



Self-avoiding closed curves in the regular and semiregular grids

Lidija Čomić^a, Paola Magillo^{b,*}

^a University of Novi Sad, Faculty of Technical Sciences, Novi Sad, Serbia

^b University of Genova, Department of Computer Science Bioengineering Robotics and Systems Engineering, Genova, Italy



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ABSTRACT

We consider closed curves in the three regular and eight semiregular grids in the plane, in which each vertex and each edge can be repeated a limited number of times. We define the conditions for such curves to be self-avoiding, and we present a linear-time algorithm to check them. We define the orientation of such curves. We propose a classification of their vertices, and we give a unifying formula relating the number of different types of vertices, valid in the regular and semiregular grids. Our results can be used in the plane tiling applications.

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1. Introduction

There are three regular grids in the plane, and eight semiregular ones (composed of regular polygons, such that the circular order of polygons is the same around each vertex), tiling the plane into regular triangles, squares or hexagons in the regular case, and into a combination of regular triangles, squares, hexagons, octagons and dodecagons in the semiregular case.

A simple closed curve in these grids is composed of grid vertices and grid edges, and it passes through each of its vertices exactly once (note that edges cannot cross, as they belong to the grid).

The curves in the hexagonal and square grids, in which each vertex is repeated at most three times and each edge is repeated at most twice, describe the boundary of a generalized polyomino. In any of the regular or semiregular grids, a generalized polyomino is the union of a finite number of polygons (basic elements of the grid), possibly disconnected and with holes (see Fig. 1). Its boundary can be described by a non-unique single self-avoiding curve obtained by connecting the boundaries of components and/or holes through chains of double edges. We define a class of such curves, the cases in which they are self-avoiding, their orientation, and we give a linear-time algorithm to check if a given closed grid curve falls in the class of our curves.

Our self-avoiding curves will be a more specific class with respect to weakly simple polygons in the ordinary (non-gridded) 2D plane. Weakly simple curves [1,9] are polygons in the plane (not constrained to a grid) with arbitrary finite number of repetitions of vertices and edges, such that it is possible to make an arbitrarily small perturbation of repeated vertices so that a simple polygon is obtained. They admit spurs (which our curves do not allow), vertices in the interior of edges (which cannot occur in our curves, as their edges are constrained to be grid edges), and an arbitrary number of repetitions of vertices and edges (while in our curves they can be repeated three and two times, respectively).

* Corresponding author.

E-mail address: paola.magillo@unige.it (P. Magillo).

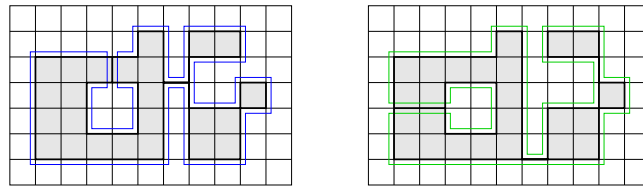


Fig. 1. A generalized polyomino in the square grid. Its boundary can be described by a (non-unique) curve with repeated vertices and edges. Two of such curves are shown.

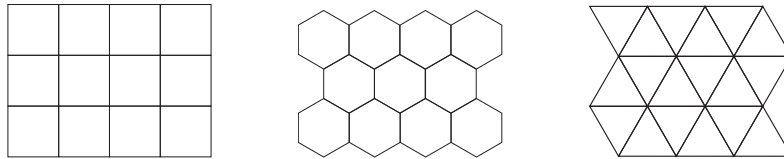


Fig. 2. The regular square, hexagonal and triangular grids (from left to right). Only some cells are shown for each grid.

Vertices at which a simple closed curve C makes a left and a right turn are called convex and concave vertices, respectively (with counterclockwise orientation for C). In each of the square [13,14] and the hexagonal [6,7] grids, there is a known combinatorial relationship between the number of convex and concave vertices in C , which has been used in the study of the tiling problem in these grids [8,20]. A similar relationship exists in the triangular grid if each vertex v in C is further classified based on the angle at v [11]. We extend and generalize this relationship from the regular to the semiregular grids, and from simple curves to self-avoiding curves with repeated vertices and edges, with a limited number of repetitions.

Our contributions are:

- The definition of self-avoiding closed curves in any of the regular or semiregular grids, in which the vertices can be repeated at most three times and the edges can be repeated at most twice; this extends and generalizes the existing one in the hexagonal and the square grids [8,20], in which the number of repetitions is at most two.
- A linear-time algorithm to check whether one of such curves is self-avoiding, directly descending from the definition.
- The definition of the orientation of such curves.
- A unifying combinatorial relationship between the weighted number of vertices in such curves, valid in all eight semiregular grids (and in the three regular ones), of the form

$$A(C) = \sum_{v \in C} d(v) = 1,$$

where the contribution $d(v)$ of vertex v depends on the interior angles of the curve at v .

This paper is organized as follows. In Section 2 we provide background notions about grids and curves. In Section 3 we introduce the definitions of our classes of closed curves with repeated vertices and edges, and the conditions for them to be self-avoiding. In Section 4 we present our algorithm to check whether a curve is self-avoiding. In Section 5 we present and prove the formula $A(C) = 1$ for our class for self-avoiding curves. Finally, Section 6 contains concluding remarks.

2. Background

We introduce some background notions on the regular and semiregular grids in 2D [15,19], on simple closed curves [4,17] and classification of their vertices in the regular grids [7,10,14].

2.1. Regular and semiregular grids

The three regular grids in the plane, i.e., the triangular, square and hexagonal grids (see Fig. 2), induce the tessellations of the plane into regular triangles, squares and hexagons, respectively.

The eight semiregular grids (see Fig. 3) induce the tessellations into two or three different types of regular polygons (triangles, squares, hexagons, octagons and dodecagons) of the same side length, such that the configuration of polygons is the same around each grid vertex. Thus, one vertex configuration in each of the semiregular grids can be obtained from the configuration of another vertex using one translation and at most one rotation [3].

The regular and semiregular grids can be labelled (coded) by a sequence of integers, denoting the numbers of vertices of the polygons around each grid vertex. The labels for hexagonal, square and triangular grids are 6.6.6, 4.4.4.4 and 3.3.3.3.3.3, respectively. Using the letters from Fig. 3, the labels for semiregular grids are A=3.3.3.3.6, B=3.3.3.4.4, C=3.3.4.3.4, D=3.4.6.4, E=3.6.3.6, F=3.12.12, G=4.6.12 and H=4.8.8.

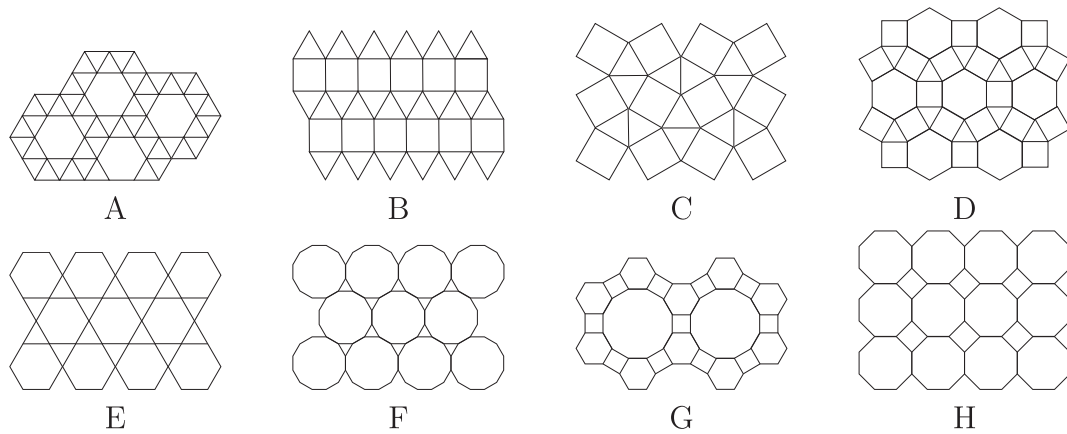


Fig. 3. The eight semiregular grids: (A) 3.3.3.3.6, (B) 3.3.3.4.4, (C) 3.3.4.3.4, (D) 3.4.6.4, (E) 3.6.3.6, (F) 3.12.12, (G) 4.6.12 and (H) 4.8.8. Only some cells of each grid are shown.

2.2. Simple closed curves

Definition 1. A closed curve C of length n in a grid is a cyclic sequence $v_0, e_0, v_1, e_1, \dots, v_{n-1}, e_{n-1}$, of grid vertices v_i and grid edges $e_i, 0 \leq i \leq n-1$, where each pair of consecutive edges e_i, e_{i+1} shares the vertex v_{i+1} (symbol $+$ is intended modulo n).

Note that C may pass several times through the same grid vertex, i.e., some vertices v_i may be repeated. For each $i = 0, 1, \dots, n-1$, three angles can be measured:

- the interior angle τ_i is the angle between edges e_{i-1}, e_i measured to the left of C ;
- the supplementary angle $2\pi - \tau_i$ is measured between the same two edges to the right of C ,
- the turning angle is $\pi - \tau_i$.

All such angles are considered as normalized to the interval $[0, 2\pi)$.

At each vertex v in C , the curve C makes a left turn, a right turn, or continues straight ahead. In the literature, vertices making a left turn are called convex, salient, positive or interior; vertices making a right turn are called concave, reentrant, negative or outer [7,10,14]. Here we will use the terms convex and concave. At convex vertices the interior angle is less than π , at concave vertices it is greater than π . At straight vertices, the interior angle is equal to π .

Definition 2. A simple closed curve is a closed curve $C : v_0, e_0, v_1, e_1, \dots, v_{n-1}, e_{n-1}$, in which there is no repetition of vertices, i.e., $v_i \neq v_j$ if $i \neq j$.

Thus, C is a discrete 1-surface [16] and it defines a (polygonal) Jordan curve.

In a simple closed curve C , the (cyclic) order, in which the vertices of C are listed, defines a consistent orientation, which leaves the interior (bounded region) on one side and the exterior (unbounded) region on the other one. We say that a simple closed curve C is oriented counterclockwise (clockwise) if it is oriented in such a way to have the interior to the left (right) side. We will use the shortcut CCW and CW for counterclockwise and clockwise, respectively.

In the square and the hexagonal grids, the number n^+ of convex vertices in a CCW oriented simple closed curve C is equal to the number n^- of concave ones plus 4 and 6, respectively [7,14,21], i.e.,

$$A(C) = \frac{1}{k}n^+ - \frac{1}{k}n^- = 1, \tag{1}$$

where $k = 4$ in the square and $k = 6$ in the hexagonal grid.

In the triangular grid, there is no (nontrivial) combinatorial relationship between the number of convex and concave vertices in C of the form $an^+ + bn^- = c$ for some values a, b and c (see [7] for an example).

To obtain a combinatorial relationship between the number of different types of vertices in a CCW oriented simple closed curve C in the triangular grid, a distinction is made between different types of convex and concave vertices. If n_3^+, n_{3+3}^+, n_3^- and n_{3+3}^- denote the number of vertices in a simple closed curve C where the interior angle is equal to $\pi/3, 2\pi/3, 5\pi/3, 4\pi/3$, respectively, then [10]

$$A(C) = \left(\frac{1}{3}n_3^+ + \frac{1}{6}n_{3+3}^+ \right) - \left(\frac{1}{3}n_3^- + \frac{1}{6}n_{3+3}^- \right) = 1. \tag{2}$$

Note that, differently from other authors, we measure the interior angle always to the left of the oriented curve C (with the orientation implied by the cyclic sequence of vertices), independently on whether C is oriented CCW or CW. Consequently, the value $A(C) = 1$ in Eqs. (1) and (2) assumes a CCW oriented curve C ; for a CW oriented one, $A(C) = -1$ holds.

