\mathbb{Z}) \cap \mathrm{H}^{1,1}(S)$ represented by an ample line bundle H on S such that $H^2 = 2d$. Any such polarized Abelian surface can be embedded in a fixed projective space $\mathbb{P}^N$ of dimension $N = 9d - 1$ by means of the linear system $|3H|$ associated to the ample line bundle $3H$. As a consequence, there exists a Zariski open subset T_{2d} of the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^N$ with Hilbert Polynomial $P(x) = 9dx^2$, such that, for any $t \in T_{2d}$, the corresponding subscheme $S_t \subseteq \mathbb{P}^N$ is an Abelian surface equipped with an ample line bundle H_t such that $(S_t, \mathrm{c}_1(H_t))$ is a polarized Abelian surface of degree $2d$. Furthermore, for any polarized Abelian surface (S,h) of degree $2d$ there exists $t \in T_{2d}$ such that $(S,h) \simeq (S_t, \mathrm{c}_1(H_t))$. By base change over T_{2d} of the universal family of the above mentioned Hilbert scheme, we get a smooth and proper family

$$f_{2d} \colon \mathcal{A}_{2d} \longrightarrow T_{2d}.$$

By [MR21, Remark 3.1], the base T_{2d} is a smooth, connected, quasi-projective variety of dimension $(N+1)^2+2$. Moreover, as the moduli space of polarized Abelian surfaces has dimension 3 (see [BL04, Remark 8.10.4]) and by its algebraic construction (see [BL04, Sections 8.7, 8.10]), we get that the ample line bundles H_t, for $t \in T_{2d}$, glue together in a relatively ample line bundle $\mathcal{H}_{2d}$ on $\mathcal{A}_{2d}$.

In light of Remark 5.1.2, we can explicitly describe the families of identity (5.3) in Theorem 5.1.1 as

$$f_i := f_{2d_i} \colon \mathcal{A}_{2d_i} \longrightarrow T_{2d_i},$$

where $H_i^2 = 2d_i$ for $i = 0, \dots, 4$.

In order to identify the ample classes $h_i \in \mathrm{NS}(S)$, we provide the following lattice theoretic result, which is precisely [MR21, Lemma 3.5] and differs from the latter only by a slight different choice of the classes defining the stabilizers involved.

Lemma 5.1.3. *Let U_1 denote one distinguished copy of the unimodular hyperbolic plane inside $U^{\oplus 3}$ and let $\{e_1, f_1\}$ denote its standard basis. Then, for any positive integers $r, s, p > 0$, the stabilizers of the classes*

$$h_1 = e_1 + r f_1, \quad h_2 = e_1 + (r-1) f_1, \quad h_3 = s e_1 + p f_1, \quad h_4 = (s-1) e_1 + p f_1$$

satisfy the following identity

$$\mathrm{SO}^+(U^{\oplus 3}) = \langle \mathrm{SO}^+(U^{\oplus 3})_{h_i} \colon i = 0, \dots, 4 \rangle. \tag{5.4}$$

Proof. The proof precisely parallels that of [MR21, Lemma 3.5], up to replacing the classes in identity (6) of that statement with h_i, for $i = 1, \dots, 4$, but we include it for the sake of completeness.

Let $L_2 \simeq U^{\oplus 2}$ be the orthogonal complement of U_1 in $U^{\oplus 3}$ and let us denote by $\{e_2, f_2, e_3, f_3\}$ its standard basis. Lemma B.5.3 (1) ensures that $\mathrm{SO}^+(U^{\oplus 3})$ is generated by $\mathrm{SO}^+(L_1)$ and the group $E_{U_1}(L_1)$ of Eichler's transvections (see Appendix B.5) defined in (B.8), hence it suffices to show that the latter are contained in the four stabilizers in identity (5.4).

The claim is straightforward for $\mathrm{SO}^+(L_1)$, which naturally embeds in $\mathrm{SO}^+(U^{\oplus 3})$ by extending each isometry by the identity on U_1. Hence, it remains to show that, for any $a \in L_1$, the Eichler transvections $t(e_1, a)$ and $t(f_1, a)$ belong to the group generated by the four stabilizers above.

By applying Lemma B.5.2 to L_1, we get that there exists an isometry $g \in \mathrm{SO}^+(L_1)$ sending a to an element of the second copy of U, spanned by e_2, f_2, which yields, by Remark B.5.1 (2),

$$t(e_1, g(a)) = t(g(e_1), g(a)) = g \circ t(e_1, a) \circ g^{-1}$$

and analogously for $t(f_1, a)$. Hence, we can proceed by proving the claim for $t(e_1, a)$ and $t(f_1, a)$, assuming that $a \in \langle e_2, f_2 \rangle$. Again by Remark B.5.1, these transvections act trivially on $\langle e_3, f_3\rangle$, hence can be considered as isometries of $\langle e_1, f_1, e_2, f_2\rangle$. By Lemma B.5.3 (2), the latter can be written as compositions of the Eichler transvections $t(e_2, e_1)$, $t(e_2, f_1)$, $t(f_2, e_1)$, $t(f_2, f_1)$. Let us consider the classes

$$h_1' := e_1 - rf_1, \quad h_2' := e_1 - (r-1)f_1, \quad h_3' := se_1 - pf_1, \quad \text{and} \quad h_4' := (s-1)e_1 - pf_1,$$

which are orthogonal to h_1, h_2, h_3 and h_4, respectively. Hence, by Remark B.5.1, the respective transvections satisfy $t(e, h_i') \in \mathrm{SO}^+(U^{\oplus 3})_{h_i}$, for any $e \in \langle e_2, f_2\rangle$. Moreover, by Remark B.5.1 (1), we get

$$\begin{aligned}
t(e_2, e_1) &= t(e_2, se_1 - pf_1 - (s-1)e_1 + pf_1) = t(e_2, h_3') \circ t(e_2, h_4')^{-1} \\
t(e_2, f_1) &= t(e_2, e_1 - rf_1 - e_1 + (r-1)f_1) = t(e_2, h_1') \circ t(e_2, h_2')^{-1} \\
t(f_2, e_1) &= t(f_2, h_3') \circ t(f_2, h_4')^{-1} \\
t(f_2, e_2) &= t(f_2, h_1') \circ t(f_2, h_2')^{-1},
\end{aligned}$$

proving the claim. □

Let us now assume that $S \simeq E \times F$, with E and F two elliptic curves, and $\mathrm{NS}(S) = \langle e, f\rangle \simeq U$, where e and f denote, respectively, the classes of E and F, as in Section 4.2.3. Let us consider three positive integers $r, s, p > 0$ and define the following ample classes

$$h_1 = e + rf, \quad h_2 = e + (r-1)f, \quad h_3 = se + pf, \quad h_4 = (s-1)e + pf. \tag{5.5}$$

Under these assumptions, Lemma 5.1.3 provides the identity

$$\mathrm{SO}^+(\mathrm{H}^2(S, \mathbb{Z})) = \langle \mathrm{SO}^+(\mathrm{H}^2(S, \mathbb{Z}))_{h_i} \colon i = 0, \ldots, 4\rangle. \tag{5.6}$$

Equalities (5.1) and (5.6) together yield equality (5.2) of Theorem 5.1.1.

Remark 5.1.4. We conclude by pointing out that, by [MR21, Corollary 3.4], equality (5.3) holds for any projective deformation $f \colon \mathcal{S} \to T$ of any polarized Abelian surface (S, h).

5.1.2 Lift of the monodromy of an Abelian surface to moduli spaces of sheaves

We will now show that the monodromy operators induced by the projective families introduced in the previous Subsection can be lifted to locally trivial monodromy operators on a moduli space $K_v(S, H)$ for a suitable $(m, k)-$triple (S, v, H). This will be proven in two special cases, which will be the only two needed in the next Section.

Remark 5.1.5. We recall that the monodromy group $\mathrm{Mon}^2(S)$ of an Abelian surface S is, by definition, a subgroup of $\mathrm{O}(\mathrm{H}^2(S,\mathbb{Z}))$, and that the latter can be identified as the subgroup of $\mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Z}))$ of isometries acting as the identity on $\mathrm{H}^0(S,\mathbb{Z}) \oplus \mathrm{H}^4(S,\mathbb{Z})$. Another remarkable subgroup of $\mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Z}))$ is given by the following: let (S,v,H) be an $(m,k)-$triple and let us consider the homomorphism of groups

$$\widetilde{\Phi}^{m,k}_{\mathrm{def},(S,v,H)} \colon \mathrm{Aut}_{\mathcal{G}^{m,k}_{\mathrm{def}}}(S,v,H) \longrightarrow \mathrm{Aut}_{\widetilde{\mathcal{H}}^{m,k}}(\widetilde{\mathrm{H}}(S,\mathbb{Z}),v,\epsilon_S)$$

induced by the action of the representation $\widetilde{\Phi}^{m,k}_{\mathrm{def}}$ on the respective isotropy groups (see Definition 4.0.1 (1)). By Definition 4.3.3 we have an inclusion of groups

$$\mathrm{Im}(\widetilde{\Phi}^{m,k}_{\mathrm{def},(S,v,H)}) \subseteq \mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Z}))_v,$$

where the latter is the subgroup of isometries fixing the vector v.

On the other hand, provided that $(m,k) \neq (1,1),(1,2)$, by Corollary 4.3.16, the representation $\mathsf{pt}^{m,k}_{\mathrm{def},(S,v,H)}$ provides us an inclusion of groups

$$\mathrm{Im}(\mathsf{pt}^{m,k}_{\mathrm{def},(S,v,H)}) \subseteq \mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H)).$$

The interplay between the two representations above, studied in Section 4.3.3, will allow us to compare the groups $\mathrm{Mon}^2(S)$ and $\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H))$.

Lemma 5.1.6. *Let S be an Abelian surface and let $v = (m,0,-mk)$ be a Mukai vector, with $m \geq 1$. Let $h \in \mathrm{NS}(S)$ be the class of a $v-$generic polarization on S and $f \colon \mathcal{S} \to T$ a projective deformation of the polarized Abelian surface (S,h) as in Remark 5.1.2. Then there exists a $v-$generic polarization H on S such that $h = \mathrm{c}_1(H)$ and*

$$\mathrm{Mon}^2_f(S) \subseteq \mathrm{Im}(\widetilde{\Phi}^{m,k}_{\mathrm{def},(S,v,H)}).$$

Proof. The claim will follow as soon as we show that the smooth and projective deformation $f \colon \mathcal{S} \to T$ can be used to produce a deformation path centered in the $(m,k)-$triple (S,v,H), with H a suitable $v-$generic polarization. Notice that, as v is of the form $(m,0,-mk)$, it shall be constant along the deformation.

By Remark 5.1.2, there exists an ample line bundle $\mathcal{H}$ on $\mathcal{S}$ such that, for every $t \in T$, its restriction $H_t := \mathcal{H}_t$ is an ample line bundle on $\mathcal{S}_t$ such that $(\mathcal{S}_t, \mathrm{c}_1(H_t))$ is a polarized Abelian surface of degree h^2. Moreover, there exists a point t_0 such that $(\mathcal{S}_{t_0}, \mathrm{c}_1(H_{t_0})) \simeq (S,h)$. Set $H := H_{t_0}$ and observe that, as the notion of $v-$genericity of a polarization only depends on its first Chern class, the ample line bundle H is $v-$generic. Hence, by Remark 4.1.2, the points $t \in T$ such that H_t is not $v-$generic define a Zariski closed proper subset Z of T. As in the proof of [OPR24, Lemma 3.6], we get that for any loop γ in T centered in t_0, there exists a loop γ' in $T' := T \setminus Z$ centered in t_0 and homotopic to γ. Hence, up to replacing T with T', we can assume that $\mathcal{H}_t$ is $v-$generic for every $t \in T$.

Consequently, if $g \in \mathrm{Mon}^2_f(S)$ is the monodromy operator in the family f along a path γ centered in t_0, then the deformation path

$$\alpha = (f\colon \mathcal{S} \to T, \mathcal{O}_\mathcal{S}, \mathcal{H}, t_0, t_0, \gamma)$$

defines a a morphism $\overline{\alpha}$ in $\mathcal{G}^{m,k}_{\mathrm{def}}$ such that $(\mathrm{id}, g, \mathrm{id}) = \widetilde{\Phi}^{m,k}_{\mathrm{def},(S,v,H)}(\overline{\alpha})$, concluding the proof. □

For later use, we provide an analogue of Lemma 5.1.6 for another special class of $(m,k)-$triples.

Lemma 5.1.7. *Let (S,h) be a polarized Abelian surface such that $\mathrm{NS}(S) = \mathbb{Z}h$, let $v = (r, mh, a)$ be a Mukai vector, and let $f\colon \mathcal{S} \to T$ be a projective deformation of (S,h) as in Remark 5.1.2. Then there exists a $v-$generic polarization H on S such that $h = \mathrm{c}_1(H)$ and*

$$\mathrm{Mon}^2_f(S) \subseteq \mathrm{Im}(\widetilde{\Phi}^{m,k}_{\mathrm{def},(S,v,H)}).$$

Proof. The proof runs exactly as the proof of Lemma 5.1.6, by using the deformation path $\alpha = (f\colon \mathcal{S} \to T, \mathcal{H}^{\otimes m}, \mathcal{H}, \gamma, t_0, t_0)$. □

Corollary 5.1.8. *Let (S,v,H) be an $(m,k)-$triple, with $(m,k) \neq (1,1),(1,2)$, and $f\colon \mathcal{S} \to T$ be a projective family as in Lemma 5.1.6 or in Lemma 5.1.7. Then there exists an injective morphism*

$$\mathrm{Mon}^2_f(S) \hookrightarrow \mathrm{Im}(\mathrm{pt}^{m,k}_{\mathrm{def},(S,v,H)}) \subseteq \mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H)).$$

Proof. Lemma 5.1.6 and Lemma 5.1.7 provide an inclusion of groups

$$\mathrm{Mon}^2_f(S) \subseteq \mathrm{Im}(\widetilde{\Phi}^{m,k}_{\mathrm{def},(S,v,H)}).$$

As the latter is made of orientation preserving isometries by Remark 4.3.6 (2), the morphism $\Psi^{m,k}$ defined in 4.3.17 provides an injective morphism

$$\begin{aligned} \mathrm{Mon}^2_f(S) &\hookrightarrow \mathrm{Im}(\Phi^{m,k}_{\mathrm{def},(S,v,H)}) \\ g &\mapsto (\mathrm{id}, g, \mathrm{id})_{|v^\perp}. \end{aligned}$$

The claim follows by applying the isomorphism of functors $\lambda_{\mathrm{def}}\colon \Phi^{m,k}_{\mathrm{def}} \to \mathrm{pt}^{m,k}_{\mathrm{def}}$ defined in Proposition 4.3.18. □

We conclude the Section by showing that, in the case in which S is the product of two elliptic curves, the classes $h_1, \ldots, h_4$ defined in (5.5) and generating $\mathrm{Mon}^2(S)$ (see Theorem 5.1.1) can be chosen to belong to the same $v-$chamber (see Section 3.1.4). As a consequence, we get that only one $(m,k)-$triple (S,v,H) is needed to generate the whole monodromy group $\mathrm{Mon}^2(S)$ using morphisms in the image of $\widetilde{\Phi}^{m,k}_{\mathrm{def},(S,v,H)}$.

In the following, we will assume that $S \simeq E \times F$, with E and F two elliptic curves, and $\mathrm{NS}(S) \simeq \langle e, f \rangle = U$, where e and f are, respectively, the classes of E and F in $\mathrm{NS}(S)$, as in Section 4.2.3.

Proposition 5.1.9. *Let $S \simeq E \times F$ be the product of two elliptic curves E, F and suppose that $\mathrm{NS}(S) \simeq \langle e, f \rangle$, with $e = \mathrm{c}_1(E)$ and $f = \mathrm{c}_1(F)$, and let $v = (m, 0, -mk)$, with $m, k \geq 1$. Then there exists an integer $t \gg 0$ and a $v-$generic polarization H such that $\mathrm{c}_1(H) = e + tf$ and*

$$\mathrm{Mon}^2(S) \subseteq \mathrm{Im}(\widetilde{\Phi}^{m,k}_{\mathrm{def},(S,v,H)}).$$

Proof. By Theorem 5.1.1, there exist four ample classes $h_1, \ldots, h_4$ on S such that

$$\mathrm{Mon}^2(S) = \langle \mathrm{Mon}^2_{f_1}(S), \ldots, \mathrm{Mon}^2_{f_4}(S) \rangle,$$

where $f_i := f_{h_i^2}$ are projective deformations of the polarized Abelian surfaces (S, h_i), for $i = 1, \ldots, 4$, as in Remark 5.1.2, and by Lemma 5.1.6, for any $i = 1, \ldots, 4$, we have an inclusion of groups

$$\mathrm{Mon}^2_{f_i}(S) \subseteq \mathrm{Im}(\widetilde{\Phi}^{m,k}_{\mathrm{def},(S,v,H_i)}),$$

where H_i is a $v-$generic polarization such that $\mathrm{c}_1(H_i) = h_i$. As in (5.5), we can choose three positive integers $r, s, p > 0$ such that

$$h_1 = e + rf, \quad h_2 = e + (r-1)f, \quad h_3 = se + pf, \quad h_4 = (s-1)e + pf.$$

By Lemma 2.38 (see also Definition 2.37) of [PR23], if we choose those integers such that r and the quotient p/s are big enough, then all the classes h_i belong to the unique $v-$chamber whose closure contains the class f. We can then choose a polarization H on S such that its class $h := \mathrm{c}_1(H)$ lies in that $v-$chamber. Consequently, the $(m, k)-$triple (S, v, H) is congruent to (S, v, H_i) for every $i = 1, \ldots, 4$ and we get a collection of isomorphisms

$$\chi^{\sharp}_{H,H_i} \colon \mathrm{Aut}_{\mathcal{G}^{m,k}_{\mathrm{def}}}(S, v, H) \longrightarrow \mathrm{Aut}_{\mathcal{G}^{m,k}_{\mathrm{def}}}(S, v, H_i)$$

that induce, via $\widetilde{\Phi}^{m,k}_{\mathrm{def}}$, the following identifications

$$\mathrm{Im}(\widetilde{\Phi}^{m,k}_{\mathrm{def},(S,v,H_i)}) = \widetilde{\Phi}^{m,k}_{\mathrm{def}}(\chi_{H,H_i})^{\sharp}(\mathrm{Im}(\widetilde{\Phi}^{m,k}_{\mathrm{def},(S,v,H)})) = \mathrm{Im}(\widetilde{\Phi}^{m,k}_{\mathrm{def},(S,v,H)}),$$

where the last equality follows from the fact that $\widetilde{\Phi}^{m,k}_{\mathrm{def}}(\chi_{H,H_i})$ is the identity morphism. □

By combining Proposition 5.1.9 and Corollary 5.1.8, we get the following

Corollary 5.1.10. *Let $S \simeq E \times F$ be the product of two elliptic curves E, F and suppose that $\mathrm{NS}(S) \simeq \langle e, f \rangle$, with $e = \mathrm{c}_1(E)$ and $f = \mathrm{c}_1(F)$, and let $v = (m, 0, -mk)$, with $m, k \geq 1$, and suppose $(m, k) \neq (1, 1), (1, 2)$. If H is a $v-$generic polarization on S whose class is contained in the unique $v-$chamber whose closure contains the class f, then there is an injective morphism of groups*

$$\mathrm{Mon}^2(S) \hookrightarrow \mathrm{Mon}^2_{\mathrm{lt}}(K_v(S, H)).$$

5.2 The group $\mathsf{N}(v^{\perp})$ as subgroup of $\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H))$

In this Section we will include the group $\mathsf{N}(K_v(S,H))$, defined in Appendix B.4, in the locally trivial monodromy group $\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H))$, by showing that its generators belong to the image of the groupoid representations defined in Section 4.3.

We start by recalling that, by Theorem 3.4.3, if (S,w,H) is a $(1,k)-$triple with $k>2$, then

$$\mathrm{Mon}^2(K_w(S,H)) = \mathsf{N}(K_w(S,H)) \simeq \mathsf{N}(w^{\perp}),$$

where the last isomorphism is given by conjugation via the isometry λ_w of Theorem 3.3.4. The main result of this section is the following

Theorem 5.2.1. *Let (S,v,H) be an $(m,k)-$triple, with $(m,k)\neq(1,1),(1,2)$. Then*

$$\mathsf{N}(K_v(S,H)) \subseteq \mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H)).$$

The proof of Theorem 5.2.1 will be addressed by showing that the generators of the group $\mathsf{N}(K_v(S,H))$ are isometries belonging to the image of the group morphism $\mathsf{pt}^{m,k}_{(S,v,H)}\colon \mathrm{Aut}_{\mathcal{G}^{m,k}}(S,v,H)\to\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H))$ (see Corollary 4.3.16), induced by the representation $\mathsf{pt}^{m,k}$ defined in Section 4.3.2, for a suitable $(m,k)-$triple (S,v,H). As the Hodge isometry $\lambda_v\colon v^{\perp}\to \mathrm{H}^2(K_v(S,H),\mathbb{Z})$ naturally conjugates $\mathsf{N}(v^{\perp})$ to $\mathsf{N}(K_v(S,H))$, the proof of Theorem 5.2.1 reduces to showing the following inclusion of groups:

$$\mathsf{N}(v^{\perp}) \subseteq \mathrm{Im}(\Phi^{m,k}_{(S,v,H)}). \tag{5.7}$$

Paralleling Markman's proof of [Mar22, Theorem 1.4] in the primitive case, up to conjugation with a morphism $\Phi^{m,k}(\eta)$, for any $\eta\in\mathrm{Hom}_{\mathcal{G}^{m,k}}((S,v,H),(S_1,v_1,H_1))$ and $(S_1,v_1,H_1)\in\mathcal{G}^{m,k}$, we can assume without loss of generality that $(S,v,H)\in\mathcal{G}^{m,k}$ is an $(m,k)-$triple with Mukai vector $v=m(1,0,-k)$.

Remark 5.2.2. (1) As recalled in Remark B.4.1 (2), if $k\geq 2$, the group $\mathsf{N}(v^{\perp})$ is generated by $\mathrm{SO}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))_v$ and by the involution $R_s\circ R_{s_1}\colon v^{\perp}\to v^{\perp}$ given by the composition of the two reflections around the $(-2)-$vector $s=(1,0,1)$ and the $(+2)-$vector $s_1=(1,0,-1)$, which sends the vector $m(1,0,k)$ to its opposite and acts as the identity on $\lambda_v(\mathrm{H}^2(S,\mathbb{Z}))$. More precisely, as its action on the discriminant $A_{v^{\perp}}$ is trivial (see Corollary B.2.3), the latter corresponds to the restriction of the product $R_s\circ R_{s_1}\colon\widetilde{\mathrm{H}}(S,\mathbb{Z})\to\widetilde{\mathrm{H}}(S,\mathbb{Z})$ of the two above-defined reflections on $\widetilde{\mathrm{H}}(S,\mathbb{Z})$, whose action is precisely $-D^{\mathrm{H}}_S$, where D^{H}_S is the cohomological action of duality defined in (4.7), and which belongs to $\mathrm{O}(\widetilde{\mathrm{H}}(S,\mathbb{Z}))_v$ by definition.

(2) On the other hand, if $k=1$, the group $\mathsf{N}(v^{\perp})$ is isomorphic to $\mathrm{SO}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))_v$, by Remark B.4.1 (3).

Inclusion (5.7) will be proven on each of the generators described in Remark 5.2.2. The first and most technical part is given by the following:

Proposition 5.2.3. *Let m, k be two positive integers, with $(m, k) \neq (1, 1), (1, 2)$, and let (S, v, H) an $(m, k)-$triple, with $S = E \times F$ the product of two elliptic curves E and F, $NS(S) \simeq \langle e, f \rangle$, with $e := c_1(E)$ and $f := c_1(F)$, $v = m(1, 0, -k)$ and $h := c_1(H) = e + tf$, with $t \gg 0$, as in Proposition 5.1.9. Then*

$$\mathrm{SO}^+(\widetilde{\mathrm{H}}(S, \mathbb{Z}))_v \subseteq \mathrm{Im}(\Phi^{m,k}_{(S,v,H)}).$$

For this purpose, we introduce the following result:

Proposition 5.2.4. *Let $S = E \times F$ be the product of two elliptic curves E and F, with $NS(S) \simeq \langle e, f \rangle$, where $e := c_1(E)$ and $f := c_1(F)$. Let $v = m(1, 0, -k) \in \widetilde{\mathrm{H}}(S, \mathbb{Z})$ a Mukai vector, with m and k two positive integers. The stabilizer $\mathrm{SO}^+(\widetilde{\mathrm{H}}(S, \mathbb{Z}))_v$ is generated by $\mathrm{SO}^+(\mathrm{H}^2(S, \mathbb{Z}))$ and the products $R_{t_1} \circ R_{t_2}$ of reflections around $(-2)-$vectors $t_i = (1, \beta_i - f, k)$, with $\beta_i \in \mathrm{NS}(S)^\perp$ a primitive class, for $i = 1, 2$.*

Let m, k be two positive integers. The starting point for the proof of Proposition 5.2.4 is the following

Lemma 5.2.5. *[Mar22, Lemma 5.3 (1), (3), Lemma 5.4] Let S be an Abelian surface and let $v = m(1, 0, -k) \in \widetilde{\mathrm{H}}(S, \mathbb{Z})$ a Mukai vector. The stabilizer $\mathrm{SO}^+(\widetilde{\mathrm{H}}(S, \mathbb{Z}))_v$ is generated by $\mathrm{SO}^+(\mathrm{H}^2(S, \mathbb{Z}))$ and the products $R_{u_1} \circ R_{u_2}$ of reflections around $(-2)-$vectors $u_i = (1, \xi_i, k)$, with $\xi_i \in \mathrm{H}^2(S, \mathbb{Z})$ a primitive class, for $i = 1, 2$.*

The first step consists of the following reduction, that allows us to assume that the primitive elements $\xi_1, \xi_2 \in \mathrm{H}^2(S, \mathbb{Z})$ in Lemma 5.2.5 span a primitive sublattice of $\mathrm{H}^2(S, \mathbb{Z})$.

