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Topical Review

A critical survey of twisted spectral triples beyond the Standard Model

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Abstract

We review the applications of twisted spectral triples to the Standard Model. The initial motivation was to generate a scalar field, required to stabilise the electroweak vacuum and fit the Higgs mass, while respecting the first-order condition. Ultimately, it turns out that the truest interest of the twist lies in a new—and unexpected—field of 1-forms, which is related to the transition from Euclidean to Lorentzian signature.

Keywords: Standard Model, spectral triple, Wick rotation

1. Introduction

From the pioneering work of [35] till the full formalism of Connes [16], noncommutative geometry provides a unified description of the Lagrangian of the Standard Model of fundamental interactions (electromagnetism, weak and strong interactions) [8, 9, 21]; minimally coupled to the Einstein–Hilbert action of General Relativity [18]; including right handed neutrinos [12]; where the Higgs boson comes out naturally on the same footing as the other bosons, i.e. as the local expression of a connection 1-form.

The approach works very well on Riemannian manifolds. The generalisation to pseudo-Riemannian geometry, in particular Lorentzian manifolds, is far from obvious (there are various attempts in this direction, see for instance [1–3, 38, 53] and reference within).

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In addition, noncommutative geometry offers possibilities to go beyond the Standard Model, by modifying the rules of the game in various ways: one may enlarge the space of fermions [51, 52], or get rid of the *first-order condition* (defined below) [13, 14], modify the real structure (also defined below) [6, 7], switch to non-associative geometry [4, 5], use some structure of Clifford bundle in order to modify some of the mathematical requirements defining a noncommutative geometry [26]. For a recent review of ‘beyond Standard Model’ propositions in the framework of noncommutative geometry, see [15].

Here we focus on another class of variations around Connes’ initial model, obtained by twisting the noncommutative geometry by an algebra automorphism [32, 34, 47].

All the possibilities above but the first are minimal extensions of the Standard Model, in that they yield an extra scalar field σ —suggested by particle physicists to stabilise the electroweak vacuum—but do not touch the fermionic content. The novelty of the twist is to generate another additional piece: a new field of 1-forms, which surprisingly turns out to be related to the transition from Euclidean to Lorentzian signature [30]. In particular, in the example of electrodynamics, this field identifies with the (dual) of the 4-momentum vector in Lorentzian signature, even though one started with a Riemannian manifold [47].

All this is explained as follows. In the next section we begin by some recalling on the spectral description of the Standard Model [12]. We stress the process of fluctuation of the metric, which is the way to generate bosonic fields in noncommutative geometry by turning the constant parameters of the model into fields.

In section 3 we describe the model of grand algebra developed in [32], which aimed at generating the extra scalar field σ , while respecting the first-order condition. The idea is to start with an algebra bigger than the one of the Standard Model, in order to have more ‘space’ to generate bosonic fields by fluctuations of the metric. This model does indeed generate the expected field σ , by letting the Majorana mass of the neutrinos fluctuate. Even though the first-order condition associated with this Majorana mass is preserved, the problem moves to the free Dirac operator: not only does the latter break the first-order condition, but its commutator with the algebra is unbounded, in contradiction with the very definition of spectral triple. This kind of problem is typically solved by twisting the spectral triple in the sense of Connes and Moscovici [24]. A twisting of the grand algebra down to the Standard Model has been worked out in [34], but we show in section 3.3 that this does not define *stricto sensu* a twisted spectral triple, for only the part of the algebra relevant for the extra scalar field has been twisted.

Applying the twist to the whole algebra suggests a general procedure to twist any graded spectral triple, as recalled in section 4. It consists in doubling the algebra one is beginning with, rather than finding it from the reduction of a bigger algebra. Such a ‘twisting up’ procedure has been studied in a couple of papers [41, 42]. There is some freedom in the construction, especially in the choice of the twisting operator whose eigenspaces determine the representation of the doubled algebra. By choosing the grading as the twisting operator, one automatically satisfies the twisted first-order condition. However, when applied to the spectral triple of the Standard Model, this twist-by-grading does not generate any extra scalar field. Some preliminary results, mentioned in section 4.3, indicate that this is a general feature of the twisting-up procedure: the twisted first-order condition and the extra scalar field are mutually exclusive. Hence the necessity to adapt to the twisted case the fluctuations without first-order condition introduced in [14]. This has been done in [49] and is summarised in section 4.3.

Section 5 deals with what might be the truest interest of the twist, namely the unexpected field of 1-forms arising from the twisted fluctuation. In the example of electrodynamics [47, 54], this field identifies with the dual of the 4-momentum in Lorentzian signature, even though one started with a Riemannian spectral triple.

2. The spectral description of the Standard Model

We begin with the definition of spectral triple, which is the central tool in Connes' noncommutative geometry, emphasising how the bosonic fields—including the Higgs field—are obtained as connection 1-forms, through the process of *fluctuation of the metric*. We then summarise the spectral description of the Standard Model.

2.1. Spectral triple

A spectral triple [16] consists of an algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ together with a selfadjoint operator D with compact resolvent, whose commutator $[D, a]$ is bounded for any $a \in \mathcal{A}$. It is graded if it comes with a selfadjoint operator Γ on $\mathcal{H}$ which squares to the identity operator $\mathbb{I}$, anticommutes with D and commutes with the algebra. A spectral triple is real [17] if in addition there is an antilinear operator J on $\mathcal{H}$ satisfying

$$J^2 = \epsilon \mathbb{I}, \quad JD = \epsilon' DJ, \quad J\Gamma = \epsilon'' \Gamma J \quad (1)$$

where $\epsilon, \epsilon', \epsilon'' = \pm 1$ define the KO -dimension $k \in [0, 7]$. This real structure implements a map $a \rightarrow a^\circ := Ja^*J^{-1}$ from $\mathcal{A}$ to the opposite algebra $\mathcal{A}^\circ$. This yields a right action of $\mathcal{A}$ on $\mathcal{H}$, $\psi a := a^\circ \psi$, which is asked to commute with the left action

$$[a, Jb^*J^{-1}] = 0 \quad \forall a \in \mathcal{A} \quad (\text{order zero condition}). \quad (2)$$

There is also a first-order condition [18],

$$[[D, a], Jb^*J^{-1}] = 0 \quad \forall a, b \in \mathcal{A} \quad (3)$$

which is there to guarantee that the operator D be a first-order differential operator.

All these properties are satisfied by the triple

$$(C^\infty(\mathcal{M}), L^2(\mathcal{M}, S), \not{D}) \quad (4)$$

where $C^\infty(\mathcal{M})$ is the (commutative) algebra of smooth functions on a closed Riemannian manifold $\mathcal{M}$ of dimension m , acting by multiplication on the Hilbert space $L^2(\mathcal{M}, S)$ of square-integrable spinors on $\mathcal{M}$, and

$$\not{D} = -i \sum_{\mu=1}^m \gamma^\mu (\partial_\mu + \omega_\mu), \quad \text{with} \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbb{I} \quad (5)$$

is the Dirac operator (ω^μ is the spin connection, γ^μ the Dirac matrices and $g_{\mu\nu}$ the Riemannian metric on $\mathcal{M}$, all given in local coordinates). For m even, this spectral triple has grading the product of the Dirac matrices (in dimension 4, the matrix γ^5) and real structure $\mathcal{J}$ the charge conjugation operator. Adding other conditions [20] (which are satisfied by the triple (4)), one gets *Connes' reconstruction theorem*, that extends Gelfand duality (between compact topological spaces and C^* -commutative algebras) beyond topology. Namely, given any real spectral triple $(\mathcal{A}, \mathcal{H}, D)$ satisfying these conditions, with $\mathcal{A}$ commutative, then there exists a closed Riemannian manifold $\mathcal{M}$ such that $\mathcal{A} \simeq C^\infty(\mathcal{M})$.

A *noncommutative geometry* is then defined as a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is non (necessarily) commutative. This gives access to new geometrical objects, that can be intended as 'spaces' whose algebra of functions $\mathcal{A}$ is not commutative.

2.2. Connection

Take a gauge theory with gauge group G . From a mathematical point of view, the fermionic fields form sections of a G -bundle $\mathcal{E}$ over the spacetime $\mathcal{M}$, while the bosonic fields are described as connections on $\mathcal{E}$.

In noncommutative geometry the spacetime $\mathcal{M}$ is substituted by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ plays the role of ‘algebra of functions’ on the noncommutative space. To understand what plays the role of a gauge bundle, recall that the set of sections of any bundle on a manifold $\mathcal{M}$ forms a finite projective $C^\infty(\mathcal{M})$ -module. Conversely, by Serre–Swan theorem, any such module actually is the module of sections of a bundle on $\mathcal{M}$. That is why, in noncommutative geometry, the role of gauge bundles is played by finite projective $\mathcal{A}$ -modules $\mathcal{E}$.