2.3. More general closed curves

Some characterizations of curves which are allowed to touch themselves, without actually crossing themselves, were given in the literature. Such curves are interesting because, like simple curves, they partition the plane into consistently defined interior and exterior regions. The terms used to denote them are various, including crossing-free, self-avoiding, weakly-simple.

In the ordinary plane (without a grid superimposed to it), formulating a correct definition of such class of curves is not trivial [1,9]. A closed curve C is simple if it is the image of a continuous function from the unit circle to points of the 2D plane. Intuitively, a closed curve C is weakly simple if it is possible to transform it into a simple curve by means of small perturbations. Restricting to polygons with at least three vertices, we can consider just perturbations of the vertex positions. The formal definition, relying on the Fréchet distance, has been provided and proven correct in [9]. The same authors provide an algorithm, working in $O(n \log n)$ time, to check whether a given polygon is weakly simple. The $\log n$ factor in the time complexity comes from the need of checking for crossings among the edges of the polygon.

The problem has been studied in the square grid as well, where a curve is constrained to follow the grid edges. This means that two edges of the curve can intersect (or touch) only at their endpoints, and that a curve is simple if and only if it traverses each vertex only once. Curves are usually represented as words on an alphabet of four symbols meaning the four directions the curve can take to move from the current vertex to the next one. A linear-time algorithm to check whether the curve encoded in a given word contains repeated vertices was presented in [5]. Brlek et al. [8] considered a problem similar to the one we consider in this paper, but restricted to the square grid. They characterized self-avoiding closed curves that are allowed to pass through the same vertex at most twice (intended as boundaries of polyominoes in the square grid), and provided an algorithm to check whether a given word represents a curve satisfying such conditions. Here, we extend this work to any of the regular and semiregular grids, also allowing vertices traversed three times by the curve (but edges traversed only twice).

Čomić and Magillo [12] proposed an algorithm to test if a given word, encoding a curve in the square grid, is self-avoiding (and to draw it if this is the case). But the considered class of curves is too wide with respect to the ones necessary for bounding polyominoes: the only restriction is that the curve cannot contain spurs, while vertices and edges can be traversed an arbitrary number of times. Therefore, our algorithm will be much simpler than that.

3. Characterization of simple and self-avoiding closed curves

We provide a characterization of self-avoiding closed curves in the three regular and eight semiregular grids, which are allowed to contain repeated vertices and edges, with a limited number of repetitions.

3.1. Characterization of self-avoiding curves with repeated vertices

We define the class of closed curves where vertices are allowed to be traversed more than once, but edges are not. Then, we will provide conditions for a curve in this class to be self-avoiding.

Definition 3. A closed RV-curve in any of the regular or semiregular grids, is a closed curve $C : v_0, e_0, v_1, e_1, \dots, v_{n-1}, e_{n-1}$, with repeated vertices, and without repeated edges.

The letters “RV” in the name stand for “Repeated Vertex”. Note that, since the edges of a closed V-curve C are all distinct, then C cannot contain spurs (vertices at which the turning angle is equal to π).

The maximum number of times a vertex can be repeated, without repeating an edge as well, depends on the grid type, and, in particular, on the degree deg of its vertices (the degree is the number of incident edges in a vertex), and it is equal to $\lfloor deg/2 \rfloor$. Therefore, it is 3 in the triangular grid (where $deg = 6$), 1 in the hexagonal grid and in grids F, G and H (where $deg = 3$), and 2 in the other grids (where $deg = 4$ or 5). We call double vertex a vertex which is repeated twice in the curve, and triple vertex a vertex which is repeated three times.

For a double vertex $v = v_i = v_j$ in the curve C , we consider the CCW order of the four edges $e_{i-1}, e_i, e_{j-1}, e_j$ around v . Similarly, for a triple vertex $v = v_i = v_j = v_k$, with i, j, k occurring in this order, modulo cyclicity, in C , we consider the CCW order of the six edges $e_{i-1}, e_i, e_{j-1}, e_j, e_{k-1}, e_k$ around v .

Definition 4. A closed RV-curve is self-avoiding if the radial CCW order of the edges in C incident to any double vertex v is one of the two configurations

+1v: $e_i, e_{j-1}, e_j, e_{i-1}$,

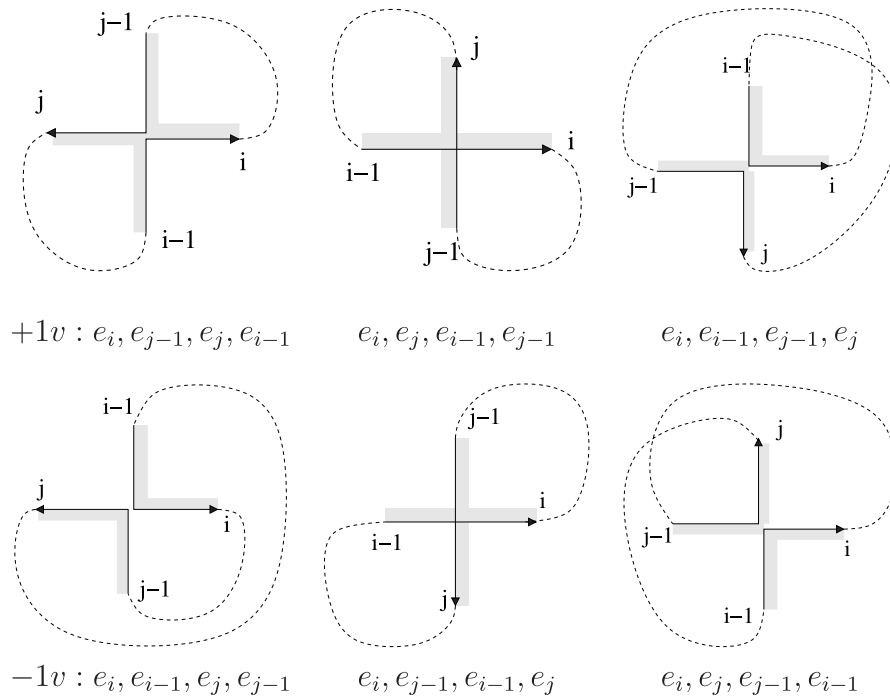


Fig. 4. Possible configurations for a double vertex. The left side of the edges is shaded, and a possible shape of the curve is shown by dotted lines. Only the configurations in the left column can appear in a self-avoiding closed RV-curve.

$-1v : e_i, e_{i-1}, e_j, e_{j-1},$

and the order of the edges incident to any triple vertex (in the triangular grid) is one of the four configurations

- $+2v' : e_i, e_{j-1}, e_j, e_{k-1}, e_k, e_{i-1},$
- $-2v' : e_i, e_{i-1}, e_k, e_{k-1}, e_j, e_{j-1},$
- $+2v'' : e_i, e_{j-1}, e_j, e_{i-1}, e_k, e_{k-1},$
- $-2v'' : e_i, e_{k-1}, e_k, e_{i-1}, e_j, e_{j-1}.$

where all orders are intended modulo cyclic permutation.

The configurations $\pm 1v$ are shown in Fig. 4, left; the configurations $\pm 2v'$ and $\pm 2v''$ are shown in Fig. 5.

In case of a double vertex v , there are other four permutations of the edges around v , but it is easy to verify that in those cases the closed RV-curve is self-crossing. With $e_i, e_j, e_{i-1}, e_{j-1}$ or $e_i, e_{j-1}, e_{i-1}, e_j$ the curve crosses itself at v , and with $e_i, e_{i-1}, e_{j-1}, e_j$ or $e_i, e_j, e_{j-1}, e_{i-1}$ the curve necessarily crosses itself at some other vertex (see Fig. 4, middle and right).

Similarly, in case of a triple vertex, it is easy to verify that the other permutations correspond to a self-crossing curve.

Let us consider the radial sectors around a repeated vertex v in one of the configurations allowed by Definition 4 (sectors with apex v and bounded by pairs of consecutive edges in the radial order around v). We label the sectors with alternating labels L (left) and R (right), starting with L for the sector bounded by e_i and by the edge following it in the configuration. In Figs. 4 (left) and 5, the sectors labelled with L are the shaded ones.

A self-avoiding closed RV-curve C divides the plane into a collection of polygonal regions, exactly one of them unbounded.

Definition 5. A self-avoiding closed RV-curve C is oriented CCW (or CW) if the unbounded region is to its right (or its left).

This rule consistently defines the orientation of a self-avoiding RV-curve; if the curve is self-crossing, its orientation cannot be defined. Assuming a CCW (CW) oriented self-avoiding closed RV-curve, the bounded regions of the partition are classified as inner or outer if they lie to the left or right (right or left) of C , respectively.

The orientation of a closed RV-curve C can be checked by applying the following ray-based procedure. Take an (oriented) edge e_i of C and a point p lying in the interior of the grid polygon to the left of e_i . Draw a ray from p to infinity in a direction such that the ray does not pass through any vertex of the curve, and count the number of intersections with

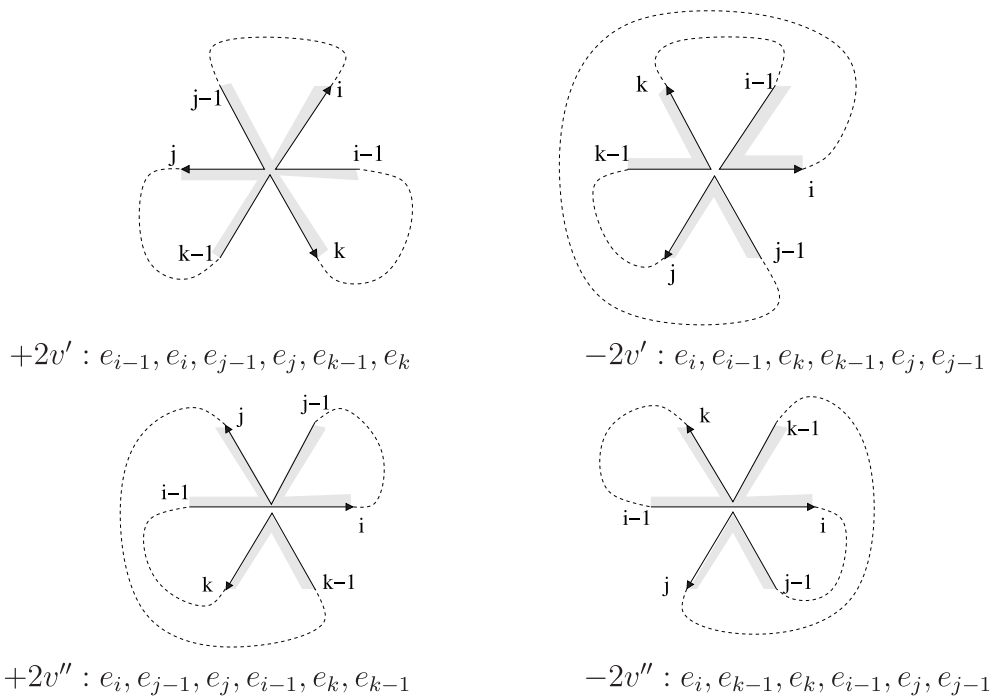


Fig. 5. Possible configurations for a triple vertex. The left side of the edges is shaded, and a possible shape of the curve is shown by dotted lines. Only the shown configurations can appear in a self-avoiding closed RV-curve.