Lemma 5.2.6. *Let $\xi_1, \xi_2 \in U^{\oplus 3}$ be two primitive elements and let $l \in \mathbb{Z}$. Then there exists a primitive element $\xi \in U^{\oplus 3}$ such that $\xi^2 = 2l$ and such that $\langle \xi_1, \xi \rangle$ and $\langle \xi_2, \xi \rangle$ are two primitive sublattices of $U^{\oplus 3}$.*

Proof. Let $S := \overline{\langle \xi_1, \xi_2 \rangle}$ be the saturation of $\langle \xi_1, \xi_2 \rangle$ in $U^{\oplus 3}$ (see Appendix B.2). As $\mathrm{rank}(S) = 2$, by Proposition B.2.5, it admits a primitive embedding in $U^{\oplus 3}$ that is unique up to isometry of L, and, by Corollary B.2.6, we can assume, up to isometry of $U^{\oplus 3}$, that $\xi_1, \xi_2 \in U^{\oplus 2}$. Let us denote by U_1 the copy of the hyperbolic plane U that is orthogonal to S, and let us denote by e_1, f_1 the standard basis of the hyperbolic plane U_1, so that $e_1^2 = f_1^2 = 0$ and $e_1 \cdot f_1 = 1$.

Set $\xi := e_1 + lf_1$ and notice that $\xi^2 = 2l$. Moreover, both the lattices $\langle \xi_1, \xi \rangle$ and $\langle \xi_2, \xi \rangle$ are primitive in $U^{\oplus 3}$. Indeed, suppose that there exist two rational numbers $\lambda, \mu \in \mathbb{Q}$ such that $v := \lambda \xi + \mu \xi_1 \in U^{\oplus 3}$. Then, $v \cdot f_1 = \lambda \xi \cdot f_1 = \lambda \in \mathbb{Z}$, so that $\mu \xi_1 = v - \lambda \xi \in U^{\oplus 3}$, from which we deduce that $\mu \in \mathbb{Z}$, since ξ_1 is primitive. Hence, $\langle \xi_1, \xi \rangle$ is primitive, and the same computation applies to show that $\langle \xi_2, \xi \rangle$ is primitive as well. $\square$

As straightforward application of Lemma 5.2.5 and Lemma 5.2.6, we get the following.

Corollary 5.2.7. *Let S be an Abelian surface and let $v = m(1,0,-k) \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$ a Mukai vector. The stabilizer $\mathrm{SO}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))_v$ is generated by $\mathrm{SO}^+(\mathrm{H}^2(S,\mathbb{Z}))$ and the products $R_{u_1} \circ R_{u_2}$ of reflections around $(-2)-$vectors $u_i = (1,\xi_i,k)$, with $\xi_i \in \mathrm{H}^2(S,\mathbb{Z})$ a primitive class, for $i = 1,2$, such that $\langle \xi_1, \xi_2 \rangle$ is primitive.*

Proof. As $R_u^2 = \mathrm{id}$ for any reflection R_u around any $(-2)-$vector u, we can write

$$R_{u_1} \circ R_{u_2} = R_{u_1} \circ R_u \circ R_u \circ R_{u_2}.$$

By Lemma 5.2.6, we can choose $u = (1,\xi,k) \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$ such that $\xi^2 = 2k-2$ and such that $\langle \xi_1, \xi \rangle$ and $\langle \xi_2, \xi \rangle$ are primitive. □

The second technical step refining Lemma 5.2.5 is given by the following result.

Lemma 5.2.8. *Let $S = E \times F$ be the product of two elliptic curves E and F and assume that $NS(S) \simeq \langle e, f \rangle$, with $e := \mathrm{c}_1(E)$ and $f := \mathrm{c}_1(F)$. Let $\xi_1, \xi_2 \in \mathrm{H}^2(S,\mathbb{Z})$ be two primitive elements such that $\xi_1^2 = \xi_2^2 = 2k-2$ and such that $\langle \xi_1, \xi_2 \rangle$ is a primitive sublattice of $\mathrm{H}^2(S,\mathbb{Z})$. Then there exist two primitive elements $\beta_1, \beta_2 \in \mathrm{NS}(S)^\perp$ and an isometry $g \in \mathrm{SO}^+(\mathrm{H}^2(S,\mathbb{Z}))$ such that $g(\xi_i) = \beta_i - f$, for $i = 1,2$.*

Proof. Let us denote $U_1 := \mathrm{NS}(S) = \langle e, f \rangle$, $L := \mathrm{H}^2(S,\mathbb{Z}) \simeq U^{\oplus 3} = U_1 \overset{\perp}{\oplus} U^{\oplus 2}$ and $S_1 := \langle \xi_1, \xi_2 \rangle$. By Corollary B.2.6, there exists a primitive embedding of S_1 in L such that the orthogonal complement $S_1^\perp$ of S_1 in L contains U_1, and that embedding is unique up to isometry of L. Hence, there exists an isometry $\varphi_1 \in \mathrm{O}(L)$ such that $\varphi_1(\xi_i) \in U_1^\perp$, for $i = 1,2$.

Let us set $\beta_i := \varphi_1(\xi_i)$, for $i = 1,2$, and $S_2 := \langle \beta_1 - f, \beta_2 - f \rangle$. Then, since S_1 is primitive, S_2 is primitive. Indeed, if there exist two rational numbers $\lambda, \mu \in \mathbb{Q}$ such that $v := \lambda(\beta_1 - f) + \mu(\beta_2 - f) \in L$, then $v = \lambda\beta_1 + \mu\beta_2 - (\lambda+\mu) f \in L$. But β_1, β_2 and f are $\mathbb{Q}-$linearly independent, hence $\lambda\beta_1 + \mu\beta_2 \in L$. Primitivity of S_1 implies primitivity of $\langle \beta_1, \beta_2 \rangle$, so that $\lambda, \mu \in \mathbb{Z}$.

Moreover, S_1 and S_2 are isometric as abstract lattices, and by Proposition B.2.5 their embedding in L is unique up to isometry of L, so there exists an isometry $\varphi \in \mathrm{O}(L)$ such that $\varphi(\xi_i) = \beta_i - f$ for $i = 1,2$.

More explicitly, the isometry sending ξ_i to $\varphi_1(\xi) - f = \beta_i$, for $i = 1,2$, provides a primitive embedding of S_1 in L whose image is S_2. Such an embedding is unique up to isometry of L, so the latter extends to an isometry $\varphi \in \mathrm{O}(L)$.

Determinant: Let us denote by K the orthogonal complement of S_2 in L, which, by Corollary B.2.6, contains a copy of the hyperbolic plane U, and let K_1 be the orthogonal complement of U in K.

We consider the isometry $\eta \in \mathrm{O}(K)$ interchanging the two isotropic generators of U and acting as the identity on K_1. We notice that $\det(\eta) = -1$ and $\mathrm{disc}(\eta) = \mathrm{id}_{A_K}$. Hence, by Corollary B.2.3, we can extend η to $\widetilde{\eta} := \eta \oplus \mathrm{id}_{S_2} \in \mathrm{O}(L)$ satisfying $\det(\widetilde{\eta}) = \det(\eta) = -1$. Consequently, if $\det(\varphi) = -1$, we can replace φ with $\widetilde{\eta} \circ \varphi \in \mathrm{SO}(L)$.

Orientation: In conclusion, we can work in a similar fashion to make φ orientation preserving. Again by Corollary B.2.6, we can orthogonally decompose $K := S_2^{\perp} = U \oplus K_1$ and we can define an isometry $\theta := -\mathrm{id}_U \oplus \mathrm{id}_{K_1} \in \mathrm{O}(K)$ such that $\mathrm{or}(\theta) = 1$, as $sgn(U) = (1,1)$, $\det(\theta) = 1$, as $\mathrm{rk}(U) = 2$, and $\mathrm{disc}(\theta) = \mathrm{id}_{A_K}$. By Corollary B.2.3, we can extend θ to $\widetilde{\theta} := \theta \oplus \mathrm{id}_{S_2}$ such that $\mathrm{or}(\widetilde{\theta}) = \mathrm{or}(\theta) = 1$ and $\det(\widetilde{\theta}) = \det(\theta) = 1$. Hence, the claim follows by setting $g := \widetilde{\theta}^{\mathrm{or}(\varphi)} \circ \varphi \in \mathrm{SO}^+(L)$. □

The proof of Proposition 5.2.4 now reduces to a combined application of Corollary 5.2.7 and Lemma 5.2.8.

Proof of Proposition 5.2.4. In order to prove the claim, we need to show that any product $R_{u_1} \circ R_{u_2}$ of reflections around $(-2)-$vectors $u_i = (1, \xi_i, k)$, with $\xi_i \in \mathrm{H}^2(S,\mathbb{Z})$ a primitive class, for $i = 1,2$, generating a primitive lattice $\langle \xi_1, \xi_2 \rangle$ as in Corollary 5.2.7 can be conjugated into the product $R_{t_1} \circ R_{t_2}$ of reflections around $(-2)-$vectors $t_i = (1, \beta_i - f, k)$, with $\beta_i \in \mathrm{NS}(S)^{\perp}$ a primitive class, for $i = 1,2$, as in Proposition 5.2.4, via an isometry $\widetilde{g} \in SO^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))_v$ defined as $\widetilde{g} := \mathrm{id}_{\mathrm{H}^0(S,\mathbb{Z})} \oplus g \oplus \mathrm{id}_{\mathrm{H}^4(S,\mathbb{Z})}$, with $g \in \mathrm{SO}^+(\mathrm{H}^2(S,\mathbb{Z}))$.

Let $u_i = (1, \xi_i, k)$, with $\xi_i \in \mathrm{H}^2(S,\mathbb{Z})$ primitive, for $i = 1,2$, be two $(-2)-$vectors as above, so that $\xi_i^2 = 2k-2$, for $i = 1,2$, and $\langle \xi_1, \xi_2 \rangle$ is primitive. By Lemma 5.2.8, there exist two elements $\beta_1, \beta_2 \in \mathrm{NS}(S)^{\perp}$ and an isometry $g \in \mathrm{SO}^+(\mathrm{H}^2(S,\mathbb{Z}))$ such that $g(\xi_i) = \beta_i - f$. By extending g to $\widetilde{g} \in SO^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))_v$ via the identity as above, we get $\widetilde{g}(u_i) = (1, \beta_i - f, k) =: t_i$ for $i = 1,2$, and a straightforward application of the identity

$$\widetilde{g} \circ R_u \circ \widetilde{g}^{-1} = R_{\widetilde{g}(u)} \qquad \text{for any } u \in \widetilde{\mathrm{H}}(S,\mathbb{Z}) \tag{5.8}$$

leads to

$$\widetilde{g} \circ R_{u_1} \circ R_{u_2} \circ \widetilde{g}^{-1} = R_{\widetilde{g}(u_1)} \circ R_{\widetilde{g}(u_2)} = R_{t_1} \circ R_{t_2},$$

concluding the proof. □

The last result allows us to reduce the first part of the proof of Proposition 5.2.3 to the following

Lemma 5.2.9. *Let m and k be two positive integers, let (S, v, H) be an $(m,k)-$triple as above, with $\mathrm{c}_1(H) = e + tf$, with $t \gg 0$, and let $t_i = (1, \xi_i, k) \in \widetilde{\mathrm{H}}(S,\mathbb{Z})$ be a $(-2)-$vector such that $\xi_i = \beta_i - f$, with $\beta_i \in \langle e, f \rangle^{\perp}$ for $i = 1,2$. Then*

$$R_{t_1} \circ R_{t_2} \in \mathrm{Im}(\widetilde{\Phi}^{m,k}_{(S,v,H)}).$$

Proof. Under the current assumptions, Lemma 4.2.5 guarantees that the derived equivalence $\mathrm{FM}_{\mathcal{E}}$ introduced in Section 4.2.3 induces a morphism

$$\mathrm{FM}_{\mathcal{E}} \in \mathrm{Hom}_{\mathcal{G}^{m,k}_{\mathrm{FM}}}((S, v, H), (S, \overline{v}, H)),$$

where we set $\overline{v} := (0, m(e+kf), m)$. Arguing verbatim as in the proof of [MR21, Proposition 4.9], we get a description of the cohomological action of $\mathrm{FM}_{\mathcal{E}}$ on the whole Mukai lattice $\widetilde{\mathrm{H}}(S,\mathbb{Z})$. Consequently, for $i = 1,2$, it holds

$$\mathrm{FM}^{\mathrm{H}}_{\mathcal{E}}(t_i) = \mathrm{FM}^{\mathrm{H}}_{\mathcal{E}}(1, \beta_i - f, k) = (0, e - kf - \beta_i, 0).$$

Setting $a_i := e - kf - \beta_i$, for $i = 1,2$, by identity (5.8) we get

$$R_{t_1} \circ R_{t_2} = (\mathrm{FM}^{H}_{\mathcal{E}})^{-1} \circ R_{(0,a_1,0)} \circ R_{(0,a_2,0)} \circ \mathrm{FM}^{\mathrm{H}}_{\mathcal{E}}.$$

Hence, we can reduce the proof to showing that

$$R_{(0,a_1,0)} \circ R_{(0,a_2,0)} \in \mathrm{Im}(\widetilde{\Phi}^{m,k}_{(S,\overline{v},H)}). \tag{5.9}$$

Let us consider a deformation path $\alpha = (f\colon \mathcal{S} \to T, \mathcal{L}, \mathcal{H}, t_1, t_2, \gamma)$, where

$$(\mathcal{S}_{t_1}, \mathcal{L}_{t_1}, \mathcal{H}_{t_1}) = (S, m(e+kf), H) \text{ and } (\mathcal{S}_{t_2}, \mathcal{L}_{t_2}, \mathcal{H}_{t_2}) = (S', mh', H'),$$

with $H' \in \mathrm{Pic}(S')$, $h' = \mathrm{c}_1(H')$ and $\mathrm{NS}(S') = \mathbb{Z}h'$, and set $v' := v_{t_2}$. Then, up to conjugation with $\widetilde{\Phi}^{m,k}_{\mathrm{def}}(\overline{\alpha})$, the statement in (5.9) is equivalent to

$$R_{(0,b_1,0)} \circ R_{(0,b_2,0)} \in \mathrm{Im}(\widetilde{\Phi}^{m,k}_{(S',v',H')}), \tag{5.10}$$

where $(0, b_i, 0) = \widetilde{\Phi}^{m,k}_{\mathrm{def}}(\overline{\alpha})(0, a_i, 0)$ - as $\widetilde{\Phi}^{m,k}_{\mathrm{def}}(\overline{\alpha})$ is a graded isomorphism of groups by definition - for $i = 1,2$. Moreover, the latter satisfies the following properties:

(1) $b_i^2 = a_i^2 = t_i^2 = -2$;

(2) $b_i \cdot h' = t_i \cdot v = 0$

for $i = 1,2$. Hence, by identifying $R_{(0,b_i,0)} = \mathrm{id}_{\mathrm{H}^0} \oplus R_{b_i} \oplus \mathrm{id}_{\mathrm{H}^4}$ with the reflection R_{b_i} in $\mathrm{H}^2(S',\mathbb{Z})$ under the natural embedding $\mathrm{O}(\mathrm{H}^2(S',\mathbb{Z})) \hookrightarrow \mathrm{O}(\widetilde{\mathrm{H}}(S',\mathbb{Z}))$, we get that $R_{b_1} \circ R_{b_2} \in \mathrm{SO}^+(\mathrm{H}^2(S',\mathbb{Z}))$, by point (1) and equality (B.3), and that

$$R_{b_1} \circ R_{b_2} \in \mathrm{SO}^+(\mathrm{H}^2(S',\mathbb{Z}))_{h'},$$

by point (2). Hence, by Remark 5.1.4 and Lemma 5.1.7, we deduce

$$R_{b_1} \circ R_{b_2} \in \mathrm{Im}(\widetilde{\Phi}^{m,k}_{\mathrm{def},(S',v',H')}),$$

proving claim (5.10) and concluding the proof. □

We are finally in the position to quickly address the proof of Proposition 5.2.3.

Proof of Proposition 5.2.3. In order to show that $\mathrm{SO}^+(\widetilde{\mathrm{H}}(S,\mathbb{Z}))_v \subseteq \mathrm{Im}(\Phi^{m,k}_{(S,v,H)})$, by Lemma 5.2.5 and Proposition 5.2.4 it is enough to prove the inclusion $\mathrm{SO}^+(\mathrm{H}^2(S,\mathbb{Z})) \subseteq \mathrm{Im}(\Phi^{m,k}_{(S,v,H)})$ and that $R_{t_1} \circ R_{t_2} \in \mathrm{Im}(\Phi^{m,k}_{(S,v,H)})$ for any $t_i = (1, \beta_i - f, k)$, with $\beta_i \in \langle e, f\rangle^{\perp}$, as in Proposition 5.2.4. The first assertion follows from Proposition

5.1.9 and the fact that the representation $\Psi^{m,k}$ acts trivially - namely, just as restriction to $v^{\perp}$ - on orientation preserving isometries. The second part of the claim follows analogously from Lemma 5.2.9. More precisely, the latter shows that the product of such reflections can be written as a composition of orientation preserving isometries (see Corollary 4.3.23 (1)) of the form $\widetilde{\Phi}^{m,k}(\eta)$ for a suitable morphism $\eta \in \mathrm{Hom}_{\mathcal{G}^{m,k}}((S_1, v_1, H_1), (S_2, v_2, H_2))$, so the representation $\Psi^{m,k}$ acts trivially on $R_{t_1} \circ R_{t_2}$ again, providing an element of $\mathrm{Im}(\Phi^{m,k}_{(S,v,H)})$. □

By Proposition 5.2.3 and Remark 5.2.2 we immediately deduce the following:

Corollary 5.2.10. *If (S, v, H) is an $(m, 1)-$triple, then inclusion (5.7) holds, namely*

$$\mathsf{N}(v^{\perp}) \subseteq \mathrm{Im}(\Phi^{m,k}_{(S,v,H)}).$$

The last ingredient needed to complete the proof of Theorem 5.2.1 is the following:

Proposition 5.2.11. *Let m and k be two positive integers, with $(m, k) \neq (1, 1), (1, 2)$, and let (S, v, H) be an $(m, k)-$triple, with $v = m(1, 0, -k)$. Then*

$$R_s \circ R_{s_1} \in \mathrm{Im}(\Phi^{m,k}_{(S,v,H)}),$$

where $s = (1, 0, 1)$, $s_1 = (1, 0, -1) \in v^{\perp}$ are as in Remark 5.2.2.

Proof. Let us start by considering the reflections R_s and R_{s_1} around the vectors $s = (1, 0, 1)$ and $s_1 = (1, 0, -1)$, respectively, defined on the whole Mukai lattice $\widetilde{\mathrm{H}}(S, \mathbb{Z})$. As recalled in Remark 5.2.2, their composition $R_s \circ R_{s_1}$ is an involution satisfying

$$R_s \circ R_{s_1} = -D^{\mathrm{H}}_S. \tag{5.11}$$

The proof will be easily concluded as soon as we show that $D^{\mathrm{H}}_S \in \mathrm{Im}(\widetilde{\Phi}^{m,k}_{(S,v,H)})$.

Let $p \in \mathbb{N}^*$ be a positive integer. We recall that, by Lemma 4.2.2 (2), the Fourier-Mukai equivalence $p\mathsf{H}\colon D^b(S) \to D^b(S)$ defines a morphism

$$p\mathsf{H} \in \mathrm{Hom}_{\mathcal{G}^{m,k}_{\mathrm{FM}}}((S, v, H), (S, v_{pH}, H)),$$

where

$$v_{pH} = (m, mph, \frac{rp^2h^2}{2} + mk)$$

and where $h := \mathrm{c}_1(H)$. More generally, the cohomological action $p\mathsf{H}^{\mathrm{H}}$ on the whole Mukai lattice $\widetilde{\mathrm{H}}(S, \mathbb{Z})$ is given by the following: for any $(r, \xi, a) \in \widetilde{\mathrm{H}}(S, \mathbb{Z})$,

$$p\mathsf{H}^{\mathrm{H}}(r, \xi, a) = (r, \xi, a).\mathrm{ch}(pH) = (r, rph + \xi, r\frac{p^2h^2}{2} + a + p\xi h). \tag{5.12}$$

By Lemma 4.2.3 (1), we can choose $p \gg 0$ such that both the Fourier-Mukai equivalences $\mathrm{FM}_{\mathcal{P}}$ and $\mathrm{FM}_{\mathcal{P}}^{\vee}$ induce isomorphisms on the corresponding moduli spaces. The composition $\mathrm{FM}_{\mathcal{P}}^{-1} \circ \mathrm{FM}_{\mathcal{P}}^{\vee}$ defines a morphism $\phi \in \mathrm{Aut}_{\mathcal{G}^{m,k}_{\mathrm{FM}}}(S, v_{pH}, H)$ such that, for any $(r, \xi, a) \in \widetilde{\mathrm{H}}(S, \mathbb{Z})$,

$$\widetilde{\Phi}^{m,k}_{\mathrm{FM}}(\phi)(r, \xi, a) = (r, -\xi, a) = D^{\mathrm{H}}_{S}(r, \xi, a), \tag{5.13}$$

by (4.7). Hence, combining (5.12) and (5.13), we get

$$\psi := p\mathsf{H} * \phi * p\mathsf{H} \in \mathrm{Aut}_{\mathcal{G}^{m,k}}((S, v, H))$$

and, for any $(r, \xi, a) \in \widetilde{\mathrm{H}}(S, \mathbb{Z}))$,

$$\begin{aligned}
\widetilde{\Phi}^{m,k}_{\mathrm{FM}}(\psi) &= \widetilde{\Phi}^{m,k}_{\mathrm{FM}}(p\mathsf{H}) \circ \widetilde{\Phi}^{m,k}_{\mathrm{FM}}(\mathrm{FM}_{\mathcal{P}}^{-1}) \circ \widetilde{\Phi}^{m,k}_{\mathrm{FM}}(\mathrm{FM}_{\mathcal{P}}^{\vee}) \circ \widetilde{\Phi}^{m,k}_{\mathrm{FM}}(p\mathsf{H})(r, \xi, a) = \\
&= \widetilde{\Phi}^{m,k}_{\mathrm{FM}}(p\mathsf{H}) \circ D^{\mathrm{H}}_{S}(r, rph + \xi, r\frac{p^2h^2}{2} + a + p\xi h) = \\
&= \widetilde{\Phi}^{m,k}_{\mathrm{FM}}(p\mathsf{H})(r, -rph - \xi, r\frac{p^2h^2}{2} + a + p\xi h) = (r, -\xi, a) = D^{\mathrm{H}}_{S}(r, \xi, a).
\end{aligned}$$

In order to conclude, we observe that

$$\mathrm{or}(\widetilde{\Phi}^{m,k}_{\mathrm{FM}}(\psi)) = \mathrm{or}(p\mathsf{H}^{\mathrm{H}}) + \mathrm{or}((\mathrm{FM}_{\mathcal{P}}^{\vee})^{\mathrm{H}}) - \mathrm{or}(\mathrm{FM}_{\mathcal{P}}^{\mathrm{H}}) + \mathrm{or}(p\mathsf{H}^{\mathrm{H}}) = 1,$$

by Corollary 4.3.23. Consequently,

$$\Phi^{m,k}(\psi) = \Psi^{m,k}(\widetilde{\Phi}^{m,k}_{\mathrm{FM}}(\psi)) = (-1)^{\mathrm{or}(\psi^{\mathrm{H}})}\psi^{\mathrm{H}}_{|v^{\perp}} = -D^{\mathrm{H}}_{S|v^{\perp}} = R_s \circ R_{s_1},$$

concluding the proof. □

We conclude the Chapter with the proof of Theorem 5.2.1, which has now become a straightforward application of Proposition 5.2.3, Proposition 5.2.11 and Remark 5.2.2.