Connections on these modules are, by definition, objects associated with a derivation. Recall that a derivation δ on an algebra $\mathcal{A}$ is a map from $\mathcal{A}$ to some $\mathcal{A}$ -bimodule Ω satisfying the Leibniz rule

$$\delta(ab) = a\delta(b) + \delta(a)b \quad \forall a, b \in \mathcal{A}. \quad (6)$$

A connection on a (right) $\mathcal{A}$ -module $\mathcal{E}$ associated with δ is a map from $\mathcal{E}$ to $\mathcal{E} \otimes_{\mathcal{A}} \Omega$ such that the following Leibniz rule holds,

$$\nabla(\eta a) = \nabla(\eta)a + \eta \otimes \delta(a) \quad \forall \eta \in \mathcal{E}, a \in \mathcal{A}, \quad (7)$$

where

$$\Omega = \left\{ \sum_i a_i \delta(b_i), a_i, b_i \in \mathcal{A} \right\} \quad (8)$$

is the $\mathcal{A}$ -bimodule generated by the derivation δ , while $\nabla(\eta)a$ is a shorthand notation for $\eta_{(0)}a \otimes \eta_{(1)}$, using Sweedler notations $\nabla\eta = \eta_{(0)} \otimes \eta_{(1)}$ with $\eta_{(0)} \in \mathcal{E}$ and $\eta_{(1)} \in \Omega$.

Example: The exterior derivative d is a derivation on the algebra $C^\infty(\mathcal{M})$. It generates the $C^\infty(\mathcal{M})$ -bimodule of section s of the cotangent bundle,

$$\Omega^1(\mathcal{M}) := \left\{ \sum_i f_i dg_i \text{ with } f_i, g_i \in C^\infty(\mathcal{M}) \right\}. \quad (9)$$

A connection on the tangent bundle TM associated with d is a map

$$\nabla : \Gamma^\infty(TM) \rightarrow \Gamma^\infty(TM) \otimes \Omega^1(\mathcal{M}), \quad (10)$$

$$\partial_\nu \mapsto \Gamma_{\mu\nu}^\rho \partial_\rho \otimes dx^\mu, \quad (11)$$

where $\Gamma^\infty(TM)$ denotes the set of smooth sections of TM . One retrieves the usual notion of connection, as a map from $\Gamma^\infty(TM) \times \Gamma^\infty(TM)$ to $\Gamma^\infty(TM)$ as

$$\nabla : (\partial_\eta, \partial_\nu) \mapsto \nabla_\eta \partial_\nu := \Gamma_{\mu\nu}^\rho \partial_\rho \otimes_{C^\infty(\mathcal{M})} \langle dx^\mu, \partial_\eta \rangle \simeq \langle dx^\mu, \partial_\eta \rangle \Gamma_{\mu\nu}^\rho \partial_\rho = \Gamma_{\eta\nu}^\rho \partial_\rho,$$

where $\langle \cdot, \cdot \rangle$ is the $C^\infty(\mathcal{M})$ -valued dual product between the cotangent and the tangent bundles and the last equation is the isomorphism between $\mathcal{E} \otimes_{C^\infty(\mathcal{M})} C^\infty(\mathcal{M})$ and $\mathcal{E}$.

2.3. Fluctuation of the metric

To understand when two algebras are ‘similar’, a relevant notion is *Morita equivalence*. This is more flexible than isomorphism of algebras for, roughly speaking, two algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if they have the same representation theory. Technically, it means that

there exists an Hermitian finite projective $\mathcal{A}$ -module $\mathcal{E}$ such that $\mathcal{B}$ is isomorphic to the algebra $\text{end}_{\mathcal{A}}(\mathcal{E})$ of $\mathcal{A}$ -linear, adjointable, endomorphisms of $\mathcal{E}$ (for details see e.g. [50] or [40]).

The idea of fluctuation of the metric comes from the following question: how does one export a real spectral triple $(\mathcal{A}, \mathcal{H}, D)$ to a Morita equivalent algebra $\mathcal{B}$? We recall the construction of [18], whose details may be found in [23] and even more details in [42].

Assume $\mathcal{E} = \mathcal{E}_R$ is a right $\mathcal{A}$ -module. The algebra $\mathcal{B}$ acts on $\mathcal{H}_R := \mathcal{E}_R \otimes_{\mathcal{A}} \mathcal{H}$ as

$$b(\eta \otimes \psi) = b\eta \otimes \psi \quad \forall b \in \mathcal{B}, \eta \in \mathcal{E}_R, \psi \in \mathcal{H}. \quad (12)$$

However, the ‘natural’ action of D on $\mathcal{H}_R$,

$$D_R(\eta \otimes \psi) := \eta \otimes D\psi, \quad (13)$$

is not compatible with the tensor product, for

$$D_R(\eta a \otimes \psi) - D_R(\eta \otimes a\psi) = -\eta \otimes [D, a]\psi \quad (14)$$

has no reason to vanish. This is cured by equipping $\mathcal{E}_R$ with a connection ∇ with value in the $\mathcal{A}$ -bimodule of generalised 1-forms

$$\Omega_D^1(\mathcal{A}) := \left\{ \sum_i a_i [D, b_i], a_i, b_i \in \mathcal{A} \right\} \quad (15)$$

generated by the derivation $\delta(\cdot) = [D, \cdot]$. Indeed, the following operator,

$$D_R(\eta \otimes \psi) := \eta \otimes D\psi + (\nabla \eta)\psi \quad (16)$$

is well defined on $\mathcal{H}_R$, and selfadjoint as soon as ∇ is an Hermitian connection. Moreover one checks that the commutator $[D_R, b]$ is bounded for any $b \in \mathcal{B}$, so that $(\mathcal{B}, \mathcal{H}_R, D_R)$ is a spectral triple. In particular, if one considers self-Morita equivalence, that is $\mathcal{B} = \mathcal{E}_R = \mathcal{A}$, then the operator (16) with ∇ Hermitian reads

$$D_R = D + A_R \quad (17)$$

with $A_R = A_R^* \in \Omega_D^1(\mathcal{A})$ a selfadjoint generalised 1-form.

A similar construction holds if one implements Morita equivalence with a left module $\mathcal{E}_L$. Then $\mathcal{H}_L = \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}_L$ is a Hilbert space and the operator

$$D_L(\psi \otimes \eta) := D\psi \otimes \eta + (\nabla^\circ \eta)\psi \quad (18)$$

with ∇° a connection with value in the bimodule

$$\Omega_D^1(\mathcal{A}^\circ) = \left\{ \sum_i a_i^\circ [D, b_i^\circ], a_i^\circ, b_i^\circ \in \mathcal{A}^\circ \right\}$$

is well defined on $\mathcal{H}_L$. For ∇° Hermitian, one obtains a spectral triple $(\mathcal{B}, \mathcal{H}_L, D_L)$. For self-Morita equivalence, one gets

$$D_L = D + A^\circ = D + \epsilon' J A_L J^{-1} \quad (19)$$

with $A^\circ \in \Omega_D^1(\mathcal{A}^\circ)$ and $A_L \in \Omega_D^1(\mathcal{A})$.

To take into account the real structure, one combines the two constructions, using an $\mathcal{A}$ -bimodule $\mathcal{E}$ to implement Morita equivalence. For self-Morita equivalence, one then obtains the operator $D' = D + A_R + \epsilon' J A_L J^{-1}$ acting on $\mathcal{H}$. Requiring this operator to be selfadjoint and J to be a real structure amounts to the existence of a generalised selfadjoint 1-form $A \in \Omega_D^1(\mathcal{A})$ such that

$$D' = D_A := D + A + \epsilon' J A J^{-1}. \quad (20)$$

Then $(\mathcal{A}, \mathcal{H}, D_A)$ is a real spectral triple. The operator D_A is called a covariant Dirac operator, and the substitution of D with a D_A is a *fluctuation of the metric*. The name is motivated by the existing relation between the Dirac operator and the metric. This relation is obvious on a spin manifold, from the very definition of the Dirac matrices ($\gamma^\nu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$), and it still makes sense for an arbitrary noncommutative geometry, via the definition of the spectral distance [22]. On a manifold, this distance gives back the geodesic distance associated with the Riemannian structure of $\mathcal{M}$, while on an arbitrary spectral triple it may be seen as a generalisation of the Wasserstein distance of order 1 in the theory of optimal transport (cf [28, 46] and references therein). By exporting a noncommutative geometry to a Morita equivalent algebra, one passes from D to the covariant operator D_A and modifies accordingly the spectral distance. In particular, for the Standard Model, such a fluctuation provides a purely metric interpretation to the Higgs field (which is one of the components of the generalised 1-form A , see below) [18, 48]. The metric interpretation of the other components of A has been worked out in [44, 48], in relation with the Carnot–Carathéodory distance in sub-Riemannian geometry.

2.4. Gauge transformation

A gauge transformation is a change of connection on the Morita-equivalence bimodule $\mathcal{E}$. In case of self-Morita equivalence, it is implemented by the conjugate action on $\mathcal{H}$ of the group $U(\mathcal{A})$ of unitaries element of $\mathcal{A}$ (i.e. $u \in \mathcal{A}$ such that $u^* u = uu^* = \mathbb{I}$):

$$\text{Ad}(u) : \psi \rightarrow u\psi u^* = u(u^*)^\circ \psi = uJuJ^{-1}\psi \quad \forall \psi \in \mathcal{H}. \quad (21)$$

This action maps the covariant Dirac operator D_A to

$$\text{Ad}(u) D_A \text{Ad}(u)^{-1} \quad (22)$$

and one checks that this operator coincides with the operator D_{A^u} , defined as in (20) with

$$A^u := u[D, u^*] + uAu^*. \quad (23)$$

This is the formula of transformation of the gauge potential in noncommutative geometry, which generalises the usual one of gauge theories.

