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Observer design for nonlinear systems with delayed output measurement[☆]

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ABSTRACT

This paper deals with nonlinear observer design for systems with a long delay in the nonlinear outputs. The main idea behind this work consists of using a dynamic extension technique to transform a system with delayed nonlinear outputs into a system with linear outputs and a delay-dependent integral term in the dynamic process. First, a general result for arbitrary nonlinear structures is proposed, and then further contributions are provided for the specific family of systems in companion form. For systems in companion form, we obtain novel high-gain observer synthesis conditions. To relax the necessary conditions related to the observer design procedure, a different state observer structure is proposed, and analytical comparisons are provided. Moreover, to relax the design conditions, we propose an alternative design approach using a specific Lyapunov–Krasovskii functional, which allows accommodating long delays in the outputs.

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1. Introduction

Delayed outputs are naturally encountered in several real-world applications, such as remote estimation (Xing et al., 2023; Zhao et al., 2023), cyberattacks detection (Jeon et al., 2020), (Huong et al., 2019; Huynh et al., 2019a, 2019b), and multiagent systems in general (Wang et al., 2022; Wooldridge, 2009). In observer design, accounting for output delays is crucial for accurate state estimation and system stability. Delays, due, for instance, to sensor limitations, communication lags, or processing time, can lead to outdated feedback, affecting observer performance. Properly accommodating these delays ensures robust state estimation, maintaining reliable system performance in practical applications like control systems and robotics. Estimating the state of a nonlinear system from delayed measurements is a challenging yet critical task in control theory. Several methods have been proposed in the literature to handle this challenge; however, many of them are limited in their ability to handle large delays (Seuret et al., 2009).

The main objective of this work is to develop a novel approach for estimating the state of a nonlinear system from delayed measurements. Specifically, we aim to propose a method that can accommodate long delays in the measurements while ensuring robust and accurate state estimation.

To deal with arbitrarily long delays in the measurement, some methods are based on the use of a chain of observers (Cacace et al., 2014; Germani et al., 2002) while other methods exploit predictors-based observers (Ahmed-Ali et al., 2013b; Khosravian et al., 2015). For some families of systems, namely feedforward systems, the problem may be solved by using the time-scaling technique as in Mazenc et al. (2017). On the other hand, different methods based on the high-gain observer methodology have been proposed for systems in triangular form with some recent improvements. However, these methods are valid for only systems with small values of the upper bound of the delay, $\tau^\star$. The aim of this paper is to overcome this limitation and propose a novel approach that will be both simple and enhance the maximum allowable value of $\tau^\star$.

This paper introduces a novel observer design methodology for systems with delayed outputs, aimed at relaxing the conservative constraints found in existing approaches. A central contribution is the use of a new output-based dynamic extension technique, preliminarily introduced in Arezki et al. (2024), which effectively transfers the time-delay from the output to the dynamics of an augmented system. This approach significantly increases the maximum allowable delay. In contrast to traditional methods that are typically restricted to systems in companion form or with

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feedforward structures, the proposed framework offers a unified design strategy applicable to systems with arbitrary structures.

A new observer structure incorporating an integral term, enabled by the dynamic extension technique, is also proposed. Rigorous analysis demonstrates that this structure considerably improves the upper bound of the maximum allowable delay, thus enhancing robustness and performance. However, this comes at the cost of requiring certain Lipschitz conditions, which is the main limitation. To address this, a relaxed observer design is proposed, based on the same extension technique but without the integral term. Additional contributions include specific results for systems in companion form, where the maximum delay is shown to explicitly depend on the tuning parameter of the high-gain observer. Lastly, to avoid conservative conditions needed for exponential convergence, a delay-dependent LMI-based design using a specific Lyapunov–Krasovskii functional is proposed. This approach allows for higher delay tolerance compared to existing methods in Targui et al. (2019) and Nechaf et al. (2023).

While the primary limitation of this approach is the computational complexity issue associated to the exponential number of LMIs to be solved, it remains the least conservative LMI-based method, as highlighted in Zemouche and Boutayeb (2013). Furthermore, alternative strategies have been proposed, for the users, to significantly reduce the number of LMIs, as suggested in Zemouche et al. (2016).

2. Problem formulation and preliminary tools

This section is devoted to formulating the observer design problem tackled in this paper, namely addressing observer design in the presence of delayed output measurements. We start by defining the class of systems under investigation and revisiting conventional observer design methods available in existing literature. To provide a strong motivation for the results developed in this paper, we give a succinct overview of the current state-of-the-art methodologies, thus establishing the basis for the development of our proposed method. Then, before dealing with the main results of the paper, we introduce the primary mathematical tools used throughout the Lyapunov stability analysis.

2.1. Problem formulation

The motivation of the work consists of developing a simple method to deal with nonlinear systems with delayed nonlinear output measurements. The aim is to establish novel design conditions accommodating high values of the maximum allowable delay in the output measurement while ensuring the exponential convergence of the observer. We stand out from the literature by proposing a simple but useful method. The class of systems we consider in this paper is described by the following equations:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = h\big(x(t-\tau(t))\big) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the control input, and $y(t) \in \mathbb{R}^p$ represents the output measurements vector. The delay $\tau(t) \geq 0$ is assumed to be known and bounded, i.e.: there exists a positive constant $\tau^\star$ such that $\tau(t) \leq \tau^\star, \forall t \geq 0$. The following assumption is made regarding the nonlinearities of the system:

Assumption 1. The functions f and h are globally Lipschitz with respect to x uniformly on $u(t)$.

The main objective consists of estimating the system state, $x(t)$, in real-time from the delayed measurements $y(t)$. While the problem in the case of delay-free outputs, is relatively easy to handle, however, the presence of the delay makes the problem challenging.

Usually in the literature, the following Luenberger state observer is proposed for the class of systems (1):

$$\dot{\hat{x}}(t) = f(\hat{x}(t), u(t)) + L\left[y(t) - h\big(\hat{x}(t-\tau(t))\big)\right], \tag{2}$$

where $\hat{x}(t)$ is the estimate of $x(t)$ and L is the observer gain to be determined. Several methods have been devoted in the literature for studying the convergence of the observer (2) for large values of the maximum allowable delay, $\tau^\star$. Except a few results for feedforward systems (Mazenc et al., 2017) and a limited number of methods addressing systems with arbitrary structures (Ahmed-Ali et al., 2013a; Khosravian et al., 2015), most of the contributions in this area concerns systems in companion form (Adil et al., 2022; Cacace et al., 2014; Germani et al., 2002; Van Assche et al., 2011; Zemouche et al., 2019; Zhang & Shen, 2017). Although recent improvements, as presented in Adil et al. (2022), have exploited the HG/LMI technique introduced in Zemouche et al. (2019), the obtained results remain conservative. Moreover, the convergence analysis, based on Lyapunov theory and the use of the observer (2), requires extensive computations and cumbersome mathematical developments. Additionally, the observer gain L explicitly amplifies the delay in the output, requiring the application of the Jensen inequality in mathematical derivations to achieve appropriate synthesis conditions, as suggested by the conditions outlined in Adil et al. (2022, Theorem 10), which are strongly interdependent, cumbersome, and conservative.

All the aforementioned drawbacks associated with the utilization of observer (2) have motivated us to devise a novel approach to address both delay and nonlinearity in output measurements. Specifically, our objective is to develop a unified observer design method that:

- Is applicable to systems of arbitrary structure without imposing specific forms;
- Involves simple mathematical derivations and relies on less conservative tools, such as the Jensen inequality;
- Avoids the need for observer (2), wherein the observer gain is directly multiplied by the delayed output, thus amplifying the effect of the delay;
- Enhances the design conditions for specific nonlinear systems compared to existing literature.

2.2. Preliminary tools

Before stating the main results of the paper, we introduce the following simple and well-known mathematical tools.

Lemma 1 (*Halanay, 1966; Mazenc et al., 2021*)**.** *Consider a continuous, piece-wise $\mathcal{C}^1$, and non-negative function ϑ defined in the interval $[-\tau^\star, +\infty)$ such that*

$$\dot{\vartheta}(t) \leq -c_1\vartheta(t) + c_2 \sup_{s\in[t-\tau^\star,t]} \vartheta(s). \tag{3}$$

Assume that $c_1 > c_2 > 0$. Then, there exist two scalars $\alpha > 0$ and $\beta > 0$ such that

$$\vartheta(t) \leq \alpha e^{-\beta t} \sup_{s\in[-\tau^\star,0]} \vartheta(s), \forall t \geq 0. \tag{4}$$

Remark 1. In this paper, we are not interested in the values of α and β introduced in Lemma 1. We only need their existence to ensure exponential stability. For further details on the construction of α and β, we refer the reader to Halanay (1966), Mazenc et al. (2021).

To handle some nonlinear terms when analyzing stability, we need the following mean value theorem for vector-valued functions.

Lemma 2 (Zemouche & Boutayeb, 2013). *Let $\Psi : \mathbb{R}^n \mapsto \mathbb{R}^q$ be a differentiable function and let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ be two vectors. Then, there exists*

$$\boldsymbol{z} \triangleq \begin{bmatrix} \boldsymbol{z}^1 \\ \boldsymbol{z}^2 \\ \vdots \\ \boldsymbol{z}^q \end{bmatrix} \in \mathbb{R}^{nq},\ \boldsymbol{z}^i \in \mathbf{Co}(x, y), i = 1, \ldots, q \tag{5}$$

where $\mathbf{Co}(x, y)$ stands for the convex hull of convex combinations of x and y, such that

$$\Psi(x) - \Psi(y) = \nabla_x^{\Psi}(\boldsymbol{z})(x - y) \tag{6}$$

where

$$\nabla_x^{\Psi}(\boldsymbol{z}) \triangleq \begin{bmatrix} \frac{\partial \Psi_1(\boldsymbol{z}^1)}{\partial x} \\ \frac{\partial \Psi_2(\boldsymbol{z}^2)}{\partial x} \\ \vdots \\ \frac{\partial \Psi_q(\boldsymbol{z}^q)}{\partial x} \end{bmatrix}. \tag{7}$$

Finally, we need the following Lemma 3.

Lemma 3. *Let $\phi : \mathbb{I} \to \mathbb{R}$ be a non-negative function, where $\mathbb{I}$ is an interval of $\mathbb{R}$. Then the following identity holds:*

$$\left[\sup_{s \in \mathbb{I}}\big(\phi(s)\big)\right]^2 = \sup_{s \in \mathbb{I}}\big(\phi^2(s)\big). \tag{8}$$

3. New observer design results

This section is devoted to the main contributions of this paper. We, first, present a preliminary result on which the main contributions are based. It will be used straightforwardly as a tool to conclude the main results.

3.1. Preliminary result as a main tool

Consider the class of systems described as follows:

$$\begin{cases} \dot{\zeta}(t) = f_\zeta\left(\zeta(t), u(t)\right) + B_\zeta \displaystyle\int_{t-\tau(t)}^{t} g(\zeta(s), u(s))\mathrm{d}s \\ y_\zeta(t) = C\zeta(t) \end{cases} \tag{9}$$

where $\zeta(t) \in \mathbb{R}^{n_\zeta}$ is the state of the system, $u(t) \in \mathbb{R}^m$ is the control input, and $y_\zeta(t) \in \mathbb{R}^{p_\zeta}$ represents the output measurements vector. The delay $\tau(t) \geq 0$ is assumed to be known and bounded, i.e.: there exists a positive constant $\tau^\star$ such that $\tau(t) \leq \tau^\star, \forall t \geq 0$; assume that $\zeta(t) = \zeta_0, \forall t \in [-\tau^\star, 0]$. Let us introduce the following main assumption of the nonlinearities of the system:

Assumption 2. The functions f_ζ and g are globally γ_{f_ζ}-Lipschitz and γ_g-Lipschitz, respectively, with respect to ζ uniformly on $u(t)$.