Proof of Theorem 5.2.1. Let (S, v, H) an $(m, k)-$triple, with $(m, k) \neq (1, 1), (1, 2)$, and let us assume that $S = E \times F$ is the product of two elliptic curves E and F and that $NS(S) \simeq \langle e, f \rangle$, with $e := \mathrm{c}_1(E)$ and $f := \mathrm{c}_1(F)$. Set $v = m(1, 0, -k)$ and $h := \mathrm{c}_1(H) = e + tf$, with $t \gg 0$, as in Proposition 5.1.9. By Proposition 5.2.3,

$$\mathrm{SO}^{+}(\widetilde{\mathrm{H}}(S, \mathbb{Z}))_v \subseteq \mathrm{Im}(\Phi^{m,k}_{(S,v,H)})$$

and, by Proposition 5.2.11, the involution $R_s \circ R_{s_1} \in \mathrm{Im}(\Phi^{m,k}_{(S,v,H)})$. By Remark 5.2.2, those are exactly the generators of the group $\mathsf{N}(v^{\perp})$, hence, inclusion (5.7) is proven. The isomorphism of functors $\lambda \colon \Phi^{m,k} \to \mathsf{pt}^{m,k}$ defined in Section 4.3.3 (see also Corollary 4.3.24) induces an identification

$$\lambda^{\sharp}_{v}(\mathrm{Im}(\Phi^{m,k}_{(S,v,H)})) = \mathrm{Im}(\mathsf{pt}^{m,k}_{(S,v,H)}),$$

as subgroups of $\mathrm{Mon}^2(K_v(S,H))$, by Corollary 4.3.16, where λ_v is the Hodge isometry of Theorem 3.3.4. The restriction of $\lambda_v^\sharp$ to $\mathsf{N}(v^\perp)$, together with inclusion (5.7), proves the thesis for any $(m,k)-$triple satisfying the current hypotheses. The result is easily extended to any $(m,k)-$triple (S',v',H'), as

$$\mathrm{Im}(\mathsf{pt}^{m,k}_{(S',v',H'))}) = \mathsf{pt}^{m,k}(\eta)^\sharp(\mathrm{Im}(\mathsf{pt}^{m,k}_{(S,v,H)}))$$

for any non-trivial element $\eta \in \mathrm{Hom}_{\mathcal{G}^{m,k}}((S,v,H),(S',v',H'))$, whose existence is guaranteed by Remark 4.3.2. $\square$

Chapter 6

The monodromy group

This Chapter addresses the proof of Theorem B.1, namely the existence of an isomorphism between the monodromy groups $\mathrm{Mon}^2(K_v(S,H))$ and $\mathrm{Mon}^2(K_w(S,H))$, and completes the proof Theorem A.1, establishing an explicit lattice theoretic description of $\mathrm{Mon}^2(K_v(S,H))$, for $(m,k)-$triples with $k > 2$.

More explicitly, we relate monodromy operators on a singular moduli space $K_v(S,H)$ of semistable sheaves on an Abelian surface to monodromy operators on a smooth moduli space $K_w(S,H)$ of the same kind. The key point is the study of the most singular locus of $K_v(S,H)$ and the action on the second integral cohomology group of its closed embedding. This study will be conducted, respectively, in Sections 6.1 and 6.2. The latter will be used in Section 6.3 to produce an injective morphism on the respective monodromy groups, which will allow us to include $\mathrm{Mon}^2(K_v(S,H))$ in $\mathrm{Mon}^2(K_w(S,H))$ as a subgroup. Combining this natural inclusion with Theorem 5.2.1, we show that such morphism is actually surjective, proving Theorems A.1 and B.1 simultaneously. Finally, in Section 6.4, we will discuss some necessary numerical assumptions on the Mukai vectors v considered, namely the hypothesis $k > 2$ throughout this Chapter.

6.1 The embedding $K_w(S,H) \to K_v(S,H)$

Let m and k be two positive integers and let (S,v,H) be an $(m,k)-$triple. We recall that, by Proposition 1.2.14, the moduli space $K_v(S,H)$ admits a stratification of singularities

$$K_v(S,H) = X_0 \supset X_1 \supset \cdots \supset X_l = K_v(S,H)^{ms}.$$

Aim of this Section is to show that the most singular locus $K_v(S,H)^{ms}$ admits a finite number of connected components, each of which is isomorphic to the associated smooth moduli space $K_w(S,H)$, allowing us to define a natural closed embedding

$$i_{w,m} \colon K_w(S,H) \longrightarrow K_v(S,H)$$

that deforms in locally trivial families.

Let us start by considering the moduli space $M_v(S,H)$ of H-semistable sheaves on S with Mukai vector v (see Section 3.1.2). As in [OPR24, Section 1.4.], we recall the main ideas of the proof of [KLS06, Theorem 4.4.] that provide a description of the most singular locus $M_v(S,H)^{ms}$ of $M_v(S,H)$.

By Theorem 3.1.13, the singular locus $M_v(S,H)^{sing}$ of $M_v(S,H)$ coincides with the variety parametrizing strictly semistable sheaves. Each point of $M_v(S,H)^{sing}$ then corresponds to the class $[F]$ of a semistable sheaf F that, admitting a non-trivial Jordan-Hölder decomposition (see Definition A.1.4 and equation (A.3)), is $S-$equivalent to a sheaf of the form $F_1 \oplus F_2$, where $[F_i] \in M_{m_i w}(S,H)$ and $m_1 + m_2 = m$. More precisely, it belongs to the image of the summation morphism

$$\begin{aligned} \sigma_{m_1,m_2} \colon M_{m_1 w}(S,H) \times M_{m_2 w}(S,H) &\longrightarrow M_v(S,H) \\ ([F_1],[F_2]) &\longmapsto [F_1 \oplus F_2], \end{aligned} \tag{6.1}$$

which is an irreducible component of the singular locus. The intersection of all such components is the locus

$$\Sigma_m := \{[E_1 \oplus \cdots \oplus E_m] \colon [E_i] \in M_w(S,H)\} \simeq \mathrm{Sym}^m(M_w(S,H)),$$

whose most singular locus - and hence, the most singular locus of $M_v(S,H)$ - is

$$Y := \{[E^{\oplus m}] \in M_v(S,H) \colon [E] \in M_w(S,H)\}, \tag{6.2}$$

which is naturally isomorphic to $M_w(S,H)$.

We now check the compatibility of this description with the fibers of the respective Yoshioka fibrations involved.

Proposition 6.1.1. *Let m and k be two positive integers and let (S,v,H) be an $(m,k)-$triple. Then*

$$K_v(S,H)^{ms} = M_v(S,H)^{ms} \cap K_v(S,H).$$

Proof. As discussed in Section 3.1.3 (see Theorem 3.1.14), the Yoshioka fibration

$$a_v \colon M_v(S,H) \longrightarrow S \times \hat{S}$$

is an isotrivial fibration. Therefore, by Grauer-Fischer Theorem ([FG65]), it is also an analytically locally trivial family, in the sense of Definition 2.2.1. Hence, by Remark 2.2.7, it induces a relative stratification

$$M_v(S,H) = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_l = M_v(S,H)^{ms},$$

such that, for every $\alpha \in S \times \hat{S}$,

$$a_v^{-1}(\alpha) = \mathcal{X}_{0,\alpha} \supseteq \cdots \supseteq \mathcal{X}_{l,\alpha}$$

is a stratification of singularities of $a_v^{-1}(\alpha)$. In particular, for $\alpha = (0, \mathcal{O}_S)$, we get a stratification of singularities of $K_v(S,H)$, whose last stratum $\mathcal{X}_{l,0} = K_v(S,H)^{ms}$ is precisely, by definition,

$$\mathcal{X}_l \cap a_v^{-1}(O, \mathcal{O}_S) = M_v(S,H)^{ms} \cap K_v(S,H). \qquad \square$$

Proposition 6.1.1 yields a more explicit description of the most singular locus of $K_v(S,H)$. In the following, for every positive integer $m \in \mathbb{N} \setminus \{0\}$, we will denote by $S[m] \times \hat{S}[m]$ the $m-$torsion points in $S \times \hat{S}$, which define a finite set of order m^8.

Proposition 6.1.2. *Let m and k be two positive integers and let (S, v, H) be an $(m, k)-$triple. Then*

$$K_v(S,H)^{ms} \simeq \bigcup_{(x,L) \in S[m] \times \hat{S}[m]} a_w^{-1}(x, L).$$

Proof. From Proposition 6.1.1 and characterization (6.2), we get

$$K_v(S,H)^{ms} = \{[E^{\oplus m}] \in K_v(S,H) \colon [E] \in M_w(S,H)\}.$$

A straightforward computation shows that the summation map (6.1) is compatible with both the Yoshioka fibrations $a_{m_i w}$ for $i = 1,2$ and, in particular, for every $E \in M_w(S,H)$, it holds $a_v(E^{\oplus m}) = a_w(E)^m$, with respect to the group structure on $S \times \hat{S}$. It follows that, given $[E] \in M_w(S,H)$, the point $[E^{\oplus m}]$ belongs to $K_v(S,H)^{ms}$ if and only if

$$a_w(E)^m = (0, \mathcal{O}_S) \in S \times \hat{S},$$

from which the claim follows. $\square$

We now recall a remarkable construction providing us a family of natural automorphisms of $K_v(S,H)$. For any $(x,L) \in S \times \hat{S}$ it is defined a natural *translation* morphism

$$\begin{aligned} \tau_{(x,L)} \colon M_v(S,H) &\longmapsto M_v(S,H) \\ F &\longmapsto \tau_{x*}(F) \otimes L \end{aligned} \tag{6.3}$$

- where we recall that $\tau_{x*} = \tau_{-x}^*$ - which naturally identifies the fibers of the Yoshioka fibration $a_v \colon M_v(S,H) \to S \times \hat{S}$ (see (3.10)). Now, let $F \in M_v(S,H)$, with $v(F) = mw = m(r, \xi, a)$ and suppose, additionally, that $(x,L) \in S[m] \times \hat{S}[m]$. Then

$$\det(\tau_{(x,L)}(F)) = \det(\tau_{x*}(F) \otimes L) \simeq \det(\tau_{x*}(F)) \otimes L^{\otimes mr} \simeq \det(\tau_{x*}(F)) \simeq \det(F),$$

since $\mathrm{rk}(F) = mr$, $\mathrm{c}_1(F) = m\xi$ and $(x,L) \in S[m] \times \hat{S}[m]$. Analogously, if $\mathcal{P} \in D^b(S \times \hat{S})$ is the Poincaré line bundle (see Sections 3.1.3 and 4.2.2), then

$$\begin{aligned} \det(\mathrm{FM}_{\mathcal{P}}(\tau_{(x,L)}(F))) &\simeq \det(\mathrm{FM}_{\mathcal{P}}(\tau_{x*}(F) \otimes L)) \simeq \det(\tau_{L^\vee}^*(\mathrm{FM}_{\mathcal{P}}(F)) \otimes \mathcal{P}_{|\{-x\} \times \hat{S}}) \simeq \\ &\simeq \tau_{L^\vee}^*(\det(\mathrm{FM}_{\mathcal{P}}(F)) \otimes \mathcal{P}_{|\{-x\} \times \hat{S}}^{\otimes ma}) \simeq \tau_{L^\vee}^*(\det(\mathrm{FM}_{\mathcal{P}}(F))), \end{aligned}$$

since $\mathrm{rk}(\mathrm{FM}_{\mathcal{P}}(F)) = ma$ and $\mathcal{P}_{|\{-x\}\times\hat{S}}$ is $m-$torsion, as x. Moreover, since $c_1(\mathrm{FM}_{\mathcal{P}}(F)) = -m\hat{\xi}$ and $L^\vee \in \hat{S}[m]$, as before, we deduce that the right hand side is isomorphic to $\det(\mathrm{FM}_{\mathcal{P}}(F))$.

Consequently, we get the following.

Corollary 6.1.3. *Let m and k be two positive integers and let (S, v, H) be an $(m, k)-$triple. Then, for any $(x, L) \in S[m] \times \hat{S}[m]$, the automorphism $\tau_{(x,L)} \in \mathrm{Aut}(M_v(S, H))$ defined in (6.3) is compatible with the Yoshioka fibration a_v, namely*

$$a_v(\tau_{(x,L)}(F)) = a_v(F)$$

for any $F \in M_v(S, H)$. Hence, for any $(y, M) \in S \times \hat{S}$, the restriction $\tau_{(x,L)|a_v^{-1}(y,M)}$ defines an automorphism of any fiber $a_v^{-1}(y, M))$.

Notation 6.1.4. With a slight abuse of notation, we will denote

$$\tau_{(x,L)} := \tau_{(x,L)|K_v(S,H)} \in \mathrm{Aut}(K_v(S, H)).$$

Moreover, set

$$G_m := \{\tau_{(x,L)} \in \mathrm{Aut}(K_v(S, H)) : (x, L) \in S[m] \times \hat{S}[m]\} \simeq S[m] \times \hat{S}[m]. \tag{6.4}$$

As a first consequence of Proposition 6.1.2, we get, by restriction, an action $G_m \subseteq \mathrm{Aut}(K_v(S, H)^{ms})$, which is transitive on the set of connected components $\pi_0(K_v(S, H)^{ms})$ of $K_v(S, H)^{ms}$.

Corollary 6.1.5. *Let m and k be two positive integers and let (S, v, H) be an $(m, k)-$triple. For any $(x, L) \in S[m] \times \hat{S}[m]$, the translation morphism $\tau_{(x,L)} \in \mathrm{Aut}(K_v(S, H))$ restricts to an automorphism*

$$\tau_{(x,L)} \in \mathrm{Aut}(K_v(S, H)^{ms}).$$

In particular, for any $F \in K_v(S, H)^{ms}$, isomorphic to $E^{\oplus m}$ for a suitable $E \in a_w^{-1}(y, M)$, with $(y, M) \in S[m] \times \hat{S}[m]$,

$$\tau_{(x,L)}(F) = \tau_{(x,L)}(E^{\oplus m}) = \tau_{(x,L)}(E)^{\oplus m},$$

where $\tau_{(x,L)}(E) \in a_w^{-1}(x + y, L + M) \subseteq K_v(S, H)^{ms}$, via the isomorphism of Proposition 6.1.2. Thus, the action of G_m is transitive on $\pi_0(K_v(S, H)^{ms})$.

Furthermore, we can use the morphisms $\tau \in G_m$ to define a collection of closed embeddings of $K_w(S, H)$ in $K_v(S, H)$, as stated below.

Corollary 6.1.6. *Let m and k be two positive integers and let (S, v, H) be an $(m, k)-$triple. For every $(x, L) \in S[m] \times \hat{S}[m]$, there is a closed embedding*

$$\begin{aligned} \tau^m_{(x,L)} : K_w(S, H) &\longrightarrow K_v(S, H)^{ms} \\ E &\longmapsto \tau_{(x,L)}(E)^{\oplus m}. \end{aligned}$$

From finiteness of $m-$torsion points on Abelian surfaces, we deduce that we have a finite number of choices - namely, m^8 - to embed $K_w(S, H)$ in $K_v(S, H)$ as one of the connected components of its most singular locus.

6.2 Action on the second integral cohomology

We now focus on the action of the translation automorphisms and of the closed embeddings defined in the previous Subsection on the second integral cohomology, in order to relate integral isometries - and, in particular, monodromy operators - defined on $K_v(S,H)$ and on $K_w(S,H)$.

The first crucial property we are going to recall is that the the group G_m defined in (6.4) is made of automorphisms acting trivially on $\mathrm{H}^2(K_v(S,H),\mathbb{Z})$, for any $(m,k)-$triple (S,v,H), with $(m,k) \neq (1,1),(1,2)$.

Proposition 6.2.1. *Let (S,v,H) be an $(m,k)-$triple.*

(1) The group $\mathrm{Aut}_0(M_v(S,H))$ of automorphisms acting trivially on $\mathrm{H}^2(M_v(S,H),\mathbb{Z})$ contains a group isomorphic to $S \times \hat{S}$. In other words, for any $(x,L) \in S \times \hat{S}$,

$$\tau_{(x,L)}^* = \mathrm{id}_{\mathrm{H}^2(M_v(S,H),\mathbb{Z})}.$$

(2) If $(m,k) \neq (1,1),(1,2)$, then there is an inclusion $G_m \subseteq \mathrm{Aut}_0(K_v(S,H))$, i.e. any $(x,L) \in S[m] \times \hat{S}[m]$,

$$\tau_{(x,L)}^* = \mathrm{id}_{\mathrm{H}^2(K_v(S,H),\mathbb{Z})}.$$

Proof. (1) The first part of the statement follows from the fact that, for any $(x,L) \in S \times \hat{S}$, the translation morphism $\tau_{(x,L)}$ is homotopic to the identity morphism on $M_v(S,H)$. Indeed, as $S \times \hat{S}$ is path-connected, we can consider any path $\gamma \in \Omega(S \times \hat{S},(0,\mathcal{O}_S),(x,L))$ and use it to define an homotopy

$$\begin{aligned} H\colon M_v(S,H) \times [0,1] &\to M_v(S,H) \\ (F,t) &\mapsto \tau_{\gamma(t)}(F). \end{aligned}$$

(2) The same strategy does not apply to the case of $K_v(S,H)$, as $S[m] \times \hat{S}[m]$ is a discrete subspace of $S \times \hat{S}$. Nonetheless, we can procede as follows. Let $i\colon K_v(S,H) \hookrightarrow M_v(S,H)$ be the natural inclusion, and notice that, for any $\tau \in G_m$, regarded as $\tau \in \mathrm{Aut}(M_v(S,H))$, it holds

$$\tau_{|K_v(S,H)}^* \circ i^* = i^* \circ \tau^* = i^*\colon \mathrm{H}^2(M_v(S,H),\mathbb{Z}) \to \mathrm{H}^2(K_v(S,H),\mathbb{Z}),$$

where the first equality follows from $\tau(K_v(S,H)) = K_v(S,H)$ and the second one by part (1). Hence, we deduce that the fixed locus $\mathrm{Fix}(\tau_{|K_v(S,H)}^*)$ of $\tau_{|K_v(S,H)}^*$ contains the image $\mathrm{Im}(i^*)$ of i^*. Since i^* is surjective (see [PR23, Corollary 5.3] and its proof, see also Theorem 3.3.4(2)), the claim follows. □

A first fundamental consequence of the inclusion $G_m \subseteq \mathrm{Aut}_0(K_v(S,H))$ is that its elements can be deformed along locally trivial deformations of $K_v(S,H)$.

Corollary 6.2.2. *Let (S, v, H) be an $(m, k)-$triple, with $(m, k) \neq (1, 1), (1, 2)$, and let X be an irreducible symplectic variety that is locally trivial deformation equivalent to $K_v(S, H)$. Then, an isomorphic copy of G_m is contained in $\mathrm{Aut}_0(X)$ and it acts transitively on the set $\pi_0(X^{ms})$, where X^{ms} is the most singular locus of X (see Proposition 1.2.14).*

Proof. By [BL22, Theorems 4.7, 6.14] (see also Remark 2.2.2 and Proposition 2.3.4), the proof of [HT13, Theorem 2.1] can be applied verbatim to show that, if X is a primitive sympletic variety, then $\mathrm{Aut}_0(X)$ is a locally trivial deformation invariant. This proves the first part of the statement, as $G_m \subseteq \mathrm{Aut}_0(K_v(S, H))$.

Moreover, for any locally trivial deformation $p\colon \mathcal{X} \to T$ of $K_v(S, H)$, by Remark 2.2.7, the closed embedding of its most singular locus $K_v(S, H)^{ms}$ induces a relative closed embedding $\iota\colon \mathcal{Y} \to \mathcal{X}$ and a locally trivial deformation $p_{|\mathcal{Y}}\colon \mathcal{Y} \to T$ of $K_v(S, H)^{ms}$. Set $X \simeq \mathcal{X}_t$ for some $t \in T$ and let $X^{ms} \simeq \mathcal{Y}_t \simeq \mathcal{X}_t^{ms}$ be its most singular locus. Since p is locally trivial and the action of $G_m \subseteq \mathrm{Aut}(K_v(S, H)^{ms})$ is analytic and continuous over the connected base B, we deduce that X^{ms} and $K_v(S, H)^{ms}$ have the same number of connected components and that G_m acts transitively on $\pi_0(X^{ms})$ as well. □

A second crucial consequence of Proposition 6.2.1 is that interchanging the connected components of $K_v(S, H)^{ms}$ via automorphisms in G_m corresponds to a trivial action on the second integral cohomology of $K_v(S, H)$, hence it does not affect the monodromy information carried by any embedding of $K_w(S, H)$ in $K_v(S, H)$, as will be explained in further details in Section 6.3.

More precisely, set

$$i_{w,m} := \tau^m_{(0,\mathcal{O}_S)}\colon K_w(S, H) \to K_v(S, H),$$

where the latter is one of the closed embeddings defined in Corollary 6.1.6. Notice that, as $\tau_{(0,\mathcal{O}_S)|K_w(S,H)} = \mathrm{id}_{K_w(S,H)}$, we are just mapping each sheaf $F \in K_w(S, H)$ to $F^{\oplus m} \in K_v(S, H)^{ms}$. On the other hand, we can study the action of the same automorphism, when restricted to another connected component of $K_v(S, H)^{ms}$. In other words, denote by

$$i_{w,m|a_w^{-1}(x,L)}\colon a_w^{-1}(x, L) \to K_v(S, H)$$

the closed embedding of $a_w^{-1}(x, L)$ in $K_v(S, H)$, induced, as in Corollary 6.1.6, by the action of the same automorphism $\tau_{(0,\mathcal{O}_S)}$ of $K_v(S, H)^{ms}$ of Corollary 6.1.5 - which is again the identity - but restricted to $a_w^{-1}(x, L)$.

Corollary 6.2.3. *Let (S, v, H) be an $(m, k)-$triple, with $(m, k) \neq (1, 1), (1, 2)$. Then, for any $(x, L) \in S[m] \times \hat{S}[m]$, the following commutation law holds:*

$$\tau^*_{(x,L)} \circ i^*_{w,m|a_w^{-1}(x,L)} = i^*_{w,m}\colon \mathrm{H}^2(K_v(S, H), \mathbb{Z}) \to \mathrm{H}^2(K_w(S, H), \mathbb{Z}).$$

Proof. By definition, there is a commutative diagram

$$\begin{tikzcd} K_w(S,H) \arrow[r, "i_{w,m}"] \arrow[d, "\tau_{(x,L)}"'] & K_v(S,H) \arrow[d, "\tau_{(x,L)}"] \\ a_w^{-1}(x,L) \arrow[r, "i_{w,m|a_w^{-1}(x,L)}"] & K_v(S,H), \end{tikzcd}$$

whose cohomological action is described as below, by Proposition 6.2.1,

$$\begin{tikzcd} \mathrm{H}^2(K_v(S,H),\mathbb{Z}) \arrow[r, "i^*_{w,m}"] & \mathrm{H}^2(K_w(S,H),\mathbb{Z}) \\ \mathrm{H}^2(K_v(S,H),\mathbb{Z}) \arrow[r, "i^*_{w,m|a_w^{-1}(x,L)}"] \arrow[u, "\tau^*_{(x,L)}=\mathrm{id}"] & \mathrm{H}^2(a_w^{-1}(x,L),\mathbb{Z}), \arrow[u, "\tau^*_{(x,L)}"'] \end{tikzcd}$$

proving the claim. □

We now compare the pullback

$$i^*_{w,m}\colon \mathrm{H}^2(K_v(S,H),\mathbb{Z}) \longrightarrow \mathrm{H}^2(K_w(S,H),\mathbb{Z})$$

with the composition

$$\mathrm{H}^2(K_v(S,H),\mathbb{Z}) \xrightarrow{\lambda_v^{-1}} v^\perp =\!=\!= w^\perp \xrightarrow{\lambda_w} \mathrm{H}^2(K_w(S,H),\mathbb{Z}),$$

where the arrows on the sides are given by the isometries of Theorem 3.3.4 (2), where $\lambda_v := \lambda_v^0$, according to Notation 3.3.5.

Proposition 6.2.4. *Let m,k be two positive integers, with $k > 1$, and let (S,v,H) be an $(m,k)-$triple. The morphism $i^*_{w,m}\colon \mathrm{H}^2(K_v(S,H),\mathbb{Z}) \to \mathrm{H}^2(K_w(S,H),\mathbb{Z})$ is a similitude of lattices satisfying*

$$i^*_{w,m} = m\lambda_w \circ \lambda_v^{-1}.$$

Proof. The proof follows as in Proposition 1.28 of [OPR24], assuming that the surface S is Abelian and replacing the moduli space $M_v(S,H)$ with $K_v(S,H)$, under the current hypothesis on the $(m,k)-$triple (S,v,H). We sketch the main ideas and we provide the references needed to approach this specific case.

For every $p > 0$, we will use the shortened notation K_{pw} for $K_{pw}(S,H)$. By [PR24, Section 4.20] and, again, by naturality of Yoshioka fibrations with respect to direct sums, for every $p > 0$ we have a morphism

$$\begin{aligned} f_p\colon K_w^p &\longrightarrow K_{pw} \\ (F_1,\dots,F_p) &\longmapsto F_1 \oplus \cdots \oplus F_p, \end{aligned}$$

that fits in the following diagram:

$$\begin{tikzcd} (pw)^\perp = w^\perp \arrow[r, "\lambda_{pw}"] \arrow[d, "(\lambda_w,\dots,\lambda_w)"'] & \mathrm{H}^2(K_{pw},\mathbb{Z}) \arrow[d, "f_p^*"] \\ \oplus_{i=1}^p \mathrm{H}^2(K_{pw},\mathbb{Z}) \arrow[r, "\sum_{i=1}^p \pi^*_{i,p}"'] & \mathrm{H}^2(K_w^p,\mathbb{Z}), \end{tikzcd} \tag{6.5}$$

where $\pi_{i,p}\colon K_w^p \longrightarrow K_w$ is the projection onto the $i-$th factor. Proceeding by induction on p as in [OPR24, Proposition 1.28] and applying Proposition 4.12 (2) of [PR24], we get commutativity of diagram (6.5), which yields, for $p = m$, the identity

$$f_m^*(\lambda_v(a)) = \sum_{i=1}^{m} \pi_{i,m}^*(\lambda_w(a), \dots, \lambda_w(a)), \tag{6.6}$$

for every $a \in v^\perp = w^\perp$. We now consider the diagonal morphism $\delta_m\colon K_w \longrightarrow K_w^m$ and we notice that, by definition, the equality $i_{w,m} = f_m \circ \delta_m$ holds. Therefore, by (6.6) we deduce, for every $a \in v^\perp$,

$$(i_{w,m}^* \circ \lambda_v)(a) = (\delta_m^* \circ f_m^* \circ \lambda_v)(a) = \delta_m^*(\sum_{i=1}^{m} \pi_{i,m}^*(\lambda_w(a), \dots, \lambda_w(a))) =$$
$$= \sum_{i=1}^{m} \lambda_w(a) = m\lambda_w(a),$$

concluding the proof. □

Let us denote by

$$i_{w,m,\mathbb{Q}}^*\colon \mathrm{H}^2(K_v(S,H),\mathbb{Q}) \to \mathrm{H}^2(K_w(S,H),\mathbb{Q}),$$

$$\lambda_{v,\mathbb{Q}}\colon v^\perp \otimes \mathbb{Q} \to \mathrm{H}^2(K_v(S,H),\mathbb{Q}),$$
$$\lambda_{w,\mathbb{Q}}\colon w^\perp \otimes \mathbb{Q} \to \mathrm{H}^2(K_w(S,H),\mathbb{Q})$$

the $\mathbb{Q}-$linear extensions of the morphisms $i_{w,m}^*$, λ_v and λ_w, respectively. By Proposition 6.2.4, together with Theorem 3.3.4, we get that $i_{w,m,\mathbb{Q}}^*$ is an injective morphism of $\mathbb{Q}-$vector spaces satisfying

$$i_{w,m,\mathbb{Q}}^* = m\lambda_{w,\mathbb{Q}} \circ \lambda_{v,\mathbb{Q}}^{-1}, \tag{6.7}$$

which is an isomorphism if, additionally, $k > 2$.