2.5. Standard Model

The spectral triple of the Standard Model [12] is the product

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes A_F, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes H_F, \quad D = \not{D} \otimes \mathbb{I}_{96} + \gamma^5 \otimes D_F \quad (24)$$

of the spectral triple (4) of a 4-dimensional Riemannian closed spin manifold $\mathcal{M}$ with a finite dimensional spectral triple

$$A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad H_F = \mathbb{C}^{96}, \quad D_F = \underbrace{\begin{pmatrix} D_0 & 0_{48} \\ 0_{48} & D_0^\dagger \end{pmatrix}}_{D_Y} + \underbrace{\begin{pmatrix} 0_{48} & D_R \\ D_R^\dagger & 0_{48} \end{pmatrix}}_{D_M} \quad (25)$$

where $\mathbb{H}$ is the algebra of quaternions and $M_3(\mathbb{C})$ the algebra of complex 3×3 matrices.

The dimension of H_F is the number of fermions in the Standard Model (including right-handed neutrinos). Its entries are labelled by a multi-index $CI \propto n$ where

- $C = 0, 1$ labels particles ($C = 0$) or anti-particles ($C = 1$);
- $I = 0, i$ with $i = 1, 2, 3$ is the lepto-colour index: it takes value $I = 0$ for a lepton and $I = 1, 2, 3$ for a quark with its three possible colours;
- $\alpha = \dot{1}, \dot{2}, 1, 2$ is the flavour index (with dot indicating the chirality):

$$\dot{1} = \nu_R, \dot{2} = e_R, 1 = \nu_L, 2 = e_L \text{ for leptons } (I = 0), \quad (26)$$

$$\dot{1} = u_R, \dot{2} = d_R, 1 = q_L, 2 = d_L \text{ for quarks } (I = i); \quad (27)$$

- $n = 1, 2, 3$ is the generation index.

The details of the representation of A_F is in [12]. The important point for our matter is that the quaternions act only on the particle subspace of H_F ($C = 0$), trivially on the lepto-colour index I , and through their fundamental representation on the last two flavour indices α ; whereas $M_3(\mathbb{C})$ acts only on antiparticle subspace of H_F ($C = 1$), trivially on the flavour index α and through their fundamental representation on the lepto-colour index i . The algebra $\mathbb{C}$ acts both on particles together with the quaternions (but on the first two flavour indices), and on anti-particles together with $M_3(\mathbb{C})$ (on $I = 0$).

The grading of the finite dimensional spectral triple is the 96×96 matrix Γ_F with entries $+1$ on left particles/right antiparticles, -1 on right particles/left antiparticles. The real structure is the matrix J_F that exchanges particles with antiparticles. The spectral triple (24) is real, with grading $\Gamma = \gamma^5 \otimes \Gamma_F$ and real structure $J = \mathcal{J} \otimes J_F$.

In the particles/antiparticles indices, the Dirac operator D_F of the finite dimensional spectral triple is the sum of a block diagonal matrix D_Y which contains the Yukawa couplings of the fermions, the Cabibbo–Kobayashi–Maskawa mixing matrix for the quarks and the Pontecorvo–Maki–Nakagawa–Sakata mixing matrix for the left-handed neutrinos, and a block off-diagonal matrix D_M which contains the Majorana masses k_R^n , $n = 1, 2, 3$ of the right-handed neutrinos and the corresponding mixing matrix (notations are those of [36], they differ from the ones of [32] and [34]).

The generalised 1-forms (15) for a product of spectral triples (24) decompose as [25]

$$A = \gamma^5 \otimes H - i \sum_{\mu} \gamma^{\mu} \otimes A_{\mu} \quad (28)$$

where H is a scalar field on $\mathcal{M}$ with values in A_F , while A_{μ} is a 1-form field on $\mathcal{M}$ with values in the Lie algebra of the group $U(A_F)$ of unitary elements of A_F (differently said: a connection 1-form on a $U(A_F)$ -bundle on $T\mathcal{M}$). In particular, for the spectral triple of the Standard Model, one has

$$U(A_F) = U(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})) \simeq U(1) \times SU(2) \times U(3), \quad (29)$$

which is reduced to the gauge group $U(1) \times SU(2) \times SU(3)$ of the Standard Model by imposing a unimodularity condition (which also guarantees that the model is anomaly free, see e.g [12, section 2.5]).

The action of this group on $\mathcal{H}$ is a matrix whose components are the hypercharges of the fermions of the Standard Model [12, proposition 2.16]. This allows to identify the basis elements of H_F with the 96 fermions of the Standard Model, and the corresponding elements in $\mathcal{H}$ with the fermionic fields. Moreover, the action of $A + JAJ^{-1}$ corresponds to the bosonic hypercharges, and allows to identify the components of A_{μ} with the bosonic fields of the

Standard Model [12, proposition 3.9]. One also checks that (23) yields the expected gauge transformation.

The interpretation of the scalar field H follows from the computation of the *spectral action* [8, 9], namely the asymptotic expansion $\Lambda \rightarrow \infty$ of $\text{Tr } f(\frac{D_A^2}{\Lambda^2})$ where f is a smooth approximation of the characteristic function of the interval $[0, 1]$. One obtains the bosonic Lagrangian of the Standard Model coupled with Einstein–Hilbert action in Euclidean signature, where H is the Higgs field. The coupling constants of the electroweak and strong interactions satisfy the relation expected in grand unified theories, and are related to the value at 0 of the function f .

The spectral action provides some relations between the parameters of the Standard Model at a putative unification scale. The physical predictions are obtained by running down the parameters of the theory under the renormalisation group equation, taking these relations as initial conditions. Assuming there is no new physics between the unification scale and the electroweak scale, one finds a value for the Higgs mass around 170 GeV, in disagreement with the measured value 125,1 GeV.

However, for a Higgs boson with mass $m_H \leq 130$ GeV, the quartic coupling λ of the Higgs field becomes negative at high energy, meaning the electroweak vacuum is meta-stable rather than stable [29]. This instability can be cured by a new scalar field σ which couples to the Higgs field. In the spectral description of the Standard Model, such a field is obtained by turning into a field the neutrino Majorana mass k_R which appears in the off-diagonal part D_R of the finite dimensional Dirac operator D_F :

$$k_R \rightarrow k_R \sigma,$$

Furthermore, by altering the running of the parameters under the equations of the group of renormalisation, this extra scalar field makes the computation of the mass of the Higgs boson compatible with its experimental value [11].

3. Grand algebra beyond the Standard Model

The point in the above is to justify the turning of the constant k_R into a field $k_R \sigma$. This cannot be obtained by fluctuation of the metric, for one checks that

$$[\gamma^5 \otimes D_M, a] = 0 \quad \forall a, b \in \mathcal{A} = C^\infty(\mathcal{M}) \otimes A_F. \quad (30)$$

In other terms, the constant k_R is transparent under fluctuation. The solution proposed in [14] is to remove the first-order condition. This gives more flexibility, and permits to obtain the extra scalar field as a *fluctuation without the first-order condition*. The latter is retrieved dynamically, by minimising the spectral action [13]. In this way the field σ is the ‘Higgs’ boson associated with the breaking of the first-order condition.

3.1. Grand algebra

At the same time, an alternative process was described in [32] where one mixes the internal degrees of freedom per generation of the finite dimensional Hilbert space H_F , that is $\mathcal{H}_F \simeq \mathbb{C}^{32}$, with the four spinorial degrees of freedom of $L^2(\mathcal{M}, S)$. This provides exactly the $4 \times 32 = 128$ degrees of freedom required to represent the ‘second next algebra’ in the classification of finite dimensional spectral triples made in [10, 19].

In this classification, the smallest algebra— $\mathbb{H} \oplus M_2(\mathbb{C})$ —is too small to accommodate the Standard Model; the second smallest one— $\mathcal{A}_{\text{SM}} = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ —reduces to the left–right algebra $\mathcal{A}_{\text{LR}} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_4(\mathbb{C})$ by imposing the grading condition, which breaks to the

algebra $\mathcal{A}_F$ of the Standard Model by the first-order condition. The next one is $M_3(\mathbb{H}) \oplus M_6(\mathbb{C})$ and has not found any physical interpretation so far. Then comes the *grand algebra* [32]

$$\mathcal{A}_G = M_4(\mathbb{H}) \oplus M_8(\mathbb{C}). \quad (31)$$

It is too big to be represented on the Hilbert space $\mathcal{H}_F$ in a way compatible with the axioms of noncommutative geometry: the latter require a space of dimension $d = 2(2a)^2$, where a is the dimension of the quaternionic matrix algebra. For $\mathcal{A}_{\text{SM}}$ one has $a = 2$, which corresponds to $d = 2(2 \cdot 2)^2 = 32$, that is the dimension of $\mathcal{H}_F$. For the grand algebra $\mathcal{A}_G$, $a = 4$ and one needs a space four times bigger.

This bigger space is obtained by allowing $C^\infty(\mathcal{M})$ to act independently on the left and right components of spinors. This permits to represent on $L^2(\mathcal{M}, S) \otimes \mathcal{H}_F$ the algebra $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ —viewed as functions on $\mathcal{M}$ with value in $\mathcal{A}_G$ —in such a way that for any $a \in C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ and $x \in \mathcal{M}$, then $a(x) \in \mathcal{A}_G$ acts on $\mathcal{H}_F$ in agreement with the classification of [10].