The global Lipschitz assumption may be relaxed if system (9) admits a positively invariant compact set Ω on which f_ζ and g are Lipschitz. In this case, a Lipschitz extension of the functions f_ζ and g can be incorporated into the observer, using techniques such as the *Hilbert projection theorem*, as demonstrated in Zemouche and Rajamani (2022).

As a preliminary result, we develop a simple state observer design method for the system (9). To this end, we consider the following state observer :

$$\dot{\hat{\zeta}}(t) = f_\zeta\left(\hat{\zeta}(t), u(t)\right) + B_\zeta \int_{t-\tau(t)}^{t} g(\hat{\zeta}(s), u(s))\mathrm{d}s$$
$$+L\left(y_\zeta(t) - C\hat{\zeta}(t)\right), \tag{10}$$

where $L \in \mathbb{R}^{n_\zeta \times p_\zeta}$ is the observer gain matrix to be determined such that the estimation error $\epsilon(t) \triangleq \zeta(t) - \hat{\zeta}(t)$ converges exponentially towards zero. Then, the estimation error dynamics is given as:

$$\dot{\epsilon}(t) = \Delta f_\zeta\left(\zeta(t), \hat{\zeta}(t), u(t)\right) - LC\epsilon(t)$$
$$+B_\zeta \int_{t-\tau(t)}^{t} \Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right)\mathrm{d}s \tag{11}$$

where

$$\Delta f_\zeta\left(\zeta(t), \hat{\zeta}(t), u(t)\right) \triangleq f_\zeta\left(\zeta(t), u(t)\right) - f_\zeta\left(\hat{\zeta}(t), u(t)\right),$$

$$\Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right) \triangleq g\left(\zeta(t), u(t)\right) - g\left(\hat{\zeta}(t), u(t)\right).$$

For the sake of obtaining convenient checkable stability conditions, we have to transform the nonlinear term $\Delta f_\zeta\left(\zeta(t), \hat{\zeta}(t), u(t)\right)$ by using Lemma 2. Then, there exists $\boldsymbol{z}_t \in \mathbf{Co}\left(\zeta(t), \hat{\zeta}(t)\right)$ as in (5) such that

$$\Delta f_\zeta\left(\zeta(t), \hat{\zeta}(t), u(t)\right) = \nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t)\epsilon(t)$$

where $\nabla_\zeta^{f_\zeta}$ is defined as in (7). Notice that here $\boldsymbol{z}_t$ depends on $u(t)$ but for the sake of brevity, we use $\boldsymbol{z}_t$ instead of $\boldsymbol{z}_t(u(t))$. It follows that the error system (11) is under the form:

$$\dot{\epsilon}(t) = \left[\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right]\epsilon(t)$$
$$+B_\zeta \int_{t-\tau(t)}^{t} \Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right)\mathrm{d}s. \tag{12}$$

Since f_ζ is $\gamma_{f_\zeta}-$Lipschitz, then there exist constant matrices $\mathcal{A}_j^\zeta \in \mathbb{R}^{n_\zeta \times n_\zeta}$ and functions $\lambda_j(\boldsymbol{z}_t), j = 1, \ldots \bar{n}_\zeta$ such that the generalized Jacobian $\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t)$ belongs to the convex polytopic set defined as:

$$\mathcal{H}_{f_\zeta} \triangleq \left\{\sum_{j=1}^{\bar{n}_\zeta} \lambda_j(\boldsymbol{z}_t)\mathcal{A}_j^\zeta, \sum_{j=1}^{\bar{n}_\zeta} \lambda_j(\boldsymbol{z}_t) = 1, \lambda_j(\boldsymbol{z}_t) \geq 0\right\} \tag{13}$$

Notice that the matrices $\mathcal{A}_j^\zeta \in \mathbb{R}^{n_\zeta \times n_\zeta}$, represent the vertices of the polytope $\mathcal{H}_{f_\zeta}$. Also, the Jacobian $\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t)$ is affine on the variables $\lambda_j(\boldsymbol{z}_t), j = 1, \ldots \bar{n}_\zeta$.

Before stating the preliminary proposition, notice that since the function g is γ_g-Lipschitz with respect to ζ, then we have

$$\left\|\Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right)\right\| \leq \gamma_g\|\epsilon(s)\|. \tag{14}$$

Now we are ready to state the main theorem based on the use of the standard quadratic Lyapunov function, i.e.: $\vartheta(\epsilon(t)) \triangleq \epsilon^\top(t)\mathcal{P}\epsilon(t)$, where $\mathcal{P} = \mathcal{P}^\top > 0$. We can use a more general Lyapunov function with a matrix $\mathcal{P}(\epsilon(t))$ depending on $\epsilon(t)$, however, we obtain non-constructive conditions difficult to deal with using numerical software algorithms. The second objective of using the standard quadratic Lyapunov function is to compare with available methods in the literature based on the same Lyapunov function.

Theorem 1. *Assume that there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$, a matrix $\mathcal{R} \in \mathbb{R}^{p_\zeta \times n_\zeta}$, and a positive scalar μ such that the following conditions hold:*

$$\left(\mathcal{A}_j^\zeta\right)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_j^\zeta - C^\top\mathcal{R} - \mathcal{R}^\top C + \mu\mathcal{P} \leq 0,$$
$$j = 1, \ldots, \bar{n}_\zeta \tag{15a}$$

$$\tau^\star < \frac{\mu\lambda_{\min}(\mathcal{P})}{2\gamma_g\|\mathcal{P}B_\zeta\|} \tag{15b}$$

Then the observer (10), with $L = \mathcal{P}^{-1}\mathcal{R}^\top$, converges exponentially.

Proof. By computing the derivative of the Lyapunov function $\vartheta(\epsilon(t)) \triangleq \epsilon^\top(t)\mathcal{P}\epsilon(t)$ along the trajectories of (12), we obtain

$$\begin{aligned}
\dot{\vartheta}(\epsilon(t)) &= \epsilon^\top(t)\left[\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)\right]\epsilon(t)\\
&\quad + 2\epsilon^\top(t)\mathcal{P}B_\zeta \int_{t-\tau(t)}^{t} \Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right)\mathrm{d}s\\
&\le \epsilon^\top(t)\left[\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)\right]\epsilon(t)\\
&\quad + 2\gamma_g\|\mathcal{P}B_\zeta\|\|\epsilon(t)\| \int_{t-\tau^\star}^{t} \|\epsilon(s)\|\mathrm{d}s.
\end{aligned} \tag{16}$$

Conditions (15a) and the convexity principle (Boyd et al., 1994) lead to

$$\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right) \le -\mu\mathcal{P}. \tag{17}$$

In addition, since $\|\epsilon(t)\| \le \sup_{s\in[t-\tau^\star,t]}\|\epsilon(s)\|$ and $\int_{t-\tau^\star}^{t}\|\epsilon(s)\|\mathrm{d}s \le \tau^\star \sup_{s\in[t-\tau^\star,t]}\|\epsilon(s)\|$, then from Lemma 3, we obtain

$$\begin{aligned}
\|\epsilon(t)\| \int_{t-\tau^\star}^{t} \|\epsilon(s)\|\mathrm{d}s &\le \tau^\star\left(\sup_{s\in[t-\tau^\star,t]}\|\epsilon(s)\|\right)^2\\
&\overbrace{=}^{\text{Lemma 3}} \tau^\star \sup_{s\in[t-\tau^\star,t]}\|\epsilon(s)\|^2\\
&\le \frac{\tau^\star}{\lambda_{\min}(\mathcal{P})} \sup_{s\in[t-\tau^\star,t]} \vartheta(\epsilon(s)).
\end{aligned} \tag{18}$$

Hence, from (17) and (18), we deduce that

$$\dot{\vartheta}(\epsilon(t)) \le -\mu\vartheta(\epsilon(t)) + \frac{2\gamma_g\|\mathcal{P}B_\zeta\|}{\lambda_{\min}(\mathcal{P})}\tau^\star \sup_{s\in[t-\tau^\star,t]}\vartheta(\epsilon(s)). \tag{19}$$

Consequently, from (15b) and Lemma 1, there exist two positive scalars α and β such that

$$\vartheta(t) \le \alpha e^{-\beta t} \sup_{s\in[-\tau^\star,0]} \vartheta(s), \forall t \ge 0, \tag{20}$$

which means that the estimation error $\epsilon(t)$ is exponentially stable. This completes the proof.

Remark 2. The condition (15a) can be converted to LMIs by using the term $\mu\mathbb{I}_{n_\zeta}$ instead of $\mu\mathcal{P}$, i.e:

$$\left(\mathcal{A}_j^\zeta\right)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_j^\zeta - C^\top\mathcal{R} - \mathcal{R}^\top C + \mu\mathbb{I}_{n_\zeta} \le 0,$$
$$j = 1, \ldots, \bar{n}_\zeta. \tag{21}$$

In this case, the inequality (15b) becomes

$$\tau^\star < \frac{\mu\lambda_{\min}(\mathcal{P})}{2\gamma_g\lambda_{\max}(\mathcal{P})\|\mathcal{P}B_\zeta\|}. \tag{22}$$

Remark 3 (*On The Feasibility Of* (15a)). The detailed feasibility analysis of the conditions (15a) in Theorem 1 is not introduced in this paper. Indeed, it is well-known in the literature that this LMI-based technique, called LPV/LMI technique introduced in Zemouche and Boutayeb (2013), is less conservative than the other LMI-based methods in the observer design context. Conditions (15a) offer the less conservative way to handle Lipschitz nonlinearities, as demonstrated analytically and numerically in Arezki et al. (2023), Zemouche and Boutayeb (2013). Moreover, it will be shown in the next sections that for some families of nonlinear systems, the feasibility of (15a) can always be guaranteed.

Remark 4 (*On The Construction Of* $\mathcal{A}_j^\zeta$). Several techniques exist for the convex decomposition of nonlinear functions. For a more comprehensive overview of this topic, we refer the reader to the books (Boyd et al., 1994; Rockafellar, 1997; Sename et al., 2013), which discuss the representation of elements within a convex set. Since the Jacobian $\nabla_\zeta^{f_\zeta}$ is bounded, there exist finite real numbers f_{ij}^1 and f_{ij}^2, $i, j = 1, \ldots, n_\zeta$ (independent from $\boldsymbol{z}^i$ and u) such that $\frac{\partial f_{\zeta_i}}{\partial \zeta_j}(\boldsymbol{z}^i, u) \in [f_{ij}^1, f_{ij}^2]$, for $i, j = 1, \ldots, n_\zeta$. Hence, we have $\bar{n}_\zeta = 2^{n_\zeta^2}$ possibilities, which is the exponential number of matrices we can build from all the bounds.