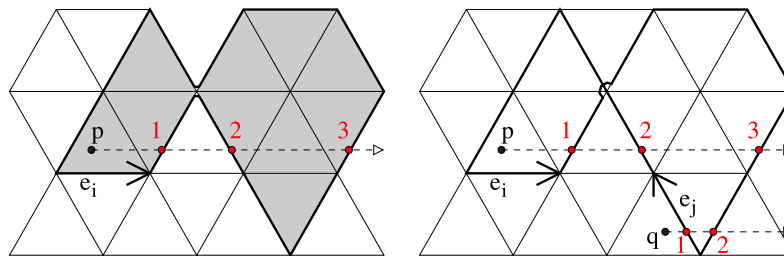


Fig. 6. Ray-based procedure to determine the orientation of a closed RV-curve. The left curve is CCW oriented because at every edge the ray has an odd number of intersections (it is shown for one edge). The right curve cannot be oriented because it is self-crossing (the number of intersection is odd for some edges and even for some others, two of them are shown).

the edges of C (see Fig. 6). If the number of intersections of the ray is odd for every edge of C , then C is CCW oriented; if the number of intersections is even for every edge, then C is CW oriented; otherwise, a consistent orientation of C cannot be defined (i.e., the region bounded by C does not lie on a unique side of C and C is self-crossing).

In the labelling of the sectors around a double or triple vertex in a CCW oriented self-avoiding RV-curve C , the ones labelled L and R belong to an inner and to an outer region of C , respectively. If C is oriented CW, the converse happens.

3.2. Characterization of self-avoiding curves with repeated edges

We extend the class of closed RV-curves to that of closed RE-curves, where edges are allowed to be traversed twice.

Definition 6. A closed RE-curve in a grid is a closed curve $C : v_0, e_0, v_1, e_1, \dots, v_{n-1}, e_{n-1}$, with each vertex repeated at most three times, and each edge repeated at most twice, and the two occurrences of a repeated edge are not consecutive in the curve (i.e., $e_i \neq e_{i+1}$ for each i).

The letters “RE” in the name stand for “Repeated Edge”. We call double edge an edge which is repeated twice in the curve.

The motivation for considering closed RE-curves is that, if self-avoiding, they can be used to describe with a single closed curve the boundary of a finite set of polygons in a regular or semiregular grid, which may be disconnected and/or

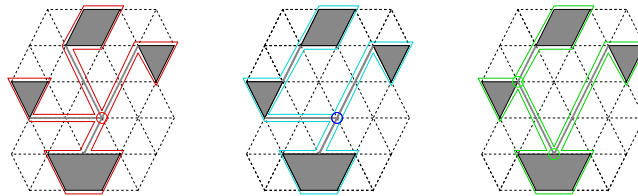


Fig. 7. A generalized polyomino (the set of dark triangles) and some closed curves bounding it. The left curve is not a closed RE-curve because one vertex is repeated four times. The other two are closed RE-curves, but only the right one will be considered in our definition of a self-avoiding closed RE-curve (Definition 8).

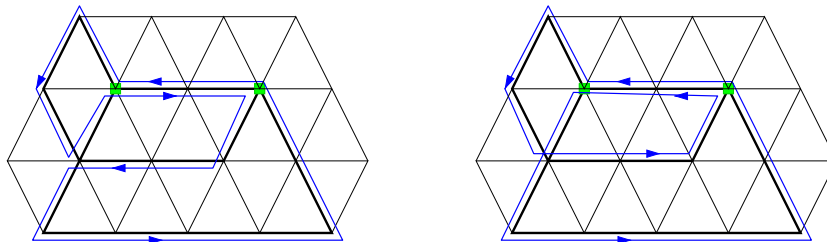


Fig. 8. Examples of double chains. In each curve there is one double chain consisting of two double edges (the endpoints of the chain are the two vertices marked with a square). The left curve traverses the edges of the double chain twice in opposite directions. The right curve traverses the edges of the double chain twice in the same direction, and is self-crossing.

with holes (see Figs. 1 and 7). Such polygons are generalized polyominoes. A generalized polyomino intuitively consists of a collection of solid connected components, and a collection of hollow connected components describing holes inside the solid ones. In its boundary curve, infinitely narrow solid (hollow) corridors, enclosed by chains of double edges, will be used to connect different solid (hollow) components to each other. For that, it is sufficient that edges can be repeated twice. Moreover, in the boundary of a generalized polyomino, repeated vertices are only needed to connect two (solid or hollow) components. So, repeating a vertex more than three times is not necessary. Fig. 7 illustrates some examples of curves bounding a generalized polyomino, showing that double edges and triple vertices are sufficient to bound it.

Note that the endpoints of a repeated edge must be repeated vertices. Consider a vertex v , endpoint of a repeated edge. The configuration at v is equal to one of $+1v, -1v, +2v', -2v', +2v'', -2v''$ (see Section 3.1) in which two edges, consecutive in the radial order around v but not consecutive in the curve C , coincide. Two edges that are consecutive along the curve C cannot coincide, because that is forbidden by Definition 6.

In the literature, self-avoiding curves containing only configurations obtained by coincidence of edges from $+1v, -1v$ have been considered in the square and the hexagonal grids [8,20]. In a recent work [12], more general self-avoiding curves in the square grid have been considered (where each edge and vertex can be repeated an arbitrary number of times), but the characterization is given by means of an algorithm.

Definition 7. A double chain in a closed RE-curve C is a pair of maximal subsequences of edges in C , such that the two sequences are disjoint in C , and both sequences traverse the same grid edges.

The two subsequences of a double chain may traverse the edges either in the same direction, or in the opposite directions (see Fig. 8).

The vertices v where four edges of the double chain are incident (i.e., two pairs of double edges), are called the internal vertices of the chain. We have that $v \equiv v_i \equiv v_j$ where the two edges e_{i-1}, e_i are the same as e_{j-1}, e_j (in any order). Note that v could be traversed by the curve another time, i.e., be a triple vertex.

Two vertices of the double chain have just two incident edges belonging to the chain (i.e., one pair of double edges), and are called the endpoints of the chain. Each endpoint u of a double chain D is incident to one repeated edge in D and

1. two (two or four in the triangular and A=3.3.3.3.6, B=3.3.3.4.4, C=3.3.4. 3.4 grids) non-repeated edges, see the vertices marked with number 1 in Fig. 9,
2. one repeated edge not belonging to D and two non-repeated edges, see the vertex marked with number 2 in Fig. 9, or
3. two repeated edges not belonging to D , see the vertex marked with number 3 in Fig. 9.

Considering the radial order of the edges incident in a vertex v , that is an endpoint of a double chain, we define the following configurations. If v is a double vertex (see Fig. 10):

$0e$: same as $\pm 1v$ where $e_i \equiv e_{j-1}$ and $e_j \equiv e_{i-1}$.

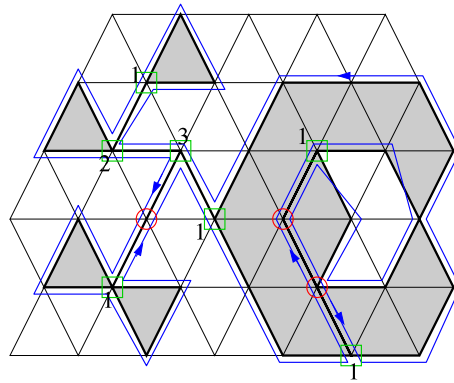


Fig. 9. The vertices marked with a square are endpoints of a double chain, those marked with a circle are intermediate vertices of a double chain. Note that an intermediate vertex may be double or triple. The numbers denote the different situations of an endpoint vertex of a double chain (see text).

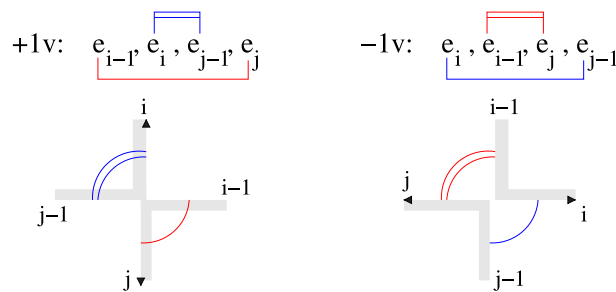


Fig. 10. Configurations with double edges obtained from $\pm 1v$ by letting some pairs of edges coincide. The two pairs that may coincide are marked. If both coincide, then the vertex is inside a double chain. If one pair coincides, the vertex is an endpoint of a double chain.

$+1e$: same as $+1v$ where $e_i \equiv e_{j-1}$ (this is equivalent to $e_j \equiv e_{i-1}$ if we exchange letters i, j in the notation)

$-1e$: same as $-1v$ where $e_i \equiv e_{j-1}$ (with same remark as above)

and if v is a triple vertex (see Figs. 11 and 12):

$+2e'$: same as $+2v'$ where $e_i \equiv e_{j-1}$ (this is equivalent to $e_k \equiv e_{i-1}$ or $e_j \equiv e_{k-1}$ if we exchange the letters in the notation)

$-2e'$: same as $-2v'$ where $e_i \equiv e_{j-1}$ (with same remark as above)

$+2e''_1$: same as $+2v''$ with $e_i \equiv e_{j-1}$ (or same as $-2v''$ with $e_i \equiv e_{k-1}$)

$-2e''_1$: same as $+2v''$ with $e_k \equiv e_{i-1}$ (or same as $-2v''$ with $e_j \equiv e_{i-1}$)

$+2e''_2$: same as $+2v''$ with $e_j \equiv e_{i-1}$ (or same as $-2v''$ with $e_k \equiv e_{i-1}$)

$-2e''_2$: same as $+2v''$ with $e_i \equiv e_{k-1}$ (or same as $-2v''$ with $e_i \equiv e_{j-1}$)

In cases $\pm 2e'$, it is possible that other one or two pairs of edges coincide. In cases $\pm 2e''_1, \pm 2e''_2$, it is possible that one other pair of edges coincide.

At a double vertex, at most two double edges may occur. The configurations with two double edges are obtained from $+1v$ or $-1v$ if the two pairs (e_i, e_{j-1}) and (e_j, e_{i-1}) coincide (if only one such pair of edges coincide, we have one double edge). The other pairs of edges cannot coincide because they are consecutive in the curve C . A double vertex in configuration $0e$ is an internal vertex of a double chain. Configurations $\pm 1e$ correspond to an endpoint of a double chain.

At a triple vertex, at most three double edges may occur. The configurations with double edges are obtained from the configurations $+2v', -2v', +2v'', -2v''$ where e_i (resp. e_j, e_k) coincides with an edge consecutive to it in the radial order, different from e_{i-1} (resp. e_{j-1}, e_{k-1}). Three coincident pairs are possible only in configurations $+2v'$ or $-2v'$ (in $+2v'$ the pairs $(e_i, e_{j-1}), (e_j, e_{k-1}), (e_k, e_{i-1})$), and similarly in $-2v'$). The configurations $+2v'', -2v''$ admit at most two coincident pairs of edges. The configurations with just two double edges, coming from $\pm 2v''$ with $e_i \equiv e_{j-1}$ and $e_j \equiv e_{i-1}$, or with $e_i \equiv e_{k-1}$ and $e_k \equiv e_{i-1}$, correspond to an internal vertex of a double chain. The other configurations correspond to endpoints of a double chain.