Assumption 6.2.5. From now on, we will work under the additional hypothesis that $k > 2$. Some consequences on the geometry of the moduli spaces under study will be addressed in Remark 6.3.1 and further motivations concerning this assumption will be discussed in Section 6.4.

Notation 6.2.6. In the following, we will use this notation: if $f\colon A \longrightarrow B$ is an isomorphism of groups, we will denote by $f^\sharp$ the map induced on the corresponding Hom sets by conjugation, i.e.

$$\begin{aligned} f^\sharp\colon \mathrm{Hom}(A,A) &\longrightarrow \mathrm{Hom}(B,B) \\ g &\longmapsto f \circ g \circ f^{-1}. \end{aligned} \tag{6.8}$$

According to the above-introduced notation, we turn our attention to the isomorphism

$$i^{\sharp}_{w,m,\mathbb{Q}} := (i^{*}_{w,m,\mathbb{Q}})^{\sharp} \colon \mathrm{O}(\mathrm{H}^2(K_v(S,H),\mathbb{Q})) \to \mathrm{O}(\mathrm{H}^2(K_w(S,H),\mathbb{Q}))$$

induced by $i^{*}_{w,m,\mathbb{Q}}$ via conjugation. A straightforward application of Proposition 6.2.4, which allows to compare the integral orthogonal groups involved, is the following.

Lemma 6.2.7. *Let m,k be two positive integers, with $k > 2$, and let (S,v,H) be an $(m,k)-$triple. The isomorphism $i^{\sharp}_{w,m,\mathbb{Q}}$ restricts to an isomorphism of groups*

$$i^{\sharp}_{w,m} \colon \mathrm{O}(\mathrm{H}^2(K_v(S,H),\mathbb{Z})) \longrightarrow \mathrm{O}(\mathrm{H}^2(K_w(S,H),\mathbb{Z}))$$

satisfying the identity $i^{\sharp}_{w,m} = \lambda^{\sharp}_{w} \circ (\lambda^{\sharp}_{v})^{-1}$, under the identification $v^{\perp} = w^{\perp}$.

Proof. The proof works verbatim as in in [OPR24, Lemma 1.30], but we include it for the sake of completeness. In fact, it quickly follows from the identity (6.7), deduced by Proposition 6.2.4, implying $(i^{*}_{w,m,\mathbb{Q}})^{-1} = \frac{1}{m}\lambda_{v,\mathbb{Q}} \circ \lambda^{-1}_{w,\mathbb{Q}}$. Therefore, for any $g \in \mathrm{O}(\mathrm{H}^2(K_v(S,H),\mathbb{Q}))$, it holds

$$\begin{aligned} i^{\sharp}_{w,m,\mathbb{Q}}(g) &= i^{*}_{w,m,\mathbb{Q}} \circ g \circ (i^{*}_{w,m,\mathbb{Q}})^{-1} = m\lambda_{w,\mathbb{Q}} \circ \lambda^{-1}_{v,\mathbb{Q}} \circ g \circ \frac{1}{m}\lambda_{v,\mathbb{Q}} \circ \lambda^{-1}_{w,\mathbb{Q}} = \\ &= \lambda_{w,\mathbb{Q}} \circ \lambda^{-1}_{v,\mathbb{Q}} \circ g \circ \lambda_{v,\mathbb{Q}} \circ \lambda^{-1}_{w,\mathbb{Q}} = \lambda^{\sharp}_{w,\mathbb{Q}} \circ (\lambda^{\sharp}_{v,\mathbb{Q}})^{-1}(g). \end{aligned}$$

Hence, $i_{w,m,\mathbb{Q}}$ sends integral isometries to integral isometries, as both $\lambda^{\sharp}_{v,\mathbb{Q}}$ and $\lambda^{\sharp}_{w,\mathbb{Q}}$ do, since $\lambda_{v^{\sharp}}$ and $\lambda^{\sharp}_{w}$ are their respective restrictions to the respective integral orthogonal groups. □

Arguing in a similar fashion, we can extend Lemma 6.2.7 to the case in which we have two $(m,k)-$triples (S_1,v_1,H_1) and (S_2,v_2,H_2), with $v_i = mw_i$ for $i = 1,2$, and compare the cohomology action of the embeddings i_{m,w_1} and i_{m,w_2}.

In the following, we will write K_{v_i} and K_{w_i} for the moduli spaces $K_{v_i}(S_i,H_i)$ and $K_{w_i}(S_i,H_i)$, respectively, for $i = 1,2$. In addition, let us denote by

$$i^{*}_{w_i,m,\mathbb{Q}} \colon \mathrm{H}^2(K_{v_i},\mathbb{Q}) \longrightarrow \mathrm{H}^2(K_{w_i},\mathbb{Q})$$

the $\mathbb{Q}-$linear extension of $i^{*}_{w_i,m}$ and let us consider the following bijective map

$$\begin{aligned} i^{\sharp}_{w_1,w_2,m,\mathbb{Q}} \colon \mathrm{O}(\mathrm{H}^2(K_{v_1},\mathbb{Q}),\mathrm{H}^2(K_{v_2},\mathbb{Q})) &\longrightarrow \mathrm{O}(\mathrm{H}^2(K_{w_1},\mathbb{Q}),\mathrm{H}^2(K_{w_2},\mathbb{Q})) \\ g &\longmapsto i^{*}_{w_2,m,\mathbb{Q}} \circ g \circ (i^{*}_{w_1,m,\mathbb{Q}})^{-1}. \end{aligned} \quad (6.9)$$

Lemma 6.2.8. *Let m, k be two positive integers, with $k > 2$, and let (S_1, v_1, H_1) and (S_2, v_2, H_2) be two $(m, k)-$triples, with $v_i = mw_i$, for $i = 1, 2$. Then the bijection $i^{\sharp}_{w_1,w_2,m,\mathbb{Q}}$ restricts to a bijection*

$$i^{\sharp}_{w_1,w_2,m} \colon \mathrm{O}(\mathrm{H}^2(K_{v_1}, \mathbb{Z}), \mathrm{H}^2(K_{v_2}, \mathbb{Z})) \longrightarrow \mathrm{O}(\mathrm{H}^2(K_{w_1}, \mathbb{Z}), \mathrm{H}^2(K_{w_2}, \mathbb{Z}))$$

satisfying, for every $g \in \mathrm{O}(\mathrm{H}^2(K_{v_1}, \mathbb{Z}), \mathrm{H}^2(K_{v_2}, \mathbb{Z}))$,

$$i^{\sharp}_{w_1,w_2,m}(g) = (\lambda_{w_2} \circ \lambda_{v_2}^{-1}) \circ g \circ (\lambda_{w_1} \circ \lambda_{v_1}^{-1})^{-1}.$$

It is immediate to notice that, if $(S_i, v_i, H_i) = (S, v, H)$ for $i = 1, 2$, then the morphism $i^{\sharp}_{w_1,w_2,m}$ described in Lemma 6.2.8 coincides with the morphism $i^{\sharp}_{w,m}$ of Lemma 6.2.7.

6.3 An isomorphism between $\mathrm{Mon}^2(K_v(S, H))$ and $\mathrm{Mon}^2(K_w(S, H))$

In this Section we will study the action of the bijection $i^{\sharp}_{w_1,w_2,m}$ introduced in Lemma 6.2.8 on parallel transport operators. The next Proposition, which is the main result of the Section, shows that the map $i^{\sharp}_{w_1,w_2,m}$ sends parallel transport operators to parallel transport operators. The outcome is that its restriction to the set of parallel transport operators provides an injection of sets.

Remark 6.3.1. We recall that, by Assumption 6.2.5, we are restricting the discussion of the last Sections to moduli spaces of sheaves $K_v(S, H)$ built from $(m, k)-$triples (S, v, H) with $k > 2$. As already pointed out in Remark 3.1.22, these yield $\mathbb{Q}-$factorial and terminal irreducible symplectic varieties. Hence, by Remark 2.2.6, all deformations of $K_v(S, H)$ are locally trivial, yielding the following identifications:

$$\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S, H)) = \mathrm{Mon}^2(K_v(S, H)) \tag{6.10}$$

and, analogously, for any $(m, k)-$triples (S_1, v_1, H_1) and (S_2, v_2, H_2),

$$\mathsf{PT}^2_{\mathrm{lt}}(K_{v_1}(S_1, H_1), K_{v_2}(S_2, H_2)) = \mathsf{PT}^2(K_{v_1}(S_1, H_1), K_{v_2}(S_2, H_2)) \tag{6.11}$$

Hence, all the statements in the next discussion will deal with classical monodromy operators and parallel transport operators in flat families, under the identifications (6.10) and (6.11).

Theorem 6.3.2. *Let m, k be two positive integers, with $k > 2$, and let (S_1, v_1, H_1) and (S_2, v_2, H_2) be two $(m, k)-$triples, with $v_i = mw_i$, for $i = 1, 2$. Then the bijection $i^{\sharp}_{w_1,w_2,m}$ of Lemma 6.2.8 restricts to an injective function*

$$i^{\sharp}_{w_1,w_2,m} \colon \mathsf{PT}^2(K_{v_1}(S_1, H_1), K_{v_2}(S_2, H_2)) \longrightarrow \mathsf{PT}^2(K_{w_1}(S_1, H_1), K_{w_2}(S_2, H_2)).$$

Proof. Let $g \in \mathsf{PT}^2(K_{v_1}(S_1, H_1), K_{v_2}(S_2, H_2)$ be a parallel transport operator. By definition, there exists a family $p\colon \mathcal{X} \to T$ of primitive symplectic varieties, two points $t_1, t_2 \in T$ such that $\mathcal{X}_{t_i} \simeq K_{v_i}(S_i, H_i)$ and a path γ from t_1 to t_2 such that $g = \mathsf{PT}_p(\gamma)$ (see Definition 2.3.12). By Remark 2.2.7 (1) there exists a relative stratification

$$\mathcal{X} = \mathcal{X}_0 \supseteq \mathcal{X}_1 \supseteq \cdots \supseteq \mathcal{X}_l =: \mathcal{Y}$$

that, restricted to each fiber $\mathcal{X}_t$ of p, corresponds to the stratification of singularities in Proposition 1.2.14. In particular, for every $t \in T$, we have $\mathcal{Y}_t \simeq \mathcal{X}_t^{ms}$ and, for $i = 1, 2$,

$$\mathcal{Y}_{t_i} \simeq K_{v_i}(S_i, H_i)^{ms} \simeq \bigcup_{x \in S_i[m], L \in \widehat{S}_i[m]} a_{w_i}^{-1}(x, L),$$

where the last isomorphism follows from Proposition 6.1.2. By isotriviality of the Yoshioka fibrations a_{w_i} for $i = 1, 2$, we can suppose that $K_{w_1}(S_1, H_1)$ and $K_{w_2}(S_2, H_2)$ belong to the same connected component $\mathcal{Z}$ of $\mathcal{Y}$. By Remark 2.2.7 (2), the restriction $p_{|\mathcal{Z}}\colon \mathcal{Z} \to T$ of p is smooth and proper and $\mathcal{Z}_{t_i} \supseteq K_{w_i}(S_i, H_i)$, for $i = 1, 2$, as connected component. By Remark 2.2.7 (3), the morphism $p_{|\mathcal{Z}}$ factors via a smooth and proper morphism $q\colon \mathcal{Z} \to \widetilde{T}$ with connected fibers and connected base $\widetilde{T}$ and a finite étale cover $h\colon \widetilde{T} \to T$. Notice that $q\colon \mathcal{Z} \to \widetilde{T}$ is a smooth deformation of IHS manifolds with both $K_{w_1}(S_1, H_1)$ and $K_{w_2}(S_2, H_2)$ as fibers.

Let $\widetilde{t_1} \in h^{-1}(t_1)$ be the point such that $\mathcal{Z}_{\widetilde{t_1}} \simeq K_{w_1}(S_1, H_1)$ and let $\widetilde{\gamma}$ be the unique lift of γ, via h, such that $\widetilde{\gamma}(0) = \widetilde{t_1}$. Set $\widetilde{t_2} := \widetilde{\gamma}(1) \in h^{-1}(t_2)$ and notice that $\mathcal{Z}_{\widetilde{t_2}} \subseteq \mathcal{Z}_{t_2}$, hence, being one of the connected components of $K_{v_2}(S_2, H_2)^{ms}$,

$$\mathcal{Z}_{\widetilde{t_2}} \simeq a_{w_2}^{-1}(x, L),$$

for some $(x, L) \in S_2[m] \times \hat{S_2}[m]$, by Proposition 6.1.2. Notice that the isomorphism $\tau_{(x,L)}\colon K_{w_2}(S_2, H_2) \to a_{w_2}^{-1}(x, L)$ (see (6.3)) defines - via pullback - a parallel transport operator, for instance in the local system $R^2 a_{w_2 *}\mathbb{Z}$. Hence,

$$\tau_{(x,L)}^* \circ \mathsf{PT}_q(\widetilde{\gamma}) \in \mathsf{PT}^2(K_{w_1}(S_1, H_1), K_{w_2}(S_2, H_2)).$$

We will now show that

$$i^{\sharp}_{w_1, w_2, m}(\mathsf{PT}_p(\gamma)) = \tau_{(x,L)}^* \circ \mathsf{PT}_q(\widetilde{\gamma}) \tag{6.12}$$

and, by Lemma 6.2.8, it suffices to prove it over $\mathbb{Q}$, namely, for their respective $\mathbb{Q}-$linear extensions.

Let $g = \mathsf{PT}_p(\gamma)$ be as above, and let $g_{\mathbb{Q}}$ be its $\mathbb{Q}-$linear extension, i.e. the parallel transport operator $\mathsf{PT}_p^{\mathbb{Q}}(\gamma)$ along γ in the local system $R^2 p_* \mathbb{Q}$. The relative closed embedding $\iota\colon \mathcal{Z} \to \mathcal{X}$ over T induces a morphism of local systems

$$\iota_{\mathbb{Q}}^*\colon h^* R^2 p_* \mathbb{Q} \longrightarrow R^2 q_* \mathbb{Q}$$

over $\widetilde{T}$, such that

$$(h^*R^2p_*\mathbb{Q})_{\widetilde{t}_i} \simeq (R^2p_*\mathbb{Q})_{h(\widetilde{t}_i)} \simeq R^2p_*\mathbb{Q}_{t_i} \simeq \mathrm{H}^2(\mathcal{X}_{t_i},\mathbb{Q}) \quad \text{for } i=1,2,$$
$$(R^2q_*\mathbb{Q})_{\widetilde{t}_i} \simeq \mathrm{H}^2(\mathcal{Z}_{\widetilde{t}_i},\mathbb{Q}) \simeq \begin{cases} \mathrm{H}^2(\mathcal{Z}_{t_1},\mathcal{Q}) \simeq \mathrm{H}^2(K_{w_1}(S_1,H_1),\mathbb{Q}) & \textit{for } i=1, \\ \mathrm{H}^2(a_{w_2}^{-1}(x,L),\mathbb{Q}) & \textit{for } i=2, \end{cases}$$

and

$$\iota^*_{\widetilde{t}_1} = i^*_{w_1,m,\mathbb{Q}}, \quad \iota^*_{\widetilde{t}_2} = (i^*_{w_2,m|a_{w_2}^{-1}(x,L)})_{\mathbb{Q}}. \tag{6.13}$$

Moreover, $i^*_{\mathbb{Q},t_1}$ is an isometry, by Proposition 6.2.4, hence $\iota^*_{\mathbb{Q}}$ is an isomorphism of local systems.

Consequently, we get the following chain of equalities:

$$\begin{aligned} i^\sharp_{w_1,w_2,m,\mathbb{Q}}(g) &= i^*_{w_2,m,\mathbb{Q}} \circ \mathsf{PT}_p(\gamma)_{\mathbb{Q}} \circ (i^*_{w_1,m,\mathbb{Q}})^{-1} = \\ &= \tau^*_{(x,L),\mathbb{Q}} \circ (i^*_{w_2,m|a_{w_2}^{-1}(x,L)})_{\mathbb{Q}} \circ \mathsf{PT}_p(\gamma)_{\mathbb{Q}} \circ (i^*_{w_1,m,\mathbb{Q}})^{-1} = \\ &= \tau^*_{(x,L),\mathbb{Q}} \circ \iota^*_{\mathbb{Q},t_2} \circ \mathsf{PT}^{\mathbb{Q}}_p(\gamma) \circ (\iota^*_{\mathbb{Q},t_1})^{-1} = \tau^*_{(x,L),\mathbb{Q}} \circ \mathsf{PT}^{\mathbb{Q}}_q(\widetilde{\gamma}), \end{aligned}$$

where the first equality follows by definition of $i^\sharp_{w_1,w_2,m}$ (see (6.9)), the second by Corollary 6.2.3, the third by (6.13) and the last by definition of parallel transport, via the isomorphism of local systems defined by $\iota^*_{\mathbb{Q}}$. Restricting to integral isometries, we get identity (6.12), thus concluding the proof. □

In the particular case of $(S_1,v_1,H_1) = (S_2,v_2,H_2) =: (S,v,H)$ we get the following

Corollary 6.3.3. *Let m,k be two positive integers, with $k>2$, and let (S,v,H) be an $(m,k)-$triple. Then the isomorphism $i^\sharp_{w,m}$ of Lemma 6.2.7 restricts to an injective morphism of groups*

$$i^\sharp_{w,m}\colon \operatorname{Mon}^2(K_v(S,H)) \longrightarrow \operatorname{Mon}^2(K_w(S,H))$$

satisfying the identity

$$i^\sharp_{w,m}(\mathsf{PT}_p(\gamma)) = \tau^*_{(x,L)} \circ \mathsf{PT}_q(\widetilde{\gamma}),$$

for every $\gamma \in \pi_1(T,p(K_v(S,H)))$ and for every deformation $p\colon \mathcal{X} \to T$ of $K_v(S,H)$, where

- *$q\colon \mathcal{Z} \to \widetilde{T}$ is the smooth deformation of $K_w(S,H)$ induced by the Stein factorization $\mathcal{Z} \xrightarrow{q} \widetilde{T} \xrightarrow{h} T$ of $p_{|\mathcal{Z}}\colon \mathcal{Z} \to T$, for any connected component $\mathcal{Z}$ of $\mathcal{X}^{ms}$;*
- *$\widetilde{\gamma}$ is the unique lift of γ via the finite étale cover h, such that $\mathcal{Z}_{\widetilde{\gamma}(0)} \simeq K_w(S,H)$;*
- *$\tau_{(x,L)} \in G_m \subseteq \operatorname{Aut}(K_v(S,H)^{ms})$ is the unique translation morphism (6.3) sending $K_w(S,H)$ to $\mathcal{Z}_{\widetilde{\gamma}(1)}$.*

We now prove the main result of this work, namely that the injective morphism

$$i^{\sharp}_{w,m}\colon \operatorname{Mon}^2(K_v(S,H)) \longrightarrow \operatorname{Mon}^2(K_w(S,H))$$

above introduced is actually an isomorphism, providing a complete description of the monodromy group of a singular moduli space of sheaves on an Abelian surface as above. In particular, for the proof of surjectivity of the morphism $i^{\sharp}_{w,m}$, a fundamental role will be played by the inclusion of groups

$$\mathsf{N}(K_v(S,H)) \subseteq \operatorname{Mon}^2_{\mathrm{lt}}(K_v(S,H)) = \operatorname{Mon}^2(K_v(S,H))$$

showed in Theorem 5.2.1, where the last equality follows again by Remark 6.3.1. The outcome is the following, which extends Markman's and Mongardi's result (see Theorem 3.4.3) to the non primitive case.

Theorem 6.3.4. *Let m, k be two positive integers, with $m \geq 1$ and $k > 2$, and let (S, v, H) be an $(m, k)-$triple. Then, there exists an isomorphism of groups*

$$i^{\sharp}_{w,m}\colon \operatorname{Mon}^2(K_v(S,H)) \longrightarrow \operatorname{Mon}^2(K_w(S,H)),$$

induced by the closed embedding $i_{w,m}\colon K_w(S,H) \to K_v(S,H)$ as a connected component the most singular locus of $K_v(S,H)$. In particular, we get an identification

$$\operatorname{Mon}^2(K_v(S,H)) = \mathsf{N}(K_v(S,H)). \tag{6.14}$$

Remark 6.3.5. The case $m = 1$ has been included for the sake of completeness and the non-trivial statement is identity (6.14), which is precisely Theorem 3.4.3. We wish to point out that, although several results of the previous Chapters apply to the the primitive case as well, the proof of Theorem 6.3.4 relies on the results of Theorem 3.4.3, hence it does not subside the latter.

Proof of Theorem 6.3.4. The key ingredient of the proof is the possibility to compare the two groups $\mathsf{N}(K_v(S,H))$ and $\mathsf{N}(K_w(S,H))$ under the identification

$$\mathrm{O}(\mathrm{H}^2(K_v(S,H),\mathbb{Z})) \simeq \mathrm{O}(\mathrm{H}^2(K_w(S,H),\mathbb{Z}))$$

given by $\lambda^{\sharp}_w \circ (\lambda^{\sharp}_v)^{-1}$ (see Theorem 3.3.4), in a way that is natural with respect to the morphism $i^{\sharp}_{w,m}$. This naturality is precisely guaranteed by Lemma 6.2.7, which states the identity

$$i^{\sharp}_{w,m} = \lambda^{\sharp}_w \circ (\lambda^{\sharp}_v)^{-1},$$

under the identification $v^{\perp} = w^{\perp}$. By Theorem 5.2.1, we have an inclusion of groups $\mathsf{N}(K_v(S,H)) \subseteq \operatorname{Mon}^2(K_v(S,H))$ and the injective morphism

$$i^{\sharp}_{w,m}\colon \operatorname{Mon}^2(K_v(S,H)) \to \operatorname{Mon}^2(K_w(S,H))$$

of Corollary 6.3.3 yields the following identification

$$i^{\sharp}_{w,m}(\mathsf{N}(K_v(S,H))) = \mathsf{N}(K_w(S,H)), \tag{6.15}$$

as both the groups involved are only defined lattice-theoretically, and as λ_v and λ_v are both isometries. The right hand side of equality (6.15) is precisely $\mathrm{Mon}^2(K_w(S,H))$, by Theorem 3.4.3, which is, on the other hand, the image of the morphism $i^{\sharp}_{w,m}$. In other words, the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Mon}^2(K_v(S,H)) \arrow[rrr, hook, "i^{\sharp}_{w,m}", "\text{Corollary 6.3.3}"'] & & & \mathrm{Mon}^2(K_w(S,H)) \arrow[d, equal, "\text{Theorem 3.4.3}"'] \\
\mathsf{N}(K_v(S,H)) \arrow[u, hook, "\cup\!\mid"', "\text{Theorem 5.2.1}"] \arrow[r, "(\lambda_v^{\sharp})^{-1}", "\sim"'] \arrow[rrr, bend right, "i^{\sharp}_{w,m}"'] & \mathsf{N}(v^{\perp}) \arrow[r, equal] & \mathsf{N}(w^{\perp}) \arrow[r, "\lambda_v^{\sharp}", "\sim"'] & \mathsf{N}(K_w(S,H))
\end{tikzcd}$$

is shown, concluding the proof. ☐

As the (locally trivial) monodromy group is a (locally trivial) deformation invariant, we can conclude the discussion by extending Theorem 6.3.4 to any irreducible symplectic variety X which is deformation equivalent to $K_v(S,H)$, for any $(m,k)-$triple (S,v,H) with $k > 2$.

Corollary 6.3.6. *Let X be an irreducible symplectic variety deformation equivalent to $K_v(S,H)$, where (S,v,H) is an $(m,k)-$triple with $m \geq 1$ and $k > 2$, and let Z be a connected component of the most singular locus of X. Then Z is an irreducible holomorphic symplectic manifold deformation equivalent to $K_w(S,H)$ and its closed embedding $i_{Z,X}\colon Z \to X$ induces an isomorphism of groups*

$$i^{\sharp}_{Z,X}\colon \mathrm{Mon}^2(X) \longrightarrow \mathrm{Mon}^2(Z).$$

Furthermore, we have an identification of groups

$$\mathrm{Mon}^2(X) = \mathsf{N}(X).$$

Proof. Let $p\colon \mathcal{X} \to T$ be a deformation of irreducible symplectic varieties and let $t_1, t_2 \in T$ such that $\mathcal{X}_{t_1} \simeq X$ and $\mathcal{X}_{t_2} \simeq K_v(S,H)$.