3.2. Main result: New observer design technique

In this section, we propose a simple observer design method for the class of systems (1) with nonlinear delayed-output measurement, which is the main motivation of this paper. As stated in Section 2.1, to handle the delay in the output measurements, several techniques have been proposed in the literature. In this paper, we propose a novel and different observer design technique. To this end, we, first, introduce the new state variable, $z(t) \in \mathbb{R}^{n_z}$, defined by:

$$\begin{cases} \dot{z}(t) = f_z(z(t), u(t)) + Y_z y(t) \\ z(0) = z_0, \end{cases} \tag{23}$$

where f_z is a known globally Lipschitz function, the matrix $Y_z \in \mathbb{R}^{n_z\times p}$ is known and constant, and $z_0 \in \mathbb{R}^{n_z}$ is a known constant vector. The idea consists in using a state augmentation approach to get a new system for which the created $z(t)$ is the output measurement. Indeed, since $f_z, u(t), Y_z$, and z_0 are all known, then the state $z(t)$ is known in real-time from the measured output $y(t)$ of the original system (1). Before introducing the main transformation, notice that from the *Newton–Leibniz* formula, $y(t)$ can be written under the form:

$$y(t) = h(x(t)) - \int_{t-\tau(t)}^{t} \frac{\partial h}{\partial x}(x(s))f\left(x(s), u(s)\right)\mathrm{d}s. \tag{24}$$

By exploiting (23) and (24), the system (1) can be transformed into the form (9) with

$$\zeta(t) \triangleq \begin{bmatrix} z(t) \\ x(t) \end{bmatrix}, y_\zeta \triangleq z(t),\ C \triangleq \begin{bmatrix}\mathbb{I}_{n_z} & 0\end{bmatrix}, \tag{25}$$

$$f_\zeta\left(\zeta(t), u(t)\right) \triangleq \begin{bmatrix} f_z(z(t), u(t)) + Y_z h(x(t)) \\ f(x(t), u(t)) \end{bmatrix}, \tag{26}$$

$$g\left(\zeta(t), u(t)\right) \triangleq \frac{\partial h}{\partial x}(x(t))f\left(x(t), u(t)\right),\ B_\zeta \triangleq \begin{bmatrix} -Y_z \\ 0 \end{bmatrix} \tag{27}$$

Then, we propose the following state observer:

$$\begin{aligned}
\dot{\hat{\zeta}}(t) = f_\zeta\left(\hat{\zeta}(t), u(t)\right) + B_\zeta \int_{t-\tau(t)}^{t} g(\hat{\zeta}(s), u(s))\mathrm{d}s&\\
+ L\left(y_\zeta(t) - C\hat{\zeta}(t)\right)&
\end{aligned} \tag{28a}$$

$$\hat{x}(t) = \begin{bmatrix}0 & \mathbb{I}_n\end{bmatrix}\hat{\zeta}(t). \tag{28b}$$

Before summarizing the result in a corollary, we need the following assumption.

Assumption 3. The function g defined in (27) is γ_g-Lipschitz with respect to $x(t)$, uniformly on $u(t)$.

Remark 5. Assumption 3 is equivalent to the boundedness of the Jacobian matrix $\frac{\partial g}{\partial x}\big(\zeta(t), u(t)\big)$, detailed below:

$$\frac{\partial g}{\partial x}\big(\zeta(t), u(t)\big) = \frac{\partial h}{\partial x}(x)\frac{\partial f}{\partial x}(x, u) + \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}(x)\right) f(x, u). \tag{29}$$

Assuming f and h to be globally Lipschitz is clearly not sufficient to ensure that g is also globally Lipschitz. To guarantee the Lipschitz continuity of g, it is additionally necessary that the term $\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x}(x)\right)\cdot f(x, u)$ is bounded. Assumption 3 can indeed be derived from certain properties of f and h in system (1), specifically the boundedness of the Jacobian matrix defined in Eq. (29). However, introducing this assumption at the beginning, when defining system (1), would make the formulation difficult to understand.

Now all the conditions to apply Theorem 1 are satisfied, we can summarize the result in the following corollary.

Corollary 1. *Consider the system (9) with the parameters and functions given in (25)–(27). Assume that Assumption 3 is satisfied and there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$, a matrix $\mathcal{R} \in \mathbb{R}^{p_\zeta \times n_\zeta}$, and a positive scalar μ such that the conditions (15a)–(15b) hold. Let $L = \mathcal{P}^{-1}\mathcal{R}^\top$ be the gain matrix of (28a). Then, the estimated state $\hat{x}(t)$ given by (28b) converges exponentially to the state $x(t)$ of the original system (1).*

Remark 6. Without loss of generality, we consider in (23) a linear function f_z depending on $z(t)$ only, i.e: $f_z(z(t), u(t)) = A_z z(t)$. Even, for simplification, we can take $f_z(z(t), u(t)) \equiv 0$. In addition, these considerations allow reducing the dimension of the corresponding polytopic set $\mathcal{H}_{f_\zeta}$, which reduces then the number of LMIs (15a) to solve. Indeed, the primary focus of the paper is not on the specific choice of $f_z(z(t), u(t))$. The unique requirement for this function is that it must be globally Lipschitz, uniformly with respect to $u(t)$, to enable the application of our LMI-based techniques. From a design perspective, the main consideration is ensuring that $f_z(z(t), u(t))$ has a small Lipschitz constant, since as shown in Zemouche and Boutayeb (2013), the feasibility of the LMI conditions depends on the Lipschitz constant or the bounds of the partial derivatives of $f_z(z(t), u(t))$. Additionally, since Eq. (23) operates as a filter, then $f_z(z(t), u(t))$ could potentially enhance the filtering of the output measurement $y(t)$, namely in the presence of measurement noises. This is not the focus of the paper.

4. Results for a particular family of systems

This section considers the high-gain observer and its robustness with respect to the delay in the output measurement. Although several techniques have been proposed in the literature for this class of systems, we show that our method is simple and applies to this class of systems under the companion form.

4.1. System description and assumptions

Without loss of generality, we consider the family of systems described by (1) with

$$\begin{aligned} f(x(t), u(t)) &= Ax(t) + Bf_x\big(x(t)\big) \\ h\big(x(t-\tau(t))\big) &= h_x\big(C_x x(t-\tau(t))\big) \\ (A)_{ij} &= \begin{cases} 1 & \text{if} \quad j = i+1 \\ 0 & \text{if} \quad j \neq i+1 \end{cases} \\ C_x &= \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^\top \end{aligned} \tag{30}$$

where f_x and h_x are $\gamma_{f_x}-$Lipschitz and $\gamma_{h_x}-$Lipschitz, respectively, with respect to their arguments.

For the sake of observability, the following assumption is necessary.

Assumption 4. There exists $\delta_h > 0$, $\delta_h \leq \gamma_{h_x}$ such that

$$\delta_h \leq \frac{\partial h_x}{\partial v}(v) \leq \gamma_{h_x}, \;\; \forall v \in \mathbb{R}. \tag{31}$$

For the sake of simplification, we assume that

$$\gamma_{h_x} \triangleq \sup_{v\in\mathbb{R}} \left(\frac{\partial h_x}{\partial v}(v)\right) = 1. \tag{32}$$

Otherwise, we use for the observer the output

$$y_{\text{new}}(t) = \frac{y(t)}{\sup_{v\in\mathbb{R}}\left(\frac{\partial h_x}{\partial v}(v)\right)} \triangleq \bar{h}_x(C_x x(t-\tau)) \tag{33}$$

where the function $\bar{h}$ satisfies

$$\frac{\delta_h}{\gamma_{h_x}} \leq \frac{\partial \bar{h}_x}{\partial v}(v) \leq 1, \;\; \forall v \in \mathbb{R}.$$

Assumption 4 is standard in observer design methods, especially for triangular systems in the high-gain observer design context. The condition $\frac{\partial h_x}{\partial v}(v) \geq \delta_h, \;\; \forall v \in \mathbb{R}$, is necessary to guarantee the observability of the system, while $\frac{\partial h_x}{\partial v}(v) \leq \gamma_{h_x}, \;\; \forall v \in \mathbb{R}$ is to guarantee that the function $h_x(.)$ is globally Lipschitz. For further details regarding this assumption, the reader is referred to Gauthier and Kupka (2001), Wang et al. (2017).

Notice that Assumption 4 implies that the function $h_x(\cdot)$ is strictly monotone increasing, and is therefore invertible. As a result, the class of systems described in (30) is equivalent, via the inverse of $h_x(\cdot)$, to systems in companion form with a linear output. However, in practice, computing the inverse of $h_x(\cdot)$ can be challenging or even infeasible in closed form. This highlights the importance of studying systems with nonlinear outputs directly, without relying on the inversion of $h_x(\cdot)$.

As in the previous section, we create the following new variable, $z(t) \in \mathbb{R}$, as in (23):

$$\begin{cases} \dot{z}(t) = \gamma y(t) \\ z(t) = z(0) = z_0, \forall t \in [-t^\star, 0]. \end{cases} \tag{34}$$

where $\gamma > 0$ is a constant scalar, which is considered a tuning parameter. Also, in this case, from the *Newton–Leibniz* formula, (24) is reduced to

$$y(t) = h(x(t)) - \int_{t-\tau(t)}^{t} \frac{\partial h_x}{\partial x_1}\big(x_1(s)\big)x_2(s)\mathrm{d}s. \tag{35}$$

Remark 7. Notice that in the case of linear output, $y(t) = C_x x(t - \tau(t))$, which is often encountered in real application models, the identity (35) becomes

$$y(t) = x_1(t) - \int_{t-\tau(t)}^{t} x_2(s)\mathrm{d}s. \tag{36}$$

In this case, Assumption 3 is clearly satisfied globally since $g(x(t)) = x_2(t)$ is linear.

Now, let us go back to (35). It is quite clear that the function

$$\phi(x_1, x_2) \triangleq \frac{\partial h_x}{\partial x_1}\big(x_1\big)x_2 \tag{37}$$

is globally Lipschitz with respect to x_2 because h_x is $\gamma_{h_x}-$Lipschitz, and then $\frac{\partial h_x}{\partial x_1}(x_1)$ is bounded. However, it is not globally Lipschitz with respect to x_1. Then, we need an additional assumption on x_2 that we will consider in the next theorem.

As in the previous section, we consider the transformed system (9) with the parameters (25)–(27) with $Y_z = \gamma$, and according to (30) as follows:

$$\begin{cases} \dot{\zeta}(t) = f_\zeta\left(\zeta(t)\right) + B_\zeta \displaystyle\int_{t-\tau(t)}^{t} g(\zeta(s))\mathrm{d}s \\ y_\zeta(t) = C\zeta(t) \end{cases} \tag{38}$$

where $g(\zeta) = \phi(\zeta_2, \zeta_3)$.

4.2. System transformation: High-gain observer

By construction of the corresponding augmented system (38), without the integral term, the triangular companion form is preserved. Then, we can apply the high-gain observer methodology. To this end, we perform a second transformation, which is usual in this context, although it is often applied to the error system. Let us introduce the following linear transformation:

$$\xi = \mathbb{T}_\theta \zeta, \text{ where } \mathbb{T}_\theta \triangleq \mathrm{diag}\left(\frac{1}{\theta}, \ldots, \frac{1}{\theta^{n+1}}\right) \tag{39}$$

which transforms (38) into

$$\begin{cases} \dot{\xi}(t) = \mathbb{T}_\theta f_\zeta\left(\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) + \mathbb{T}_\theta B_\zeta \displaystyle\int_{t-\tau(t)}^{t} g\left(\mathbb{T}_{\frac{1}{\theta}}\xi(s)\right)\mathrm{d}s \\ y_\zeta(t) = C\mathbb{T}_{\frac{1}{\theta}}\xi(t) \end{cases} \tag{40}$$

To make the developments easy to follow and for any convenience, we express the system (40) in the following detailed form:

$$\begin{cases} \dot{\xi}(t) = \begin{bmatrix} \frac{\gamma}{\theta} h_x(\theta^2\xi_2(t)) \\ \theta[0 \ \ A]\xi(t) \end{bmatrix} + \mathbb{T}_\theta \begin{bmatrix} 0 \\ B \end{bmatrix} f_x\left([0 \ \ \mathbb{I}_n]\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) \\ \qquad + \dfrac{1}{\theta} B_\zeta \displaystyle\int_{t-\tau(t)}^{t} \phi\big(\theta^2\xi_2(s), \theta^3\xi_3(s)\big)\mathrm{d}s \\ y_\zeta(t) = \theta C\xi(t) = \theta\xi_1(t) \end{cases} \tag{41}$$

Before summarizing the result, let us define the function $\breve{g}$ as the Lipschitz extension of ϕ introduced in (37):

$$\breve{g}(z_1, z_2) = \phi\big(z_1, \pi_{\mathcal{I}}(z_2)\big) \tag{42}$$

where $\pi_{\mathcal{I}}(z_2)$ stands for the *Hilbert* projection of z_2 on $\mathcal{I}$ for any closed interval $\mathcal{I} \subset \mathbb{R}$.