In all the configurations $\pm 1e, \pm 2e'$ (if only one or two pairs of edges coincide), $\pm 2e''_1, \pm 2e''_2$, we can label the radial sectors around v with L (left) or R (right) based on the labels they have in the configurations $\pm 1v, \pm 2v', \pm 2v''$ they come

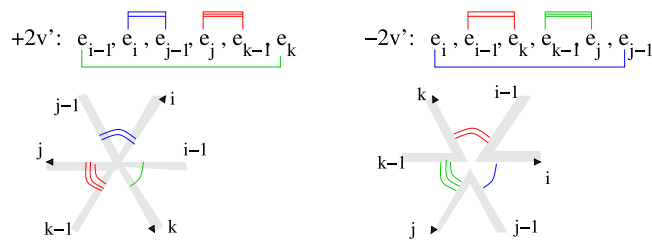


Fig. 11. Configurations with double edges obtained from $\pm 2v'$ by letting some pairs of edges coincide. The three pairs that may coincide are marked. One, two, or three of them may coincide, and the vertex is an endpoint of a double chain.

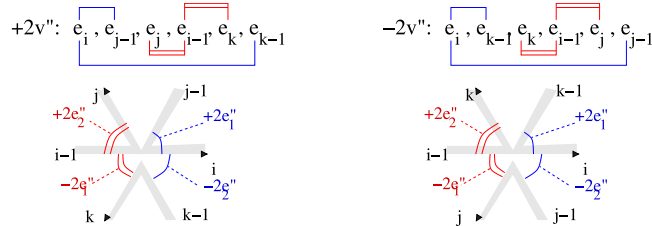


Fig. 12. Configurations with double edges obtained from $\pm 2v''$ by letting some pairs of edges coincide. Four pairs may coincide, but not all of them at the same time. The two pairs involving e_i (or e_{i-1}) cannot coincide simultaneously.

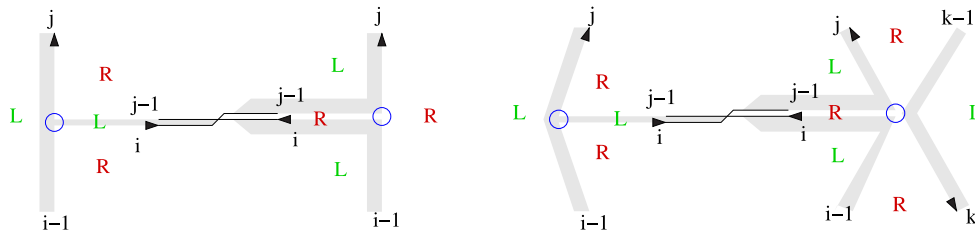


Fig. 13. Some forbidden situations for the two endpoint vertices of a double chain.

from. In this way, every infinitely narrow sector enclosed by a pair of double edges gets labelled as L or R. In configuration $0e$ it is not possible to uniquely define a labelling. The same happens in configurations $\pm 2e'$ if three pairs of edges coincide, because this configuration may come from $+2v'$ or from $-2v'$, with opposite labelling.

Definition 8. A closed RE-curve C is self-avoiding if

- each double or triple vertex of C , without double edges incident into it, is in one of the configurations $\pm 1v, \pm 2v', \pm 2v''$,
- each double or triple vertex of C , with double edges incident into it, is in one of the configurations $\{0e, \pm 1e, \pm 2e', \pm 2e_1', \pm 2e_2'\}$,
- no triple vertex of C has three double edges incident into it,
- denoting with u, w the endpoints of a double chain, the radial sectors around u and w , enclosed by the double edges of the chain, are either both labelled L, or both labelled R.

The first condition ensures that each self-avoiding RV-curve is also as self-avoiding RE-curve. The third condition forbids ambiguous situations at a triple vertex. Ambiguous situations at a double vertex (i.e., configuration $0e$) are not forbidden, because they can be disambiguated by looking at the two endpoints of the double chain containing such vertex, as ensured by the last condition.

Intuitively, the last condition in Definition 8 requires that the infinitely narrow corridor enclosed within a double chain must entirely lie to the same side with respect to the curve C . The corridor is either solid, and connects two connected components of the inner region bounded by C , or it is hollow, and connects two connected components of the outer region bounded by C (one of which may be the unbounded region). If the two endpoints have different labels, the corridor should be solid and hollow at the same time, and this is impossible in a self-avoiding closed curve. Some of the forbidden situations are illustrated in Fig. 13, and in such cases the curve C necessarily crosses itself.

Definition 8 implies that the two passages of the curve C through a double edge are in mutually opposite directions. This is easily verified looking at the allowed configurations.

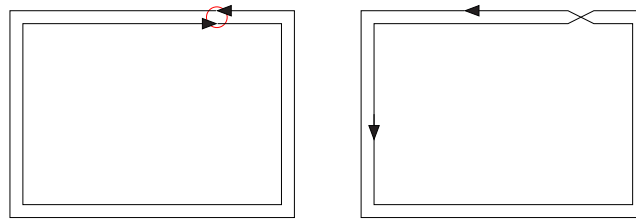


Fig. 14. In a single self-avoiding closed RE-curve, it is not possible that all vertices lying on the boundary of the unbounded region are of type 1. The curve would not be connected (left) or it would cross itself (right).

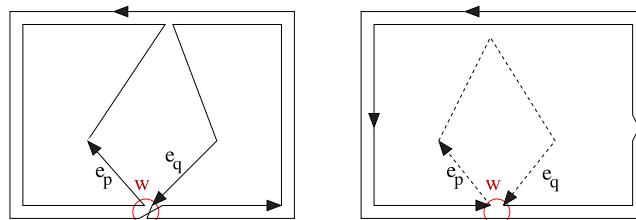


Fig. 15. Left: Denoting by e_p, e_q the two extra edges of a vertex w of type 2, if we leave w from e_p and follow the curve, the next passage through w must enter w from e_q , or the curve would be disconnected and/or cross itself. Right: by removing the portion of the curve from e_p to e_q (shown dotted), the vertex w becomes of type 1.

In order to define the orientation of a self-avoiding closed RE-curve, we first introduce the following property.

Proposition 1. *Given a self-avoiding closed RE-curve C , the unbounded region delimited by C has at least one edge on its boundary, which is not a double edge in C .*

Proof. The proof is by contradiction. A vertex on the boundary of the unbounded region must have two pairs of double edges, bounding the unbounded region. We classify such vertices into type 1 and type 2. A vertex v is of type 1 if such two pairs of double edges are the only incident edges of v (v is in configuration $0e$). Otherwise v is of type 2.

If there are no vertices of type 2, then either C is two disjoint curves, or C is self-crossing (see Fig. 14, left). Let the set of vertices of type 2 be non-empty. A vertex w of type 2 must be traversed by the curve three times, so there must be exactly other two curve edges incident into w . Because C satisfies Definition 8, these two additional edges e_p, e_q must be consecutive in the radial order around w (otherwise the two double edges incident in w would not bound the unbounded region), and are traversed by the curve in opposite directions (once entering and once leaving w , i.e., either $w \equiv v_p \equiv v_{q+1}$, as in Fig. 15, or $w \equiv v_{p+1} \equiv v_q$).

Let us start from the edge, among e_p, e_q , which is going away from w , e.g. e_p (as in Fig. 15), and follow C until we return at w . The edge, by which we return at w , must be e_q , otherwise C is disconnected and/or self-crossing (see Fig. 15, left).

If we remove the portion of C going from e_p to e_q , we obtain a curve C' where vertex w is of type 1 (see Fig. 15, right) and an edge of C' is on the boundary of the unbounded region if and only if it was on the boundary of the unbounded region in C . If we apply this construction to all vertices of type 2, we will arrive at a curve where all vertices on the boundary of the unbounded region are of type 1. But we have already proved that this case does not correspond to a self-avoiding closed RE-curve.

The CCW and CW orientations of a self-avoiding closed RE-curve C are defined in the same way as for self-avoiding closed RV-curves. This definition is correct, because Proposition 1 guarantees that there is at least one non-repeated edge on the boundary of the unbounded region.

The CCW or CW orientation of a self-avoiding closed RE-curve C can be checked by applying the ray-based procedure, similarly to self-avoiding closed RV-curves, but considering only the non-repeated edges in C , and the intersection of the ray with each repeated edge must be counted twice (see Fig. 16).

4. Algorithm for checking whether an RE-curve is self-avoiding

From Definition 8, we can draw an algorithm to detect, in $O(n)$ time, whether a closed curve C , given in one of the regular or semiregular grids, is a self-avoiding RE-curve or not, because it is either not an RE-curve, or it is self-crossing.

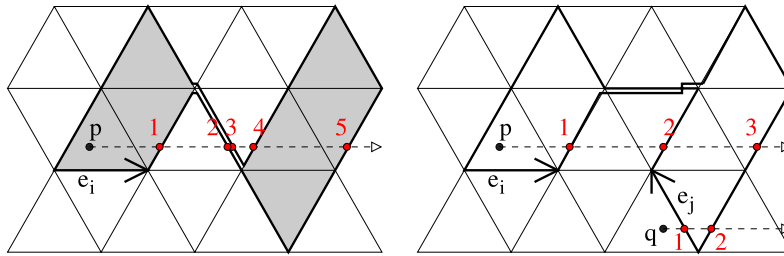


Fig. 16. Ray-based procedure to determine the orientation of a self-avoiding closed RE-curve. The left curve is CCW oriented because at every non-repeated edge the ray has an odd number of intersections (it is shown for one edge). The right curve cannot be oriented because it is self-crossing (two edges are shown, one with an odd and the other one with an even number of intersections).

The algorithms defined in the literature to check whether a polygon is weakly simple run in $O(n \log n)$ time, where n is the length of the polygon, because they need first to embed the polygon into a graph. Moreover, they need a careful processing of spurs and forks.

Here, the input closed curve is already embedded in the grid, forks are not possible, and spurs are forbidden by our [Definition 6](#).

The steps of the algorithm are:

1. initialization,
2. check that C is an RE-curve,
3. check configurations of vertices,
4. check double chains.

4.1. Initialization

As auxiliary data structures, the algorithm uses two arrays associated with a vertex v . Such arrays have as many entries as the degree of the vertices in the grid, are indexed on the directions of the edges of the grid incident in v , and radially sorted around v .

- An entry of the first array stores the list of the indexes of the edges of the curve traversing v and lying in that direction as seen from v .
- An entry of the second array stores, if the corresponding direction of the first array has two elements (indexes of the two edges of the curve forming a double edge), the label L (left) or R (right) of the radial sector enclosed by the two coincident edges.

In the first step, the algorithm traverses the given closed curve C and initializes the first array of each vertex v of the grid, which is traversed by C .

This step performs $O(n)$ operations. Note that we do not need to sort the edges in the arrays, because the number of directions is fixed by the grid.

4.2. Check that it is an RE-curve

For each vertex, we examine the first array:

- If the total number of indexes in the lists forming the first array is more than six, or if some direction has more than two indexes, then we report that C is not an RE-curve.
- If some direction has two edge indexes (v has a double edge) and such two indexes are consecutive, then we report that C is not an RE-curve (it contains a spur).

This step performs $O(1)$ steps for each vertex of the curve C , because the number of directions is constant.