Within the tensorial notation of section 2.5, the components $M_4(\mathbb{H})$ and $M_8(\mathbb{C})$ of the grand algebra are 2×2 matrices Q, M with entries in $M_2(\mathbb{H})$ and $M_4(\mathbb{C})$ that act on $\mathcal{H}_F$ as $\mathcal{A}_{\text{SM}}$. The difference with the spectral triple of the Standard Model is that, once tensorised by $C^\infty(\mathcal{M})$, the 2×2 matrices Q, M have a non-trivial action on the spinorial degrees of freedom of $L^2(\mathcal{M}, S)$. We denote the latter by two indices: $s = l, r$ for the left/right components of spinors; $\dot{s} = \dot{0}, \dot{1}$ for the particle/antiparticle subspaces.

In [32] one makes $C^\infty(\mathcal{M}) \otimes M_4(\mathbb{H}) \ni Q$, resp. $C^\infty(\mathcal{M}) \otimes M_8(\mathbb{C}) \ni M$, act non trivially on the $\dot{s}$, resp s , index. Omitting all the indices on which the action is trivial,

$$Q = \begin{pmatrix} Q_{\dot{0}\alpha}^{\dot{0}\beta} & Q_{\dot{0}\alpha}^{\dot{1}\beta} \\ Q_{\dot{1}\alpha}^{\dot{0}\beta} & Q_{\dot{1}\alpha}^{\dot{1}\beta} \end{pmatrix}_{\dot{s}t}, \quad M = \begin{pmatrix} M_{rl}^{rJ} & M_{rl}^{lJ} \\ M_{ll}^{rJ} & M_{ll}^{lJ} \end{pmatrix}_{st}, \quad (32)$$

where β, J, t and $\dot{t}$ are summation indices within the same range as α, l, s, t (the indices after the closing parenthesis are those labelling the matrix entries).

Since γ^5 acts non trivially on the spinorial chiral index, the grading condition forces M to be diagonal in the st indices: $M_{rl}^{lJ} = M_{lr}^{rJ} = 0$. Since Γ_F is non trivial only in the flavour index α , in which the remaining entries $M_{ll}^{lJ}, M_{rr}^{rJ} \in M_4(\mathbb{C})$ act trivially, the grading does not induce any further breaking in the complex sector. On the contrary, since γ^5 is trivial in the $\dot{s}$ index but quaternions act non trivially on the α index, the grading forces Q to be diagonal in the flavour index, with components $Q_{L\dot{s}}^{\dot{0}}, Q_{R\dot{s}}^{\dot{1}} \in C^\infty(\mathcal{M}) \otimes M_2(\mathbb{H})$ acting on the left/right subspaces of the internal Hilbert space $\mathcal{H}_F$. In other terms, the grading condition breaks the grand algebra in

$$\mathcal{A}'_G = (M_2(\mathbb{H})_L \oplus M_2(\mathbb{H})_R) \oplus (M_4(\mathbb{C})_l \oplus M_4(\mathbb{C})_r). \quad (33)$$

To guarantee the first-order condition for the Majorana component $\gamma^5 \otimes D_R$ of the Dirac operator, a solution is to further break $\mathcal{A}'_G$ to

$$\mathcal{A}''_G = (\mathbb{H}_L \oplus \mathbb{H}'_L \oplus \mathbb{C}_R \oplus \mathbb{C}'_R) \oplus (\mathbb{C}_l \oplus M_3(\mathbb{C})_l \oplus \mathbb{C}_r \oplus M_3(\mathbb{C})_r) \quad (34)$$

with $\mathbb{C}_R = \mathbb{C}_r = \mathbb{C}_l$. In the first term, the unprimed algebras act on the particle subspace $\dot{s} = \dot{0}$, while the primed ones act on the antiparticle subspace $\dot{s} = \dot{1}$. A fluctuation of the metric of $\gamma^5 \otimes D_R$ then yields an extra scalar field σ , which lives in the difference between $\mathbb{C}_R$ and $\mathbb{C}'_R$, and fixes the Higgs mass as expected [33]. In this grand algebra model, the fermionic content is not altered, since the total Hilbert space $\mathcal{H}$ is untouched. One also checks the order zero condition.

The first-order condition for the free part $\not{D} \otimes \mathbb{I}$ of the Dirac operator forces the components acting on the chiral spinorial index to be equal, as well as those acting on the particle/antiparticle index, meaning $\mathbb{H}'_L = \mathbb{H}_L$, $\mathbb{C}'_R = \mathbb{C}_R$ and $M_3(\mathbb{C})_l = M_3(\mathbb{C})_r$. Thus $\mathcal{A}''_G$ reduces to

$\mathbb{H}_L \oplus \mathbb{C}_R \oplus M_3(\mathbb{C})$, namely the algebra of the Standard Model. The field σ thus appears as the Higgs field related to the breaking of the first-order condition by $\not{D} \otimes \mathbb{I}$, whereas in [14] it is related with the first-order condition for $\gamma^5 \otimes D_R$. By enlarging the algebra, one has moved the symmetry breaking from the internal space to the manifold.

However, the price to pay for a non trivial action on spinors is the unboundedness of the commutator of $\not{D} \otimes \mathbb{I}$ with the grand algebra: for $a = f \otimes m \in C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ one has

$$[\not{D} \otimes \mathbb{I}, a] = [\not{D}, f] \otimes m = -i[\gamma^\mu \partial_\mu, f] \otimes m - i[\gamma^\mu \omega_\mu, f] \otimes m. \quad (35)$$

The second term is always bounded. By the Leibniz rule, the first one is

$$-i[\gamma^\mu, f] \partial_\mu - i\gamma^\mu (\partial_\mu f), \quad (36)$$

which is bounded iff the component ∂_μ vanishes. Since the only matrix that commutes with all the Dirac matrices is the identity matrix, the commutator (35) is bounded if and only if f acts on $L^2(\mathcal{M}, S)$ as a multiple of the identity matrix, that is on the same way on the left and right components of spinors.

3.2. Twisted spectral triples

Mixing the spinorial and internal degrees of freedom of the Hilbert space $\mathcal{H}$ —in order to represent an algebra bigger than the one of the Standard Model—turns out to be incompatible with the very definition of spectral triple. As explained at the end of the preceding section, this does not depend on the details of the representation: as soon as the grand algebra acts non trivially on spinors, then the commutator with the free part of the Dirac operator is unbounded [45], no matter if the representation is (32) or not.

The unboundedness of the commutator is the kind of problems addressed by Connes and Moscovici when they define *twisted spectral triples* in [24]. Their motivation had little to do with physics, but were purely mathematical (building spectral triples with type III algebras). Given a triple $(\mathcal{A}, \mathcal{H}, D)$, instead of asking the commutators $[D, a]$ to be bounded, one asks the boundedness of the twisted commutators

$$[D, a]_\rho := Da - \rho(a)D \quad (37)$$

for some fixed automorphism $\rho \in \text{Aut}(\mathcal{A})$.

The whole process of fluctuation of the metric has been adapted to the twisted case in [41, 42]. One obtains the covariant Dirac operator

$$D_{A_\rho} := D + A_\rho + JA_\rho J^{-1} \quad (38)$$

where A_ρ is an element of the set of twisted 1-forms

$$\Omega_D^1(\mathcal{A}, \rho) := \left\{ \sum_i a_i [D, Jb_i^* J^{-1}]_{\rho^\circ}, a_i, b_i \in \mathcal{A} \right\} \quad (39)$$

with $\rho^\circ := \rho(a^*)^\circ$ is the automorphism of the opposite algebra $\mathcal{A}^\circ$ induced by ρ . There is also twisted version of the first-order condition [34, 41]

$$[[D, a]_\rho, Jb^* J^{-1}]_{\rho^\circ} = 0 \quad \forall a, b \in \mathcal{A}. \quad (40)$$

A gauge transformation is implemented by the twisted action of the operator Adu (22),

$$\rho(\text{Adu}) D_{A_\rho} \text{Adu}^{-1}, \quad (41)$$

with $\rho(Adu) := \rho(u)J\rho(u)J^{-1}$. Such a transformation maps D_{A_ρ} to $D_{A_\rho^u}$ where

$$A_\rho^u = \rho(u)[D, u^*]_\rho + \rho(u)A^\rho u^*. \quad (42)$$

This is the twisted version of the gauge transformation (23).