Define also the matrix $\mathcal{A}_\gamma(.)$ as

$$\mathcal{A}_\gamma(\boldsymbol{v}) \triangleq \begin{bmatrix} 0 & \begin{bmatrix} \gamma\boldsymbol{v} & 0_{1\times n-1} \end{bmatrix} \\ 0_{n\times 1} & A \end{bmatrix}, \ \forall \boldsymbol{v} \in \mathbb{R}. \tag{43}$$

The dynamics of the estimation error, $\tilde{\xi}(t) \triangleq \xi(t) - \hat{\xi}(t)$, is given by:

$$\begin{aligned} \dot{\tilde{\xi}}(t) = {} & \theta\left[\mathcal{A}_\gamma\left(\frac{\partial h_x}{\partial x_1}(w(t))\right) - LC\right]\tilde{\xi}(t) \\ & + \left(\psi(\xi(t)) - \psi(\hat{\xi}(t))\right) \\ & + \frac{1}{\theta}B_\zeta \int_{t-\tau(t)}^{t} \left[g\left(\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) - \breve{g}\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right)\right]\mathrm{d}s \end{aligned} \tag{44}$$

where

$$\psi(\xi(t)) \triangleq \mathbb{T}_\theta \begin{bmatrix} 0 \\ B \end{bmatrix} f_x\left([0 \ \ \mathbb{I}_n]\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) \tag{45}$$

and $h_x(\theta^2\xi_2(t)) - h_x(\theta^2\hat{\xi}_2(t)) = \theta^2\frac{\partial h_x}{\partial v}(w(t))\tilde{\xi}_2(t)$, $w(t) \in \mathbf{Co}\left(\theta^2\xi_2(t), \theta^2\hat{\xi}_2(t)\right)$, from the differential mean value theorem in Lemma 2 applied to the scalar function h_x.

In addition, since f_x is γ_{f_x}−Lipschitz and from the structure of ψ in (45), there exists a constant $\kappa_f \geq \gamma_{f_x}$, independent from θ, such that

$$\left\|\psi(\xi(t)) - \psi(\hat{\xi}(t))\right\| \leq \kappa_f\|\tilde{\xi}(t)\|. \tag{46}$$

Since $\theta^3\xi_3(t) = x_2(t) \in \mathcal{I}$ and the *Hilbert* projection preserves the Lipschitz constant in $\mathbb{R}^2$ and from the structure of g in (41), there exists $\kappa_g \geq \gamma_g$ such that

$$\left\|g\left(\mathbb{T}_{\frac{1}{\theta}}\xi(t)\right) - \breve{g}\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right)\right\| \leq \kappa_g\theta^3\|\tilde{\xi}(t)\|. \tag{47}$$

Theorem 2. *Assume that the component $x_2(t)$ of the system (1), with (30), belongs to a compact interval $\mathcal{I} \subset \mathbb{R}$ and that the function ϕ defined in (37) is Lipschitz in $\mathbb{R}\times\mathcal{I}$. Assume there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\zeta\times n_\zeta}$, a matrix $\mathcal{R} \in \mathbb{R}^{p\times(n+1)}$, and positive scalars μ and γ such that the following conditions hold:*

$$\begin{aligned} & \left(\mathcal{A}_\gamma(\ell)\right)^\top\mathcal{P} + \mathcal{P}\mathcal{A}_\gamma(\ell) - C^\top\mathcal{R} - \mathcal{R}^\top C + \mu\mathbb{I}_{n+1} \leq 0, \\ & \ell \in \{\delta_h, 1\} \end{aligned} \tag{48a}$$

$$\theta > \max\left(1, \frac{2\kappa_f\lambda_{\max}(\mathcal{P})}{\mu}\right) \tag{48b}$$

$$\tau^\star < \frac{\lambda_{\min}(\mathcal{P})\left[\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\right]}{2\theta^2\gamma\kappa_g\lambda_{\max}(\mathcal{P})\|\mathcal{P}C^\top\|} \tag{48c}$$

where κ_f and κ_g are defined in (46) and (47), respectively. Then the output $\hat{x}(t)$ of the following observer

$$\begin{aligned} \dot{\hat{\xi}}(t) = {} & \mathbb{T}_\theta f_\zeta\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right) + \frac{1}{\theta}B_\zeta\int_{t-\tau(t)}^{t}\breve{g}\left(\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right)\mathrm{d}s \\ & + L\left(y_\zeta(t) - C\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t)\right) \end{aligned} \tag{49a}$$

$$\hat{x}(t) = \begin{bmatrix} 0_{n\times 1} & \mathbb{I}_n \end{bmatrix}\mathbb{T}_{\frac{1}{\theta}}\hat{\xi}(t) \tag{49b}$$

with $L = \mathcal{P}^{-1}\mathcal{R}^\top$, converges exponentially to the state $x(t)$ of the original system (1) with the particular parameters in (30).

Proof. It is sufficient to show that $\tilde{\xi}(t) \triangleq \xi(t) - \hat{\xi}(t)$ is exponentially stable. After computing the derivative of $\vartheta(\tilde{\xi}(t)) \triangleq \tilde{\xi}^\top(t)\mathcal{P}\tilde{\xi}(t)$ along the trajectories of (44), and by considering the bounds (46)–(47) and writing $B_\zeta = -\gamma C^\top$, we get

$$\begin{aligned} \dot{\vartheta}(\tilde{\xi}(t)) \leq {} & \tilde{\xi}^\top(t)\left[\left(\mathcal{A}_\gamma\left(\frac{\partial h_x}{\partial x_1}(w(t))\right) - LC\right)^\top\mathcal{P}\right. \\ & \left. + \mathcal{P}\left(\mathcal{A}_\gamma\left(\frac{\partial h_x}{\partial x_1}(w(t))\right) - LC\right)\right]\tilde{\xi}(t) \\ & + 2\theta^2\gamma\kappa_g\|\mathcal{P}C^\top\|\|\tilde{\xi}(t)\|\int_{t-\tau^\star}^{t}\|\tilde{\xi}(s)\|\mathrm{d}s \\ & + 2\kappa_f\lambda_{\max}(\mathcal{P})\|\tilde{\xi}(t)\|^2. \end{aligned} \tag{50}$$

It follows from (48a) and the convexity principle that

$$\begin{aligned} \dot{\vartheta}(\tilde{\xi}(t)) \leq {} & -\frac{\left(\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\right)}{\lambda_{\max}(\mathcal{P})}\vartheta(\tilde{\xi}(t)) \\ & + \tau^\star\frac{2\theta^2\gamma\kappa_g\|\mathcal{P}C^\top\|}{\lambda_{\min}(\mathcal{P})}\sup_{s\in[t-\tau^\star,t]}\vartheta(\tilde{\xi}(s)) \end{aligned} \tag{51}$$

Then, according to Lemma 1 and $\theta \geq 1$, the exponential convergence of $\tilde{\xi}(t)$ is inferred if

$$\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P}) > 0$$

and

$$\frac{\left(\mu\theta - 2\kappa_f \lambda_{\max}(\mathcal{P})\right)}{\lambda_{\max}(\mathcal{P})} > \tau^\star \frac{2\theta^2 \gamma \kappa_g \|\mathcal{P}C^\top\|}{\lambda_{\min}(\mathcal{P})}$$

which are equivalent to (48b) and (48c), respectively.

Remark 8. Notice that for systems with linear output, i.e: $h_x(x_1) = C_x x$, the boundedness of $x_2(t)$ is not needed because the function g does not depend on x_1, then it is globally Lipschitz on x_1.

Remark 9. Depending on the user's needs and the model at hand, the condition (48b) may be relaxed by applying the HG/LMI-based observer proposed in Zemouche et al. (2019). This leads to a smaller value of θ, which allows improving the bound of the tolerated value of $\tau^\star$.

Remark 10 (*On The Transformation* (39))**.** The general method described in Corollary 1 can be applied to systems in companion form. However, for these systems, it is possible to introduce the parameterization (39), as is common in the high-gain observer methodology. This allows obtaining the structure (44). Then, since (48a) are always feasible, the function ψ can be dominated for θ large enough as in (48b). Such a parameterization is without benefit for systems with arbitrary structures, even for linear systems. This becomes obvious upon closer examination of the computations. It is worth noting that, even with the parameterization using θ, Corollary 1 can still be applied to the transformed system (40) while guaranteeing the feasibility of the corresponding LMIs (15a). However, by dominating the entire function ψ, only two LMI conditions need to be solved. This issue has been explored in greater detail in Zemouche et al. (2019) and Arezki et al. (2023).

Remark 11. It is worth noting that the results can be generalized to consider multiple nonlinearities in (30) as follows

$$f_x(x) \triangleq \begin{bmatrix} f_{x_1}(x_1) \\ f_{x_2}(x_1, x_2) \\ \vdots \\ f_{x_{n-1}}(x_1, \dots, x_{n-1}) \\ f_{x_n}(x) + u(t) \end{bmatrix}$$

without encountering fundamental difficulties.

5. Relaxation of Assumption 3

The main drawback of the previous results is the Lipschitz assumption of the function g defined in (27), respectively, according to the corresponding case. In the case of Section 4.2, this requires the boundedness of the state $x_2(t)$ of the original system and the Lipschitz assumption on the function ϕ defined in (37). Although these conditions vanish in the case of linear outputs, however, such assumptions are conservative in general. Nevertheless, to avoid the previous strong assumptions, we can proceed differently. The idea consists of applying the *Newton–Leibniz* formula on the error system instead of on the output as in (24) and (35). To this end, we transform the original system (1) into the following one:

$$\begin{cases} \dot{\zeta}(t) = f_\zeta\left(\zeta(t), \zeta(t-\tau(t)), u(t)\right) \\ y_\zeta(t) = C\zeta(t) \end{cases} \tag{52}$$

by considering the transformations

$$\zeta(t) \triangleq \begin{bmatrix} z(t) \\ x(t) \end{bmatrix}, C \triangleq \begin{bmatrix} \mathbb{I}_{n_z} & 0 \end{bmatrix}, \tag{53}$$

$$\begin{aligned} f_\zeta\left(\zeta(t), \zeta(t-\tau(t)), u(t)\right) &\triangleq \begin{bmatrix} Y_z h(x(t-\tau(t))) \\ f(x(t), u(t)) \end{bmatrix} \\ &= A_\zeta f(x(t), u(t)) - B_\zeta h(x(t-\tau(t))) \end{aligned} \tag{54}$$

where B_ζ is defined as in (27), $A_\zeta = \begin{bmatrix} 0 & \mathbb{I}_n \end{bmatrix}^\top$, and for simplicity, the vector $z(t)$ is defined by

$$\begin{cases} \dot{z}(t) = Y_z y(t) \\ z(0) = z_0. \end{cases} \tag{55}$$

To avoid the need for Assumption 3, we apply the following observer without integral term:

$$\dot{\hat{\zeta}}(t) = f_\zeta\left(\hat{\zeta}(t), \hat{\zeta}(t-\tau(t)), u(t)\right) + L\left(y_\zeta(t) - C\hat{\zeta}(t)\right). \tag{56}$$

Before stating the results, to avoid repetitions and cumbersome notations, it is worth noting that the developments of this section are introduced without detailing the computations. For the sake of organization and comparison with the results from the previous section, we present the findings of this section on a case-by-case basis, starting with the general case, followed by the specific case of companion systems.

5.1. General case

As a result, system (52) constitutes a time-delay system with output measurements free from delay. Consequently, the Luenberger observer (56) is employed without an integral term. Notably, contrarily to the standard observer structure discussed in Section 2.1, the observer gain, L, does not directly affect the original output measurements by multiplication, as the delay is shifted to the function f_ζ. This technique, unlike the results in Adil et al. (2022), reduces the observer's sensitivity to time-delay and simplifies the complexity of both mathematical derivations and design conditions. The analytical comparisons presented in the next section validate this assertion. These comparisons show also that despite the simplicity of the observer (56), the observer (28) with integral term leads to a less conservative bound on the maximum allowable value of the delay.