4.3. Check vertex configurations

In this step, each vertex v traversed by C is classified as ‘simple’ (i.e., traversed once), ‘chain’ (i.e., intermediate vertex of a double chain), ‘repeated’ (double or triple vertex with no double edge) or ‘endpoint’ (double or triple vertex, which is an endpoint of one or more double chains). In the last case, the second array is initialized for v and v is saved in a list of vertices that will be examined in the last step of the algorithm. All the intersections, with the exception of those occurring within a double chain, are also detected in this step.

For each vertex, we examine the first array:

- If the total number of indices in the lists is two, then v is ‘simple’.
- If all directions have at most one index (v has no double edge), then we check whether the cyclic sequence of edge indices around v is compatible with one of the configurations $\pm 1v, \pm 2v', \pm 2v''$; if it is not, then we report that C is self-intersecting. Otherwise, v is ‘repeated’.
- Now, v has some double edges. First we check whether v is an intermediate vertex of a double chain. If there are two directions, one having indexes i, j and the other one $i - 1, j - 1$, for some i, j , then C is self-crossing. If there are two directions, one having indexes $i, j - 1$ and the other one $i - 1, j$, for some i, j , then v is ‘chain’.
- Now we check whether v is an endpoint of a double chain. For each direction, whose list contains two indices p, q , the radial order of the two coincident edges may be e_p, e_q or e_q, e_p . Considering both possible permutations of e_p, e_q , we check whether the radial sequence of edge indices around v is compatible with configurations $\pm 1e, \pm 2e', \pm 2e''_1, \pm 2e''_2$. If it is not, then we report that C is self-intersecting. Otherwise, v is ‘endpoint’. For each direction having a double edge, we save in the second array the label L or R, that the matching configuration gives to that pair of double edges. We add v to an initially empty list of double chain endpoints.

The matching configuration is easily searched by using a table. In case of double edges, the number of permutations to be considered is limited because there are at most six edges at v , which implies at most three double edges. Therefore also this step spends $O(1)$ time for each vertex of C .

4.4. Check double chains

We scan the list containing the endpoint vertices of the double chains. Let v be one of them.

- Each non-empty element of the second array of v corresponds to a pair of coincident edges whose indexes h, k are stored in the same position (i.e., direction) in the first array, and contains a label. Without loss of generality, let the label be L (left) and let the first array contain (in other directions) $h - 1$ and $k + 1$ (this means that e_h leaves v and e_k enters v).
- We traverse the curve C from e_h to the next vertex w which is classified as endpoint (it is the other endpoint of the double chain). Let m be the index of the edge of the double chain entering w . We check the label stored in the second array of w , for the direction containing m in the first array, is L. If not, then we report that C is self-intersecting.

If C has passed all the checks, then C is self-avoiding.

This step traverses each double chain, therefore its time complexity is in $O(n)$.

5. Formula for self-avoiding curves in the semi-regular grids

We first unify the known Formulas (1) and (2), relating the number of convex and concave vertices in a curve, and reviewed in Section 2.2, to a form which is valid in all the three regular grids. Then, we present and prove our new and more general formula, valid in all regular and semiregular grids.

5.1. General formula for the three regular grids

We first introduce a notation that allows generalizing Formulas (1) in the square and hexagonal grids and (2) in the triangular grid. Let v_i be a vertex of a simple closed curve C in one of these grids and τ_i be the interior angle at v_i . We define the contribution $d(v_i)$ of a vertex v_i to $A(C)$ as

$$d(v_i) = \frac{\pi - \tau_i}{2\pi}. \tag{3}$$

It is easy to verify that $d(v_i) = \pm 1/4$ for $\tau_i = \pi/2$ and $3\pi/2$ (square grid), $d(v_i) = \pm 1/6$ for $\tau_i = 2\pi/3$ and $4\pi/3$ (hexagonal grid), and $d(v_i) = \pm 1/3, \pm 1/6$ for $\tau_i = \pi/3, 2\pi/3, 5\pi/3$ and $4\pi/3$ (triangular grid). With this notation, for a CCW oriented simple closed curve C in the square, hexagonal and triangular grids we have

$$A(C) = \sum_{v_i \in C} d(v_i) = \sum_{\tau} n_{\tau} \frac{\pi - \tau}{2\pi} = 1 \tag{4}$$

where n_{τ} denotes the number of vertices forming an interior angle equal to τ . For a CW oriented curve, $A(C) = -1$.

5.2. General formula in the semi-regular grids

In the semiregular grids, the set of possible convex angles T^+ and the set of possible concave angles T^- contain seven elements each, i.e., $T^+ = \{\pi/3, \pi/2, 2\pi/3, 3\pi/4, 5\pi/6\}$, and $T^- = \{2\pi - \tau | \tau \in T^+\}$. For the set T^+ :

- Angle $\pi/3$ occurs when the curve turns around an equilateral triangle, leaving the triangle to its left.

- Angle $\pi/2$ occurs when the curve turns around a square, leaving the square to its left.
- Angle $2\pi/3$ occurs when the curve turns around a regular hexagon, or around a pair of mutually edge-adjacent equilateral triangles, leaving such polygon(s) to its left.
- Angle $3\pi/4$ occurs when the curve turns around a regular octagon, leaving such polygon to its left.
- Angle $5\pi/6$ occurs when the curve turns around a regular dodecagon, or around a pair of mutually edge-adjacent equilateral triangle and square, leaving such polygon(s) to its left.

The concave angles of the set T^- occur when the curve turns around the same polygons, leaving them to its right.

Definition 9. Let $C : v_0, e_0, v_1, e_1, \dots, v_{n-1}, e_{n-1}$ be a closed curve in any of the regular or semiregular grids. The contribution $d(v)$ of a vertex v in C is the sum, over all passages of C through v (i.e., for all $0 \leq i < n$ such that $v_i \equiv v$), of the contributions $d(v_i)$ as defined in Eq. (3):

$$d(v) = \sum_{v_i \in C, v_i \equiv v} d(v_i)$$

Proposition 2. If C is a CCW oriented simple closed curve, or a self-avoiding RE-curve then, denoting with n_τ the number of vertices in C with interior angle equal to τ ,

$$A(C) = \sum_{v \in C} d(v) = \sum_{l=0}^{n-1} d(v_l) = \sum_{\tau \in T} n_\tau \frac{\pi - \tau}{2\pi} = 1.$$

If C is CW oriented, $A(C) = -1$ holds.

5.3. Proof for simple closed curves

We prove that Proposition 2 holds for a (CCW oriented) simple closed curve $C : v_0, e_0, v_1, \dots, e_{n-1}$ in any of the regular or semiregular grids.

Proof. The contribution $d(v)$ of each vertex $v \equiv v_i$ in C is equal to the turning angle $\pi - \tau_i$ divided by 2π . It is well-known that the sum of turning angles over the vertices of a simple closed polygonal curve C is equal to 2π . Thus,

$$A(C) = \sum_{v_i \in C} d(v_i) = \sum_{v_i \in C} \frac{\pi - \tau_i}{2\pi} = \frac{1}{2\pi} \sum_{v_i \in C} (\pi - \tau_i) = \frac{1}{2\pi} \cdot 2\pi = 1.$$

5.4. Proof for self-avoiding closed RV-curves

We prove that Proposition 2 holds for a (CCW oriented) self-avoiding closed RV-curve $C : v_0, e_0, v_1, \dots, e_{n-1}$ in any of the regular or semiregular grids.

The proof is in two steps: first for a curve with double vertices only, and then for a curve with triple vertices as well.

5.4.1. Proof for self-avoiding closed RV-curves with double vertices only

Proof. The proof is by induction on the number N of double vertices in C .

For $N = 0$, the statement has been proven in Section 5.3.

For $N > 0$, let $v = v_i = v_j, 0 \leq i < j \leq n - 1$, be a double vertex in C . We split C at v into two curves $C' : v, e_i, \dots, e_{j-1}$ and $C'' : v, e_j, \dots, e_{i-1}$. The curves C', C'' have less than N double vertices, thus the inductive hypothesis holds for them. We denote with d, d' and d'' the contributions of a vertex to $A(C), A(C')$ and $A(C'')$, respectively.

For each vertex $u \neq v$ in $C, d(u) = d'(u) + d''(u)$. Note that each such u will belong to exactly one of C', C'' , so one of $d'(u), d''(u)$ is 0. Therefore $A(C) = A(C') + A(C'') + d(v) - (d'(v) + d''(v))$. Let us examine the contributions of v . The configuration of v is either $+1v$ or $-1v$.

Case $+1v : e_i, e_{j-1}, e_j, e_{i-1}$. Fig. 17 shows a possible shape of C and the two curves C', C'' resulting from splitting. The curves C' and C'' are both CCW oriented, therefore $A(C') = A(C'') = 1$. We denote the angles around v as in Fig. 17.

In C , the interior angle at $v = v_i$ (angle $\widehat{e_{i-1}, e_i}$) is $2\pi - \alpha$ and at $v = v_j$ ($\widehat{e_{j-1}, e_j}$) is $2\pi - \gamma$. Thus $d(v) = \frac{\pi - (2\pi - \alpha) + \pi - (2\pi - \gamma)}{2\pi} = \frac{-2\pi + \alpha + \gamma}{2\pi}$.

In the union of C', C'' the interior angles are $\widehat{e_{j-1}, e_i} = \beta$ and $\widehat{e_{i-1}, e_j} = \delta$. So, $d'(v) + d''(v) = \frac{\pi - \beta + \pi - \delta}{2\pi} = \frac{2\pi - \beta - \delta}{2\pi}$.

Remembering that $\alpha + \beta + \gamma + \delta = 2\pi$, the following equation holds:

$$A(C) = A(C') + A(C'') + \frac{-2\pi + \alpha + \gamma}{2\pi} - \frac{2\pi - \beta - \delta}{2\pi} =$$

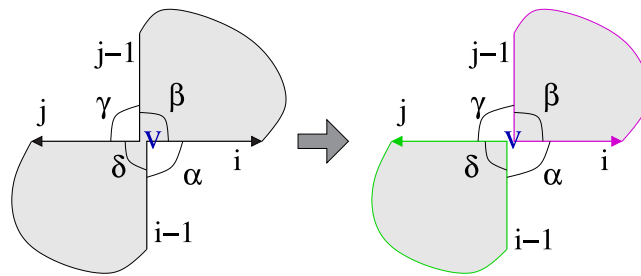


Fig. 17. A sample curve with a double vertex v in configuration $+1v$ and splitting the curve C at v into C' (magenta) and C'' (green). The edges are labelled by their indices in C . The angles at v are shown. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

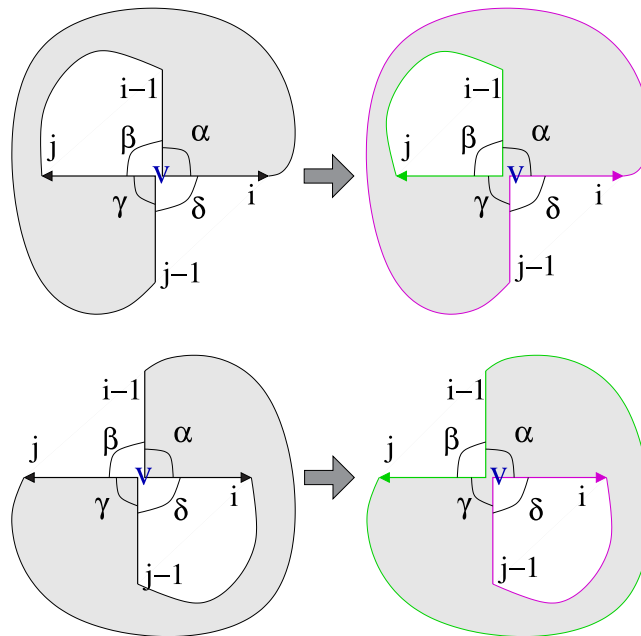


Fig. 18. A sample curve with a double vertex v in configuration $-1v$ and splitting the curve C at v into C' (magenta) and C'' (green). The edges are labelled by their indices in C . The angles at v are shown. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$= 1 + 1 + \frac{-2\pi - 2\pi + \alpha + \gamma + \beta + \gamma}{2\pi} = 2 + \frac{-4\pi + 2\pi}{2\pi} = 2 - 1 = 1.$$

Case $-1v : e_i, e_{i-1}, e_j, e_{j-1}$.