3.3. Twisting the grand algebra

To resolve the unboundedness of the commutator arising in the grand algebra model, the idea is to find an automorphism of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ such that the twisted commutator (37) of any element $(Q, M) \in C^\infty(\mathcal{M}) \otimes \mathcal{A}_G$ with $\not{D} \otimes \mathbb{I}$ be bounded. This must be true in particular for $(Q, 0)$ and $(0, M)$. Repeating the computation (35) and (36), and taking into account only the spinorial indices $s, \dot{s}$ (since $\not{D} \otimes \mathbb{I}$ acts as the identity on all the other indices, the corresponding sector of the algebra must be invariant under the automorphism, for $\mathbb{I}a - \rho(a)\mathbb{I} = 0$ iff $a = \rho(a)$), one finds that ρ should be such that

$$\gamma^\mu Q - \rho(Q)\gamma^\mu = 0 \text{ and } \gamma^\mu M - \rho(M)\gamma^\mu = 0 \quad \forall \mu = 1, \dots, \dim \mathcal{M} \quad (43)$$

for any $Q \in M_4(\mathbb{H}) \otimes C^\infty(\mathcal{M})$ and $M \in M_8(\mathbb{C}) \otimes C^\infty(\mathcal{M})$. By easy computation, using the explicit form of the γ matrices in the chiral basis,

$$\gamma^\mu = \begin{pmatrix} 0_2 & \sigma^\mu \\ \bar{\sigma}^\mu & 0_2 \end{pmatrix}_{st} \quad \sigma^\mu = \{\mathbb{I}, \sigma^i\}, \bar{\sigma}^\mu = \{\mathbb{I}, i\sigma^i\}, \quad (44)$$

where σ^i are the Pauli matrices, one checks that any two 4×4 complex matrices A, B such that $A\gamma^\mu = \gamma^\mu B$ for any μ are necessarily of the form

$$A = \begin{pmatrix} \lambda \mathbb{I}_2 & 0_2 \\ 0_2 & \lambda' \mathbb{I}_2 \end{pmatrix} \quad B = \begin{pmatrix} \lambda' \mathbb{I}_2 & 0_2 \\ 0_2 & \lambda \mathbb{I}_2 \end{pmatrix} \quad \text{for some } \lambda, \lambda' \in \mathbb{C}. \quad (45)$$

Thus (43) implies that both M and Q must be trivial in the $\dot{s}$ index, diagonal in the chiral index s , with ρ the automorphism that exchanges the left and right components. Therefore the representation (32) of the grand algebra is not suitable to build a twisted spectral triple.

In order to find a good representation, remember that the field σ has its origin in the two copies $\mathbb{C}_R, \mathbb{C}'_R$ of $\mathbb{C}$ in $\mathcal{A}'_G$ (34), which come from the non-trivial action of $C^\infty(\mathcal{M}) \otimes M_4(\mathbb{H})$ on the $\dot{s}$ index. Since the latter is no longer allowed, it seems natural to make $C^\infty(\mathcal{M}) \otimes M_4(\mathbb{H})$ act non trivially on the chiral index s . On the contrary, the complex sector plays no obvious role in the generation of the field σ , so one lets $C^\infty(\mathcal{M}) \otimes M_8(\mathbb{C})$ act trivially on both the $s, \dot{s}$ indices. On the other indices, the action of $M_4(\mathbb{H}), M_8(\mathbb{C})$ is as in the Standard Model. The grading condition now breaks $M_4(\mathbb{H})$ to $\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r$ but leaves $M_8(\mathbb{C})$ untouched. Reducing the latter ‘by hand’ to $M_4(\mathbb{C})$, one gets the algebra [34]

$$\mathcal{B}' = \mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r \oplus M_4(\mathbb{C}). \quad (46)$$

Let ρ be the automorphism of $C^\infty(\mathcal{M}) \otimes \mathcal{B}'$ that flips the chiral spinorial degrees of freedom,

$$\rho(q_L^l, q_L^r, q_R^l, q_R^r, m) := (q_L^r, q_L^l, q_R^r, q_R^l, m) \quad (47)$$

where each of the q is a quaternionic function with value in its respective copy of $\mathbb{H}$ and $m \in C^\infty(\mathcal{M}) \otimes M_4(\mathbb{C})$. Then

$$(C^\infty(\mathcal{M}) \otimes \mathcal{B}', L^2(\mathcal{M}, S) \otimes \mathbb{C}^{32}, \not{D} \otimes \mathbb{I}) \quad (48)$$

is a twisted spectral triple which satisfies the first-order condition [34, proposition 3.4].

Regarding the Majorana Dirac operator, let us consider the subalgebra of $\mathcal{B}'$

$$\tilde{\mathcal{B}} = \mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{C}_R^l \oplus \mathbb{C}_R^r \oplus (\mathbb{C} \oplus M_3(\mathbb{C})). \quad (49)$$

Given two of its elements $(q_L^l, q_L^r, c_R^l, c_R^r, c, m)$, $(r_L^l, r_L^r, d_R^l, d_R^r, d, n)$ with $c, d, c_R^l, c_R^r, d_R^l, d_R^r$ complex functions, $q_L^l, q_L^r, r_L^l, r_L^r$ quaternionic functions and m, n functions with values in $M_3(\mathbb{C})$, denoting π' the representation of $\mathcal{B}'$ in the spectral triple (48), one finds that

$$[\gamma^5 \otimes D_R, \pi'(q_L^l, q_L^r, c_R^l, c_R^r, c, m)]_\rho, \pi'(r_L^l, r_L^r, d_R^l, d_R^r, d, n)]_\rho \quad (50)$$

vanishes as soon as $c = c_R^l$ and $d = d_R^l$ (or $c = c_R^r$ and $d = d_R^r$).

In [34], this was improperly interpreted as a breaking of $\mathcal{B}'$ to

$$\mathcal{B} = \mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{C}_R^l \oplus \mathbb{C}_R^r \oplus M_3(\mathbb{C}). \quad (51)$$

acting as $\tilde{\mathcal{B}}$ with $\mathbb{C} = \mathbb{C}_R^l$, namely the representation π of $\mathcal{B}$ is

$$\pi(q_L^l, q_L^r, c_R^l, c_R^r, m) := \pi'(q_L^l, q_L^r, c_R^l, c_R^r, c_R^l, m). \quad (52)$$

But ρ exchanges the left/right components in the quaternionic sector only, so that

$$\pi'(\rho(q_L^l, q_L^r, c_R^l, c_R^r, c_R^l, m)) = \pi'(q_L^r, q_L^l, c_R^r, c_R^l, c_R^l, m) \quad (53)$$

is not the representation (52) of any element in $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ (the latter requires the identification of the first and third complex functions, whereas in (53) the second and third are identified), unless $c_R^r = c_R^l$. This means that the breaking from $\mathcal{B}'$ to $\mathcal{B}$ is not compatible with the twist unless $\mathbb{C} = \mathbb{C}_R^l$ identifies with $\mathbb{C}_R^r$. In that case, $\mathcal{B}'$ actually breaks to $\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{C} \oplus M_3(\mathbb{C})$. This algebra contains only one copy of $\mathbb{C}$ and so does not generate the field σ by twisted fluctuation of $\gamma^5 \otimes D_R$.

In other terms, the model developed in [34] does not allow to generate the extra scalar field while preserving the first-order condition (even in a twisted form), as opposed to what was claimed. The error is due to not noticing that the reduction from $\tilde{\mathcal{B}}$ to $\mathcal{B}$, imposed by the twisted first-order condition of the Majorana Dirac operator, is not invariant under the twist. So it does not make sense to try to build a spectral triple with $C^\infty(\mathcal{M}) \otimes \mathcal{B}$.

Nevertheless all the expressions computed in [34] of the form

$$T\pi'(a) - \pi'(\rho(a))T \quad (54)$$

for $T = \not{D} \otimes \mathbb{I}$ or $\gamma^5 \otimes D_R$ are algebraically correct. The point is that they are twisted commutators (37) for a in $C^\infty(\mathcal{M}) \otimes \tilde{\mathcal{B}}$, but not for a in $C^\infty(\mathcal{M}) \otimes \mathcal{B}$. Indeed, although (53) does define a representation of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$,

$$\hat{\pi}(q_L^l, q_L^r, c_R^l, c_R^r, m) := \pi'(q_L^r, q_L^l, c_R^r, c_R^l, c_R^l, m), \quad (55)$$

there is no automorphism η of $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ such that $\hat{\pi}$ would equal $\pi \circ \eta$. What the results of [34] show is that starting with the twisted spectral triple

$$(C^\infty(\mathcal{M}) \otimes \tilde{\mathcal{B}}, L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \not{D} \otimes \mathbb{I} + \gamma^5 \otimes D_F), \quad (56)$$

whose Majorana part violates the twisted first-order condition, then a twisted fluctuation of the Dirac operator by the subalgebra $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ yields the field σ . Minimising the spectral action (suitably generalised to the twisted case) breaks the algebra to the one of the Standard Model, which satisfies the first-order condition.

As noticed at the end of [41], an alternative way to interpret (54) for a in $C^\infty(\mathcal{M}) \otimes \mathcal{B}$ is to view it as a twisted commutator for the represented algebra. Namely defining the inner automorphism $\alpha_U(\mathcal{B}) := UBU^*$ of $\mathcal{B}(\mathcal{H}) \supset \mathcal{B}$ that exchanges the l, r components in the particle sector $C=0$ of $\mathcal{H}_F$ (it is implemented by the unitary $U = \gamma^0 \otimes P + \mathbb{I} \otimes (\mathbb{I} - P)$ with P the projection on the particle subspace of $\mathcal{H}_F$), then (54) reads as

$$T\pi(a) - \alpha_U(\pi(a))T \quad \text{for } a \in C^\infty(\mathcal{M}) \otimes \mathcal{B}. \quad (57)$$

It is not yet clear whether this observation is of interest.