From Lemma 2, there exist $\boldsymbol{z}_t$ and $\bar{\boldsymbol{z}}_t$ in $\mathbb{R}^n$ such that

$$\Delta f_\zeta = A_\zeta \nabla_x^f(\boldsymbol{z}_t)\epsilon_x(t) - B_\zeta \nabla_x^h(\bar{\boldsymbol{z}}_t)\epsilon_x(t-\tau(t)) \tag{57}$$

where $\epsilon_x(s) \triangleq x(s) - \hat{x}(s)$ and

$$\Delta f_\zeta \triangleq f_\zeta\left(\zeta(t), \zeta(t-\tau(t)), u(t)\right) - f_\zeta\left(\hat{\zeta}(t), \hat{\zeta}(t-\tau(t)), u(t)\right).$$

By taking

$$\nabla_x^{f,h}(\boldsymbol{z}_t, \bar{\boldsymbol{z}}_t) \triangleq \left(A_\zeta \nabla_x^f(\boldsymbol{z}_t) - B_\zeta \nabla_x^h(\bar{\boldsymbol{z}}_t)\right)\begin{bmatrix} 0 & \mathbb{I}_n \end{bmatrix} \tag{58}$$

the estimation error dynamics is expressed as follows:

$$\begin{aligned} \dot{\epsilon}(t) &= \left[\nabla_x^{f,h}(\boldsymbol{z}_t, \bar{\boldsymbol{z}}_t) - LC\right]\epsilon(t) \\ &+ B_\zeta \nabla_x^h(\bar{\boldsymbol{z}}_t)\begin{bmatrix} 0 & \mathbb{I}_n \end{bmatrix}\left(\epsilon(t) - \epsilon(t-\tau(t))\right) \end{aligned} \tag{59}$$

where $\epsilon(s) \triangleq \zeta(s) - \hat{\zeta}(s)$. Therefore, we can summarize the result in the following well-structured Proposition 1. Before stating the proposition, notice that by construction of the extended function f_ζ in (54), the Jacobian $\nabla_x^{f,h}(\boldsymbol{z}_t, \bar{\boldsymbol{z}}_t)$, introduced in (58), belongs to the convex bounded set $\mathcal{H}_{f_\zeta}$ defined in (13).

Proposition 1. *Assume that there exist a symmetric positive definite matrix $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$, a matrix $\mathcal{R} \in \mathbb{R}^{p_\zeta \times n_\zeta}$, and a positive scalar μ such that the conditions (15a) hold. Then the observer (56), with $L = \mathcal{P}^{-1}\mathcal{R}^\top$, corresponding to (52) and (54)–(55), converges* *exponentially if the maximum value of the delay, $\tau^\star$, satisfies the following bound:*

$$\tau^\star < \frac{\mu\lambda_{\min}(\mathcal{P})}{2\gamma_h\|\mathcal{P}B_\zeta\|\sqrt{\|Y_z\|^2\gamma_h^2+\gamma_f^2+\|LC\|}}. \tag{60}$$

Proof. From the convexity principle, inequalities (15a) imply the following inequality:

$$\begin{aligned}&\left(\nabla_x^{f,h}(\boldsymbol{z}_t,\bar{\boldsymbol{z}}_t)-LC\right)^\top\mathcal{P}\\&+\mathcal{P}\left(\nabla_x^{f,h}(\boldsymbol{z}_t,\bar{\boldsymbol{z}}_t)-LC\right)\le-\mu\mathcal{P}.\end{aligned} \tag{61}$$

It follows that the derivative of the Lyapunov function

$$\vartheta(\epsilon(t))\triangleq\epsilon^\top(t)\mathcal{P}\epsilon(t)$$

satisfies the inequality

$$\dot\vartheta(\epsilon(t))\le-\mu\vartheta(\epsilon(t))+\Theta\left(\mathcal{P},L,Y_z\right)\tau^\star\sup_{s\in[t-2\tau^\star,t]}\vartheta(\epsilon(s)) \tag{62}$$

where

$$\Theta\left(\mathcal{P},L,Y_z\right)\triangleq\frac{2\gamma_h\|\mathcal{P}B_\zeta\|\sqrt{\|Y_z\|^2\gamma_h^2+\gamma_f^2+\|LC\|}}{\lambda_{\min}(\mathcal{P})}. \tag{63}$$

The bound (62)–(63) is obtained after developing the computations, applying Lemma 3, and then by using the *Newton–Leibniz* formula on the term $\epsilon(t-\tau(t))-\epsilon(t)$. The details to prove (62)–(63) are omitted to avoid cumbersome repetitions. Consequently, from Lemma 1, the estimation error converges exponentially if

$$\Theta\left(\mathcal{P},L,Y_z\right)\tau^\star<\mu \tag{64}$$

which is equivalent to (60).

Remark 12. The bound (60) depends on the observer gain L, contrarily to the bound (15b). To maximize $\tau^\star$ in (60), one could introduce additional LMI constraints to (15a) to limit $\|LC\|$. If we focus on the dependence on $\|LC\|$, to maximize the bound (60), we minimize the matrix $(LC)^\top LC$. To do this, we introduce a scalar variable $\alpha>0$ and the constraint $\mathcal{P}>\mathbb{I}_n$. Then, we have, $(LC)^\top(LC)\le(LC)^\top\mathcal{P}(LC)=(C^\top\mathcal{R})\mathcal{P}^{-1}(\mathcal{R}^\top C)$. Hence, if $(C^\top\mathcal{R})\mathcal{P}^{-1}(\mathcal{R}^\top C)<\alpha\mathbb{I}_n$, which is equivalent, by Schur lemma, to

$$\begin{bmatrix}-\alpha\mathbb{I}_n & C^\top\mathcal{R}\\ \mathcal{R}^\top C & -\mathcal{P}\end{bmatrix}<0, \tag{65}$$

we obtain $(LC)^\top(LC)<\alpha\mathbb{I}_n$, which allows minimizing $\|LC\|$ by minimizing α. In addition, to control the norm of $\mathcal{P}$, we can introduce an additional constraint such as $\mathcal{P}<\beta\mathbb{I}_n$ with $\beta>0$. Then, we can minimize $\max(\alpha,\beta)$, for instance, subject to (15a), (65), $\mathcal{P}>\mathbb{I}_n$, and $\mathcal{P}<\beta\mathbb{I}_n$. This issue is beyond the scope of this paper; we provide only brief guidelines for cases requiring optimization of the bound (60).

5.2. Companion system (30)

For particular structures of systems, the bound (60) can be simplified. This is mainly the case of systems under the companion form (30). Without expanding the developments, by following similar steps as in Section 3.2 and Section 5, we summarize the results. First, the transformation (39) converts the system (52) into the following one:

$$\begin{cases}\dot\xi(t)=\mathbb{T}_\theta f_\zeta\left(\mathbb{T}_{\frac{1}{\theta}}\xi(t),\mathbb{T}_{\frac{1}{\theta}}\xi(t-\tau(t))\right)\\ y_\zeta(t)=\theta C\xi(t)\end{cases} \tag{66}$$

with the corresponding observer

$$\begin{aligned}\dot{\hat\xi}(t)=&\,\mathbb{T}_\theta f_\zeta\left(\mathbb{T}_{\frac{1}{\theta}}\hat\xi(t),\mathbb{T}_{\frac{1}{\theta}}\hat\xi(t-\tau(t))\right)\\&+L\left(y_\zeta(t)-\theta C\hat\xi(t)\right)\end{aligned} \tag{67a}$$

$$\hat x(t)=\begin{bmatrix}0_{n\times1} & \mathbb{I}_n\end{bmatrix}\mathbb{T}_{\frac{1}{\theta}}\hat\xi(t) \tag{67b}$$

where $z(t)$ is defined in this case by (34).

By considering the same notations as in Section 4, with the new observer (67a), and taking into account (32), we get the following proposition.

Proposition 2. *Let h_x satisfy Assumption 4 and (32), and f_x is γ_{f_x}−Lipschitz. Assume there exist a symmetric positive definite matrix $\mathcal{P}\in\mathbb{R}^{n_\zeta\times n_\zeta}$, a matrix $\mathcal{R}\in\mathbb{R}^{p\times n+1}$, and positive scalars μ and γ such that the conditions (48a)–(48b) hold. If the following condition is satisfied*

$$\tau^\star<\frac{\lambda_{\min}(\mathcal{P})\left[\mu\theta-2\kappa_f\lambda_{\max}(\mathcal{P})\right]}{\theta^2\sqrt{8}\gamma\lambda_{\max}(\mathcal{P})\|\mathcal{P}C^\top\|\max(1,|l_2|)} \tag{68}$$

where l_2 is the second component of the gain L, then the state $\hat x(t)$ given by (67b), with $L=\mathcal{P}^{-1}\mathcal{R}^\top$, converges exponentially to the state $x(t)$ of the original system (1) with the parameters and functions in (30).

Proof. To avoid cumbersome repetitions, the proof is simplified. It uses the results and definitions of the previous sections without repetitions. The estimation error dynamic satisfies the following equation:

$$\begin{aligned}\dot{\tilde\xi}(t)=&\,\theta\left[\mathcal{A}_\gamma\left(\frac{\partial h_x}{\partial x_1}(w(t))\right)-LC\right]\tilde\xi(t)\\&+\left(\psi(\xi(t))-\psi(\hat\xi(t))\right)\\&+\gamma\theta^2\frac{\partial h_x}{\partial x_1}(w(t))C^\top\int_{t-\tau(t)}^t\overbrace{\left[\tilde\xi_3(s)-l_2\tilde\xi_1(s)\right]}^{\dot{\tilde\xi}_2(s)}\mathrm{d}s\end{aligned} \tag{69}$$

where ψ is defined in (45) and satisfies the Lipschitz inequality (46), and $\tilde\xi(t)=\xi(t)-\hat\xi(t)$. Then, by expanding the derivative of the Lyapunov function $\vartheta(\tilde\xi(t))=\tilde\xi(t)^\top\mathcal{P}\tilde\xi(t)$, and from (48a)–(46) and (48b), we obtain:

$$\begin{aligned}\dot\vartheta(\tilde\xi(t))\le&-\frac{\left(\mu\theta-2\kappa_f\lambda_{\max}(\mathcal{P})\right)}{\lambda_{\max}(\mathcal{P})}\vartheta(\tilde\xi(t))\\&+\tau^\star\frac{2\sqrt{2}\theta^2\gamma\|\mathcal{P}C^\top\|\max(1,|l_2|)}{\lambda_{\min}(\mathcal{P})}\sup_{s\in[t-\tau^\star,t]}\vartheta(\tilde\xi(s))\end{aligned} \tag{70}$$

which implies that the estimation error is asymptotically stable if $\tau^\star$ satisfies (68). This ends the proof.

Remark 13. Note that if Assumptions 1, 2, 3, and 4 are not satisfied uniformly with respect to u, then the Lipschitz constants will depend on u. Hence, one of the solution consists in restricting the control input u to a compact or bounded set $\mathcal{U}\subset\mathbb{R}^m$. For instance, in the case of Assumption 3, instead of γ_g we will have $\gamma_g(u)$. Then, we can define a uniform Lipschitz constant over this bounded set as: $\gamma_g=\sup_{u\in\mathcal{U}}\gamma_g(u)$.

6. Comments and discussion

This section is devoted to constructive discussions and insightful comments regarding the results presented in the previous sections, particularly in contrast to established methodologies within the field. The main goal is to clarify the contributions and demonstrate the advantages of both the output-based dynamic extension technique and the new state observer structure, incorporating an integral term.