A possible shape of C and the two curves resulting from splitting are shown in Fig. 18. The curves C', C'' have opposite orientations, so $A(C') = -A(C'')$.

We denote the angles around v as in Fig. 18. In C , the interior angle at $v = v_i(\widehat{e_{i-1}, e_i})$ is α and the one at $v = v_j(\widehat{e_{j-1}, e_j})$ is γ . Thus, $d(v) = \frac{2\pi - \alpha - \gamma}{2\pi}$.

In the union of C', C'' the interior angles are $\widehat{e_{i-1}, e_j} = \alpha + \gamma + \delta$ and $\widehat{e_{j-1}, e_i} = \alpha + \beta + \gamma$. So, $d'(v) + d''(v) = \frac{2\pi - (2\alpha + \beta + 2\gamma + \delta)}{2\pi}$.

Remembering that $\alpha + \beta + \gamma + \delta = 2\pi$ the following equation holds:

$$\begin{aligned} A(C) &= A(C') + A(C'') + \frac{2\pi - \alpha - \gamma}{2\pi} - \frac{2\pi - 2\alpha - \beta - 2\gamma - \delta}{2\pi} \\ &= 0 + \frac{\alpha + \beta + \gamma + \delta}{2\pi} = 0 + \frac{2\pi}{2\pi} = 1. \end{aligned}$$

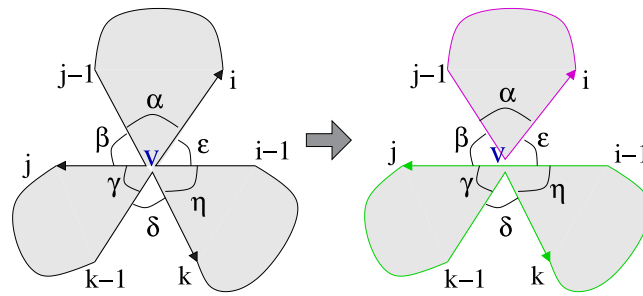


Fig. 19. Possible shape for a curve C with a vertex v in configuration $+2v'$: $e_{i-1}, e_i, e_{j-1}, e_j, e_{k-1}, e_k$, and splitting C at v into C' (magenta) and C'' (green). The angles at v are shown. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

5.4.2. Proof for self-avoiding closed RV-curves with triple vertices as well

Proof. The proof is by induction on the number N of triple vertices.

If $N = 0$ the thesis has been proven.

If $N > 0$ let $v = v_i = v_j = v_k$, $0 \leq i < j < k \leq n - 1$ be a triple vertex in C . We split C at v into the curves $C' : v, e_i, \dots, e_{j-1}$ and $C'' : v, e_j, \dots, e_{i-1}$, where v is a double vertex in one of C', C'' , and a simple vertex in the other one. Both curves have less than N triple vertices, so the inductive hypothesis holds. We distinguish among the possible configurations for v .

Case $+2v' : e_i, e_{j-1}, e_j, e_{k-1}, e_k, e_{i-1}$. Fig. 19 shows a possible shape of the curve C . The curves C', C'' are both CCW oriented, thus $A(C') = A(C'') = 1$. We call $\alpha, \beta, \gamma, \delta, \eta, \epsilon$ the angles at v as in Fig. 19.

The interior angles at v in C are $\widehat{e_{j-1}, e_j} = 2\pi - \beta$, $\widehat{e_{k-1}, e_k} = 2\pi - \delta$, and $\widehat{e_{i-1}, e_i} = 2\pi - \epsilon$, so $d(v) = \frac{3\pi - (2\pi - \beta + 2\pi - \delta + 2\pi - \epsilon)}{2\pi} = \frac{-3\pi + \beta + \delta + \epsilon}{2\pi}$.

The interior angles in the union of C', C'' are $\widehat{e_{j-1}, e_i} = \alpha$, $\widehat{e_{k-1}, e_k} = 2\pi - \delta$, and $\widehat{e_{i-1}, e_j} = \gamma + \delta + \eta$, so $d'(v) + d''(v) = \frac{3\pi - (\alpha + 2\pi - \delta + \gamma + \delta + \eta)}{2\pi} = \frac{\pi - \alpha - \gamma - \eta}{2\pi}$

Remembering that $\alpha + \beta + \gamma + \delta + \eta + \epsilon = 2\pi$, the following equation holds:

$$\begin{aligned} A(C) &= A(C') + A(C'') + \frac{-3\pi + \beta + \delta + \epsilon}{2\pi} - \frac{\pi - \alpha - \gamma - \eta}{2\pi} \\ &= 2 + \frac{-4\pi + \alpha + \beta + \gamma + \delta + \eta + \epsilon}{2\pi} = 2 + \frac{-4\pi + 2\pi}{2\pi} = 2 - 1 = 1. \end{aligned}$$

Case $-2v' : e_i, e_{i-1}, e_k, e_{k-1}, e_j, e_{j-1}$. Fig. 20 shows possible shapes of the curve C . The curves C', C'' have opposite orientation, so $A(C') = -A(C'')$. We call $\alpha, \beta, \gamma, \delta, \eta, \epsilon$ the angles at v , as in Fig. 20.

The interior angles at v in C are $\widehat{e_{i-1}, e_i} = \alpha$, $\widehat{e_{k-1}, e_k} = \gamma$, and $\widehat{e_{j-1}, e_j} = \eta$, thus $d(v) = \frac{3\pi - (\alpha + \gamma + \eta)}{2\pi}$.

The interior angles in the union of C', C'' are $\widehat{e_{j-1}, e_i} = 2\pi - \epsilon$, $\widehat{e_{k-1}, e_k} = \gamma$, and $\widehat{e_{i-1}, e_j} = \alpha + \eta + \epsilon$, thus $d'(v) + d''(v) = \frac{3\pi - (2\pi - \epsilon + \gamma + \alpha + \eta + \epsilon)}{2\pi} = \frac{\pi - \alpha - \gamma - \eta}{2\pi}$.

The following relation holds:

$$\begin{aligned} A(C) &= A(C') + A(C'') + \frac{3\pi - \alpha - \gamma - \eta}{2\pi} - \frac{\pi - \alpha - \gamma - \eta}{2\pi} \\ &= 0 + \frac{2\pi - \alpha - \gamma - \eta + \alpha + \gamma + \eta}{2\pi} = 0 + 1 = 1. \end{aligned}$$

Case $+2v'' : e_i, e_{j-1}, e_j, e_{i-1}, e_k, e_{k-1}$. Possible shapes of the curve C are shown in Fig. 21. The two curves C', C'' are CCW oriented, so $A(C') = A(C'') = 1$. We denote the angles around v as in Fig. 21.

The interior angles at v in C are $\widehat{e_{j-1}, e_j} = 2\pi - \beta$, $\widehat{e_{i-1}, e_i} = \alpha + \beta + \gamma$, and $\widehat{e_{k-1}, e_k} = \eta$, and $d(v) = \frac{3\pi - (2\pi - \beta + \alpha + \beta + \gamma + \eta)}{2\pi} = \frac{\pi - \alpha - \gamma - \eta}{2\pi}$.

The interior angles in the union of C', C'' are $\widehat{e_{j-1}, e_i} = \alpha$, $\widehat{e_{i-1}, e_j} = \gamma$ and $\widehat{e_{k-1}, e_k} = \eta$, and $d'(v) + d''(v) = \frac{3\pi - \alpha - \gamma - \eta}{2\pi}$.

The following relation holds:

$$\begin{aligned} A(C) &= A(C') + A(C'') + \frac{\pi - \alpha - \gamma - \eta}{2\pi} - \frac{3\pi - \alpha - \gamma - \eta}{2\pi} \\ &= 2 + \frac{-2\pi - \alpha - \gamma - \eta + \alpha + \gamma + \eta}{2\pi} = 2 - \frac{2\pi}{2\pi} = 2 - 1 = 1. \end{aligned}$$

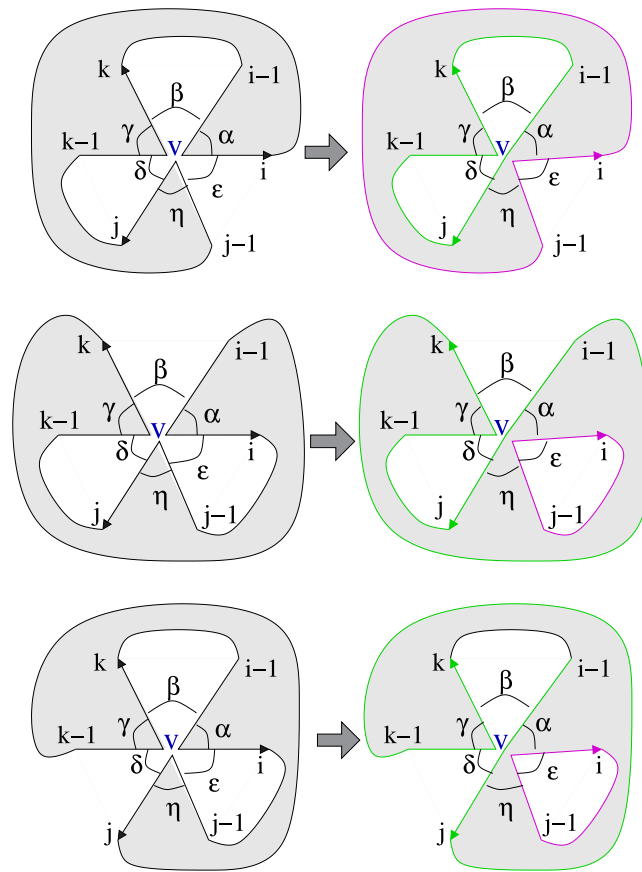


Fig. 20. Possible shapes for a curve C with a vertex in configuration $-2v'$, and splitting C into C' (magenta) and C'' (green). The angles at v are shown. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

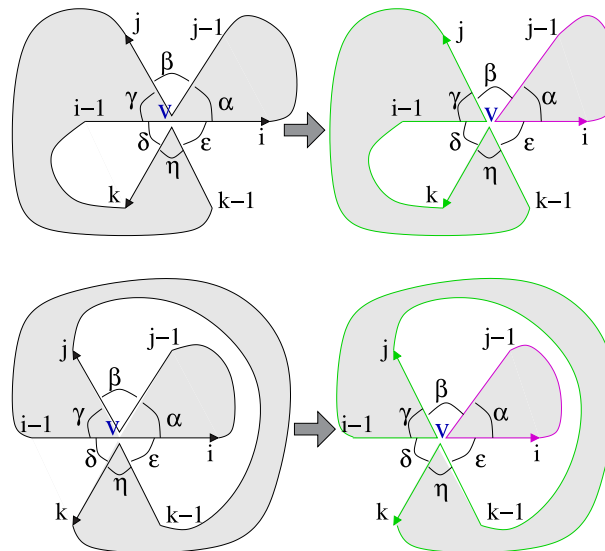


Fig. 21. Possible shapes for a curve C with a vertex in configuration $+2v''$, and the same curve split into C' (magenta) and C'' (green). The angles at v are shown. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