The first assertion was already essentially contained in the proof of Theorem 6.3.2 and it is a straightforward consequence of Remark 2.2.7 (3). In fact, for any connected component $\mathcal{Z}$ of the most singular locus of $\mathcal{X}$, the restriction $p_{|\mathcal{Z}}\colon \mathcal{Z} \to T$ factors through a finite étale cover $h\colon \widetilde{T} \to T$ and a smooth deformation of IHS manifolds $q\colon \mathcal{Z} \to \widetilde{T}$. Moreover, there exists a point $\widetilde{t_1} \in h^{-1}(t_1)$ such that $\mathcal{Z}_{\widetilde{t_1}}$ is isomorphic to a connected component Z of X^{ms} and there exists a point $\widetilde{t_2} \in h^{-1}(t_2)$ such that $\mathcal{Z}_{\widetilde{t_2}} \simeq K_w(S,H)$, thus proving the first assertion.

For the proof of the second part of the statement, being the (locally trivial) monodromy group a (locally trivial) deformation invariant, it is sufficient to define the morphism $i^\sharp_{Z,X}$ in such a way that its action on monodromy operators is natural with respect to the action of the morphism $i^\sharp_{w,m}$ described in Corollary 6.3.3.

For every $g \in \mathrm{Mon}^2(X)$, we can represent it as $g = \mathsf{PT}_{p'}(\gamma)$, where $p' \colon \mathcal{X}' \to T'$ is a flat family, with a point $t' \in T'$ such that $\mathcal{X}'_{t'} \simeq X$ and a loop γ centered at t'. As before, let $\mathcal{Z}'$ be the connected component of $(\mathcal{X}')^{ms}$ such that $\mathcal{Z}'_{t'} \simeq Z \simeq \mathcal{Z}_{t_1}$ and let $q' \colon \mathcal{Z}' \to \widetilde{T}'$ be the smooth deformation given by the Stein factorization of the restriction of p' to $\mathcal{Z}'$. Let $\widetilde{t'} \in (h')^{-1}(t)$ be the point in $\widetilde{T}$ such that $\mathcal{Z}'_{\widetilde{t'}} \simeq Z$, let $\widetilde{\gamma}$ be the unique lift of γ via h' such that $\widetilde{\gamma}(0) = \widetilde{t'}$ and notice that $\mathsf{PT}'_q(\widetilde{\gamma}) \in \mathsf{PT}^2(Z, \mathcal{Z}'_{\widetilde{\gamma}(1)})$, where $\mathcal{Z}'_{\widetilde{\gamma}(1)} \subseteq \mathcal{Z}_{t'}$ is a connected component of X^{ms}. By Corollary 6.2.2, there exists a group $G_m \subseteq \mathrm{Aut}(X^{ms})$ acting transitively on the set of connected components of X^{ms}, hence there exists an isomorphism $\tau \colon Z \to \mathcal{Z}'_{\widetilde{\gamma}(1)}$ such that $\tau^* \in \mathsf{PT}^2(\mathcal{Z}'_{\widetilde{\gamma}(1)}, Z)$. We set

$$i^\sharp_{Z,X}(g) := \tau^* \circ \mathsf{PT}_{q'}(\widetilde{\gamma}) \in \mathrm{Mon}^2(Z)$$

and we observe that this assignment is well defined and compatible with the composition law defined in Remarks 2.3.13 and 2.1.7 and, by construction, it satisfies the following identity

$$i^\sharp_{Z,X}(g) \circ i^*_{Z,X} = i^*_{Z,X} \circ g \colon \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(Z, \mathbb{Z}), \tag{6.16}$$

where $i_{Z,X}$ is the closed embedding of Z in X. Moreover, for any family $p \colon \mathcal{X} \to T$ as the one introduced at the beginning of the proof, with $t_1, t_2 \in T$ such that $\mathcal{X}_{t_1} \simeq X$ and $\mathcal{X}_{t_2} \simeq K_v(S, H)$, we can pick a smooth path δ in T from t_1 to t_2. Analogously, we can consider the smooth family $q \colon \mathcal{Z} \to \widetilde{T}$, with $\widetilde{t_i} \in h^{-1}(t_i)$ such that $\mathcal{Z}_{\widetilde{t_1}} \simeq Z$ and $\mathcal{Z}_{\widetilde{t_2}} \simeq K_w(S, H)$, and the unique lift $\widetilde{\delta}$ of δ via h such that $\widetilde{\delta}(0) = \widetilde{t_1}$. Notice that $\mathcal{Z}_{\widetilde{\delta}(1)}$ is, by construction, one of the connected components of $K_v(S, H)^{ms}$. Hence, by Proposition 6.1.2, there exists $(x, L) \in S[m] \times \hat{S}[m]$ such that $\mathcal{Z}_{\widetilde{\delta}(1)} \simeq a_w^{-1}(x, L)$, so let $\tau_{(x,L)} \in G_m$ be the automorphism of $K_v(S, H)^{ms}$ sending $K_w(S, H)$ to $a_w^{-1}(x, L)$. Then, the parallel transport operators $\mathsf{PT}_p(\delta) \in \mathsf{PT}^2(X, K_v(S, H))$ and $\mathsf{PT}_q(\widetilde{\delta}) \in \mathsf{PT}^2(Z, a_w^{-1}(x, L))$ in the two families defined above fit in the following commutation identity by construction:

$$i^*_{w,m} \circ \mathsf{PT}_p(\delta) = \tau^*_{(x,L)} \circ \mathsf{PT}_q(\delta) \circ i^*_{Z,X} \colon \mathrm{H}^2(X, \mathbb{Z}) \to \mathrm{H}^2(K_w(S, H), \mathbb{Z}). \tag{6.17}$$

Putting identities (6.16) and (6.17) together, we deduce that $i^\sharp_{Z,X}$ fits in the following commutative diagram:

$$
\begin{tikzcd}
\mathrm{Mon}^2(X) \arrow[r, "i^{\sharp}_{Z,X}"] \arrow[dd, "\mathsf{PT}_p(\delta)^{\sharp}"'] & \mathrm{Mon}^2(Z) \arrow[d, "\mathsf{PT}_q(\widetilde{\delta})^{\sharp}"] \\
 & \mathrm{Mon}^2(a_w^{-1}(x,L)) \arrow[d, "\tau^{\sharp}_{(x,L)}"] \\
\mathrm{Mon}^2(K_v(S,H)) \arrow[r, "i^{\sharp}_{w,m}"] & \mathrm{Mon}^2(K_w(S,H)),
\end{tikzcd}
\tag{6.18}
$$

where we are using Notation 6.2.6 for the isomorphisms $\mathsf{PT}_p(\delta)$, $\mathsf{PT}_q(\delta)$ and $\tau^*_{(x,L)}$. Since all the vertical arrows are isomorphisms by construction and $i^{\sharp}_{w,m}$ is an isomorphism by Theorem 6.3.4, the claim follows. □

Several consequences regarding the geometry of singular moduli spaces of sheaves on Abelian surfaces can be derived from this result. In the next Chapter, we will discuss both lattice-theoretic aspects, related to classification problems and Torelli Theorems (Section 7.1), and more explicit geometric applications, such as the SYZ conjecture (Section 7.2).

6.4 The cases $k = 1$ and $k = 2$

Before addressing some consequences of the main result of this Chapter, we conclude the discussion giving some motivations for the hypothesis $k > 2$ on the $(m,k)-$triples considered along Sections 6.2-6.3 (see Assumption 6.2.5).

We recall that, by Corollary 6.1.6, for any $(m,k)-$triple (S,v,H) with $v = mw$ a primitive Mukai vector and m,k two any positive integers, we can embed $K_w(S,H)$ in $K_v(S,H)$ as a connected component of its most singular locus. This still holds, in particular, for $k = 1,2$, but in these cases the associated primitive moduli spaces $K_w(S,H)$ and the corresponding maps $\lambda_w \colon w^{\perp} \to \mathrm{H}^2(K_w(S,H),\mathbb{Z})$ have an exceptional behavior (see Remark 3.1.19 and Table 3.2 for an overview).

Remark 6.4.1 (The case $k = 1$)**.** If $k = 1$, the moduli space $K_w(S,H)$ is isomorphic to a point and the map λ_w is the $0-$map. An analogue of Proposition 6.2.4 still holds, leading to the trivial statement $i^*_{w,m} = 0$, which suggests that no information about the locally trivial monodromy group of $K_v(S,H)$ can be recovered from the monodromy group of its most singular locus (which is, in fact, trivial), as motivated by the following example.

Example 6.4.2. If $(m,k) = (2,1)$, by Theorem 3.1.20 and Theorem 3.1.24, the moduli space $K_v(S,H)$ is an irreducible symplectic variety that admits a symplectic resolution $\widetilde{K_v(S,H)}$, deformation equivalent to the O'Grady's $6-$dimensional example OG_6 (see also part 4 of Section 1.1.2). In particular, the identifications of Remark

6.3.1 no longer hold in this case (see also Remark 3.1.22). The locally trivial monodromy group of $K_v(S,H)$ has been proven to be maximal, by [MR21, Proposition 4.14, Corollary 4.12]:

$$\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H)) = \mathrm{O}^+(\mathrm{H}^2(K_v(S,H),\mathbb{Z})). \tag{6.19}$$

In particular, by Remark 6.4.1, the only admissible group morphism

$$\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H)) \longrightarrow \mathrm{Mon}^2(K_w(S,H)) \simeq \{0\}$$

is the $0-$map.

Remark 6.4.3 (The case $k=2$)**.** If $k=2$, then $K_w(S,H)$ is a $K3$ surface and, in particular, it is a Kummer surface, by [Yos99b, Theorem 3.2]. Hence, by [Bor72],

$$\mathrm{Mon}^2(K_w(S,H)) = \mathrm{O}^+(\mathrm{H}^2(K_w(S,H),\mathbb{Z})),$$

but there is no natural way of comparing the orthogonal groups $\mathrm{O}(\mathrm{H}^2(K_v(S,H),\mathbb{Z}))$ and $\mathrm{O}(\mathrm{H}^2(K_w(S,H),\mathbb{Z}))$ using the isometries provided by Theorem 3.3.4, as explained in the following. Proposition 6.2.4 provides the identity

$$i^*_{w,m} = m\lambda_w \circ \lambda_v^{-1},$$

but in this case, invertibility of λ_w fails. In fact, by Theorem 3.3.4, the isometry $\lambda_w \colon w^\perp \to \mathrm{H}^2(K_w(S,H))$ is just injective and cannot be surjective, as $\mathrm{rk}(w^\perp)=7$ and $\mathrm{rk}(\mathrm{H}^2(K_w(S,H)))=22$.

We recall that, by Theorem 5.2.1, the inclusion

$$\mathrm{Mon}^2_{\mathrm{lt}}(K_v(S,H)) \supseteq \mathsf{N}(K_v(S,H)) \tag{6.20}$$

holds for any $(m,k)-$triple with $(m,k)\neq(1,1),(1,2)$. This means that the only excluded cases are precisely the degenerate ones given by points and K3 surfaces, but the inclusion still holds for any singular moduli space defined by $(m,k)-$triples with $m>1$ and $k\in\{1,2\}$.

Identity (6.19) of Example 6.4.2 shows that inclusion (6.20) can be proper, unlike the $k>2$ case. Together with Remarks 6.4.1 and 6.4.3, this suggests that a description of the locally trivial monodromy group of moduli spaces $K_v(S,H)$ given by $(m,k)-$triples with $k\in\{1,2\}$ might be different and may be achieved using different techniques. For this reasons, the cases $k=1$ and $k=2$ have been excluded from the last part of the work.

Chapter 7

Some Consequences and Applications

The discussion in the previous Chapters has led to a description of the monodromy group of any irreducible symplectic variety that is deformation equivalent to a singular moduli space of sheaves on an Abelian surface. In this Chapter, we present some consequences of the description obtained.

In Section 7.1, we discuss some straightforward applications of the main Theorem in its lattice-theoretic formulation (Theorem A.1), particularly in relation to the issue of the classification of these varieties. In Section 7.2, we present an explicit geometric application of the main result in its intrinsic formulation (Theorem B.1), building on the following general philosophy, that runs through several aspects of Algebraic Geometry. Roughly speaking, some birational properties are characterized by the presence of special classes of divisors, that are often preserved under the monodromy action. Hence, the study of a specific birational property can be reduced to the study of the monodromy orbits of the special classes of divisors associated to that property. The description of the locally trivial monodromy group turns out to be crucial, and in this case Theorem B.1 allows us to further reduce the problem to the smooth case. An example of application of this approach is given by the issue of the existence of a Lagrangian fibration on a fixed variety X, provided that there exists a nef and isotropic divisor. In fact, we will conclude the discussion with a proof of the SYZ Conjecture for all those irreducible symplectic varieties X that are deformation equivalent to singular moduli spaces of sheaves on Abelian surfaces as in Theorem A.1.

7.1 Lattice-theoretic issues and Global Torelli Theorem

We start with some straightforward consequences of Theorem 6.3.4, pointing out that, for any $(m,k)-$triple (S,v,H) the monodromy description of the moduli space

$K_v(S,H)$ only depends on the purely lattice-theoretic data k, exactly as their cohomological description in Theorem 3.3.4.

Remark 7.1.1. An immediate consequence of Theorem 6.3.4 is that, if (S,v,H) is an $(m,k)-$triple with $m \geq 1$ and $k > 2$, whereas the deformation class of the moduli space $K_v(S,H)$ is uniquely determined by both m and k (see Corollary 3.3.7), the isomorphism class of its monodromy group $\mathrm{Mon}^2(K_v(S,H))$ is uniquely determined by k. Indeed, if $m \neq m'$ are two positive integers and $v = mw$ and $v' = m'w$, then $i_{w,m}$ and $i_{w,m'}$ induce a natural isomorphism of groups

$$(i^\sharp_{w,m'})^{-1} \circ i^\sharp_{w,m} \colon \mathrm{Mon}^2(K_v(S,H)) \longrightarrow \mathrm{Mon}^2(K_{v'}(S,H)).$$

Another straightforward application of Theorem 6.3.4, in particular, of identity (6.14), is that we can deduce the monodromy index of the varieties under study (see Remark 3.4.4).

Corollary 7.1.2. *Let m,k be two positive integers, with $m \geq 1$ and $k > 2$, and let (S,v,H) be an $(m,k)-$triple. If X is an irreducible symplectic variety deformation equivalent to $K_v(S,H)$, then*

$$[\mathrm{O}^+(\mathrm{H}^2(X,\mathbb{Z}))\colon \mathrm{Mon}^2(X)] = 2^{\rho(k)}.$$

In particular, we get that the monodromy group of any irreducible symplectic variety X as above is never maximal, exactly as for those of Kummer type. We can rephrase Corollary 7.1.2 in terms of the formulation of Global Torelli Theorem for this deformation class, building on the discussion in Remark 2.3.23.

Corollary 7.1.3. *Let m,k be two positive integers, with $m \geq 1$ and $k > 2$, and let (S,v,H) be an $(m,k)-$triple. If X is an irreducible symplectic variety deformation equivalent to $K_v(S,H)$, there are exactly $2^{\rho(k)}$ irreducible symplectic varieties deformation equivalent to X, that are Hodge-isometric, but not bimeromorphic to X, for any generic weight 2 Hodge-isometry class of $\mathrm{H}^2(X,\mathbb{Z})$.*

In particular, Classic Bimeromorphic Global Torelli Theorem (Conjecture 2.0.3) never holds for this deformation class.

7.2 The SYZ Conjecture

In this Section we present a geometric application of the main results of this work, providing a proof of the SYZ conjecture for singular moduli spaces of sheaves on Abelian surfaces. The SYZ conjecture plays a significant role in the study of irreducible symplectic varieties, as it establishes a deep connection between algebraic and symplectic geometry and has important implications for their classification. In simple terms, it predicts that nef isotropic line bundles are related to the existence of Lagrangian fibrations (see Section 7.2.1). In the smooth case of irreducible holomorphic symplectic manifolds, the latter has been established for any known deformation class. For singular symplectic varieties the problem is widely open, but a substantial contribution has been made in the case of primitive symplectic varieties by the work of [OO25], where the conjecture is proven for singular moduli spaces of sheaves on K3 surfaces. We devote this Section to provide an adaptation of that proof to the Abelian case (see Section 7.2.2), relying on the monodromy description achieved in the previous Chapter.

7.2.1 Lagrangian fibrations

In this Section, we collect some essential notions and results concerning a special class of fibrations that is particularly relevant to the study and classification of symplectic varieties, namely Lagrangian fibrations. We then formulate the SYZ conjecture and provide an overview of the current state of the art.

Definition 7.2.1. Let $n \in \mathbb{N} \setminus \{0\}$, let (X, σ) be a primitive symplectic variety (see Definition 1.2.9 (3)) of dimension $2n$ and let X_{reg} be its smooth locus.

(1) A subvariety $Z \subseteq X$ of dimension n is called *Lagrangian* if $Z \cap X_{\mathrm{reg}} \neq \emptyset$ and $\sigma|_{X_{\mathrm{reg}} \cap Z_{\mathrm{reg}}} = 0$.

(2) A *Lagrangian fibration on* X is a surjective morphism $f \colon X \to B$ onto a normal Kähler space, with connected fibers and such that the general fiber of f is a Lagrangian subvariety of X.

A description of the main properties of Lagrangian fibrations on primitive symplectic varieties, together with a sufficient characterization, is given by the following Theorem, which generalizes to the singular setting several foundational results due to Matsushita.

Theorem 7.2.2. *Let X be a primitive symplectic variety of dimension $2n$ and let $f \colon X \to B$ be a surjective morphism with connected fibers onto a normal Kähler space of dimension $0 < \dim B < 2n$. Then $f \colon X \to B$ is a Lagrangian fibration. Furthermore:*

(1) The base B is a $\mathbb{Q}-$factorial projective klt variety of Picard rank 1. If, additionally, X is irreducible symplectic, then B is Fano and, if B is smooth, then $B \simeq \mathbb{P}^n$.

(2) The general fiber of f is an Abelian variety of dimension n completely contained in X_{reg}.

(3) The map f is equidimensional and all irreducible components of each fiber are Lagrangian subvarieties.

Proof. See [Sch20, Theorem 3], [KL25, Theorem 2.8]. □

We recall that primitive symplectic varieties, being compact Kähler spaces, have a well defined Kähler cone, which is an open cone in $\mathrm{H}^{1,1}(X, \mathbb{R})$ (see Section 1.2). A class $\alpha \in \mathrm{H}^{1,1}(X, \mathbb{R})$ is said to be *nef* if it belongs to the closure of the Kähler cone.

Let X be a primitive symplectic variety and let q_X denote its Beauville-Bogomolov-Fujiki form (see part 2 of Section 1.2.2). If $f \colon X \to B$ is a Lagrangian fibration on X and $b = \mathrm{c}_1(f^*\mathcal{O}_B(1))$ is the class of a fiber of f, then b is nef and $q_X(b) = 0$. The SYZ conjecture claims that the converse holds.

Conjecture 7.2.3. (SYZ Conjecture for primitive symplectic varieties) Let X be a primitive symplectic variety and let L be a line bundle on X. If L is nef and $q_X(L) = 0$, then there exists a Lagrangian fibration $f\colon X \to B$ such that $L = f^*\mathcal{O}_B(1)$.

In that case, we will say that L *induces a Lagrangian fibration* on X.

In the smooth case, the SYZ conjecture has been proven true for any known deformation class, thanks to the work of [BM14], [Mar19], [Mat17], [Wie16], [Yos16], [Wie18], [MR21] and [MO22]. In the singular setting, the first distinguished class of primitive symplectic varieties for which the latter has been established is the one of singular moduli spaces of sheaves on K3 surfaces, by [OO25]. In the next Section, we will show that the SYZ conjecture holds also in the Abelian case, by applying the general theory developed by [OO25] for primitive symplectic varieties and exploiting the monodromy description provided in Chapter 5.

7.2.2 The SYZ conjecture for singular moduli spaces of sheaves on Abelian surfaces

In this Section we will provide a proof for the SYZ conjecture for irreducible symplectic varieties that are deformations of moduli spaces of sheaves $K_v(S, H)$, for any $(m, k)-$triple (S, v, H), with $m \geq 1$ and $k > 2$.

For those deformation classes for which the SYZ conjecture has been established, a description of the monodromy orbit of primitive isotropic elements of the Beauville-Bogomolov-Fujiki lattice has turned out to be crucial. Building on the strategy developed by [OO25] for moduli spaces of sheaves on K3 surfaces, we use the relation between $\mathrm{Mon}^2(K_v(S, H))$ and $\mathrm{Mon}^2(K_w(S, H))$ to reduce this problem to the above-mentioned classification in the smooth case, provided by [Wie18, Section 5].

The goal of this section is the following result:

Theorem 7.2.4. *Let $m \geq 1$ and $k > 2$ be two integers, let (S, v, H) be an $(m, k)-$triple and let X be an irreducible symplectic variety deformation equivalent to $K_v(S, H)$. If L is a nef line bundle on X such that $q_X(L) = 0$, then there exists a Lagrangian fibration $f\colon X \to B$ such that $L = f^*\mathcal{O}_B(1)$.*

Deformations of Lagrangian fibrations

The first key ingredient for the proof of Theorem 7.2.4 is a result of [OO25] concerning locally trivial deformations of Lagrangian fibrations on primitive symplectic varieties.

Definition 7.2.5. (1) A *locally trivial family of Lagrangian fibrations* is a locally trivial family $p\colon \mathcal{X} \to T$ of primitive symplectic varieties (see Definition 2.2.1 (3)) such that there exists a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{B} \arrow[dl, "s"] \\ & T & \end{tikzcd}$$

where f is a $T-$morphism, s is projective and, for any $t \in T$, the restriction $f_t \colon \mathcal{X}_t \to \mathcal{B}_t$ is a Lagrangian fibration. We will denote by $p \colon \mathcal{X}/\mathcal{B} \to T$ a locally trivial deformation of Lagrangian fibrations as above.

(2) If X_1 and X_2 are two primitive symplectic varieties, two Lagrangian fibrations $f_i \colon X_i \to B_i$ on X_i, for $i = 1, 2$, are said to be *locally trivial deformation equivalent* if there exists a locally trivial deformation of Lagrangian fibrations $p \colon \mathcal{X}/\mathcal{B} \to T$ and two points $t_1, t_2 \in T$ such that $f_{t_i} = f_i$ for $i = 1, 2$.

The following generalization to the singular setting of several results due to Matsushita will play a central role in the proof of Theorem 7.2.4.

Theorem 7.2.6. *For $i = 1, 2$, let X_i be a $\mathbb{Q}-$factorial and terminal primitive symplectic variety and let $L_i \in \mathrm{NS}(X_i)$ be a nef divisor such that $q_{X_i}(L_i) = 0$. Suppose that L_1 induces a Lagrangian fibration on X_1. If there exists a parallel transport operator $g \in \mathsf{PT}^2(X_1, X_2)$ such that $g(L_1) = L_2$, then*

(1) L_2 induces a Lagrangian fibration on X_2;

(2) X_1 and X_2 are (locally trivial) deformation equivalent as Lagrangian fibrations.

Proof. See [OO25, Theorem 3.1]. □

Here, the terminology "locally trivial" has been again omitted, as we are again restricting to deal with $\mathbb{Q}-$factorial and terminal symplectic varieties (see Remarks 3.1.22 and 6.3.1). Theorem 7.2.6 will be used to construct a Lagrangian fibration on any irreducible symplectic variety as in Theorem 7.2.4 as deformation of a *Beauville-Mukai system*.

Beauville-Mukai systems

Let $m \geq 1$ and $k > 2$ be two positive integers and let us consider an $(m, k)-$triple (S, v, H), with S an Abelian surface such that $\mathrm{NS}(S) = \mathbb{Z}h$, with h the class of a polarization H of degree $h^2 = 2k$ and $v = (0, mh, 0)$ a Mukai vector. Mapping each sheaf $F \in K_v(S, H)$ to its Fitting support (see [Eis95, Section V.1.3]) defines a surjective morphism (see [PR23, Section 3.1.2])

$$f_v \colon K_v(S, H) \longrightarrow |mH| \simeq \mathbb{P}^{m^2k-1}. \tag{7.1}$$

For any smooth and irreducible curve $C \in |mH|$, the fiber $f_v^{-1}(C)$ coincides with an Abelian subvariety of the Jacobian $\mathrm{Jac}^d(C)$, with $d = m^2k$ (see [Wie18, Section 6.20]). By Stein's factorization Theorem, the morphism f_v has connected fibers and, by Theorem 7.2.2, it is a Lagrangian fibration.

Definition 7.2.7 (Beauville-Mukai system of $(m,k)-$type)**.** Let (S,v,H) be an $(m,k)-$triple as above. A Lagrangian fibration $f_v\colon K_v(S,H) \to |mH|$ defined as in (7.1) will be called *Beauville-Mukai system of $(m,k)-$type*.

If $m=1$, then $v=w=(0,h,0)$ yields a Lagrangian fibration $f_w\colon K_w(S,H) \to |H|$ on an irreducible holomorphic symplectic manifold of type Kum^{k-1}, known as *Beauville-Mukai system of generalized Kummer type* (see [Wie18, Section 6]).

Remark 7.2.8. It is straightforward from Theorem 3.2.1 that, for any $(m,k)-$triple (S,v,H) with $m \geq 1$ and $k>2$, any irreducible symplectic variety X that is deformation of a moduli space $K_v(S,H)$ is again deformation equivalent to a Beauville-Mukai system of $(m,k)-$type, in the sense of Definition 2.2.1 (4).