3.4. Twisting down

In the light of the preceding section, the conclusion of [34] should be corrected: twisted spectral triples do resolve the unboundedness of the commutator arising in the grand algebra model, but the extra scalar field breaks the first-order condition, even in its twisted form. The latter is retrieved dynamically by minimising the spectral action.

Therefore, twisting the grand algebra down to the Standard Model produces results similar to the ones of [14]. This raises questions on the interest of the twist. As explained in section 5, there is an added value in twists, even if not the one expected! But before coming to that, let us try to generalise the twisting of the grand algebra to arbitrary spectral triples.

4. Minimal twist

4.1. Twisting up

The algebra $\mathcal{B}$ is not invariant under the twisting automorphism ρ because the grand algebra has been only partially twisted: only the quaternionic sector acts non-trivially on the chiral index s . If one also makes the complex sector non trivial on the chiral index, then the grading condition breaks the grand algebra to

$$(\mathbb{H}_L^l \oplus \mathbb{H}_L^r \oplus \mathbb{H}_R^l \oplus \mathbb{H}_R^r) \oplus (M_4^l(\mathbb{C}) \oplus M_4^r(\mathbb{C})), \quad (58)$$

which is invariant under ρ . This is twice the left–right algebra $\mathcal{A}_{LR}$ of section 3.1, which is broken to the algebra $\mathcal{A}_{SM}$ of the Standard Model by the first-order condition of $\gamma^5 \otimes D_F$.

This suggests another approach to twisting the Standard Model while preserving the first-order condition. Rather than twisting down a bigger algebra to $\mathcal{A}_{SM}$, one may double $\mathcal{A}_{SM}$ to

$$\mathcal{A}_{SM} \otimes \mathbb{C}^2 \simeq \mathcal{A}_{SM} \oplus \mathcal{A}_{SM}, \quad (59)$$

then make each copy of $\mathcal{A}_{SM}$ act independently on the left/right components of spinors, and finally twist the commutator to avoid unboundedness problems.

This is a ‘twisting up’ procedure, in which the idea is to use the flexibility introduced by twisted spectral triples to enlarge the algebra—hopefully preserving the grading and the first-order conditions—rather than using these conditions to constrain a bigger algebra. The rule of the game is to leave the Hilbert space and the Dirac operator untouched, in order not to alter the fermionic content of the model. As a side remark, there exist some models in noncommutative geometry that introduce new fermions, as mentioned in the introduction, but since there is no phenomenological indications of new fermions so far, we limit ourselves to models that preserve the fermionic sector.

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the idea is thus to build a twisted spectral triple $(\mathcal{A}', \mathcal{H}, D), \rho$ with the same Hilbert space and Dirac operator, in such a way that the initial triple is retrieved as a ‘non-twisted’ limit of the twisted one. This led in [41] to define the *minimal twist* of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ by a unital algebra $\mathcal{B}$ as a twisted spectral triple $(\mathcal{A} \otimes \mathcal{B}, \mathcal{H}, D), \rho$ such that the representation of $\mathcal{A} \otimes \mathbb{I}_{\mathcal{B}}$ coincides with the initial representation of $\mathcal{A}$.

One may think of other ways to implement the idea of ‘non-twisted limit’, for instance by simply asking that $\mathcal{A}'$ contains $\mathcal{A}$ as a subalgebra invariant under the twist. More elaborate procedure for untwisting a twisted spectral triple have been proposed, for instance in [7, 39].

An advantage of minimal twists is to allow to play with the Standard Model, remaining close to it. For almost commutative geometries—i.e. the product of a manifold by a finite

dimensional spectral triple as in (24)—then the only possible minimal twist by a finite dimensional algebra is with $\mathcal{B} = \mathbb{C}^l \otimes \mathbb{C}^2$, with ρ the flip automorphism of $\mathbb{C}^2$ and $l \in \mathbb{N}$ a measure of the non irreducibility of the representation of $\mathcal{A}_F$ on $\mathcal{H}_F$ [41, proposition 4.4].

4.2. Twist by grading

The twisting up procedure is easily applicable to any graded spectral triple $(\mathcal{A}, \mathcal{H}, D)$. Indeed, by definition, the grading Γ commutes with the representation of $\mathcal{A}$, so the latter actually is the direct sum of two independent—commuting—representations of $\mathcal{A}$ on the eigenspaces $\mathcal{H}_+$, $\mathcal{H}_-$ of Γ ,

$$\pi_+(a) = \frac{1}{2}(\mathbb{I} + \Gamma)a, \quad \pi_-(a) = \frac{1}{2}(\mathbb{I} - \Gamma)a. \quad (60)$$

In other words, decomposing $\mathcal{H}$ as the sum of the two eigenspaces of Γ , the representation of $\mathcal{A}$ is block diagonal. Thus there is enough space on $\mathcal{H}$ to represent $\mathcal{A} \otimes \mathbb{C}^2$ as

$$\pi((a, a')) = \pi_+(a) + \pi_-(a') \quad \forall (a, a') \in \mathcal{A} \otimes \mathbb{C}^2. \quad (61)$$

Let

$$\rho((a, a')) = (a', a) \quad \forall (a, a') \in \mathcal{A} \otimes \mathbb{C}^2 \quad (62)$$

denote the flip automorphism. Then the triple

$$(\mathcal{A} \otimes \mathbb{C}^2, \mathcal{H}, D), \rho \quad (63)$$

with representation (61) is a graded twisted spectral triple [41, proposition 3.8]. In addition, if the initial triple is real with real structure J , then the latter is also a real structure for the twisted spectral triple (61). In particular the twisted first-order condition is automatically satisfied.

This *twist by grading* procedure associates a twisted partner to any graded spectral triple, preserving a first-order condition. This seems the ideal way to twist the Standard Model. Unfortunately, this does not generate the extra scalar field. Indeed, one has that Γ_F anticommutes independently with D_Y and D_M (see e.g. [32, section 4.1] for the computation in tensorial notations) so in particular $\gamma^5 \otimes D_M$ anticommutes with $\Gamma = \gamma^5 \otimes \Gamma_F$. This means that

$$(\gamma^5 \otimes D_M) \pi_+(a) = \pi_-(a) (\gamma^5 \otimes D_M) + \frac{1}{2}(\mathbb{I} - \Gamma) [\gamma^5 \otimes D_M, a], \quad (64)$$

$$(\gamma^5 \otimes D_M) \pi_-(a) = \pi_+(a) (\gamma^5 \otimes D_M) + \frac{1}{2}(\mathbb{I} + \Gamma) [\gamma^5 \otimes D_M, a]. \quad (65)$$

So

$$\begin{aligned} [\gamma^5 \otimes D_M, \pi((a, a'))]_\rho &= (\gamma^5 \otimes D_M) (\pi_+(a) + \pi_-(a')) - (\pi_+(a') + \pi_-(a)) (\gamma^5 \otimes D_M), \\ &= [\gamma^5 \otimes D_M, a] + [\gamma^5 \otimes D_M, a']. \end{aligned} \quad (66)$$

The right hand side is zero since $\gamma^5 \otimes D_M$ commutes with the representation of $\mathcal{A}$. Therefore $\gamma^5 \otimes D_M$ twist-commutes with the representation of $\mathcal{A} \otimes \mathbb{C}^2$. Hence the twist by grading does not modify the situation: $\gamma^5 \otimes D_M$ is transparent under twisted fluctuations, just like it was under usual fluctuations.

4.3. Twisted fluctuation without the first-order condition

The twist by grading is not the only possibility for twisting up the Standard Model. As explained in [41, below proposition 3.8], in order to minimally twist a spectral triple $(\mathcal{A}, \mathcal{H}, D)$

by $\mathbb{C}^2$, one may repeat the construction of the precedent section using, instead of the grading Γ , any operator $\tilde{\Gamma}$ that

- squares to $\mathbb{I}$ and commutes with $\mathcal{A}$: this condition is sufficient to guarantee that π_+, π_- in (60) are two representations of $\mathcal{A}$ commuting with each other, and it becomes necessary as soon as $\mathcal{A}$ is unital;
- is selfadjoint: this is to guarantee that π_+ and π_- are involutive representations;
- has both eigenvalues $+1, -1$ of non-zero multiplicity, so that neither π_+ nor π_- is zero.

But there is no need for $\tilde{\Gamma}$ to anticommute with the Dirac operator. This means that $\tilde{\Gamma}$ is not necessarily a grading for the spectral triple.

A classification of all such twisting operators $\tilde{\Gamma}$ for almost commutative geometries is on its way [37]: the conditions necessary to make the construction work actually reduce to a couple of relations on D_F only. The anticommutation with the Dirac operator seems to be required to have the twisted first-order condition (but this has yet to be proved in full generality). This would imply that the extra scalar field and the twisted first-order condition be mutually exclusive.

Therefore it becomes relevant to extend to the twisted case the results of [14] regarding inner fluctuations without the first-order condition. This has been done in [49], where it was shown that the removal of the twisted first-order condition yields a second order term in the twisted fluctuation (38), which is a straightforward adaptation of the term worked out in the non-twisted case.

Following this path, a minimal twist of the Standard Model has been worked out in great details in [36], that does not preserve the twisted first-order condition and generates the extra scalar field. The gauge part of this model is similar to the Standard Model's, and the Higgs sector is made of two Higgs doublets which are expected to combine in a single doublet in the action. There is the extra scalar field with two components σ_l, σ_r acting independently on the chiral components of spinors, and finally, there is also an unexpected new field of 1-forms X_μ , whose interpretation is discussed in the next section.