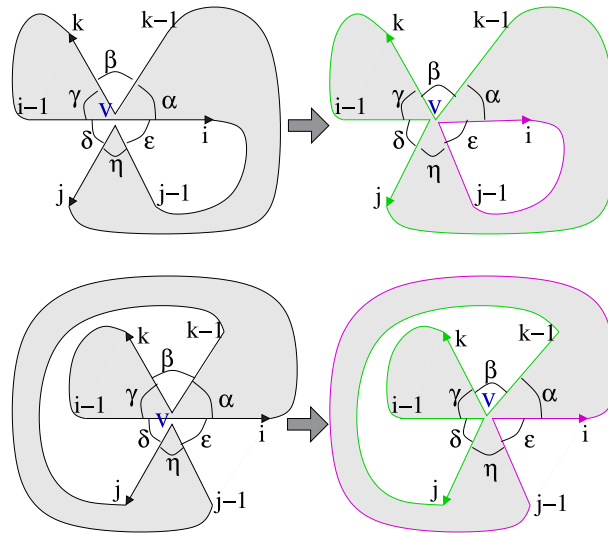


Fig. 22. Possible shapes for a curve C with a vertex in configuration $-2v''$, and the same curve split into C' (magenta) and C'' (green). The angles at v are shown. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Case $-2v'' : e_i, e_{k-1}, e_k, e_{i-1}, e_j, e_{j-1}$. Fig. 22 shows the possible shapes of the curve C . The two curves C', C'' , obtained from splitting C , have opposite orientations, so $A(C') = -A(C'')$. We denote the angles around v as in Fig. 22.

The interior angles of C at v are $\widehat{e_{i-1}, e_i} = \alpha + \beta + \gamma$, $\widehat{e_{j-1}, e_j} = \eta$, and $\widehat{e_{k-1}, e_k} = 2\pi - \beta$. So $d(v) = \frac{3\pi - \alpha - \beta - \gamma - \eta - 2\pi + \beta}{2\pi} = \frac{\pi - \alpha - \gamma - \eta}{2\pi}$.

The interior angles at v in C' and C'' are $\widehat{e_{j-1}, e_i} = 2\pi - \epsilon$, $\widehat{e_{i-1}, e_j} = 2\pi - \delta$, and $\widehat{e_{k-1}, e_k} = 2\pi - \beta$. So $d'(v) + d''(v) = \frac{3\pi - 2\pi + \epsilon - 2\pi + \delta - 2\pi + \beta}{2\pi} = \frac{-3\pi + \beta + \delta + \epsilon}{2\pi}$.

Remembering that $\alpha + \beta + \gamma + \delta + \epsilon + \eta = 2\pi$, the following relation holds:

$$\begin{aligned} A(C) &= A(C') + A(C'') + \frac{\pi - \alpha - \gamma - \eta}{2\pi} - \frac{-3\pi + \beta + \delta + \epsilon}{2\pi} = \\ &= 0 + \frac{4\pi - \alpha - \gamma - \eta - \beta - \delta - \epsilon}{2\pi} = \frac{4\pi - 2\pi}{2\pi} = 1. \end{aligned}$$

5.5. Proof for self-avoiding closed RE-curves

The proof is by induction on the number N of double chains.

If $N = 0$, the curve is a closed RV-curve and the thesis has been proven in Section 5.4.

For $N > 0$, let us consider a double chain D . By deleting the double chain D , what remains of C consists of two self-avoiding closed RE-curves C', C'' , each with fewer double chains. By induction, the thesis is true for C' and C'' .

We denote with letter d the contribution of a vertex to $A(C)$ and with \underline{d} the contribution of the same vertex to $A(C') + A(C'')$.

We first prove the following lemmas.

Lemma 1. For each internal vertex v of the double chain D , $\underline{d}(v) = d(v)$.

Proof. If v was a double vertex in C , the two passages C through v have interior angles τ_i, τ_j with $\tau_i + \tau_j = 2\pi$, therefore $d(u) = \frac{\pi - \tau_i + \pi - \tau_j}{2\pi} = \frac{2\pi - 2\pi}{2\pi} = 0$. Neither C' nor C'' contain v , thus $\underline{d}(v) = 0$ and therefore $\underline{d}(v) = d(v)$.

If v was a triple vertex, then it must be in configuration $\pm 2e'_i$ with $e_i \equiv e_{j-1}$ and $e_j \equiv e_{i-1}$, or with $e_i \equiv e_{k-1}$ and $e_k \equiv e_{i-1}$. Without loss of generality, let us assume the first case. In the passages through v as v_i and v_j , C forms two interior angles τ_i, τ_j with $\tau_i + \tau_j = 2\pi$, so the contributions to $d(v)$ of such two passages cancel out, and $d(v) = \frac{\pi - \tau_k}{2\pi}$. After deleting D , v appears in exactly one of C', C'' with interior angle τ_k . Therefore $\underline{d}(v) = \frac{\pi - \tau_k}{2\pi}$.

Let u be one of the endpoints of the double chain D . Let C' be the curve containing u after the deletion of D from C .

Lemma 2. For each endpoint u of the double chain D :

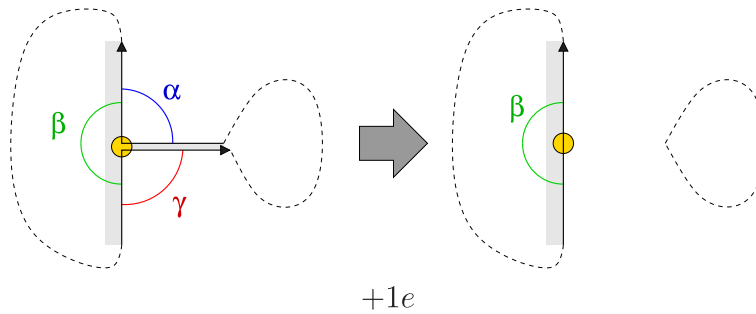


Fig. 23. Endpoint vertex of a double chain with indicated angles for configuration $+1e$ where the radial sector enclosed by the two coincident edges belonging to D is labelled L (left). The curve C and the two curves resulting after deleting the double chain are shown.

1. if the radial sector around u in C , corresponding to the pairs of coincident edges belonging to D , is labelled with L (left), then $\underline{d}(u) = d(u) + 1/2$;
2. if the above mentioned sector is labelled with R (right), then $\underline{d}(u) = d(u) - 1/2$.

Proof. In case 1, the deleted double chain D was an infinitely narrow solid corridor connecting two solid components of the internal region of C . The vertex u was either double or triple in C .

If u was a double vertex, the configuration at u is $+1e$. We denote the angles around u with α, β, γ , with $\alpha + \beta + \gamma = 2\pi$ as in Fig. 23.

The interior angles of the two passages of C at u are $\alpha + \beta$ and $\beta + \gamma$, thus $d(u) = \frac{\pi - \alpha - \beta + \pi - \beta - \gamma}{2\pi} = \frac{\pi - 2\pi + \pi - \beta}{2\pi} = \frac{-\beta}{2\pi}$.

Only one of C' and C'' passes through u with interior angle equal to β , thus $\underline{d}(u) = \frac{\pi - \beta}{2\pi} = \frac{-\beta}{2\pi} + 1/2 = d(u) + 1/2$.

If u was a triple vertex, the configuration at u is one of $+2e', +2e''_1, +2e''_2$. We denote the angles around u with $\alpha, \beta, \gamma, \delta, \eta$, with $\alpha + \beta + \gamma + \delta + \eta = 2\pi$ as in Fig. 24.

Case of configuration $+2e'$. The interior angles of the three passages of C at u are $\beta + \gamma + \delta + \eta, \alpha + \beta + \delta + \eta, \alpha + \beta + \gamma + \delta$ (note that it is possible that $\beta = 0$ and/or $\delta = 0$). Thus $d(u) = \frac{\pi - \beta - \gamma - \delta - \eta + \pi - \alpha - \beta - \delta - \eta + \pi - \alpha - \beta - \gamma - \delta}{2\pi} = \frac{3\pi - 2\pi - 2\pi - \beta - \delta}{2\pi} = \frac{-\pi - \beta - \delta}{2\pi}$.

After splitting C into C', C'' , it may happen that each of C' and C'' passes through u once, or that one of them passes through u twice while the other one does not contain u (some possible configurations are shown in Fig. 24). In any case, the interior angles of the two passages are $\alpha + \beta + \delta + \eta$ and $\beta + \gamma + \delta$. Thus $\underline{d}(u) = \frac{\pi - \alpha - \beta - \delta - \eta + \pi - \beta - \gamma - \delta}{2\pi} = \frac{2\pi - 2\pi - \beta - \delta}{2\pi} = \frac{-\beta - \delta}{2\pi} = d(u) + 1/2$.

Case of configuration $+2e''_1$. The interior angles of the three passages of C at u are $\beta + \gamma + \delta + \eta, \alpha + \beta$, and δ (note that it is possible that $\gamma = 0$ but not $\beta = 0$, otherwise u would be an internal vertex and not an endpoint of D). Thus $d(u) = \frac{\pi - \beta - \gamma - \delta - \eta + \pi - \alpha - \beta + \pi - \delta}{2\pi} = \frac{3\pi - \beta - \delta}{2\pi}$.

After splitting, u may belong to both C', C'' (one passage in each) or to just one of them (with two passages). In any case, the interior angles of the two passages are β and δ , thus $\underline{d}(u) = \frac{\pi - \beta + \pi - \delta}{2\pi} = \frac{2\pi - \beta - \delta}{2\pi} = d(u) + 1/2$.

Case of configuration $+2e''_2$. The interior angles of the three passages of C at u are $\beta, \delta + \eta, \alpha + \beta + \gamma + \delta$ (note that it is possible that $\gamma = 0$ but not $\delta = 0$, otherwise u would be an internal vertex and not an endpoint of D). Thus $d(u) = \frac{\pi - \beta + \pi - \delta - \eta + \pi - \alpha - \beta - \gamma - \delta}{2\pi} = \frac{3\pi - \beta - \delta - \eta - \alpha - \beta - \gamma - \delta}{2\pi} = \frac{3\pi - \beta - \delta}{2\pi}$.

After splitting, in any case, the interior angles of the two passages are β and δ , thus the passages are as above and $\underline{d}(u) = d(u) + 1/2$.

In case 2 of the lemma, the deleted double chain D was an infinitely narrow hollow corridor connecting a hole of the internal region of C with the unbounded region, or two holes. The vertex u was either double or triple in C .

If u was a double vertex, the configuration at u is $-1e$. We denote the angles around u as in Fig. 25. The interior angles of the two passages of C at u are α and β . Thus $d(u) = \frac{\pi - \alpha + \pi - \beta}{2\pi} = 1 - \frac{\alpha + \beta}{2\pi}$.

After splitting, the interior angle of unique passage at u (occurring either in C' or in C'') is $\alpha + \beta$. Thus $\underline{d}(u) = \frac{\pi - \alpha - \beta}{2\pi} = 1/2 - \frac{\alpha + \beta}{2\pi} = d(u) - 1/2$.