If (S,v,H) is an $(m,k)-$triple as above, with $v=(0,mh,0)$, then the class $b=(0,0,1)\in \widetilde{\mathrm{H}}(S,\mathbb{Z})$ naturally belongs to $v^\perp$. Its image $\lambda_v(b)\in \mathrm{H}^{1,1}(K_v(S,H),\mathbb{Z})\simeq \mathrm{Pic}(K_v(S,H))$ via the Hodge isometry $\lambda_v\colon v^\perp \to \mathrm{H}^2(K_v(S,H))$ (see Theorem 3.3.4) is an isotropic class which defines the Lagrangian fibration f_v. This is the content of the following Lemma, which is an adaptation to the Abelian case of Lemma 4.3 of [OO25].

Lemma 7.2.9. *Let $f_v\colon K_v(S,H)\to |mH|\simeq \mathbb{P}^n$ be a Beauville-Mukai system of $(m,k)-$type, with $n=m^2k-1$, and set $b=(0,0,1)\in v^\perp$. Then*

$$f_v^*\mathcal{O}_{\mathbb{P}^n}(1)=\lambda_v(b).$$

Proof. The considered Beauville-Mukai system of $(m,k)-$type fits in the following commutative diagram

$$\begin{array}{ccc} K_w(S,H) & \xhookrightarrow{i_{w,m}} & K_v(S,H) \\ {\scriptstyle f_w}\downarrow & & \downarrow{\scriptstyle f_v} \\ \mathbb{P}^{k-1}\simeq |H| & \xrightarrow[\nu_m]{} & |mH|\simeq \mathbb{P}^n, \end{array}$$

where $i_{w,m}$ is the embedding of $K_w(S,H)$ into $K_v(S,H)$ defined in Section 6.1, the morphism f_w is the Beauville-Mukai system of Kummer type associated to the $(1,k)-$triple (S,w,H) and ν_m is the map induced by the m-th Veronese embedding. By commutativity of the diagram, we get

$$(i_{w,m}^*\circ f_v^*)(\mathcal{O}_{\mathbb{P}^n}(1))=(f_w^*\circ \nu_m^*)(\mathcal{O}_{\mathbb{P}^n}(1)).$$

By definition, we have $\nu_w^*\mathcal{O}_{\mathbb{P}^n}(1)=\mathcal{O}_{\mathbb{P}^{k-1}}(m)=m\mathcal{O}_{\mathbb{P}^{k-1}}(1)$ and, by [Wie18, Proposition 6.29 (iii)], it holds $f_w^*\mathcal{O}_{\mathbb{P}^{k-1}}(1)=\lambda_w(b)$. Putting identities together we get

$$(i_{w,m}^*\circ f_v^*)(\mathcal{O}_{\mathbb{P}^n}(1))=m\lambda_w(b).$$

By Lemma 6.2.4, it holds $i_{w,m}^*=m\lambda_w\circ\lambda_v^{-1}$, hence the claim follows. $\square$

Deformations of Beauville-Mukai systems

With the notions collected so far, we are finally in the position to prove Theorem 7.2.4, by adapting the proof of [OO25, Proposition 6.2] to our setting.

Proof of Theorem 7.2.4. Let $m \geq 1$ and $k > 2$ be two positive integers and let X be an irreducible symplectic variety deformation equivalent to $K_v(S,H)$ for an $(m,k)-$triple (S,v,H). Let Z be a connected component of its most singular locus, which, by Corollary 6.3.6, is an irreducible holomorphic symplectic manifold deformation equivalent to $K_w(S,H)$.

Let $l = \mathrm{c}_1(L) \in \mathrm{NS}(X)$ be the class of an isotropic nef line bundle L on X. By Lemma 6.2.4 and Corollary 6.3.6, the pullback

$$i^*_{Z,X}\colon \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(Z,\mathbb{Z})$$

of the closed embedding $i_{Z,X}\colon Z \hookrightarrow X$ is m times a Hodge isometry. Hence, there exists a unique class $l_0 \in \mathrm{NS}(Z)$ of divisibility $\mathrm{div}(l_0) = \mathrm{div}(l)$ such that $i^*_{Z,X}(l) = ml_0$. By Proposition 6.29 and Proposition 6.32 of [Wie18], there exists a Beauville-Mukai system of Kummer type $(1,k)$

$$f_{w'}\colon Z' := K_{w'}(S',H') \to |H'| \simeq \mathbb{P}^{k-1}$$

and a parallel transport operator $g_0 \in \mathsf{PT}^2(Z,Z')$ such that $g_0(l_0) = f^*_{w'}\mathcal{O}_{\mathbb{P}^{k-1}}(1)$. Let us now set $v' = mw'$, $n = m^2k - 1$ and let us consider the Beauville-Mukai system of $(m,k)-$type

$$f_{v'}\colon X' := K_{v'}(S',H') \longrightarrow |mH'| \simeq \mathbb{P}^n.$$

We will now construct a parallel transport operator $g \in \mathsf{PT}^2(X,X')$ such that the following diagram

$$\begin{array}{ccc} \mathrm{H}^2(X,\mathbb{Z}) & \xrightarrow{g} & \mathrm{H}^2(X',\mathbb{Z}) \\ {\scriptstyle i^*_{Z,X}}\big\downarrow & & \big\downarrow{\scriptstyle i^*_{w',m}} \\ \mathrm{H}^2(Z,\mathbb{Z}) & \xrightarrow{g_0} & \mathrm{H}^2(Z',\mathbb{Z}) \end{array} \tag{7.2}$$

is commutative. Let $g' \in \mathsf{PT}^2(X',X)$ be a parallel transport operator and let $g'_0 \in \mathsf{PT}^2(Z',Z)$ be the corresponding parallel transport operator, under the injective map defined by Theorem 6.3.2 and Corollary 6.3.6. More precisely, suppose that $g' = \mathsf{PT}_p(\gamma)$, with $p\colon \mathcal{X} \to T$ a deformation of primitive symplectic varieties, with $t,t' \in T$ such that $\mathcal{X}_t \simeq X$ and $\mathcal{X}_{t'} \simeq X'$, and γ a continuous path in T from t' to t. Let $\mathcal{Z} \xrightarrow{q} \widetilde{T} \xrightarrow{h} T$ be the Stein factorization of $p_{|\mathcal{Z}}\colon \mathcal{Z} \to T$, for any connected component $\mathcal{Z}$ of $\mathcal{X}^{ms}$ (see Remark 2.2.7 (3)), yielding a smooth family $q\colon \mathcal{Z} \to \widetilde{T}$ of IHS manifolds with both Z and Z' as fibers. Let $\widetilde{t'} \in h^{-1}(t')$ be the unique point such that $\mathcal{Z}_{\widetilde{t'}} \simeq Z'$ and let $\widetilde{\gamma}$ be the unique lift of γ via the finite étale cover h, such that $\widetilde{\gamma}(0) = \widetilde{t}'$. Finally, let $\tau \in G_m$ be the automorphism of X^{ms} sending Z to $\mathcal{Z}_{\widetilde{\gamma}(1)}$ (see Corollaries 6.1.5 and 6.2.2). Then

$$g_0' := \tau^* \circ \mathsf{PT}_q(\widetilde{\gamma}) \in \mathsf{PT}^2(Z', Z). \tag{7.3}$$

Let us consider $g_0' \circ g_0 \in \mathrm{Mon}^2(Z)$ and $(i^\sharp_{Z,X})^{-1}(g_0' \circ g_0) \in \mathrm{Mon}^2(X)$. Then

$$g := (g')^{-1} \circ (i^\sharp_{Z,X})^{-1}(g_0' \circ g_0) \in \mathsf{PT}^2(X, X')$$

is such that

$$\begin{aligned} i^*_{w',m} \circ g = i^*_{w',m} &\circ (g')^{-1} \circ (i^\sharp_{Z,X})^{-1}(g_0' \circ g_0) = \\ &= (\tau^* \circ \mathsf{PT}_q(\widetilde{\gamma}))^{-1} \circ i^*_{Z,X} \circ (i^\sharp_{Z,X})^{-1}(g_0' \circ g_0) = \\ &= (g_0')^{-1} \circ i^*_{Z,X} \circ (i^\sharp_{Z,X})^{-1}(g_0' \circ g_0) = \\ &= (g_0')^{-1} \circ i^\sharp_{Z,X} \circ \circ (i^\sharp_{Z,X})^{-1}(g_0' \circ g_0) \circ i^*_{Z,X} = \\ &= (g_0')^{-1} \circ g_0' \circ g_0 \circ i^*_{Z,X} = g_0 \circ i^*_{Z,X}, \end{aligned}$$

where the second equality follows from (6.17), the third from (7.3) and the fourth from (6.16). Commutativity of diagram (7.2) is then proven, and from the latter we deduce that

$$i^*_{w',m}(g(l)) = g_0(i^*_{Z,X}(l)) = g_0(ml_0) = mf^*_{w'}\mathcal{O}_{\mathbb{P}^{k-1}}(1) = i^*_{w',m}(f^*_{v'}\mathcal{O}_{\mathbb{P}^n}(1)),$$

where the last equality follows from Lemma 7.2.9. By injectivity of $i^*_{w',m}$, we obtain

$$g(l) = f^*_{v'}\mathcal{O}_{\mathbb{P}^n}(1).$$

By Theorem 7.2.6, we conclude that L induces a Lagrangian fibration $f\colon X \to B$ on X, that is deformation equivalent to the Beauville-Mukai system $f_{v'}\colon X' \to |mH'|$, in the sense of Definition 7.2.5. □

Remark 7.2.10. Notice that the proof of Theorem 7.2.4 can be applied in particular to the case in which X is already a Beauville-Mukai system, to show that, for any fixed pair (m,k), with $m \geq 1$ and $k > 2$, all Beauville-Mukai systems of $(m,k)-$type are deformation equivalent as Lagrangian fibrations. In light of this, Definition 7.2.7 precisely identifies a (locally trivial) deformation equivalence class of Lagrangian fibrations.

7.2.3 Polarization type

Let (S,v,H) be an $(m,k)-$triple with $m \geq 1$ and $k > 2$, and consider an irreducible symplectic variety X that is deformation equivalent to a moduli space $K_v(S,H)$.

Theorem 7.2.4 guarantees that any nef and isotropic line bundle L on X induces a Lagrangian fibration $f\colon X \to B$ that is deformation equivalent to a suitable Beauville-Mukai system (Definition 7.2.7). The *polarization type* (Definition 7.2.12) of a Lagrangian fibration is a locally trivial deformation invariant and its computation for Beauville-Mukai systems is due to [Wie18]. Hence, as a consequence of Theorem 7.2.4, we get a description of the polarization type on any Lagrangian fibration on X, induced by L, as above.

We recall that, for any Abelian variety F of dimension n and any positive definite line bundle L on F - generally called *polarization* on F, with a slight abuse of notation - there exists an $n-$uple $(d_1, \dots, d_n)$ of positive integers, such that $d_i | d_{i+1}$ for any $i = 1, \dots, n-1$, uniquely determined by $\operatorname{Im} c_1(L)$ (see [BL04, Section 3.1]). Then one can define the *polarization type* of L (see [BL04, Section 4.1]) as

$$\underline{d}(L) = \underline{d}(c_1(L)) := (d_1, \dots, d_n). \tag{7.4}$$

By Theorem 7.2.2 (2), we deduce that the general fiber F of a Lagrangian fibration is an Abelian variety, even when X is not necessarily projective. The following results of [Wie16, Section 4] yield a well defined notion of *polarization type* of a Lagrangian fibration on a primitive symplectic variety, starting from a polarization on a smooth fiber.

Proposition 7.2.11. *Let $f\colon X \to B$ be a Lagrangian fibration on a primitive symplectic variety X.*

(1) For any smooth fiber F of f, there exists a Kähler class $\omega \in \mathrm{H}^{1,1}(X, \mathbb{R})$ whose restriction $\omega_{|F}$ is integral and primitive, i.e. it is contained in $\mathrm{H}^2(F, \mathbb{Z})$ and indivisible, thus defining a polarization on F.

(2) The polarization type $\underline{d}(\omega_{|F})$ of $\omega_{|F}$ does not depend on the smooth fiber F.

Proof. (1) This is the content of [Wie16, Proposition 4.3], whose proof works verbatim in our setting as well, as the Kähler cone of a primitive symplectic variety is an open cone in $\mathrm{H}^{1,1}(X, \mathbb{R})$, which admits a pure weight 2 Hodge structure, by Remark 1.2.10.

(2) The statement is precisely [Wie16, Proposition 4.10], which again applies to our setting, by comparing [Wie16, Lemma 4.4 and Section 4] with [OO25, Lemma 2.3 and Section 2.1]. In particular, this follows from the fact that the polarization type of a Lagrangian fibration $f\colon X \to B$ defined in [OO25, Definition 2.4] and [Wie16, Definition 4.8] coincides with

$$\underline{d}(f) := \underline{d}(\omega_{|F}), \tag{7.5}$$

for any smooth fiber F of f (see [Wie18, Definition 3.2, Theorem 3.3]). □

Definition 7.2.12. Let $f\colon X \to B$ be a Lagrangian fibration on a primitive symplectic variety X. We define the *polarization type* of f as the polarization type

$$\underline{\mathrm{d}}(f) := \underline{\mathrm{d}}(\omega_{|F}),$$

for any smooth fiber F of f, as in equation (7.5), where $\omega \in \mathrm{H}^{1,1}(X,\mathbb{R})$ is a Kähler satisfying condition (1) of Proposition 7.2.11 with respect to F.

A key feature of the polarization type as defined above is its invariance under locally trivial deformations of Lagrangian fibrations, as stated below.

Theorem 7.2.13. *The polarization type of a Lagrangian fibration is invariant under locally trivial deformations of Lagrangian fibrations, in the sense of Definition 7.2.5.*

Proof. See [Wie16, Theorem 4.9], see also [Kim25, Corollary 3.32]. □

As an application of Theorem 7.2.4, Theorem 7.2.13 and Wieneck's computation ([Wie18]) in the smooth case, we get the following.

Corollary 7.2.14. *Let X be an irreducible symplectic variety that is deformation equivalent to a moduli space of sheaves $K_v(S,H)$, with (S,v,H) an $(m,k)-$triple, with $m \geq 1$ and $k > 2$. Then, any nef and isotropic line bundle L on X, of divisibility $d := \mathrm{div}(\mathrm{c}_1(L))$, induces a Lagrangian fibration $f\colon X \to B$ with polarization type*

$$\underline{\mathrm{d}}(f) = (1,\ldots,1,d,\frac{m^2k}{d}).$$

Proof. By Theorem 7.2.4, the line bundle L induces a Lagrangian fibration $f\colon X \to B$ that is deformation equivalent to a Beauville-Mukai system of $(m,k)-$type

$$f_{v'}\colon K_{v'}(S',H') \to \mathbb{P}^n,$$

where $n = m^2k-1$ and $\mathrm{div}(\mathrm{c}_1(f_{v'}^*\mathcal{O}_{\mathbb{P}^n}(1))) = \mathrm{div}(\mathrm{c}_1(L)) = d$ (see also Lemma 7.2.9). Wieneck's computation ([Wie18, Theorem 6.27, Proposition 6.29 (iv)]) of the polarization type of Beauville-Mukai systems of Kummer type $(1,k)$ applies to the $(m,k)-$type as well, yielding $\underline{\mathrm{d}}(f_{v'}) = (1,\ldots,1,d,\frac{m^2k}{d})$. The proof is now concluded, as $\underline{\mathrm{d}}(f) = \underline{\mathrm{d}}(f_{v'})$, by Theorem 7.2.13. □

Appendix

Background notions and complements

Appendix A

Moduli spaces of sheaves: general theory and construction

This Appendix is meant to provide a brief overview of the construction and the foundational results concerning moduli spaces of coherent sheaves on polarized projective varieties, following [HL97] as main reference.

This theory was pioneered in the 1960s by David Mumford, whose case of study was a parametrizing space for vector bundles of fixed rank and degree over a smooth projective curve. This theory was later generalized for sheaves on higher dimensional varieties by Takemoto, Gieseker, Maruyama, and Simpson during the 1980s. Nowadays it remains a central and highly active field of research, as it provide a powerful systematic method for constructing and investigating new examples of higher-dimensional algebraic varieties with specific geometric properties.

The ideal goal of this theory is to parametrize coherent sheaves on a fixed variety X, aiming to produce a parametrizing space M that not only carries the structure of a variety, but also reflects some geometric aspects of the underlying variety X. This can be done in a reasonable way only by putting some constraints on the sheaves to be parametrized, namely fixing some numerical invariants and restricting to those sheaves satisfying some suitable *stability condition*.

A.1 Stability conditions for coherent sheaves

Although we will be primarily interested in the notion of Gieseker stability, for later use we introduce also the notion of Mumford-Takemoto stability, which was the first one to be introduced, as naive generalization of the stability notion introduced in the case of curves. Moreover, this notion is strongly related to the differential counterpart of this theory related to the Calabi conjecture, predicting the existence of an optimal metric solving the Yang-Mills equations, known as *Kähler-Einstein metric*. This relation is established by *Kobayashi-Hitchin correspondence*, by the work of Narasimhan-Seshadri, Donaldson and Uhlenbeck-Yau, which relates the existence of such a metric to the notion of Mumford-Takemoto (poly)stability of the tangent bundle.

Let X be a complex smooth projective variety and let H be a *polarization* on X, i.e. an ample line bundle $H \in \mathrm{Pic}(X)$. In the following, we will often refer to the pair (X, H) as a smooth *polarized* variety.

Definition A.1.1 (Mumford-Takemoto stability)**.** Let F be a coherent sheaf on X of dimension $\dim(Supp(F)) = \dim X =: d$.

(1) We define the $H-$*slope* of F as the rational number

$$\mu_H(F) := \frac{c_1(F) \cdot H^{d-1}}{\mathrm{rk}(F)}.$$

(2) We say that F is μ_H-*stable* (respectively, μ_H-*semistable*) if, for any subsheaf $E \subseteq F$ such that $\mathrm{rk}(E) < \mathrm{rk}(F)$, it holds $\mu_H(E) < \mu_H(F)$ (respectively, $\mu_H(E) \leq \mu_H(F)$).

(3) If F is a μ_H-semistable sheaf, any subsheaf $E \subseteq F$ as above realizing the identity $\mu_H(E) = \mu_H(F)$ will be called μ_H-*destabilizing* sheaf. If any μ_H-destabilizing sheaf E exists, then F will be called *strictly* μ_H-*semistable*.

While Mumford-Takemoto stability seems more natural and closely paralleling the theory of stability for curves, the notion of stability which is best suited for the construction of moduli spaces as projective schemes is Gieseker stability. In fact, the latter allows to include also rank 0 sheaves in the discussion. Before introducing the notion of Gieseker stability, we recall that, for any coherent sheaf F on a polarized variety (X, H), its *Hilbert polynomial* is the polynomial $P_H(F) \in \mathbb{Q}[t]$ defined, for any $m \in \mathbb{Z}$, by

$$P_H(F)(m) := \chi(F \otimes \mathcal{O}_X(H)^{\otimes m}) = \sum_{i=0}^{\dim(X)} (-1)^i h^i(X, F \otimes \mathcal{O}_X(H)^{\otimes m}). \tag{A.1}$$

By [HL97, Lemma 1.2.1], if F is a coherent sheaf of dimension d, then its Hilbert polynomial can be uniquely written in the form

$$P_H(F)(m) = \sum_{i=0}^{d} \alpha_i^H(F) \frac{m^i}{i!}. \tag{A.2}$$

If $F \neq 0$, then the coefficient $\alpha_d^H(F)$ in (A.2) is always positive, and might depend on both the rank of F and the degree of X with respect to the polarization H. We define the *reduced Hilbert polynomial* of F as the polynomial $p_H(F) \in \mathbb{Q}[t]$ given by

$$p_H(F)(m) := \frac{P_H(F)(m)}{\alpha_d^H(F)},$$

for any $m \in \mathbb{Z}$. Considering the lexicographic order $\leq$ on $\mathbb{Q}[t]$, we can give the following definition.

Definition A.1.2 (Gieseker stability)**.** Let (X, H) be a smooth polarized variety.

(1) A coherent sheaf F on X of pure dimension d is said to be *(Gieseker) $H-$stable* (respectively, *(Gieseker) $H-$semistable*) if, for any proper subsheaf $E \subsetneq F$, it holds $p_H(E) < p_H(F)$ (respectively, $p_H(E) \leq p_H(F)$).

(2) If F is a $H-$semistable sheaf, any proper subsheaf $E \subsetneq F$ as above realizing the identity $p_H(E) = p_H(F)$ will be called *$H-$destabilizing* sheaf. If any $H-$destabilizing sheaf E exists, then F will be called *strictly $H-$semistable*.

Remark A.1.3. We highlight some basic properties of stable and semistable sheaves, according to both Mumford-Takemoto and Gieseker stability conditions.

(1) Every $H-$stable sheaf F is simple, i.e. $\mathrm{End}(F) \simeq \mathbb{C}$ (see [HL97, Corollary 1.2.8]).

(2) If $X = C$ is a smooth projective curve, then Riemann-Roch Theorem shows that the two above-introduced stability conditions coincide, namely a coherent sheaf F on C is μ_H-(semi)stable if and only if it is $H-$(semi)stable.

(3) Any line bundle is both μ_H-stable and $H-$stable by definition.

(4) The direct sum of two line bundles of different degrees is not even $H-$semistable (see [HL97, Example 1.2.10]).

(5) If $X = C$ is a projective curve of genus $g \geq 2$, and F_1 and F_2 are two non-isomorphic stable vector bundles, then any non-trivial extension $0 \to F_1 \to F \to F_2 \to 0$ provides a sheaf F that is semistable, but not stable (see [HL97, Example 1.2.10]).

(6) If F is a pure coherent sheaf of dimension $\dim(X)$, then the following chain of implications holds (see [HL97, Lemma 1.2.13]):

$$\begin{gathered} F \text{ is } \mu_H-\text{stable} \implies F \text{ is } H-\text{stable} \implies \\ \implies F \text{ is } H-\text{semistable} \implies F \text{ is } \mu_H-\text{semistable}. \end{gathered}$$

All the reversed implications hold, for instance, if $\mathrm{rk}(F)$ and $\deg(F)$ are coprime (see [HL97, Lemma 1.2.14]).

From a classificatory point of view, stable and semistable sheaves play a relevant role, due to the existence of two special filtrations. As motivating example, let us start by recalling that vector bundles on the projective line $\mathbb{P}^1$ are completely classified by Birkhoff-Grothendieck Theorem, which establishes that the latter can be uniquely decomposed as direct sums of line bundles of the form $\mathcal{O}_{\mathbb{P}^1}(a)$ for suitable integers $a \in \mathbb{Z}$. Moreover, those summands arise as the factors of a not unique split filtration of special vector bundles.

More generally, given a smooth polarized variety (X, H) and a non-trivial pure-dimensional sheaf F on X, it admits a unique finite increasing filtration $HN_\bullet(F)$, called *Harder-Narasimhan filtration* for F, whose factors

$$gr_i^{HN}(F) := HN_i(F)/HN_{i-1}(F)$$

are $H-$semistable sheaves with strictly decreasing reduced Hilbert polynomials (see [HL97, Definition 1.3.2]). In light of this, we can think of semistable sheaves as building blocks for arbitrary pure dimensional sheaves. In turn, any semistable sheaf admits a further filtration that identifies stable sheaves as their ultimate building blocks.

Definition A.1.4. Let F be a $H-$semistable sheaf on X. A *Jordan-Hölder filtration* of E is an increasing filtration

$$0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_l = F$$

such that the factors $gr_i(F) := F_i/F_{i-1}$ are $H-$stable sheaves with $p_H(gr_i(F)) = p_H(F)$ for any $i = 1, \ldots, l$.

By [HL97, Proposition 1.5.2], Jordan-Hölder filtrations always exist, and the sheaf

$$gr^{JH}(F) = \bigoplus_{i=0}^{l} gr_i(F) \tag{A.3}$$

is unique up to isomorphism. We can therefore give the following definition.

Definition A.1.5. (1) Two $H-$semistable sheaves F_1 and F_2 with $p_H(F_1) = p_H(F_2)$ are said to be $S-$*equivalent* if $gr^{JH}(F_1) \simeq gr^{JH}(F_2)$.

(2) A $H-$semistable sheaf F is called $H-$*polystable* if it is the direct sum of $H-$stable sheaves.

Remark A.1.6. Notice that if F is either $H-$polystable or $H-$stable - in which case $F \simeq gr^{JH}(F)$ - then S-equivalence relation coincides with isomorphism equivalence relation. In other words, every $S-$quivalence class of $H-$semistable sheaves contains exactly one polystable sheaf up to isomorphism.

The $S-$equivalence relation will play a fundamental role in the construction of the moduli space of $H-$semistable sheaves on X.

A.2 The moduli problem

The key strategy for the construction of a moduli space parametrizing semistable sheaves with fixed numerical invariants on a smooth projective complex variety is given by Grothendieck's approach via *moduli functors*.

Moduli problem: given a collection $\mathcal{A}$ of algebraic objects with a suitable equivalence relation $\sim$, construct an algebraic object M whose points are in bijection with ${\mathcal{A}}\!/\!{\sim}$ in a natural way, by means of a *moduli functor*.

In the context of our interest, the collection $\mathcal{A}$ will be given by $H-$semistable sheaves with fixed Hilbert polynomial P_H and we will be primarily interested in families of such sheaves. For this purpose, let us recall that, given a morphism $p\colon \mathcal{X} \to T$ of finite type of Noetherian schemes, a *flat family of coherent sheaves* on the fibers of p is a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ which is flat over T, i.e. such that $\mathcal{F}_x$ is a flat $\mathcal{O}_{T,p(x)}-$module for any $x \in \mathcal{X}$. If $\mathcal{X} = X \times T$ and $p = \pi_T$ is the projection on the second factor, then $\mathcal{F}$ is a *$T-$flat family of coherent sheaves on* X, as $X_t := \mathcal{X}_t \simeq X$ for any $t \in T$.