5. Twist and change of signature

At this point of our journey through twisted spectral triples, one seems to be back to the starting point: twisted spectral triples solve the unboundedness of the commutator of the grand algebra with $\not{D} \otimes \mathbb{I}$, but they do not permit to generate the extra scalar field, unless one violates the twisted first-order condition. What is then their added value?

The interest of the twist is not so much in the generation of the extra scalar field than in the new field of 1-form X_μ mentioned above. This field was already observed in [34], and its appearance actually does not depend on the details of the model [45]: it is intrinsic to minimal twists of almost commutative geometries. Even in the simplest case of a minimally twisted four dimensional manifold (without any product by a finite dimensional structure), a twisted fluctuation of the Dirac operator $\not{D}$ yields a field of 1-forms, in contrast with the non twisted case where $\not{D}$ does not fluctuate.

The physical sense of this fluctuation remained obscure, until it was confronted with an observation made in [30]: a twist induces on the Hilbert space a new inner product with *Lorentzian signature*. Furthermore, this product permits to define a twisted version of the fermionic action. In some example detailed below, in this action formula the field X_μ identifies with the (dual of) the 4-momentum in Lorentzian signature [47].

5.1. Twisted inner product

A gauge transformation (22), $D_A \rightarrow \text{Ad}(u) D_A \text{Ad}(u)^{-1}$, preserves the selfadjointness of the covariant Dirac operator D_A , for $\text{Ad}(u)^{-1} = Ju^* J^{-1} u^* = \text{Ad}(u)^*$. A twisted gauge transformation (41)

$$D_{A_\rho} \rightarrow \rho(\text{Ad}(u)) D_{A_\rho} \text{Ad}(u)^{-1} \quad (67)$$

does not. Is there some selfadjointness which is preserved by (67)?

There is a natural inner product associated with a twisted spectral triple, as soon as the twisting automorphism ρ extends to an inner automorphism of $\mathcal{B}(\mathcal{H})$:

$$\rho(\mathcal{O}) = R\mathcal{O}R^\dagger \quad \forall \mathcal{O} \in \mathcal{B}(\mathcal{H}) \quad (68)$$

for some unitary operator R on $\mathcal{H}$. Namely, the ρ -inner product [30]

$$\langle \Psi, \Phi \rangle_\rho := \langle \Psi, R\Phi \rangle. \quad (69)$$

Since $\langle \Psi, \mathcal{O}\Phi \rangle_\rho = \langle \rho(\mathcal{O})^\dagger \Psi, \Phi \rangle_\rho$, the adjoint of $\mathcal{O}$ with respect to this new product is

$$\mathcal{O}^+ := \rho(\mathcal{O})^\dagger. \quad (70)$$

If the unitary R commutes or anticommutes with the real structure, then $\rho(\text{Ad}(u))$ as defined before (42) coincides with $R\text{Ad}(u)R^*$ (making the notation $\rho(\text{Ad}(u))$ unambiguous). In addition,

$$(\text{Ad}(u)^{-1})^+ = (RJu^* J^{-1} u^* R^*)^\dagger = RuJuJ^{-1} R^* = \rho(\text{Ad}(u)). \quad (71)$$

Therefore a twisted gauge transformation (67) preserves the selfadjointness with respect to the ρ -inner product.

Example: The minimal twist of a Riemannian spin manifold $\mathcal{M}$ of even dimension $2m$ is

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, S), \quad D = \not{D}; \quad \rho \quad (72)$$

with twisting automorphism the flip $\rho(f, g) = (g, f)$ for $f, g \in C^\infty(\mathcal{M})$. The representation is

$$\pi(f, g) = \begin{pmatrix} f\mathbb{I}_{2^{m-1}} & 0 \\ 0 & g\mathbb{I}_{2^{m-1}} \end{pmatrix} \quad \forall (f, g) \in \mathcal{A}. \quad (73)$$

The flip ρ extends to the inner automorphism of $\mathcal{B}(\mathcal{H})$ that exchanges the element on the diagonal and on the off-diagonal, implemented for instance by $R = \gamma^0$ the first Dirac matrix. Then the ρ -product (69)

$$\langle \Psi, \Phi \rangle_\rho = \int_{\mathcal{M}} \Psi^\dagger \gamma^0 \Phi d^4x \quad (74)$$

coincides pointwise with the Krein product for the space of spinors on a *Lorentzian manifold* (only pointwise, for the manifold on which one integrates is still Riemannian).

This example points towards a link between twists and a kind of transition from Euclidean to Lorentzian signatures: by fluctuating a twisted Riemannian manifold, one ends up preserving a Lorentzian product! However, the twist is not an implementation of Wick rotation in non-commutative geometry (for this, see [27]): a twisted fluctuation (67) does not turn the operator D_{A_ρ} , selfadjoint for the initial (Euclidean) inner product, into an operator $D_{A'_\rho}$ selfadjoint for the Lorentzian product.³ A better understanding of the link between twist and Lorentzian signature follows from the study of the fermionic action.

³ If one were starting with an operator selfadjoint for the twisted product, much in the vein of [53], then this selfadjointness would be preserved by twisted fluctuation.

5.2. Fermionic action

Given a real spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the fermionic action for the covariant operator D_A is [12]

$$S^f(D_A) = \mathfrak{A}_{D_A}(\tilde{\xi}, \tilde{\xi}) \quad (75)$$

with $\tilde{\xi}$ the Grassman variables associated to $\xi \in \mathcal{H}^+ = \{\xi \in \mathcal{H}, \Gamma\xi = \xi\}$ and

$$\mathfrak{A}_{D_A}(\xi, \xi') = \langle J\xi, D_A\xi' \rangle \quad (76)$$

the antisymmetric bilinear form defined by D_A and the real structure J . The latter is needed to make the form antisymmetric (hence applicable on Grassman variables). One restricts to the eigenspace $\mathcal{H}^+$ of the grading because of the fermion doubling [43]. This also makes sense physically, for $\mathcal{H}^+$ is the subspace of $\mathcal{H}$ generated by the elements $\psi \otimes \Psi$ with a well defined chirality (that is $\psi \in L^2(\mathcal{M}, S)$ and $\Psi \in \mathcal{H}_F$ are eigenvectors of γ^5 , Γ_F with the same eigenvalue).

For a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D), \rho$ as in section 5.1, the fermionic action is [30]

$$S^f(D_{A_\rho}) = \mathfrak{T}_{D_{A_\rho}}(\tilde{\xi}, \tilde{\xi}) \quad (77)$$

for $\xi \in \mathcal{H}_r := \{\xi \in \mathcal{H}, R\xi = \xi\}$, $\tilde{\cdot}$ the Grassmann variables and

$$\mathfrak{T}_{D_{A_\rho}}(\xi, \xi') := \langle J\xi, R D_{A_\rho} \xi' \rangle.$$

One inserts the matrix R in the bilinear form in order to make the action (77) invariant under a twisted gauge transformation (41) (the same is true in case there is no first-order condition [49]). The restriction to $\mathcal{H}_r$ guarantees that the bilinear form be antisymmetric.

5.3. Twisted fluctuation as Lorentzian 4-momentum

We begin with the minimal twist (72) of a 4-dimensional manifold. The $+1$ eigenspace of $R = \gamma^0$ is spanned by Dirac spinors of the form $\xi = \begin{pmatrix} \zeta \\ \zeta \end{pmatrix}$ with ζ a Weyl spinor. A selfadjoint twisted fluctuation (38) sends $\not{D}$ to the covariant operator

$$\not{D}_{A_\rho} = \not{D} - iX_\mu \gamma^\mu, \quad (78)$$

parametrised by the 1-form field

$$X_\mu = f_\mu \gamma^5 \quad \text{with} \quad f_\mu \in C^\infty(\mathcal{M}, \mathbb{R}). \quad (79)$$

The twisted fermionic action is [47, Proposition 3.5]

$$S^f(\not{D}_{A_\rho}) = 2 \int_{\mathcal{M}} d\mu \, \tilde{\zeta}^\dagger \sigma_2 \left(if_0 - \sum_{j=1}^3 \sigma_j \partial_j \right) \tilde{\zeta}. \quad (80)$$

The integrand reminds of the Weyl Lagrangian—which lives in Lorentzian signature

$$i\psi_l^\dagger \tilde{\sigma}_M^\mu \partial_\mu \psi_l \quad \text{where} \quad \tilde{\sigma}_M^\mu := \{\mathbb{I}_2, -\sigma_j\}, \quad (81)$$

except that the ∂_0 derivative is missing. It can be restored assuming that ζ is a plane wave function of energy f_0 (in unit $\hbar = 1$) with spatial part $\zeta(\mathbf{x})$, that is

$$\zeta(x_0, \mathbf{x}) = e^{if_0 x_0} \zeta(\mathbf{x}). \quad (82)$$

Then the integrand reads (modulo an irrelevant factor 2) as $\tilde{\zeta}^\dagger \sigma_2 \tilde{\sigma}_M^\mu \partial_\mu \tilde{\zeta}$. However, this cannot be identified with the Weyl Lagrangian (81) because of the extra σ_2 matrix which forbids the simultaneous identification of $\tilde{\zeta}$ with ψ_l and $\tilde{\zeta}^\dagger \sigma_2$ with $i\psi_l^\dagger$. In other terms, there are not

enough degrees of freedom to identify the fermionic action of a twisted manifold with the Weyl Lagrangian.