6.1. On the general case

It is quite clear that although the estimation strategy proposed in Section 3 needs the Lipschitz assumption on the function g defined in (27), it leads, however, to a simpler and higher bound on $\tau^\star$. For an explicit comparison, let us define $\tau_1^\star$ and $\tau_2^\star$ the maximum bounds obtained by (15b) and (60), respectively. Then, we have $\tau_1^\star < \tau_2^\star$ if

$$\frac{\gamma_g}{\gamma_h} > \sqrt{\|Y_z\|^2\gamma_h^2 + \gamma_f^2 + \|LC\|} \tag{71}$$

which is not easy to evaluate in general. Indeed, this depends on the values of γ_h, γ_f, $\|Y_z\|$, and $\|LC\|$. For instance, for systems with linear outputs, the inequality (71) cannot be satisfied because $\gamma_g \leq \gamma_h\gamma_f$. This means that for this class of systems with linear outputs, which is often encountered in real-world applications, the bound obtained by the strategy of Section 3.2 is less conservative than (60). In addition, the upper bound (60) depends on the observer gain L, which is not convenient for the user. Indeed, the higher is the norm $\|LC\|$ the lower is the bound of $\tau^\star$ in (60). Further results and analytical comparisons are given in the next sections.

Additionally, it is worth noting that an alternative bound to (64) can be derived without resorting to the state augmentation technique, as commonly seen in the literature (Ahmed-Ali et al., 2013b; Van Assche et al., 2011; Zemouche et al., 2019). However, employing the state augmentation strategy offers notable advantages: it streamlines the developments and simplifies the problem to systems with linear outputs. Conversely, the bound (64) is derived without the utilization of the well-known *Jensen inequality*, resulting, in general, in more conservative upper bounds. Moreover, without the dynamic extension approach, when utilizing the observer (2), the gain L multiplies the original output, $y(t)$, thus increasing the impact of delay on estimation accuracy.

6.2. Case of the companion form

This category of systems holds particular significance, as it has attracted considerable attention in the literature, with numerous studies dedicated to developing methodologies for addressing high values of $\tau^\star$ in output measurements (Ahmed-Ali et al., 2013a; Cacace et al., 2014; Germani et al., 2002; Hassan et al., 2011; Khosravian et al., 2015; Van Assche et al., 2011; Zhang & Shen, 2017). Recently, a novel approach exploiting the standard high-gain observer and exploiting the HG/LMI technique has emerged, as showcased in Adil et al. (2022). The authors demonstrated notable enhancements in the maximum allowable value of $\tau^\star$. Therefore, the outcomes presented in this section for this class of systems will be compared to the work in Adil et al. (2022).

6.2.1. On the maximum bound in (68) and (48c)

Here, we provide the value of θ for which the bounds in (68) and (48c) reach their maximum. We begin with the bound in (68). Let us introduce the following function:

$$\rho_{68}(\theta) \triangleq \frac{\lambda_{\min}(\mathcal{P})\big[\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\big]}{\theta^2\sqrt{8}\gamma\lambda_{\max}(\mathcal{P})\|\mathcal{P}C^\top\|\max(1,|l_2|)}.$$

It is sufficient to study the positive function $\rho_{68}(\theta)$ on the interval $\left[\max\left(1, \frac{2\kappa_f\lambda_{\max}(\mathcal{P})}{\mu}\right), +\infty\right[$ to find its maximum value. By simple calculations, we deduce that the derivative $\frac{d\rho_{68}}{d\theta}(.) = \rho_{68}'(.)$ is positive on the interval $\left[\max\left(1, \frac{2\kappa_f\lambda_{\max}(\mathcal{P})}{\mu}\right), \frac{4\kappa_f\lambda_{\max}(\mathcal{P})}{\mu}\right]$ and negative on the interval $\left[\frac{4\kappa_f\lambda_{\max}(\mathcal{P})}{\mu}, +\infty\right[$, with $\rho_{68}'\left(\frac{4\kappa_f\lambda_{\max}(\mathcal{P})}{\mu}\right) = 0$. This means that the maximum value of $\tau^\star$ in (68) is obtained for $\theta^\star = \frac{4\kappa_f\lambda_{\max}(\mathcal{P})}{\mu}$. That is, we have

$$\tau^\star \leq \tau_{68}^\star = \rho_{68}\left(\frac{4\kappa_f\lambda_{\max}(\mathcal{P})}{\mu}\right) = \frac{\mu^2\lambda_{\min}(\mathcal{P})}{16\sqrt{2}\gamma\kappa_f\lambda_{\max}^2(\mathcal{P})\|\mathcal{P}C^\top\|\max(1,|l_2|)}. \tag{72}$$

Of course, this is the case when $\kappa_f\lambda_{\max}(\mathcal{P}) > \frac{\mu}{4}$. Otherwise, the maximum value $\tau_{68}^\star$ will be

$$\tau_{68}^\star = \rho_{68}(1) = \frac{\lambda_{\min}(\mathcal{P})\big[\mu - 2\kappa_f\lambda_{\max}(\mathcal{P})\big]}{\sqrt{8}\gamma\lambda_{\max}(\mathcal{P})\|\mathcal{P}C^\top\|\max(1,|l_2|)}. \tag{73}$$

We do the same to obtain the value of θ for which the bound in (48c) reach its maximum. We consider the function

$$\rho_{48}(\theta) \triangleq \frac{\lambda_{\min}(\mathcal{P})\big[\mu\theta - 2\kappa_f\lambda_{\max}(\mathcal{P})\big]}{2\theta^2\gamma\kappa_g\lambda_{\max}(\mathcal{P})\|\mathcal{P}C^\top\|}$$

and for the same reasons as for ρ_{68}, we get the following maximum bound in this case:

$$\tau^\star \leq \tau_{48}^\star = \rho_{48}\left(\frac{4\kappa_f\lambda_{\max}(\mathcal{P})}{\mu}\right) = \frac{\mu^2\lambda_{\min}(\mathcal{P})}{16\gamma\kappa_g\kappa_f\lambda_{\max}^2(\mathcal{P})\|\mathcal{P}C^\top\|}. \tag{74}$$

However, if $\kappa_f\lambda_{\max}(\mathcal{P}) > \frac{\mu}{4}$, the bound $\tau_{48}^\star$ should be the following one:

$$\tau_{48}^\star = \rho_{48}(1) = \frac{\lambda_{\min}(\mathcal{P})\big[\mu - 2\kappa_f\lambda_{\max}(\mathcal{P})\big]}{2\gamma\kappa_g\lambda_{\max}(\mathcal{P})\|\mathcal{P}C^\top\|}. \tag{75}$$

6.2.2. Comparison between (68) and (48c)

This subsection is devoted to analytical comparisons between the bound (48c) derived from employing the new observer structure with an integral term, and the bound (68) obtained by using the observer (67). To this end, we compute the ratio, $\ell_{68/48}$, between the bounds (68) and (48c) as

$$\ell_{68/48} \triangleq \frac{\rho_{68}(\theta)}{\rho_{48}(\theta)} = \frac{\kappa_g}{\sqrt{2}\max(1,|l_2|)}. \tag{76}$$

We remind that κ_g represents the Lipschitz constant of the function $g(\zeta) = \phi(\zeta_2, \zeta_3)$, where ϕ is defined in (37). Clearly, the bound (48c) surpasses (thus, is higher than) that of (68), if $\ell_{68/48} < 1$. However, this condition relies on the values of κ_g and l_2. According to (32), we have

$$\kappa_g = \sqrt{\sup_{\zeta_2\in\mathbb{R},\zeta_3\in\mathcal{I}}\left(\left|\frac{\partial^2 h_x}{\partial\zeta_2^2}(\zeta_2)\zeta_3\right|^2 + \left|\frac{\partial h_x}{\partial\zeta_2}(\zeta_2)\right|^2\right)} = \sqrt{1 + \sup_{\zeta_2\in\mathbb{R},\zeta_3\in\mathcal{I}}\left(\left|\frac{\partial^2 h_x}{\partial\zeta_2^2}(\zeta_2)\zeta_3\right|^2\right)}. \tag{77}$$

Therefore, $\ell_{68/48} < 1$ if the following inequality holds:

$$\sup_{\zeta_2\in\mathbb{R},\zeta_3\in\mathcal{I}}\left|\frac{\partial^2 h_x}{\partial\zeta_2^2}(\zeta_2)\zeta_3\right| < 2\,[\max(1,|l_2|)]^2 - 1 = \max\left(1, 2|l_2|^2 - 1\right). \tag{78}$$

Otherwise, if $\ell_{68/48} > 1$, the bound (68) surpasses (48c). In general, direct comparison between the two bounds is challenging. However, the advantage of (48c) over (68) stems from the fact that (68) explicitly depends the observer gain component $|l_2|$, which has the potential to be high. More notably, in systems with linear output – common in the techniques developed for this class of systems – the bound (48c) consistently exceeds that of (68). This is obvious because, in the case of linear output, $\kappa_g = 1$ due to $\frac{\partial^2 h_x}{\partial \zeta_2^2} \equiv 0$. Consequently, (78) holds for any l_2.

6.2.3. Comparison with the bound in Adil et al. (2022)

To provide a comprehensive comparison between (48c) and the bounds obtained in Adil et al. (2022), it is necessary to establish an analytical connection between the solutions of the LMI condition (18) outlined in Adil et al. (2022, Theorem 10) and those of our LMI conditions (48a). Although the observer presented in Adil et al. (2022, Theorem 10) operates within a dimension of n, contrarily to our observer, which operates in a dimension of $n+1$, our design methods offer distinct advantages, notably:

- The LMI conditions presented in Adil et al. (2022, Eq.(18), Theorem 10) depend on the maximum allowable delay value, $\tau^\star$, presenting a notable limitation that necessitates the utilization of specific numerical algorithms to solve the resulting inequalities. Furthermore, compared to our LMI conditions (48a), those in Adil et al. (2022, Eq.(18), Theorem 10) exhibit a certain conservatism from feasibility point of view, while notably, our LMI conditions remain consistently feasible for the considered class of triangular systems.
- In contrast to our approach, the LMIs proposed in Adil et al. (2022, Eq.(18), Theorem 10) incorporate an additional decision variable beyond the Lyapunov matrix and the observer gain, which contributes to the determination of the bounds on $\tau^\star$. This introduces an additional conservatism since this decision variable may decrease significantly the maximum allowable delay value, $\tau^\star$.
- In contrast to our method, the approach in Adil et al. (2022) gives two different bounds on $\tau^\star$, namely τ_1 and τ_2, where $\tau^\star < \min(\tau_1, \tau_2)$. However, the bound τ_1 depends on $\lambda\theta - 2\kappa_f \lambda_{\max}(\mathcal{P})$ while τ_2 depends on $\frac{1}{\lambda\theta - 2\kappa_f \lambda_{\max}(\mathcal{P})}$. Hence, regardless of the value of θ, whether it is close to $\lambda\theta - 2\kappa_f \lambda_{\max}(\mathcal{P})$ or not, it affects the value of $\tau^\star$ since either τ_1 or τ_2 is very small.

6.3. On the performance of the new observer structure

At first glance, the integral term in the observer (10) might suggest that the estimation error is propagated and accumulated more significantly than in an observer (56) without the integral term. However, a closer examination of the estimation error dynamics in both cases, particularly for triangular/companion form, as illustrated in Eqs. (44) and (69), shows that the integral term also appears in the dynamics of the estimation error in the case without the integral term. This arises because the observer without the integral term evaluates the nonlinear function at $\hat{\xi}(t-\tau(t))$, instead of $\hat{\xi}(t)$ in the case with integral term. Such an integral term is unavoidable due to the presence of time-delay. The inclusion of an integral term in the observer does not imply inferior performance compared to an observer without it. On the contrary, the additional term facilitates better compensation for delays, as shown analytically in the previous sections across various scenarios and families of systems.