Given a smooth polarized complex variety (X, H) and a polynomial $P \in \mathbb{Q}[t]$, we define a *moduli functor*

$$\underline{\mathcal{M}}_P(X, H)\colon Sch(\mathbb{C})^{op} \longrightarrow Set \tag{A.4}$$

as follows:

* For any complex scheme T we set

$$\underline{\mathcal{M}}_P(X,H)(T) := \left\{ \begin{array}{c} \mathcal{F} \in Coh(X \times T)\ T-\textit{flat such that}\ \mathcal{F}_t := \mathcal{F}_{X_t}\ \textit{is} \\ H-\textit{semistable with Hilbert polynomial}\ P_H(X_t) = P \end{array} \right\}\Big/\!\sim,$$

  where $\mathcal{F} \sim \mathcal{G}$ if and only if there exists a line bundle $L \in \mathrm{Pic}(T)$ such that $\mathcal{G} \simeq \mathcal{F} \otimes \pi_T^* L$.

* For any morphism $f\colon S \to T$ we define $\underline{\mathcal{M}}_P(X,H)(f)$ to be the morphism induced by the pull-back $(\mathrm{id}_X \times f)^*$ on coherent sheaves on $X \times T$.

Definition A.2.1. The moduli functor $\underline{\mathcal{M}}_P(X, H)$ is called:

(1) *corepresentable* if there exists an object $M_P(X,H) \in Ob(Sch(\mathbb{C}))$ and a natural transformation of functors

$$\alpha\colon \underline{\mathcal{M}}_P(X,H) \longrightarrow \mathrm{Hom}_{Sch(\mathbb{C})}(\cdot, M_P(X,H))$$

such that any natural transformation $\alpha'\colon \underline{\mathcal{M}}_P(X,H) \to \mathrm{Hom}_{Sch(\mathbb{C})}(\cdot, M')$ factors through a unique natural transformation

$$\beta\colon \mathrm{Hom}_{Sch(\mathbb{C})}(\cdot, M') \to \mathrm{Hom}_{Sch(\mathbb{C})}(\cdot, M_P(X,H)).$$

(2) *representable* if it is corepresentable and the natural transformation α as above is an isomorphism.

The object $M_P(X,H)$ is said to *corepresent* (respectively, *represent*) the functor $\underline{\mathcal{M}}_P(X,H)$ and is called *coarse* (respectively, *fine*) *moduli space of sheaves of semistable sheaves on X with Hilbert polynomial P* and it is unique up to isomorphism.

Let us denote, for simplicity, $\underline{\mathcal{M}}_P := \underline{\mathcal{M}}_P(X, H)$ and $M_P := M_P(X, H)$.

Remark A.2.2. (1) If the functor $\underline{\mathcal{M}}_P$ is corepresentable, then $\underline{\mathcal{M}}_P(\mathrm{Spec}(\mathbb{C}))$ corepresents it, hence $M_P \simeq \underline{\mathcal{M}}_P(\mathrm{Spec}(\mathbb{C}))$ and its points are in bijection with equivalence classes of $H-$semistable sheaves on X with Hilbert polynomial P.

(2) If $\underline{\mathcal{M}}_P$ is representable, then we get an isomorphism

$$\alpha_{M_P} \colon \underline{\mathcal{M}}_P(M_P) \to \mathrm{Hom}_{Sch(\mathbb{C})}(M_P, M_P)$$

and we denote by $[\mathcal{U}_P] := \alpha_{M_P}^{-1}(\mathrm{id}_{M_P})$ the unique equivalence class in $\underline{\mathcal{M}}_P(M_P)$ corresponding to the identity morphism via α_{M_P}. The flat family $\mathcal{U}_P \in Coh(X \times M_P)$ is called *universal family* and is unique up to a twist by a line bundle on M_P. A remarkable property of the latter is that, by construction, for any family $\mathcal{F}$ of semistable sheaves on $X \times T$ as above, there exists a *classifying morphism*

$$\varphi_{\mathcal{F}} \colon T \to M_P$$

such that $(\varphi_{\mathcal{F}} \times \mathrm{id}_X)^* \mathcal{U}_P$ and $\mathcal{F}$ are equivalent.

In simple terms, a universal family is a family of $H-$semistable sheaves on X with $P_H = P$, universally parametrized by M_P.

(3) Analogously, one can define a *moduli functor* $\underline{\mathcal{M}}_P^s(X, H)$ for $H-$stable sheaves, which is a subfunctor of $\underline{\mathcal{M}}_P(X, H)$. If they are corepresentable, then the moduli space $M_P^s(X, H)$ of $H-$stable sheaves on X with Hilbert polynomial P is an open subset of $M_P(X, H)$.

If the moduli functor $\underline{\mathcal{M}}_P(X, H)$ defined in (A.4) is representable, then the existence of the moduli space of $H-$semistable sheaves with Hilbert polynomial P is guaranteed. Unfortunately, the latter is, in general, only corepresentable and almost never representable (see [HL97, Lemma 4.1.2]), mainly due to the existence of strictly semistable sheaves. Some criteria for representability of $\underline{\mathcal{M}}_P^s(X, H)$ are provided in [HL97, Section 4.6] and will be discussed at the end of this section.

Surprisingly, the following fundamental result holds:

Theorem A.2.3. *Let (X, H) be a smooth polarized complex variety and let $P \in \mathbb{Q}[t]$. Then a coarse moduli space $M_P(X, H)$ of $H-$semistable sheaves on X with Hilbert polynomial P exists and it is a complex projective scheme.*

Furthermore, its points correspond to $S-$equivalence classes of $H-$semistable sheaves and there exists an open subset $M_P^s(X, H) \subseteq M_P(X, H)$ parametrizing (isomorphism classes of) $H-$stable sheaves.

Proof. See [HL97, Theorem 4.3.4]. □

A.3 A glimpse of the construction

For the purposes of this work, we restrict ourselves to outlining the main steps of the proof of Theorem A.2.3, introducing some constructions and formalisms that will prove useful in the subsequent discussion. For further details, we refer to [HL97, Chapter 4].

Let X be a smooth projective complex variety, let H be a polarization on X and let $P \in \mathbb{Q}[t]$ be a polynomial.

Step 1. *The family of $H-$semistable sheaves on X with Hilbert polynomial P is bounded.*

We recall that a family $\mathcal{A}$ of isomorphism classes of coherent sheaves on X is *bounded* if there exists a complex scheme S of finite type and a coherent $\mathcal{O}_{X\times S}-$module $\mathcal{F}$ such that $\mathcal{A}$ is contained in the set $\{\mathcal{F}_s := \mathcal{F}|_{X\times\{s\}} : s \text{ closed point in } S\}$.

Boundedness of the family as above is stated in Theorem 3.3.7 of [HL97] and is a consequence of Kleiman's criterion for bounded families ([HL97, Theorem 1.7.8]) and a boundedness result due to Le Potiers-Simpson ([HL97, Theorem 1.7.8]).

An important consequence of boundedness, related to the notion of *Castelnuovo-Mumford regularity* (see [HL97, Section 1.7]) is the following.

Corollary A.3.1. *There exists a positive integer $m \gg 0$ such that, for any $H-$semistable sheaf F with $P_H(F) = P$, the following holds:*

(1) the sheaf $F(m) := F \otimes \mathcal{O}_X(m)$ is globally generated;

(2) $P_H(F)(m) = h^0(X, F(m))$;

(3) the natural evaluation map $\rho_m \colon \mathrm{H}^0(X, F(m)) \otimes \mathcal{O}_X(-m) \to F$ is surjective.

Proof. See [HL97, Lemma 1.7.2, Lemma 1.7.6]. □

Step 2. *Let F an $H-$semistable sheaf with $P_H(F) = P$ as above and set $V := \mathbb{C}^{P(m)}$ and $\mathcal{H} := V \otimes_{\mathbb{C}} \mathcal{O}_X(-m)$. For any choice of an isomorphism $V \simeq \mathrm{H}^0(X, F(m))$, the sheaf F identifies a closed point*

$$[\mathcal{H} \twoheadrightarrow F] \in Quot_X(\mathcal{H}, P)$$

in the projective scheme parametrizing isomorphism classes of quotients of $\mathcal{H}$ with Hilbert polynomial P. Classes of quotients identified by stable and, respectively, semistable sheaves define two open subsets $R^s \subseteq R \subseteq Quot_X(\mathcal{H}, P)$.

If F is as above and $\sigma \colon V \xrightarrow{\sim} \mathrm{H}^0(X, F(m))$ is an isomorphism, then the composition of $\sigma \otimes \mathrm{id}_{\mathcal{O}_X(-m)}$ with the natural evaluation map ρ_m in Corollary A.3.1 defines a surjective map

$$\rho \colon \mathcal{H} \twoheadrightarrow F.$$

For any fixed coherent $\mathcal{O}_X-$module $\mathcal{H}$, one can analogously (see A.4) define a moduli functor $\underline{Quot}_X(\mathcal{H}, P)$ to parametrize flat families of quotients of $\mathcal{H}$ on X with

Hilbert polynomial P. By [HL97, Theorem 2.2.4] (see also [Gro60]), there exists a complex projective scheme $Quot_X(\mathcal{H}, P)$, called Grothendieck's *Quot scheme*, representing the functor $\underline{Quot}_X(\mathcal{H}, P)$ and parametrizing isomorphism classes of quotients of $\mathcal{H}$ with Hilbert polyonomial P. Moreover, on $X \times Quot_X(\mathcal{H}, P)$, there exists a *universal quotient*

$$\mathcal{H}_Q := \pi^*_{Quot_X(\mathcal{H},P)}\mathcal{H} \twoheadrightarrow \mathcal{Q}. \tag{A.5}$$

Example A.3.2. If $\mathcal{H} = \mathcal{O}_X$ and $P \equiv n$ for $n \in \mathbb{N}$, then $Quot_X(\mathcal{O}_X, n) \simeq Hilb^n(X)$ is the Hilbert scheme of points on X, parametrizing $0-$dimensional closed subschemes in X of length n. The isomorphism is given by identifying any such quotient $\mathcal{Z}$ with its support $supp(\mathcal{Z})$.

Hence, any isomorphism class $[\rho\colon \mathcal{H} \twoheadrightarrow F]$ of quotients as above determines a closed point in $Quot_X(\mathcal{H}, P)$. Additionally, as stability and semistability are open properties (see [HL97, Definition 2.1.9, Proposition 2.3.1]), classes determined by stable or - respectively - semistable sheaves define two open subsets of $Quot_X(\mathcal{H}, P)$ denoted - respectively - R^s and R, for which the following natural inclusions hold: $R^s \subseteq R \subseteq Quot_X(\mathcal{H}, P)$.

Step 3. *Construction of $M_P^{(s)}(X, H)$ as projective GIT quotient $R^{(s)} /\!\!/ PGL(V)$.*

The open subset $R \subseteq Quot_X(\mathcal{H}, P)$ parametrizes $H-$semistable sheaves on X with Hilbert polynomial P, but with an ambiguity arising from the choice of an isomorphism $\sigma\colon V \xrightarrow{\sim} \mathrm{H}^0(X, F(m))$. In other words, the group $PGL(V)$ acts on $Quot_X(\mathcal{H}, P)$ from the right by composition: $[\rho] \cdot \sigma := [\rho \circ \sigma]$ for any $\sigma \in PGL(V)$ and $R^{(s)}$ is invariant under this action. In this setting, a *good quotient* of the open set $R^{(s)}$ under this action is provided by the machinery of *Geometric Invariant Theory* (*GIT*) in the projective setting. While referring to [MF82] or [Hos24] for a detailed discussion concerning Geometric Invariant Theory, we limit ourselves to outlining the key **ingredients** needed for the construction:

* a projective scheme X - in this case, $X = \overline{R}$ is the closure of R in $Quot_X(\mathcal{H}, P)$.
* a *reductive* group G acting on X, namely an algebraic group G whose connected unipotent normal subgroups are all trivial - in this case, $G = PGL(V)$.
* a $G-$*linearized* ample line bundle L on X, namely an ample line bundle L on X equipped with an isomorphism of $\mathcal{O}_{X\times G}-$modules

  $$\Phi\colon \sigma^* L \to \pi_X^* L$$

  - where $\sigma\colon X \times G \to X$ is the $G-$action and $\pi_X\colon X \times G \to X$ is the projection on the second factor - inducing a $G-$action on L commuting with the projection $L \to X$ and such that the action on the fibers $L_{\sigma(x,g)} \to L_x$ is linear for any $x \in X, g \in G$. In this case, by [HL97, Proposition 2.2.5], a $PGL(V)-$linearized ample line bundle on $Quot_X(\mathcal{H}, P)$ (and on R) is provided by

  $$L_\ell := \det(\pi_{Quot_X(\mathcal{H},P)*}(\mathcal{Q} \otimes \pi_{X*}\mathcal{O}_X(\ell))),$$

for $\ell \gg 0$, where $\mathcal{Q}$ is the universal quotient defined in (A.5).

and the **outcome** of the latter ([HL97, Theorem 4.2.10]):

* two open subsets $X^s(L), X(L) \subseteq X$ parametrizing points that are *stable* or, respectively, *semistable* with respect to the $G-$action on L (see [HL97, Definition 4.2.9]), characterized by Hilbert-Mumford criterion ([HL97, Theorem 4.2.11]) - in this case, this notion of (semi)stability coincides with $H-$(semi)stability on $S-$equivalence classes ([HL97, Theorem 4.3.3]) and $X^{(s)}(L) = R^{(s)}$.
* a *good quotient* $\pi\colon X(L) \twoheadrightarrow X /\!\!/ G := \mathrm{Proj}((\oplus_{n\geq 0} \mathrm{H}^0(X, L^{\otimes n}))^G)$ for the $G-$action on X, namely an affine, $G-$invariant, surjective morphism π inducing a quotient topology on $X /\!\!/ G$ and an isomorphism $\mathcal{O}_{X/\!\!/G} \simeq (\pi_* \mathcal{O}_{X(L)})^G$.
* a *geometric quotient* $\pi\colon X^s(L) \twoheadrightarrow Y^s$ onto an open subset $Y^s \subseteq X /\!\!/ G$, i.e. a good quotient whose fibers are $G-$orbits.

Projective GIT, together with [HL97, Lemma 4.3.1] yields:

(1) The *moduli space* $M_P(X, H)$ *of* $H-$*semistable sheaves on* X *with Hilbert polynomial* P, corepresenting the functor $\underline{\mathcal{M}}_P(X, H)$ defined in (A.4), together with a good quotient $R \twoheadrightarrow R /\!\!/ PGL(V) = M_P(X, H)$. It is a complex projective scheme and its closed points correspond to $S-$equivalence classes of $H-$semistable sheaves.

(2) The *moduli space* $M^s_P(X, H)$ *of* $H-$*stable sheaves on* X *with Hilbert polynomial* P, corepresenting the functor $\underline{\mathcal{M}}^s_P(X, H)$ defined in Remark A.2.2 (3), together with a geometric quotient $R^s \twoheadrightarrow M^s_P(X, H)$. It is an open subset of $M_P(X, H)$ and its closed points correspond to isomorphism classes of $H-$stable sheaves.

A.3.1 Universal and quasi-universal families

In particular, we get that the moduli functors $\underline{\mathcal{M}}^{(s)}_P(X, H)$ are always corepresented by the moduli spaces $M^{(s)}_P(X, H)$. On the other hand, representability is a more delicate issue: as soon as there exists strictly semistable sheaves - equivalently, the inclusion $M^s_P(X, H) \subsetneq M_P(X, H)$ is strict - there is no hope for the existence of a universal family, due to the occurring of pathologies such as the *jump phenomenon* (see [HL97, Lemma 4.1.2]). Moreover, even if $M^s_P(X, H) = M_P(X, H)$, a universal family is not expected to exist. Nonetheless, slightly weakening this definition, we get the notion of *quasi-universal family* and it turns out that such objects, that will play a remarkable role in the next discussion, always exist on $M_P(X, H)$.

Definition A.3.3. Let (X, H) be a smooth polarized complex variety, let us set $M^s_P := M^s_P(X, H)$ and let $\mathcal{E} \in Coh(X \times M^s_P)$ be a flat family of stable sheaves on X parametrized by M^s_P.

(1) The family $\mathcal{E}$ is called *quasi-universal* if, for any $T-$flat family $\mathcal{F}$ of stable sheaves on X with Hilbert polynomial P, parametrized by a complex scheme T, there exists a locally free $\mathcal{O}_T-$module W such that $\mathcal{F} \otimes \pi_T^* W \simeq \phi^*_{\mathcal{F}} \mathcal{E}$, where

$\Phi_{\mathcal{F}} \in \mathrm{Hom}_{Sch(\mathbb{C})}(T, M^s_P)$ is the classifying morphism induced by T (see Definition A.2.1(1)). The rank $\mathrm{rk}(W)$ is called *similitude* of the quasi-universal family.

(2) A *universal family* is a quasi-universal family of similitude 1.

Remark A.3.4. (1) By [HL97, Proposition 4.6.2], a quasi-universal family $\mathcal{E}$ always exists on $X \times M^s_P(X, H)$.

(2) An existence criterion for a universal family $\mathcal{E}$ on $X \times M^s_P(X, H)$ is provided by [HL97, Theorem 4.6.5, Corollary 4.6.6], as follows. For any pure $d-$dimensional coherent sheaf F on X, we can rewrite the expression (A.2) of its Hibert polynomial in order to find integral coefficients $a_0, \dots, a_d$ such that

$$P_H(F)(m) = \sum_{i=0}^{d} a_i \binom{m+i-1}{i}.$$

If $\gcd(a_0, \dots, a_d) = 1$, then there exists a universal family $\mathcal{E}$ on $X \times M_P(X, H)$.

(3) In the case in which X is a K3 or Abelian surface and we restrict to check the existence of a universal family on $X \times M^s_c(X, H)$, where $M^s_c(X, H) \subseteq M^s_P(X, H)$ is the closed subscheme parametrizing stable sheaves with Chern character $c = (r, \xi, a) \in \widetilde{\mathrm{H}}(X, \mathbb{Z})$ compatible with P (see Remark 3.1.6), the condition above amounts to requiring that $\gcd(r, \xi, a) = 1$ (see [HL97, Corollary 4.6.7]). In particular, this implies that, with this fixed numerical invariants, there are no properly $H-$semistable sheaves on X, i.e. $M^s_c(X, H) = M_c(X, H)$.

A.3.2 Local structure and dimension estimates

In [HL97, Section 4.5], the relation between the local structure of the moduli space of stable sheaves and the local deformation and obstruction theory of a sheaf (see [HL97, Appendix 2.A]) is made explicit, yielding some smoothness criteria and dimension bounds for $M_P(X, H)$, for any smooth polarized complex variety (X, H).

Proposition A.3.5. *Let (X, H) be a smooth polarized variety and let $F \in M_P(X, H)$ be a stable sheaf.*

(1) The Zariski tangent space of $M_P(X, H)$ at F is canonically given by

$$T_F M_P(X, H) \simeq \mathrm{Ext}^1(F, F). \tag{A.6}$$

(2) If $\mathrm{Ext}^2(F, F) = 0$, then $M_P(X, H)$ is smooth at F and, in general, it holds

$$\mathrm{ext}^1(F, F) - \mathrm{ext}^2(F, F) \leq \dim_F M_P(X, H) \leq \mathrm{ext}^1(F, F).$$

Proof. See [HL97, Theorem 4.5.1, Corollary 4.5.2]. □

A refinement of the statement above is provided by the following argument. By the universal property of the moduli space, it is defined a determinant map

$$\det\colon M_P(X,H) \to \mathrm{Pic}(X),$$

whose tangent map can be described, under the canonical isomorphism (A.6) of Proposition A.3.5, in terms of the *trace map* $\mathrm{tr}\colon \mathcal{E}nd(F) \to \mathcal{O}_X$, inducing morphisms of complex vector spaces

$$\mathrm{tr}^i\colon \mathrm{Ext}^i(F,F) \simeq \mathrm{H}^i(X, \mathcal{E}nd(F)) \to \mathrm{H}^i(X, \mathcal{O}_X) \tag{A.7}$$

for $i = 0, \dots, \dim X$. In the following, we will denote by $\mathrm{Ext}^i(F,F)_0 := \ker(\mathrm{tr}^i)$ the kernel of the trace map on each $i-$th cohomology level, and by $\mathrm{ext}^i(F,F)_0$ its dimension.

Theorem A.3.6. *Let (X,H) be a smooth polarized variety and let $F \in M_P(X,H)$ be a stable sheaf. The tangent map of* $\det\colon M_P(X,H) \to \mathrm{Pic}(X)$ *at F is given by*

$$\det_F = \mathrm{tr}^1\colon \mathrm{Ext}^1(F,F) \to \mathrm{H}^1(X,\mathcal{O}_X) \simeq T_{[\det(F)]}\mathrm{Pic}(X), \tag{A.8}$$

under the canonical isomorphism (A.6) and

$$\mathrm{tr}^2\colon \mathrm{Ext}^2(F,F) \to \mathrm{H}^2(X,\mathcal{O}_X) \simeq \mathrm{Ext}^2(\det(F),\det(F))$$

preserves the obstruction spaces. Suppose additionally that $\mathrm{rank}(F) = r > 0$ *and* $\det(F) = \mathcal{Q}$ *and let us denote by $M(\mathcal{Q})$ the fiber of* $\det$ *over the point $[\mathcal{Q}]$.*

(1) Then $T_F M(\mathcal{Q}) \simeq \mathrm{Ext}^1(F,F)_0$.

(2) If the obstruction space $\mathrm{Ext}^2(F,F)_0$ vanishes, then both $M_P(X,H)$ and $M(\mathcal{Q})$ are smooth at F.

(3) Moreover, it holds

$$\mathrm{ext}^1(F,F)_0 - \mathrm{ext}^2(F,F)_0 \le \dim_F M(\mathcal{Q})) \le \mathrm{ext}^1(F,F)_0.$$

Proof. See [HL97, Theorem 4.5.3]. □

Remark A.3.7. If X is a smooth surface, then the dimension bound can be made more explicit, recalling that any stable sheaf F is simple (Remark A.1.3 (1)):

$$\mathrm{ext}^1(F,F)_0 - \mathrm{ext}^2(F,F)_0 = \chi(\mathcal{O}_X) - \sum_{i=0}^{2}(-1)^i \mathrm{ext}^i(F,F). \tag{A.9}$$

A more precise description is given in Section 3.1, in the case in which X is either a projective $K3$ surface or an Abelian surface.

A.3.3 Relative moduli spaces of sheaves

We conclude by outlining a construction that will play a central role in Section 4.1, when dealing with deformations of moduli spaces of sheaves. Indeed, it turns out that the construction above-sketched generalizes to yield *relative moduli spaces of sheaves*, i.e. moduli spaces of semistable sheaves on the fibers of a projective morphism.

Theorem A.3.8. *Let $f\colon \mathcal{X} \to T$ be a projective morphism of smooth varieties defined over $\mathbb{C}$ with connected fibers and let $\mathcal{H}$ be a relatively ample line bundle on $\mathcal{X}$ - i.e., for any $t \in T$, the restriction $\mathcal{H}_t$ is an ample line bundle on $\mathcal{X}_t$ - and let $P \in \mathbb{Q}[t]$ be a polynomial. Then there exists a projective morphism*

$$M_P(X/T, \mathcal{H}) \to T$$

corepresenting the functor

$$\underline{\mathcal{M}}_P(X/T, \mathcal{H})\colon Sch(T)^{op} \to Sets$$

associating to any $T-$scheme S of finite type the set of isomorphism classes of $S-$flat families of semistable sheaves on the fibers of the morphism $f_S\colon \mathcal{X}_S := \mathcal{X} \times_T S \to S$ with Hilbert polynomial P. Moreover, there exists an open subscheme $M_P^s(\mathcal{X}/T, \mathcal{H}) \subseteq M_P(\mathcal{X}/T, \mathcal{H})$ corepresenting the subfunctor $\underline{\mathcal{M}}_P^s(X/T, \mathcal{H}) \subseteq \underline{\mathcal{M}}_P(X/T, \mathcal{H})$. In particular, for any closed point $t \in T$, it holds

$$M_P(\mathcal{X}/T, \mathcal{H})_t \simeq M_P(\mathcal{X}_t, \mathcal{H}_t) \text{ and } M_P^s(\mathcal{X}/T, \mathcal{H})_t \simeq M_P^s(\mathcal{X}_t, \mathcal{H}_t).$$

Proof. See [HL97, Theorem 4.3.7]. □

Appendix B

Some lattice theory results

This Appendix is devoted to recall some lattice theory results that will be useful throughout this work. In the following, we will let $(\Lambda, \cdot)$ be an *even lattice*, i.e, a finitely generated free $\mathbb{Z}-$module Λ equipped with a non-degenerate symmetric bilinear pairing $\cdot\colon \Lambda \times \Lambda \to \mathbb{Z}$, such that the associated quadratic form q_Λ takes values in $2\mathbb{Z}$.

If Λ and Λ' are two such lattices and the respective bilinear forms are clear from the context and if $v \in \Lambda$, we will use the following notations:

* $\mathrm{O}(\Lambda, \Lambda')$ for the group of isometries from Λ to Λ';
* $\mathrm{O}(\Lambda)$ for the group of isometries from Λ to itself;
* $\mathrm{O}(\Lambda)_v$ for the subgroup of $\mathrm{O}(\Lambda)$ of isometries fixing the element v.