This can be cured by doubling the manifold. Namely one considers the product

$$(C^\infty(\mathcal{M}) \otimes \mathbb{C}^2, L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^2, \not{D} \otimes \mathbb{I}_2). \quad (83)$$

of $\mathcal{M}$ by a finite dimensional spectral triple $(\mathbb{C}^2, \mathbb{C}^2, 0)$. Its minimal twist is

$$\mathcal{A} = (C^\infty(\mathcal{M}) \otimes \mathbb{C}^2) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^2, \quad D = \not{D} \otimes \mathbb{I}_2 \quad (84)$$

with representation

$$\pi(a, a') = \begin{pmatrix} f\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & f'\mathbb{I}_2 & 0 & 0 \\ 0 & 0 & g'\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & g\mathbb{I}_2 \end{pmatrix} \quad a = (f, g), a' = (f', g') \in \mathcal{A} \quad (85)$$

and twist $\rho(a, a') = (a', a)$. The latter is implemented by the unitary $R = \gamma^0 \otimes \mathbb{I}_2$, whose $+1$ eigenspace $\mathcal{H}_r$ is now spanned by $\{\xi \otimes e, \phi \otimes \bar{e}\}$ where $\{e, \bar{e}\}$ is a basis of $\mathbb{C}^2$ and

$$\xi = \begin{pmatrix} \zeta \\ \zeta \end{pmatrix}, \quad \phi = \begin{pmatrix} \varphi \\ \varphi \end{pmatrix} \quad (86)$$

are Dirac spinors with ζ and φ Weyl spinors. A selfadjoint twisted fluctuation of D ,

$$D_{A_\rho} = D - iX_\mu \gamma^\mu \otimes \mathbb{I}_2 + g_\mu \gamma^\mu \otimes \Gamma_F \quad (87)$$

with Γ_F the grading of the finite dimensional spectral triple [47, proposition 4.3], is parametrised by the same field X_μ as before and a second 1-form field

$$g_\mu \mathbb{I}_4 \quad \text{with} \quad g_\mu \in C^\infty(\mathcal{M}). \quad (88)$$

For a vanishing g_μ , the fermionic action is the integral of [47, proposition 4.4]

$$\mathcal{L} := 4\bar{\varphi}^\dagger \sigma_2 \left(if_0 - \sum_{j=1}^3 \sigma_j \partial_j \right) \tilde{\zeta}. \quad (89)$$

One retrieves the Weyl Lagrangian (81) by identifying the physical Weyl spinors as $\psi_l := \tilde{\zeta}$ and $\psi_l^\dagger := -i\bar{\varphi}^\dagger \sigma_2$, then assuming ψ_l be of the form (82), that is $\partial_0 \psi_l = if_0 \psi_l$. Thus the fermionic action for a twisted doubled Riemannian manifold describes a plane wave solution of Weyl equation, in Lorentzian signature, whose 0th component of the 4-momentum is $p_0 = -f_0$. The result also holds for the right-handed Weyl equation (see [47, proposition 4.5]).

A similar analysis holds for the spectral triple of electrodynamics proposed in [54]. Its minimal twist is

$$\mathcal{A}_{ED} = (C^\infty(\mathcal{M}) \otimes \mathbb{C}^2) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^4, \quad D = \not{D} \otimes \mathbb{I}_4 + \gamma^5 \otimes D_{\mathcal{F}}$$

where the internal Dirac operator and the representation are

$$D_{\mathcal{F}} = \begin{pmatrix} 0 & d & 0 & 0 \\ \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & d & 0 \end{pmatrix}, \quad \pi(a, a') = \begin{pmatrix} f\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f'\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f'\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f\mathbb{I}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g'\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g\mathbb{I}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g'\mathbb{I}_2 \end{pmatrix}$$

with $d \in \mathbb{C}$, $a = (f, g)$, $a' = (f', g')$ in $C^\infty(\mathcal{M}) \otimes \mathbb{C}^2$. The twist $\rho(a, a') = (a', a)$ extends to an inner automorphism of $\mathcal{B}(\mathcal{H})$ generated by the unitary $\gamma^0 \otimes \mathbb{I}_4$. Its $+1$ -eigenspace is generated by

$$\xi_1 \otimes e_l, \quad \xi_2 \otimes e_r, \quad \phi_1 \otimes \bar{e}_l, \quad \phi_2 \otimes \bar{e}_r, \quad (90)$$

where ξ_k, ϕ_k ($k = 1, 2$) are Dirac spinors of the form (86) while $\{e_l, e_r, \bar{e}_l, \bar{e}_r\}$ is an orthonormal basis of $\mathbb{C}^4$.

A selfadjoint twisted fluctuation of D is parametrised by the same two 1-form fields as before [47, Proposition 5.3]

$$D_{A_\rho} = D - iX_\mu \gamma^\mu \otimes \mathbb{I}' + g_\mu \gamma^\mu \otimes \mathbb{I}'' \quad (91)$$

where $\mathbb{I}' = \text{diag}(1, -1, 1, -1)$, $\mathbb{I}'' = \text{diag}(1, 1, -1, -1)$ (the part $\gamma^5 \otimes D_F$ is transparent under twisted fluctuation: there is no Higgs field in classical electrodynamics!). Under a gauge transformation (41), one has that f_μ is invariant while g_μ transforms as the $U(1)$ gauge potential of electrodynamics.

The spectral action is the integral of [47, proposition 5.12]

$$\mathcal{L}_\rho^f = \bar{\varphi}_1^\dagger \sigma_2 \left(if_0 - \sum_j \sigma_j \mathcal{D}_j \right) \tilde{\zeta}_1 - \bar{\varphi}_2^\dagger \sigma_2 \left(if_0 + \sum_j \sigma_j \mathcal{D}_j \right) \tilde{\zeta}_2 + \left(\bar{d} \bar{\varphi}_1^\dagger \sigma_2 \tilde{\zeta}_2 + d \bar{\varphi}_2^\dagger \sigma_2 \tilde{\zeta}_1 \right) \quad (92)$$

where $\mathcal{D}_\mu = \partial_\mu - ig_\mu$ is the covariant derivative associated to the electromagnetic 4-potential. Defining the physical spinors as

$$\psi = \begin{pmatrix} \psi_l \\ \psi_r \end{pmatrix} := \begin{pmatrix} \tilde{\zeta}_1 \\ \tilde{\zeta}_2 \end{pmatrix}, \quad \psi^\dagger = (\psi_l^\dagger, \psi_r^\dagger) := (-i\bar{\varphi}_1^\dagger \sigma_2, i\bar{\varphi}_2^\dagger \sigma_2) \quad (93)$$

then assuming that $\partial_0 \psi = if_0 \psi$ and imposing $d = -im$ with $m > 0$ to be purely imaginary, the Lagrangian (92) reads

$$\mathcal{L}_M = i\psi_l^\dagger \left(\mathcal{D}_0 - \sum_j \sigma_j \mathcal{D}_j \right) \psi_l + i\psi_r^\dagger \left(\mathcal{D}_0 + \sum_j \sigma_j \mathcal{D}_j \right) \psi_r - m (\psi_l^\dagger \psi_r + \psi_r^\dagger \psi_l). \quad (94)$$

This is the Dirac Lagrangian in Minkowski spacetime, for a mass m , in the temporal gauge (that is $\mathcal{D}_0 = \partial_0$). Hence the fermionic action for the minimal twist of the spectral triple of electrodynamics describes a plane wave solution of the Dirac equation in Lorentz signature, with 0th component of the 4-momentum $p_0 = -f_0$.

By implementing the action of boosts on $L^2(\mathcal{M}, S) \otimes \mathbb{C}^2$, one is able to identify the other components of the fluctuation f_μ with the other components of the 4-momentum. However this is obtained at the level of the equation of motion, not for the Lagrangian density (see [47, section 6.1]).

6. Conclusion and outlook

The idea of using twisted spectral triples in high-energy physics was born with the hope of generating the extra scalar field needed to stabilise the electroweak vacuum (and to fit the Higgs mass), respecting the axioms of noncommutative geometry. More specifically it was thought that the first-order condition could be twisted, rather than abandoned. We have shown in this note that this is not possible. This moves the interest of the twist towards what seemed at first sight a side effect, namely the non-zero twisted fluctuation of the free Dirac operator $\not{D}$. It yields a new field of 1-forms, whose physical meaning becomes clear by computing the

fermionic action. For the minimal twist of a doubled manifold, and the minimal twist of the spectral triple of electrodynamics, this field identifies with (the dual of) the 4-momentum in Lorentzian signature. Preliminary computations indicate that a similar result also holds for the twist of the Standard Model presented in [36].

It remains to be understood why one ends up in the temporal gauge. And more importantly, does the identification between twisted fluctuation of $\not{D}$ and the 4-momentum still makes sense for the bosonic part of the action, given by the spectral action? Not to mention that the definition of the latter in a twisted context has not been established yet [31].

Data availability statement

No new data were created or analysed in this study.

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