If u was a triple vertex in C , then the configuration at u is one of $-2e', -2e''_1, -2e''_2$. We denote the angles around u with $\alpha, \beta, \gamma, \delta, \eta$, with $\alpha + \beta + \gamma + \delta + \eta = 2\pi$ as in Fig. 26.

Case of configuration $-2e'$. The three passages of C at u have interior angles equal to α, γ, η (note that it may be $\beta = 0$ and/or $\gamma = 0$). Thus $d(u) = \frac{\pi - \alpha + \pi - \gamma + \pi - \eta}{2\pi} = \frac{3\pi - \alpha - \gamma - \eta}{2\pi} = 3/2 - \frac{\alpha + \gamma + \eta}{2\pi}$.

After splitting, the interior angles of the two passages at u are γ and $\alpha + \eta$. Thus $\underline{d}(u) = \frac{\pi - \gamma + \pi - \alpha - \eta}{2\pi} = 1 - \frac{\alpha + \gamma + \eta}{2\pi} = d(u) - 1/2$.

Case of configuration $-2e''_1$. The three passages of C at u have interior angles equal to $\alpha, \gamma + \delta + \eta$ and $\alpha + \beta + \gamma + \eta$ (note that it may be $\gamma = 0$ but not $\beta = 0$, or u would not be a double chain endpoint). Thus $d(u) = \frac{\pi - \alpha + \pi - \gamma - \delta + \pi - \alpha - \beta - \gamma - \eta}{2\pi} = \frac{3\pi - 2\pi - \alpha - \gamma - \eta}{2\pi} = 1/2 - \frac{\alpha + \gamma + \eta}{2\pi} = 1/2 - \frac{\alpha + \eta}{2\pi}$.

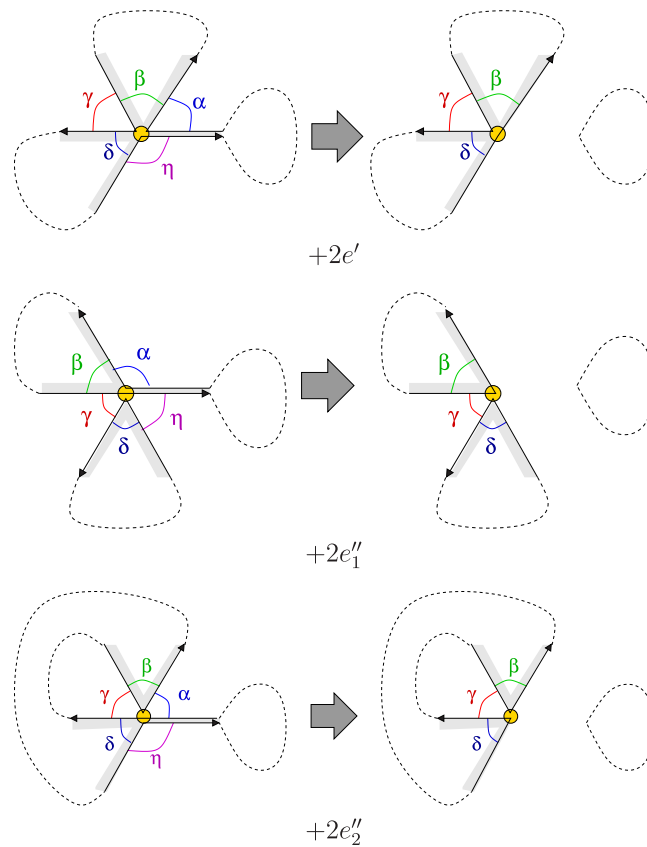


Fig. 24. Endpoint vertex of a double chain with indicated angles for configurations $+2e'$, $+2e''_1$ and $+2e''_2$, where the radial sector enclosed by the two coincident edges belonging to D is labelled L (left). The curve C and the two curves resulting after deleting the double chain are shown.

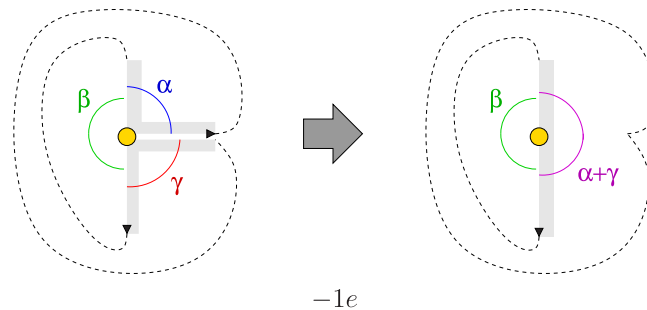


Fig. 25. Endpoint vertex of a double chain with indicated angles for configuration $-1e$ where the radial sector enclosed by the two coincident edges belonging to D is labelled R (right). The curve C and the two curves resulting after deleting the double chain are shown.

After splitting, the interior angles of the two passages at u are $\alpha + \beta + \gamma + \eta$ and $\alpha + \gamma + \delta + \eta$. Thus $\underline{d}(u) = \frac{\pi - \alpha - \beta - \gamma - \eta + \pi - \alpha - \gamma - \delta - \eta}{2\pi} = \frac{2\pi - \alpha - \gamma - \eta}{2\pi} = 1 - \frac{\alpha + \gamma + \eta}{2\pi} = d(u) - 1/2$.

Case of configuration $-2e''_2$. The three passages of C at u have interior angles equal to $\alpha + \beta + \gamma$, $\alpha + \gamma + \delta + \eta$, η (note that it may be $\gamma = 0$ but not $\delta = 0$). Thus $d(u) = \frac{\pi - \alpha - \beta - \gamma + \pi - \alpha - \gamma - \delta - \eta + \pi - \eta}{2\pi} = \frac{3\pi - 2\pi - \alpha - \gamma - \eta}{2\pi} = 1/2 - \frac{\alpha + \eta}{2\pi}$.

After splitting, the interior angles of the two passages at u are $\alpha + \gamma + \delta + \eta$ and $\alpha + \beta + \gamma + \eta$. Thus $\underline{d}(u) = \frac{\pi - \alpha - \gamma - \delta - \eta + \pi - \alpha - \beta - \gamma - \eta}{2\pi} = \frac{2\pi - 2\pi - \alpha - \gamma - \eta}{2\pi} = -\frac{\alpha + \eta}{2\pi} = d(u) - 1/2$.

Now we are able to prove the thesis of Proposition 2 in case $N > 0$. After deleting the double chain and splitting C into C' and C'' , the contribution of all vertices, with the exception of the two endpoints v, w of D , is equal in $A(C)$ and in $A(C') + A(C'')$. This is trivial for the vertices not traversed by D and follows from Lemma 1 for the intermediate vertices of D . So $A(C) = A(C') + A(C'') + d(v) + d(w) - \underline{d}(v) - \underline{d}(w)$.

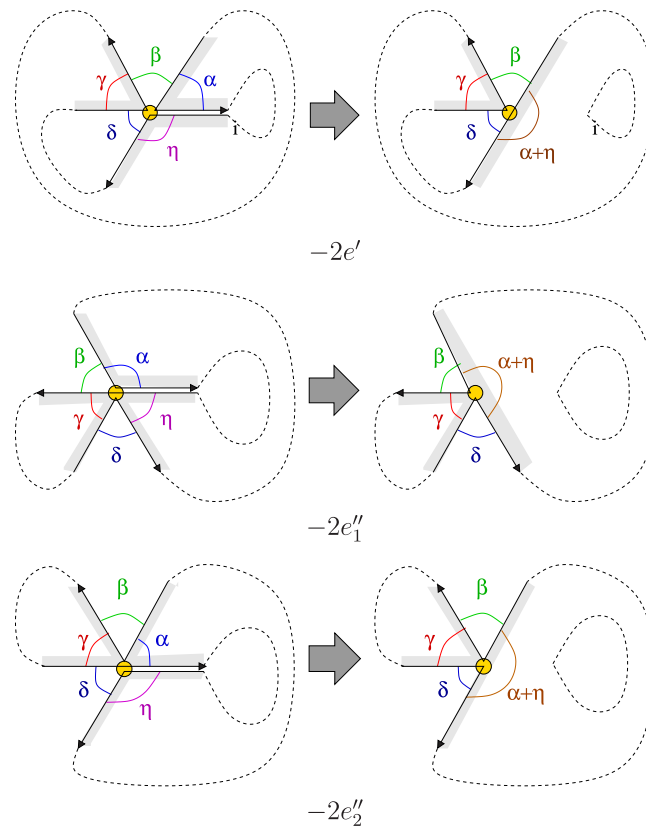


Fig. 26. Endpoint vertex of a double chain with indicated angles, for configurations $-2e'$, $-2e''_1$ and $-2e''_2$, where the radial sector enclosed by the two coincident edges belonging to D is labelled R (right). The curve C and the two curves resulting after deleting the double chain are shown.

Because C is a self-avoiding closed RE-curve, the infinitely small radial sector bounded by the two coincident edges of D have the same label at u and w . In case this label is L (left), the curves C' , C'' have the same CCW orientation as C , so by inductive hypothesis $A(C') = A(C'') = 1$. By Lemma 2, $\underline{d}(v) = d(v) + 1/2$ and $\underline{d}(w) = d(w) + 1/2$. Therefore $A(C) = 1 + 1 + d(v) + d(w) - (d(v) + 1/2) - (d(w) + 1/2) = 2 - 1/2 - 1/2 = 2 - 1 = 1$.

In case the label is R (right), one of the curves, let it be C' , has the same CCW orientation as C , and the other one, let it be C'' , has CW orientation. So by inductive hypothesis $A(C') = 1$ and $A(C'') = -1$. By Lemma 2, $\underline{d}(v) = d(v) - 1/2$ and $\underline{d}(w) = d(w) - 1/2$. Therefore $A(C) = 1 - 1 + d(w) - (d(v) - 1/2) - (d(w) - 1/2) = 0 + 1/2 + 1/2 = 1$.

6. Summary and future work

We defined self-avoiding closed RE-curves in the regular and semiregular grids (i.e., under the constraint that each vertex can be repeated at most three times, each edge can be repeated at most two times, and no consecutive edges coincide). This extends and generalizes the existing definition of self-avoiding closed curves in the hexagonal and square grids (given in terms of the word coding the curve), in which each vertex and each edge can be repeated at most two times [8,20]. We defined the orientation of such curves and proposed a linear-time algorithm for testing if a closed grid curve is a self-avoiding RE-curve.

We intend these curves as boundaries of generalized polyominoes, i.e., finite sets of grid polygons, possibly disconnected and with holes. Distinct components and holes are connected through double chains of edges traversed in opposite directions to produce a single self-avoiding boundary curve. A similar notion is that of a weakly simple polygon, which is not restricted to a grid and can have spurs and vertices inside edges [9].

We proposed a classification of the vertices on such curves and we showed a combinatorial relationship between the number of vertices of different types. Our classification and the resulting relationship extend and generalize the existing ones for the simple closed curves in the regular grids [6,7,10,13,14] to a wider class of non-simple curves in both regular and semiregular grids.

In the square and the hexagonal grids, the relationship between the number of convex and concave vertices has been used in the study of planar tilings by exact polyominoes [2,8,18]. We hope that the combinatorial relationship given here will contribute to similar applications in the triangular and semiregular grids.

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Data availability

No data was used for the research described in the article.

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