The novel output-based dynamic extension technique introduced in this paper is designed to address long delays in system outputs. Notably, this extension works as a filter, reducing sensitivity to measurement noise, among other benefits. Furthermore, the paper provides practical options for application-oriented users: if the delay in the system output meets the sufficient conditions required by the observer without an integral term, that observer can be employed. Otherwise, users can opt for the novel observer structure with an integral term. Both observers use the dynamic extension method, significantly enhancing their performance.

Remark 14. Note that implementing the observer with integral term in discrete time is nontrivial, often requiring high sampling rates and substantial memory to accurately approximate the integral over the interval $[t-\tau(t), t]$ using numerical methods such as the trapezoidal rule, Simpson's rule, and others. In particular, accurate numerical integration over a moving time window requires: a high sampling frequency to reduce discretization error; storage of past signal values over the entire delay window; and computation of the integral at each time step, which increases the computational complexity.

7. Further result

This section presents a new approach for handling large output delays by solving delay-dependent LMIs, avoiding restrictive constraints on $\tau^\star$ such as (15b). We consider the general case, yielding results applicable to any specific system family. Using a Lyapunov–Krasovskii functional, the method improves existing LMI-based designs for similar systems (Nechaf et al., 2023; Targui et al., 2019).

7.1. Lyapunov-Krasovskii analysis

Consider the system (9) with the parameters and functions given in (25)–(27) and the observer (28), with the function g satisfying Assumption 3. This leads to the estimation error dynamic (12). By using Lemma 2, there exists $\bar{\boldsymbol{z}}_t \in \mathbf{Co}\left(\zeta(t), \hat{\zeta}(t)\right)$ as in (5) such that $\Delta g\left(\zeta(s), \hat{\zeta}(s), u(s)\right) = \nabla_\zeta^g(\bar{\boldsymbol{z}}_s)\epsilon(s)$, where ∇_ζ^g is defined as in (7). Then, the error dynamic (12) is rewritten under the following form:

$$\dot{\epsilon}(t) = \left[\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right]\epsilon(t) + B_\zeta \int_{t-\tau(t)}^{t} \nabla_\zeta^g(\bar{\boldsymbol{z}}_s)\epsilon(s)\mathrm{d}s. \tag{79}$$

Since the function g satisfies Assumption 3, then there exist constant matrices $\mathcal{B}_j^g$ of appropriate dimensions, and functions $\bar{\lambda}_j(\bar{\boldsymbol{z}}_t)$, $j = 1, \dots \bar{n}_g$ such that the generalized Jacobian $\nabla_\zeta^g(\bar{\boldsymbol{z}}_s)$ belongs to the convex polytopic set defined as:

$$\mathcal{H}_g \triangleq \left\{\sum_{j=1}^{\bar{n}_g} \bar{\lambda}_j(\bar{\boldsymbol{z}}_s)\mathcal{B}_j^g, \sum_{j=1}^{\bar{n}_g} \bar{\lambda}_j(\bar{\boldsymbol{z}}_s) = 1, \bar{\lambda}_j(\bar{\boldsymbol{z}}_s) \geq 0\right\}. \tag{80}$$

Now, consider the Lyapunov–Krasovskii functional

$$\begin{aligned} &\vartheta\left(\epsilon(t)\right) = \epsilon^\top(t)\mathcal{P}\epsilon(t) \\ &+ \int_{-\tau^\star}^{0}\int_{v}^{0} \epsilon(t+s)^\top \Pi(t+s)\epsilon(t+s)\mathrm{d}s\mathrm{d}v \end{aligned} \tag{81}$$

with $\Pi(s) \triangleq \left(\nabla_\zeta^g(\bar{\boldsymbol{z}}_s)\right)^\top \mathbb{S}\, \nabla_\zeta^g(\bar{\boldsymbol{z}}_s)$, where $\mathcal{P}$ and $\mathbb{S}$ are two symmetric and positive definite matrices of appropriate dimensions. By expanding the derivative of ϑ along the trajectories of (79), we get

$$\begin{aligned} \dot{\vartheta}(\epsilon(t)) = \epsilon^\top(t)&\left[\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)\right]\epsilon(t) \\ &+ 2\int_{t-\tau(t)}^{t} \epsilon^\top(t)\mathcal{P}B_\zeta\nabla_\zeta^g(\bar{\boldsymbol{z}}_s)\epsilon(s)\mathrm{d}s + \tau^\star\epsilon(t)^\top\Pi(t)\epsilon(t) \\ &- \int_{t-\tau^\star}^{t} \epsilon^\top(s)\Pi(s)\epsilon(s)\mathrm{d}s \end{aligned} \tag{82}$$

where we used $\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\tau^\star}^{0}\int_{v}^{0}\varphi(t+s)\mathrm{d}s\mathrm{d}v = \tau^\star\varphi(t) - \int_{t-\tau^\star}^{t}\varphi(\nu)\mathrm{d}\nu$ for any continuous function $\varphi(.)$ defined on $[-\tau^\star, +\infty[\to \mathbb{R}$. Now, by using the Young inequality on the first integral term in (82) and the fact that $\tau(t) \leq \tau^\star$, we obtain

$$2\int_{t-\tau(t)}^{t} \epsilon^\top(t)\mathcal{P}B_\zeta\nabla_\zeta^g(\bar{\boldsymbol{z}}_s)\epsilon(s)\mathrm{d}s \leq \tau^\star\epsilon^\top(t)\left(\mathcal{P}B_\zeta\right)^\top\mathbb{S}^{-1}\mathcal{P}B_\zeta\epsilon(t)$$

$$+\int_{t-\tau^\star}^{t} \epsilon^\top(s)\Pi(s)\epsilon(s)\mathrm{d}s.$$

It follows that

$$\dot{\vartheta}(\epsilon(t)) \le \epsilon^\top(t)\mathbb{M}\left(\mathcal{P}, \mathbb{S}, t\right)\epsilon(t) \tag{83}$$

where

$$\mathbb{M}\left(\mathcal{P}, \mathbb{S}, t\right) \triangleq \left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right) + \tau^\star \mathcal{P}B_\zeta \mathbb{S}^{-1}\left(\mathcal{P}B_\zeta\right)^\top + \tau^\star \Pi(t).$$

As a consequence, we deduce that $\dot{\vartheta}(\epsilon(t)) < 0, \forall \epsilon(t) \neq 0$ if

$$\mathbb{M}\left(\mathcal{P}, \mathbb{S}, t\right) < 0, \forall t \ge 0. \tag{84}$$

7.2. Delay-dependent unified LMIs

This section is devoted to a new LMI-based method, where the provided LMIs are delay-dependent and free of any extra constraint on the maximum allowable delay $\tau^\star$. The result is summarized in the following theorem.

Theorem 3. *Assume that there exist two symmetric and positive definite matrices $\mathcal{P} \in \mathbb{R}^{n_\zeta \times n_\zeta}$ and $\mathbb{S} \in \mathbb{R}^{n_s \times n_s}$, and a matrix $\mathcal{R} \in \mathbb{R}^{p_\zeta \times n_\zeta}$ such that the LMI conditions (85) hold. Then the observer (28), with $L = \mathcal{P}^{-1}\mathcal{R}^\top$, corresponding to the system (9) with the parameters and functions given in (25)–(27) converges asymptotically.*

$$\begin{bmatrix} \left(\mathcal{A}_j^\zeta\right)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_j^\zeta - C^\top\mathcal{R} - \mathcal{R}^\top C & \left(\mathcal{P}B_\zeta\right)^\top & \mathcal{B}_k^{g\top}\mathbb{S} \\ \mathcal{P}B_\zeta & -\frac{1}{\tau^\star}\mathbb{S} & 0 \\ \mathbb{S}\mathcal{B}_k^g & 0 & -\frac{1}{\tau^\star}\mathbb{S} \end{bmatrix} < 0, j = 1, \dots, \bar{n}_\zeta; k = 1, \dots, \bar{n}_g. \tag{85}$$

Proof. Once the Lyapunov–Krasovskii functional is expanded to derive inequality (84), the subsequent steps of the proof unfold straightforwardly. Specifically, invoking the Schur Lemma confirms the satisfaction of inequality (84) provided the following conditions are met:

$$\begin{bmatrix} \Pi(1,1) & \left(\mathcal{P}B_\zeta\right)^\top & \left(\nabla_\zeta^g(\bar{\boldsymbol{z}}_t)\right)^\top \mathbb{S} \\ \mathcal{P}B_\zeta & -\frac{1}{\tau^\star}\mathbb{S} & 0 \\ \mathbb{S}\,\nabla_\zeta^g(\bar{\boldsymbol{z}}_t) & 0 & -\frac{1}{\tau^\star}\mathbb{S} \end{bmatrix} < 0 \tag{86}$$

for all $t \ge 0$, where

$$\Pi(1,1) \triangleq \left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right)^\top \mathcal{P} + \mathcal{P}\left(\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) - LC\right).$$

Therefore, using the convexity principle and introducing the change of variable $\mathcal{R} = L^\top\mathcal{P}^{-1}$, the inequality (86) is fulfilled if (85) holds true. Consequently, we infer that $\dot{\vartheta}(\epsilon(t)) < 0, \forall\epsilon(t) \neq 0$, implying asymptotic stability of the estimation error around the origin.

7.3. Decoupled LMIs

The design approach outlined in Theorem 3 is notably straightforward, requiring only the fulfillment of a set of LMI conditions without imposing any additional restrictive constraints on the delay. Although the LMIs (85) are delay-dependent, they allow optimizing the maximum allowable delay value, $\tau^\star$. On the other hand, we can convert the global LMIs (85) into two LMIs, where the first one is delay-independent and the second one is delay-dependent. Indeed, imposing the inequality (21) and using the Schur Lemma, we deduce that (85) is satisfied if in addition to (21), the following LMIs hold:

$$\begin{bmatrix} -\mu\mathbb{I}_{n_\zeta} & \left(\mathcal{P}B_\zeta\right)^\top & \mathcal{B}_k^{g\top}\mathbb{S} \\ \mathcal{P}B_\zeta & -\frac{1}{\tau^\star}\mathbb{S} & 0 \\ \mathbb{S}\mathcal{B}_k^g & 0 & -\frac{1}{\tau^\star}\mathbb{S} \end{bmatrix} < 0, k = 1, \dots, \bar{n}_g. \tag{87}$$

For an optimal value of $\tau^\star$, the LMIs (21) and (87) are to be solved simultaneously. Inequality (87) acts as an additional constraint to (21), that must be satisfied, replacing inequality (22). However, compared to (22), (87) offer two significant advantages:

- They can be solved simultaneously with (21), facilitating the determination of the optimal value of $\tau^\star$.
- Even if (87) are solved a posteriori after solving (21), they are less conservative than (22), as they incorporate an additional decision variable – the matrix $\mathbb{S}$ – derived from the delay-dependent Lyapunov–Krasovskii functional.