B.1 Isometries as kernels of natural group morphisms

In this work we will be interested in special groups of isometries that can be characterized as kernels of some natural group homomorphisms, which we will recall in the following section.

B.1.1 Determinant character

If $g \in \mathrm{O}(\Lambda)$ is an isometry, we define its *determinant* $\det(g)$ as the determinant of the matrix associated to g with respect to any $\mathbb{Z}-$basis of Λ. This yields a natural *determinant character*

$$\det\colon \mathrm{O}(\Lambda) \longrightarrow \{1, -1\}.$$

We will denote by $\mathrm{SO}(\Lambda) := \ker(\det)$ the group of isometries of Λ of determinant 1.

B.1.2 Orientation character

The second fundamental character that can be naturally associated to any isometry is related to the notion of *orientation*. For a more precise discussion on this theory, we refer the reader to [Mar11, Section 4] and [Mar08, Section 4.1].

Let $(\widetilde{\Lambda}, \cdot)$ be a lattice of signature $sgn(\widetilde{\Lambda}) = (p, q)$, with $p, q \geq 1$ and let us consider the *big positive cone*

$$\widetilde{\mathcal{C}} := \{\alpha \in \widetilde{\Lambda} \otimes_{\mathbb{Z}} \mathbb{R} \colon \alpha \cdot \alpha > 0\}.$$

(a) If $p = 1$, then $\widetilde{\mathcal{C}}$ has two connected components and we define an *orientation character*

$$\mathrm{or} \colon \mathrm{O}(\widetilde{\Lambda}) \longrightarrow \mathbb{Z}/2\mathbb{Z},$$

by setting $\mathrm{or}(g) = 0$ if g is an isometry mapping each connected component to itself and $\mathrm{or}(g) = 1$ otherwise. The choice of any connected component of $\widetilde{\mathcal{C}}$ determines an *orientation on* $\widetilde{\Lambda}$.

(b) If $p > 1$, by [Mar11, Lemma 4.1], the cone $\widetilde{\mathcal{C}}$ is a deformation retract of $\mathbb{R}^p \setminus \{0\}$, hence homotopic to the sphere S^{p-1}, and the action of $\widetilde{\Lambda}$ on the top cohomology group of of $\widetilde{\mathcal{C}}$ determines an *orientation character*

$$\mathrm{or} \colon \mathrm{O}(\widetilde{\Lambda}) \longrightarrow \mathbb{Z}/2\mathbb{Z},$$

defined as follows: as $\mathrm{H}^{p-1}(\widetilde{\mathcal{C}}, \mathbb{Z}) \simeq \mathrm{H}^{p-1}(S^{p-1}, \mathbb{Z}) \simeq \mathbb{Z}$, we can choose a generator ϵ of the latter and notice that, if $g \in \mathrm{O}(\widetilde{\Lambda})$, then it induces an action $(g_{\mathbb{R}|\widetilde{\mathcal{C}}})^*$ on $\mathrm{H}^{p-1}(\widetilde{\mathcal{C}}, \mathbb{Z})$ that either preserves ϵ or maps it to its opposite. We then set

$$\mathrm{or}(g) = \begin{cases} 0 & \text{if } (g_{\mathbb{R}|\widetilde{\mathcal{C}}})^*(\epsilon) = \epsilon \\ 1 & \text{if } (g_{\mathbb{R}|\widetilde{\mathcal{C}}})^*(\epsilon) = -\epsilon. \end{cases}$$

A choice of a generator ϵ as above, or, equivalently, of an ordered basis of a positive $p-$dimensional real subspace of $\widetilde{\Lambda} \otimes_{\mathbb{Z}} \mathbb{R}$, determines an *orientation on* $\widetilde{\Lambda}$.

In any of the two cases above, we define the group of *orientation preserving isometries of* $\widetilde{\Lambda}$ as

$$\mathrm{O}^+(\widetilde{\Lambda}) := \ker(\mathrm{or})$$

and, similarly, we define $\mathrm{SO}^+(\widetilde{\Lambda}) := \mathrm{O}^+(\widetilde{\Lambda}) \cap \mathrm{SO}(\widetilde{\Lambda})$. Analogously, if $(\widetilde{\Lambda}_1, \epsilon_1)$ and $(\widetilde{\Lambda}_2, \epsilon_2)$ are two oriented lattices as above, we get an *orientation map*

$$\mathrm{or} \colon \mathrm{O}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2) \longrightarrow \mathbb{Z}/2\mathbb{Z} \tag{B.1}$$

and we can define the set of *orientation preserving isometries* as $\mathrm{O}^+(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2) := \mathrm{or}^{-1}(0)$.

We also point out that, if $\Lambda \subseteq \widetilde{\Lambda}$ is a sublattice of $\widetilde{\Lambda}$ of signature (p', q'), with $p', q' \geq 1$, inducing a natural embedding $\mathrm{O}(\Lambda) \subseteq \mathrm{O}(\widetilde{\Lambda})$ by extending to the identity on $\Lambda^{\perp}$, then the orientation character of $\widetilde{\Lambda}$ restricts to Λ as the natural orientation character of Λ.

B.1.3 Discriminant

The non-degeneracy of the bilinear form $\cdot$ gives a canonical embedding of Λ in its *dual lattice* $\Lambda^\vee := \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

* We define the *discriminant group* of Λ as the quotient $A_\Lambda := \Lambda^\vee / \Lambda$, which is a finite group of order $|\det(\Lambda, \cdot)|$, where $\det(\Lambda, \cdot)$ is the determinant of the matrix associated to the bilinear form $\cdot$ with respect to any $\mathbb{Z}-$basis of Λ.

* A lattice Λ is called *unimodular* if $A_\Lambda \simeq \{0\}$ or, equivalently, if $\det(\Lambda) = \pm 1$.

* Any isometry $g \in \mathrm{O}(\Lambda)$ induces an automorphism $\overline{g} \in \mathrm{Aut}(A_\Lambda)$ that preserves the discriminant quadratic form $\overline{q_\Lambda}\colon A_\Lambda \to \mathbb{Q}/2\mathbb{Z}$ induced by q_Λ. This naturally defines a group morphism

$$\mathrm{disc}\colon \mathrm{O}(\Lambda) \longrightarrow \mathrm{O}(A_\Lambda), \tag{B.2}$$

  where $\mathrm{O}(A_\Lambda)$ is subgroup of $\mathrm{Aut}(A_\Lambda)$ of automorphisms preserving $\overline{q_\Lambda}$.

* We will denote by $\widetilde{\mathrm{O}}(\Lambda) := \ker(\mathrm{disc})$ the group of isometries of Λ acting trivially on its discriminant group, and similarly we will set $\widetilde{\mathrm{O}}^+(\Lambda) := \widetilde{\mathrm{O}}(\Lambda) \cap \mathrm{O}^+(\Lambda)$, $\widetilde{\mathrm{SO}}(\Lambda) := \widetilde{\mathrm{O}}(\Lambda) \cap \mathrm{SO}(\Lambda)$ and $\widetilde{\mathrm{SO}}^+(\Lambda) := \widetilde{\mathrm{O}}(\Lambda) \cap \mathrm{SO}^+(\Lambda)$.

B.2 Primitive embeddings

In this Subsection we will collect basic results concerning primitive embeddings of even lattices into even unimodular lattices and some extendibility criteria for isometries that will lay the groundwork for the proof of Proposition 5.2.4. For a more detailed discussion, we refer to [Nik79] (see also [Kon21, Section 1.3.1] and [Huy16, Chapter 14]).

Let L be an even lattice and let us consider an embedding $S \overset{i}{\hookrightarrow} L$ of lattices.

* The embedding i (or - respectively - the sublattice S, identified with its image via i) is called *primitive* if the quotient $L/i(S)$ is torsion free.

* The *saturation* $\overline{S}$ of S in L is the smallest primitive sublattice of L containing S, i.e. $\overline{S} := (S \otimes_{\mathbb{Z}} \mathbb{Q}) \cap L$. Notice that we have a natural chain of embeddings $S \subseteq \overline{S} \subseteq L$ and that the index $[\overline{S} \colon S]$ is finite. Furthermore, it holds $\mathrm{rk}(S) = \mathrm{rk}(\overline{S})$ and a set of generators for $\overline{S}$ can be provided by a suitable $\mathbb{Q}-$linear combination of a set of generators for S.

The sublattice S is primitive if and only if $S = \overline{S}$. In that case, its orthogonal complement $K := S^\perp$ in L is also a primitive sublattice of L. Let us assume that L is unimodular and let us consider the quotient $H = L/S \oplus K$. Then H is a finite Abelian group and an isotropic subgroup of $A_S \oplus A_K$ with respect to $\overline{q_S} \oplus \overline{q_K}$. Furthermore, it can be checked that the restrictions $p_S|_H \colon H \to A_S$ and $p_K|_H \colon H \to A_K$ of the natural projections are both isomorphisms (see [Kon21, Lemma 1.31]). In particular,

set
$$\gamma_{SK} := p_K \circ (p_S|_H)^{-1} \colon A_S \to A_K,$$
we have the following results.

Theorem B.2.1. *Let L be an even, unimodular lattice, S a primitive sublattice of L and $K = S^\perp$ its orthogonal complement in L. Then*
$$A_S \underset{\gamma_{SK}}{\simeq} A_K \quad \text{and} \quad \overline{q_S} = -\overline{q_K} \circ \gamma_{SK}.$$

Proof. See [Nik79, Proposition 1.6.1], [Kon21, Theorem 1.32]. □

A straightforward application of Theorem B.2.1 provides a criterion for the extendibility of an isometry $\varphi \colon S_1 \to S_2$ between two primitive sublattices of L to an isometry of the whole lattice L, i.e. the existence of an isometry $\widetilde{\varphi} \in \mathrm{O}(L)$ such that $\widetilde{\varphi}|_{S_1} = \varphi$.

Proposition B.2.2. *Let $S_1, S_2 \subseteq L$ be two primitive sublattices and let $K_i := S_i^\perp$ for $i = 1, 2$. An isometry $\varphi \in \mathrm{O}(S_1, S_2)$ can be extended to an isometry of L if and only if there exist and isometry $\psi \in \mathrm{O}(K_1, K_2)$ such that*
$$\overline{\psi} \circ \gamma_{S_1K_1} = \gamma_{S_2K_2} \circ \overline{\varphi}.$$

Proof. See [Nik79, Proposition 1.6.1], [Kon21, Corollary 1.33]. □

In particular, we get the following folklore result:

Corollary B.2.3. *Let $S \subseteq L$ a primitive sublattice of an even unimodular lattice L and let $K := S^\perp$. If $\varphi \in \mathrm{O}(S)$ is an isometry such that $\mathrm{disc}(\varphi) = \pm\mathrm{id}_{A_S}$, then it can be extended to $\widetilde{\varphi} \in \mathrm{O}(L)$ such that $\widetilde{\varphi}|_K = \pm\mathrm{id}_K$.*

In order to apply Proposition B.2.2, a description of the image of the discriminant map defined in (B.2) might be crucial. We address this problem in the case of an even indefinite lattice, in relation with the question of its uniqueness up to isomorphism.

Proposition B.2.4. *Let T be an indefinite even lattice of signature (t_+, t_-), with discriminant quadratic form $\overline{q} = \overline{q_T}$. Suppose that*
$$rk(T) \geq l(A_T) + 2,$$
where $l(A_T)$ is the minimum number of generators of the finite Abelian group A_T. Then any even lattice of signature (t_+, t_-) and discriminant quadratic form $\overline{q}$ is isomorphic to T and the natural discriminant map $\mathrm{disc} \colon \mathrm{O}(T) \to \mathrm{O}(A_T)$ is surjective.

Proof. See [Nik79, Theorem 1.14.2], [Kon21, Proposition 1.37]. □

Proposition B.2.4 can be read by saying that any even indefinite lattice T satisfying $\mathrm{rk}(T) \geq l(A_T) + 2$ is *unique in its genus*. Two lattices Λ_1 and Λ_2 are said to have the same *genus* if $\Lambda_1 \otimes \mathbb{Z}_p \simeq \Lambda_2 \otimes \mathbb{Z}_p$ for all primes p and $\Lambda_1 \otimes \mathbb{R} \simeq \Lambda_2 \otimes \mathbb{R}$, where all the isomorphism are compatible with the natural quadratic forms.

By definition, lattices with the same genus have the same signature, and, by [Nik79, Corollary 1.9.4], the converse holds for two even lattices Λ_1, Λ_2, provided that $(A_{\Lambda_1}, q_{A_{\Lambda_1}}) \simeq (A_{\Lambda_2}, q_{A_{\Lambda_2}})$.

A consequence of uniqueness in genus of indefinite even lattices (Proposition B.2.4) and the exendibility criteria of Proposition B.2.2 is the uniqueness of their primitive embeddings in suitable ambient lattices, up to automorphisms of the overlattice. We will be particularly interested in primitive embeddings in unimodular lattices, which is the case of the following result.

Proposition B.2.5. *Let S be an even lattice of rank r. For any integer $r' \geq r$, there exists a primitive embedding $S \hookrightarrow U^{\oplus r'}$. Moreover, if $r' > r$, then the embedding is unique up to isometry of $U^{\oplus r'}$.*

Proof. See [Huy16, Proposition 1.8]. □

As an immediate consequence of Proposition B.2.5 we get the following result, which will be frequently used in Section 5.2.

Corollary B.2.6. *Let r, s be two strictly positive integers and let S be an even lattice of rank r. Then, the orthogonal complement of S in $U^{\oplus(r+s)}$ contains $U^{\oplus s}$.*

Proof. By Proposition B.2.5, there exists a primitive embedding $S \hookrightarrow U^{\oplus r}$. As $U^{\oplus(r+s)}$ decomposes as $U^{\oplus r} \overset{\perp}{\oplus} U^{\oplus s}$, the latter provides a primitive embedding of S in $U^{\oplus(r+s)}$ such that $S^{\perp}$ contains $U^{\oplus s}$. Again by Proposition B.2.5, as $r < r+s$, such an embedding is unique up to isometry of $U^{\oplus(r+s)}$. □

B.3 Existence and uniqueness of lattices

For the sake of completeness, we include the following classical result due to Milnor concerning the existence and uniqueness up to isomorphism of even unimodular lattices, that will be used to characterize the lattices studied throughout this work.

Theorem B.3.1. *Let (p, q) be a pair of non-negative integers. Then there exists an even, unimodular lattice of signature (p, q) if and only if $p - q \equiv 0 \mod 8$. If $p, q > 0$, then the lattice is unique.*

In particular, if Λ is an even, indefinite, unimodular lattice of signature (p, q), then $\tau := p - q \equiv 0 \mod 8$ and

$$\Lambda \simeq \begin{cases} E_8^{\oplus \frac{\tau}{8}} \oplus U^{\oplus q} & \text{if } \tau \geq 0 \\ E_8(-1)^{\oplus \frac{-\tau}{8}} \oplus U^{\oplus p} & \text{if } \tau < 0. \end{cases}$$

Proof. See [Mil58] and [Huy16, Theorem 1.1, Corollary 1.3]. □

B.4 Weyl groups of reflections

In this Section we introduce two special groups of reflections that will play a central role in Sections 5.2 and 6.3. Let $k \geq 1$ and let Λ be an even lattice of signature (p, q), with $p, q \geq 3$, and with cyclic discriminant group of order $2k$. For any $u \in \Lambda$ such that $u \cdot u = \pm 2$, let

$$\begin{aligned} R_u \colon \Lambda &\longrightarrow \Lambda \\ v &\longmapsto v - 2\frac{v \cdot u}{u \cdot u} u \end{aligned}$$

be the reflection in u and define the isometry $\rho_u := -\frac{u \cdot u}{2} R_u$. The group

$$\mathsf{W}(\Lambda) := \langle \rho_u \colon u \in \Lambda, u \cdot u = \pm 2 \rangle$$

is a normal subgroup of finite index of $\mathrm{O}(\Lambda)$, often called *Weyl group of reflections*. Notice that the action of the previously defined group morphisms on the generators of $\mathsf{W}(\Lambda)$ is the following (see, [Mar11, Lemma 4.1], [Mar08, Lemma 4.10] [Mar22, Section 1.1]):

$$\mathrm{or}(\rho_u) = 0, \qquad \det(\rho_u) = \frac{u \cdot u}{2}, \qquad \mathrm{disc}(\rho_u) = -\frac{u \cdot u}{2}. \tag{B.3}$$

From the first equality of (B.3) we deduce that $\mathsf{W}(\Lambda) \subseteq \mathrm{O}^+(\Lambda)$ and from the last equality we get the following discriminant character

$$\mathrm{disc} \colon \mathsf{W}(\Lambda) \longrightarrow \{-1, 1\} \subseteq \mathrm{O}(A_\Lambda),$$

where A_Λ is a cyclic group of order $2k$. Again by (B.3), we get that, for any generator ρ_u of Λ, it holds $\det \cdot \mathrm{disc}(\rho_u) = -1$. We define the following group of isometries

$$\mathsf{N}(\Lambda) := \ker(\det \cdot \mathrm{disc}), \tag{B.4}$$

which is an index 2 subgroup of $\mathsf{W}(\Lambda)$, generated by compositions $\rho_{u_1} \circ \rho_{u_2}$, with $u_i \in \Lambda$ such that $u_i \cdot u_i = \pm 2$ for $i = 1, 2$.

Remark B.4.1. Let $\widetilde{\Lambda} = \widetilde{\mathrm{H}}(S, \mathbb{Z})$ be the Mukai lattice of an Abelian surface S (see Section 3.1), isometric to $U^{\oplus 4}$. Let $v = m(1, 0, -k) \in \widetilde{\Lambda}$, with $m \geq 1$ and $k \geq 2$, and let $\Lambda = v^\perp$, which is even lattice of signature $(3, 4)$, isometric to $U^{\oplus 3} \oplus \langle -2k \rangle$.

(1) By [Mar08, Lemma 4.10], we get that $\mathsf{W}(\Lambda)$ is isomorphic to the subgroup of $\mathrm{O}^+(\Lambda)$ of isometries that can be extended to the whole Mukai lattice $\widetilde{\Lambda}$ or, equivalently, to the image of the stabilizer $\mathrm{O}(\widetilde{\Lambda})_v$ under the homomorphism

$$\begin{aligned} \psi \colon \mathrm{O}(\Lambda) &\longrightarrow \mathrm{O}^+(\Lambda) \\ g &\longmapsto (-1)^{\mathrm{or}(g)} g, \end{aligned}$$

where we used the natural inclusion $\mathrm{O}(\widetilde{\Lambda})_v \subseteq \mathrm{O}(\Lambda)$. Notice that, as $sgn(\Lambda) = (3,4)$, the isometry $-\mathrm{id}_\Lambda$ is orientation reversing.

In simple terms, $\mathsf{W}(\Lambda)$ can be characterized as the group of orientation preserving isometries of Λ acting as $\pm\mathrm{id}$ on the discriminant group (see also Corollary B.2.3).

(2) Section 5 and the proof of Theorem 1.4 of [Mar22] (see also [Mar10, Lemma 8.4]) provide a set of generators for $\mathsf{N}(\Lambda)$, namely

$$\mathsf{N}(\Lambda) = \langle \mathrm{SO}^+(\widetilde{\Lambda})_v, R_s \circ R_{s_1} \rangle, \tag{B.5}$$

where $s = (1,0,1), s_1 = (1,0,-1) \in \Lambda$. Indeed, by Lemma 5.3 and Corollary 5.1 of [Mar22], we have an equality

$$\mathrm{SO}^+(\widetilde{\Lambda})_v = \langle R_{u_1} \circ R_{u_2}, R_{t_1} \circ R_{t_2} \colon u_i, t_i \in \Lambda, u_i^2 = 2, t_i^2 = -2, \text{ for } i = 1,2 \rangle.$$

Consequently, for any $g \in \mathrm{SO}^+(\widetilde{\Lambda})_v$, it holds $\det \cdot \mathrm{disc}(g) = 1$ and $\det(g) = 1$, from which we deduce that $\mathrm{SO}^+(\widetilde{\Lambda})_v$ is an index 2 subgroup of $\mathsf{N}(\Lambda)$. On the other hand, $\det \cdot \mathrm{disc}(R_s \circ R_{s_1}) = 1$ and $\det(R_s \circ R_{s_1}) = -1$, showing equality (B.5).

(3) If we let $\widetilde{\Lambda}$, v and Λ be as above, and we assume that $k = 1$, then A_Λ is a cyclic group of order 2, hence, by degeneracy of the discriminant character and Corollary B.2.3, we get the following chain of isomorphism

$$\mathsf{W}(\Lambda) \simeq \widetilde{\mathrm{O}}^+(\Lambda) \simeq \mathrm{O}^+(\Lambda) \simeq \mathrm{O}^+(\widetilde{\Lambda})_v,$$

implying

$$\mathsf{N}(\Lambda) \simeq \widetilde{\mathrm{SO}}^+(\Lambda) \simeq \mathrm{SO}^+(\Lambda) \simeq \mathrm{SO}^+(\widetilde{\Lambda})_v. \tag{B.6}$$

B.5 Eichler's transvections

We conclude this Appendix by reporting some results that are particularly relevant for even indefinite lattices containing some copies of the unimodular hyperbolic plane. We introduce some special generators for their orthogonal groups, together with a criterion establishing whether, given two elements of such a lattice $(L, \cdot)$, there exists an isometry of L moving the first into the second. As shown below, this can be done as soon as L contains at least two copies of the unimodular hyperbolic plane U and provided that the two vectors under study, divided by their divisibility, belong to the same class in the discriminant group of L. We recall that, for any $v \in L$, we can define its *divisibility* as the unique positive generator of the ideal $v \cdot L \subseteq \mathbb{Z}$.

The above-mentioned criterion is proved using a special class of isometries, called *Eichler's transvections*, introduced in [Eic52, Chapter 3] (see also [GHS09, Section 3.1]) and defined as follows. Given an isotropic element $e \in L$ and $a \in e^\perp$, the *Eichler's transvection* $t(e,a)$ with respect to e and a is the isometry acting, for any $v \in L$, as

$$t(e,a)(v) = v - (a \cdot v)e + (e \cdot v)a - \frac{1}{2}(a \cdot a)(e \cdot v)e. \tag{B.7}$$

Remark B.5.1. We notice that, by definition, any transvection of the form $t(e,a)$, with $e \in L$ isotropic and $a \in e^{\perp}$, acts as the identity on $e^{\perp} \cap a^{\perp}$. Moreover, Eichler's transvections satisfy the following properties: let $e \in L$ be an isotropic element. Then

(1) $t(e,a) \circ t(e,b) = t(e,a+b)$ for any $a,b \in e^{\perp}$. In particular, $t(e,a)^{-1} = t(e,-a)$;

(2) $g \circ t(e,a) \circ g^{-1} = t(g(e),g(a))$ for any $a \in e^{\perp}$ and $g \in \mathrm{O}(L)$;

(3) $t(\lambda e,a) = t(e,\lambda a)$ and $t(e,\lambda e) = \mathrm{id}$, for any $a \in e^{\perp}$ and $\lambda \in \mathbb{Z} \setminus \{0\}$;

(4) $t(e,a) = R_a \circ R_{a+\frac{1}{2}(a\cdot a)e}$ for any $a \in e^{\perp} \setminus \{0\}$.

In particular, by (4) (see also [GHS09, Lemma 3.1 and (10)]), we get that $t(e,a) \in \widetilde{\mathrm{SO}}^+(L)$ for any $a \in e^{\perp} \setminus \{0\}$.

Let us assume that L contains at least one copy of the unimodular hyperbolic plane U and let us denote by L_1 its orthogonal complement in L. We define

$$E_U(L_1) := \langle t(e,a) \colon e \in U, a \in L_1 \rangle, \tag{B.8}$$

which is a subgroup of $\widetilde{\mathrm{SO}}^+(L)$.

Lemma B.5.2 (Eichler's criterion)**.** *Let L be an even lattice, let us suppose that $L = U_1 \oplus L_1$ and $L_1 = U_2 \oplus L_2$, where U_1 and U_2 denote, respectively, two distinguished copies of the unimodular hyperbolic plane U in L. If $v_1, v_2 \in L$ are two primitive elements such that*

(1) $v_1^2 = v_2^2$,

(2) $[v_1/\mathrm{div}(v_1)] = [v_2/\mathrm{div}(v_2)]$ in A_L,

then there exists a transvection $t \in E_{U_1}(L_1)$ such that $t(v_1) = v_2$.

Proof. See [GHS09, Proposition 3.3 (i)]. □

Another important application of Eichler's transvections is given by the following result.

Lemma B.5.3. *(1) Let $U^{\oplus 2}$ be two copies of the unimodular hyperbolic plane, with standard basis $\{e_1, f_1, e_2, f_2\}$ - so that $e_i^2 = f_i^2 = e_i \cdot f_j = 0$ for $i,j = 1,2$, $i \neq j$, and $e_i \cdot f_i = 1$ for $i = 1,2$. Then*

$$\mathrm{SO}^+(U^{\oplus 2}) = \langle t(e_2,e_1), t(e_2,f_1), t(f_2,e_1), t(f_2,f_1) \rangle.$$

(2) Let $L = U \oplus L_1$ an even lattice. Then

$$\mathrm{O}^+(L) = \langle \mathrm{O}^+(L_1), E_U(L_1) \rangle \quad \textit{and} \quad \mathrm{SO}^+(L) = \langle \mathrm{SO}^+(L_1), E_U(L_1) \rangle.$$

Proof. See [GHS09, Lemma 3.2] for part (1) and [GHS09, Proposition 3.3 (iii)] for part (2). □

The above-introduced results will be particularly relevant for the discussion in Section 5.1.1.

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