7.4. On the computational complexity of (85)

The presence of two Jacobian matrices in Theorem 3 significantly increases the computational complexity of the polytopic LMI-based approach, leading to an exponential growth in the number of LMIs to be solved. This is the main limitation of the proposed approach since we have $2^{(n+n_z)^2+pn}$ LMIs to solve with the same Lyapunov matrix $\mathcal{P}$. However, as emphasized and demonstrated in prior works (Rajamani et al., 2020; Zemouche & Boutayeb, 2013), using the standard LMI-based methods, based on the S-Procedure Lemma or Young's inequality, to dominate nonlinearities results in a single LMI, but this LMI is often overly conservative and infeasible for systems with Lipschitz constants greater than one (Zemouche & Boutayeb, 2013). This highlights the importance of seeking a balance between computational feasibility and the conservativeness of the results. It is also worth noting that, in many real-world applications, systems frequently feature linear outputs and some nonlinear components with constant partial derivatives. These characteristics can greatly reduce the number of LMIs that need to be solved, reducing computational complexity. Moreover, the polytopic approach has already proven its effectiveness in the past and has been applied to various real-world systems and experimental models, and the method has also been generalized to different classes of systems and control problems (see Guerra et al. (2024), Nemati et al. (2019), Phanomchoeng et al. (2011), Rajamani et al. (2020) and references therein). In addition, improvements have been proposed in the previous literature to overcome the challenges associated with this approach, such as computational complexity (Zemouche et al., 2016), in particular by proposing a significant reduction in the number of LMIs to solve. Hence, in case the exponential number of LMIs is an obstacle for the numerical software, we can reduce the number of LMIs by following the techniques given in Zemouche et al. (2016) or convert our polytopic LMIs into a single, but more restrictive LMI.

In the general case, without assuming a specific system structure, the feasibility of (85) cannot be guaranteed; only necessary conditions can be explicitly identified. In our case, the detectability of the pairs $\left(\mathcal{A}_j^\zeta, C\right)$, which is embedded in the LMI formulation, is a necessary condition for the feasibility of the LMIs (85). Indeed, (85) is feasible if the first diagonal block is negative definite, i.e.:

$$\left(\mathcal{A}_j^\zeta\right)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_j^\zeta - C^\top\mathcal{R} - \mathcal{R}^\top C < 0, j = 1, \dots, \bar{n}_\zeta$$

which can be satisfied if necessarily the pairs $\left(\mathcal{A}_j^\zeta, C\right)$ are detectable. The Lipschitz assumptions (Assumptions 1 and 3) are also required since they are used in the construction of the bounded matrices $\mathcal{A}_j^{\zeta}$ and $\mathcal{B}_k^{g}$, respectively. The conservatism of these LMI conditions stems from two main factors: (1) the use of a constant Lyapunov matrix, $\mathcal{P}$; and (2) the possible loss of linear observability on the bounding polytopes. These limitations are common across all LMI-based approaches. Notice that no existing LMI-based observer design method in the literature can guarantee the feasibility of LMIs for an arbitrary nonlinear system structure. Feasibility can only be guaranteed for specific structures, such as the triangular form in Section 4.

7.5. Numerical design procedure

The formulation in (21)–(22) has the advantage of decoupling the feasibility of the set of LMIs from the maximum allowable delay, while the LMIs (85) depend on the maximum allowable delay, $\tau^\star$. Since $\tau^\star$ is embedded within (85) and these LMIs are solved offline, the gridding method can be employed on $\tau^\star$ to determine an optimal value, $\tau^\star_{\text{opt}}$, for which the LMIs (85) remain feasible. The resulting LMIs are then feasible for any time-delay $\tau(t)$ where $\tau^\star \leq \tau^\star_{\text{opt}}$. The gridding method is required because the inequality (85) is not linear on $\tau^\star$, due to the coupling $\frac{1}{\tau^\star}\mathbb{S}$. Then, it cannot be optimized directly by using LMI solvers. The procedure is outlined in the well-structured Algorithm 1.

Algorithm 1: Numerical design algorithm

Require: LMI (85) nonlinearly dependent on $\tau^\star$, denoted LMI($\tau^\star$), and a range for $\tau^\star$

Ensure: The maximum admissible value of $\tau^\star$, denoted $\tau^\star_{\text{opt}}$

1: **Initialization:**
   Define the range of $\tau^\star$: $[\tau_{\min}, \tau_{\max}]$.
   Choose the number of grid points N and compute the grid:
   $\{\tau_1^\star, \tau_2^\star, \dots, \tau_N^\star\}$ where $\tau_i^\star = \tau_{\min} + (i-1)\frac{\tau_{\max}-\tau_{\min}}{N-1}$.
   Initialize $\tau^\star_{\text{opt}} \leftarrow \tau_{\min}$.

2: **Grid Search:**
   foreach $\tau_i^\star$ *in the grid* **do**
      Substitute $\tau_i^\star$ into the LMI (85) to get LMI($\tau_i^\star$);
      Solve the linearized LMI($\tau_i^\star$) using an LMI solver;
      if *LMI (85) is feasible at* $\tau_i^\star$ **then**
         $\tau^\star_{\text{opt}} \leftarrow \tau_i^\star$;

3: **Return:** $\tau^\star_{\text{opt}}$

To enhance the possibility of obtaining a high value for $\tau^\star_{\text{opt}}$, it is convenient to select a sufficiently large $\tau_{\max}$. Additionally, increasing the number of grid points, N, improves the accuracy of the algorithm. For application-oriented users, this means that depending on their specific model and the delays affecting their outputs, they can use the solutions obtained from (85) with an optimal value $\tau^\star_{\text{opt}}$ to design their observer, provided the delay satisfies $\tau^\star \leq \tau^\star_{\text{opt}}$.

8. Illustrative example

This section presents an illustrative example to validate the proposed observer design method and demonstrate the superiority of the established results. Specifically, we propose a two-dimensional bioreactor model previously explored in the literature. This model has been extensively employed in numerous studies on this subject, including those by Adil et al. (2022), Van Assche et al. (2011), Targui et al. (2019), and Nechaf et al. (2023). Its simplicity and wide use make it a convenient benchmark for illustrating the effectiveness of our methods and enhancing consistent and meaningful comparisons with prior research. The model is described by the following equations:

$$\dot{x} = \begin{bmatrix} -l_1 & c_1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ c_2\sin(x_2) + c_3\cos(x_2) + c_4 u \end{bmatrix} \tag{88}$$
$$y(t) = x_1(t-\tau(t)),$$

Table 1
Comparison between different methods.

Method	$\tau^\star$
(Van Assche et al., 2011, Theorem 1)	< 0.01
(Adil et al., 2022, Theorem 17)	< 0.02
Theorem 1–LMI conditions (15)	≈ 0.03
(Targui et al., 2019, Theorem 2)	≈ 1
(Nechaf et al., 2023, Theorem 3)	≈ 23.60
Theorem 3–LMI conditions (85)	≈ 34.50

where the input $u(t)$ is given by $u(t) = \sin(0.35t)$ and the constants l_1 and $c_i, i = 1, \dots, 4$, are given by $l_1 = 1$, $c_1 = 1$, $c_2 = c_3 = 0.02$, and $c_4 = 8$. In this case, the function g in (27) is reduced to $g(\zeta, u) = -l_1\zeta_2 + c_1\zeta_3$, which satisfies Assumption 3 with Lipschitz constant $\gamma_g = \sqrt{l_1^2 + c_1^2} = \sqrt{2}$. We use the output augmentation technique by using the new variable $z(t)$ introduced in (34). Then, we have

$$B_\zeta = \begin{bmatrix} -\gamma & 0 & 0 \end{bmatrix}^\top, \; f_\zeta(\zeta, u) = \begin{bmatrix} \gamma\zeta_2 & f(\zeta, u) \end{bmatrix}^\top$$

with $f(\zeta, u) \triangleq \begin{bmatrix} -l_1\zeta_2 + c_1\zeta_3 \\ c_2\sin(\zeta_3) + c_3\cos(\zeta_3) + c_4 u \end{bmatrix}$. Therefore, the Jacobian of the function f_ζ is given by

$$\nabla_\zeta^{f_\zeta}(\boldsymbol{z}_t) = \begin{bmatrix} 0 & \gamma & 0 \\ 0 & -l_1 & c_1 \\ 0 & 0 & c_2\cos(\boldsymbol{z}_t) - c_3\sin(\boldsymbol{z}_t) \end{bmatrix}.$$

Also, the Jacobian of the scalar function g is given by $\nabla_\zeta^{g}(\bar{\boldsymbol{z}}_t) = \mathcal{B}_1^{g} = \begin{bmatrix} 0 & -l_1 & c_1 \end{bmatrix}$. Utilizing the LMI conditions (15a) subject to (15b) outlined in Theorem 1, and the LMIs (85) of Theorem 3, we obtain the results presented in Table 1, contrasting them with the methodologies introduced in prior works such as (Adil et al., 2022; Nechaf et al., 2023; Targui et al., 2019; Van Assche et al., 2011). These references explore the same example to compare the efficiency of the approaches proposed therein.

It is quite clear from Table 1 that whether based fully on LMIs or requiring extra constraints on $\tau^\star$, our methods significantly enhance existing techniques in the literature. For $\tau^\star = 5s$ and $\gamma = 1$, the observer gain obtained from the LMIs (85) is $L = \begin{bmatrix} 5.1004 & 1.0124 & 0.7387 \end{bmatrix}^\top$. To demonstrate the efficiency of the proposed output-based dynamic extension technique, we introduce Gaussian noise with a mean of zero and a standard deviation of σ to disturb the output measurement. Fig. 1 presents results for $\sigma = 10$. Notably, even with a significant value of σ, the new observer structure, augmented with an integral term, shows good performance, showcasing robustness to measurement noise.

In the simulation, to simplify the computation of the integral term, we considered a constant delay, $\tau(t) = \tau^\star$, and assumed that $g(\hat{\zeta}(t), u(t)) = g(\hat{\zeta}(0), u(0))$ for all $t \in [-\tau^\star, 0]$. Hence, we have $\int_{-\tau^\star}^{0} g(\hat{\zeta}(s), u(s))\,ds = \tau^\star g(\hat{\zeta}(0), u(0))$, thus avoiding the need for numerical integration during the initialization phase. Moreover, in our example, the function g is independent of the control input u, i.e., $g(\hat{\zeta}(s), u(s)) = g(\hat{\zeta}(s))$. Then, for the online computation of the integral $\int_{t-\tau^\star}^{t} g(\hat{\zeta}(s))\,ds$, we used a simple and effective implementation in Simulink based on the identity $\int_{t-\tau^\star}^{t} g(\hat{\zeta}(s))\,ds = \int_{0}^{t} g(\hat{\zeta}(s))\,ds - \int_{0}^{t-\tau^\star} g(\hat{\zeta}(s))\,ds$. This approach eliminates the need for explicitly storing or buffering past values of $g(\hat{\zeta}(s))$. The implementation consists of three main components: a standard `Integrator` block to compute $\int_{0}^{t} g(\hat{\zeta}(s))\,ds$, a `Transport Delay` block to generate a delayed version of this integral, and a `Subtract` block to compute the difference.

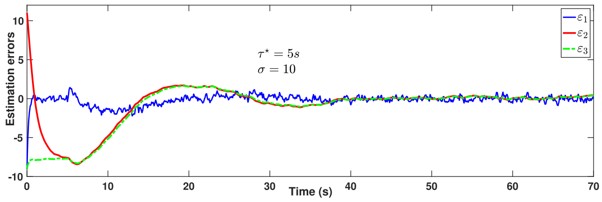

Fig. 1. Behavior of the estimation errors for $\tau^\star = 5s$.

The resulting signal corresponds to the integral over the moving window $[t - \tau^\star, t]$, which is directly used in the observer equations.

9. Conclusion

In this paper, we proposed several observer design techniques for nonlinear systems with delayed and nonlinear outputs. Using state augmentation and output transformation, the delay and nonlinearities are transferred to the system dynamics by creating a new output measurement and extending the system. For systems in companion form, novel synthesis conditions are derived that are less conservative than existing results. An alternative technique is also introduced to avoid the additional conservative Lipschitz assumption, and is thoroughly compared with both the relaxed Lipschitz method and existing approaches. Finally, based on a Lyapunov–Krasovskii functional, we propose an LMI-based method ensuring asymptotic convergence of the observer. Future work will extend this idea to unknown input and adaptive observers for systems with unknown parameters.

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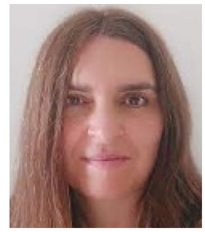

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