



Classical and Quantum KMS States on Spin Lattice Systems

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Received: 24 June 2024 / Accepted: 15 April 2025

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Abstract: We study the classical and quantum KMS conditions within the context of spin lattice systems. Specifically, we define a strict deformation quantization (SDQ) for a \mathbb{S}^2 -valued spin lattice system over \mathbb{Z}^d generalizing the renowned Berezin SDQ for a single sphere. This allows to promote a classical dynamics on the algebra of classical observables to a quantum dynamics on the algebra of quantum observables. We then compare the notion of classical and quantum thermal equilibrium by showing that any weak*-limit point of a sequence of quantum KMS states fulfils the classical KMS condition. In short, this proves that the semiclassical limit of quantum thermal states describes classical thermal equilibrium, strengthening the physical interpretation of the classical KMS condition. Finally we provide two sufficient conditions ensuring uniqueness of classical and quantum KMS states: The latter are based on an version of the Kirkwood-Salzburg equations adapted to the system of interest. As a consequence we identify a mild condition which ensures uniqueness of classical KMS states and of quantum KMS states for the quantized dynamics for a common sufficiently high temperature.

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1. Introduction

The description of thermal equilibrium is a well-established and extensively studied topic in classical and quantum statistical mechanics [15,28,35,54]. Adopting the algebraic approach, where observables of the physical system of interest are modelled by a C^* -algebra \mathfrak{A} , classical and quantum thermal equilibrium are characterized by two slightly different yet related conditions, called classical and quantum Kubo-Martin-Schwinger (KMS) conditions, cf. [32,33].

Specifically, a quantum system is described in term of an abstract non-commutative C^* -algebra \mathfrak{A} —through all paper we will only be interested in algebras with a unit. Time evolution is modelled by a strongly continuous one-parameter group $t \mapsto \tau_t$ of $*$ -automorphisms on \mathfrak{A} with infinitesimal generator δ . Within this setting a state $\omega \in S(\mathfrak{A})$ —that is, a linear, positive and normalized functional $\omega: \mathfrak{A} \rightarrow \mathbb{C}$ — is called (β, δ) -**KMS quantum state**, $\beta \in [0, \infty)$, if

$$\omega(\alpha\tau_{i\beta}(b)) = \omega(b\alpha), \tag{1}$$

for all pairs $\alpha, b \in \mathfrak{A}$ of analytic elements for τ , cf. [15]. The quantum (β, δ) -KMS condition (1) selects those states on \mathfrak{A} which are interpreted as describing thermal equilibrium with respect to τ at a fixed inverse temperature β , cf. [33]—here $\beta = 0$ corresponds to infinite temperature.

The description of thermal equilibrium for a classical system is slightly different, cf. [1,2,22]. In this scenario the observables of a classical physical system are described by a commutative Poisson C^* -algebra \mathfrak{A} . We recall that a **Poisson structure** over a commutative C^* -algebra \mathfrak{A} is given by a bilinear map $\{, \}: \mathfrak{A} \times \mathfrak{A} \rightarrow \mathfrak{A}$ defined on a dense $*$ -subalgebra $\mathfrak{A} \subset \mathfrak{A}$ which fulfils:

$$\begin{aligned} \{a, b\} &= -\{b, a\}, & \{a, b\}^* &= \{a^*, b^*\}, & \{a, bc\} &= \{a, b\}c + b\{a, c\}, \\ \{a, \{b, c\}\} &= \{\{a, b\}, c\} + \{b, \{a, c\}\}, \end{aligned}$$

for all $a, b, c \in \mathfrak{A}$. Given $\beta \in [0, +\infty)$ and a $*$ -derivation $\delta: \mathfrak{A} \rightarrow \mathfrak{A}$, a state $\omega \in S(\mathfrak{A})$ is called (β, δ) -**KMS classical state** if

$$\omega(\{a, b\}) = \beta\omega(b\delta(a)), \tag{2}$$

for all $a, b \in \mathfrak{A}$. Once again δ is regarded as the infinitesimal generator of the time evolution on \mathfrak{A} and β is interpreted as an inverse temperature.

The quantum KMS condition has received a lot of attention and has been investigated in several scenarios [15,35,54]. In particular the physical content of Eq. (1) has been investigated, cf. [33,50], providing concrete justifications for its interpretation. Conversely, there are fewer investigations on the classical KMS condition (2). The latter has been introduced in [32] and further developed in [1,2,26,31,49] in the context classical system of infinitely many particles. The classical KMS condition has been investigated also in the context of pure Poisson geometry in [6–9,14,24], moreover, its relation with the Dobrushin-Lanford-Ruelle (DLR) [19–21,38] probabilistic approach to classical thermal equilibrium was investigated in [23] while its connection to the notion of Gibbs measures for non-linear Hamiltonian systems was studied in [4,5,16,47].

The physical justification of the classical KMS condition (2) seems to be less studied. In particular, in [32] condition (2) has been formally derived by considering a suitable semiclassical limit of the quantum KMS condition (31). The first goal of this paper is to

provide a mathematically rigorous version of this derivation within the setting of strict deformation quantization, cf. Theorem 3.5.

Strict (or C^* -algebraic) deformation quantization (SDQ) provides a mathematically rigorous setting to study the quantization of a classical system [39,52]. This framework is not only suitable to investigate the semiclassical limit of states of a quantum system with a fixed, but arbitrary number of degrees of freedom cf. [40,43], but it can also be applied to describe the macroscopic properties of quantum systems over an infinitely extended lattice [22,41,42,56,57] or the semi-classical properties of condensates [48].

From a mathematical point of view, a SDQ requires the notion of bundle of C^* -algebra which we briefly recall following [40, App. C.19], [12, Section IV.1.6]. Setting $\overline{\mathbb{Z}_+/2} := \mathbb{Z}_+/2 \cup \{\infty\}$ and given a collection $\{\mathfrak{A}_j\}_{j \in \overline{\mathbb{Z}_+/2}}$ of C^* -algebras, we denote by $\prod_{j \in \overline{\mathbb{Z}_+/2}} \mathfrak{A}_j$ the associated **full C^* -direct product**, which is the C^* -algebra made by sequences $(\alpha_j)_{j \in \overline{\mathbb{Z}_+/2}}$, $\alpha_j \in \mathfrak{A}_j$, such that $\|(\alpha_j)_{j \in \overline{\mathbb{Z}_+/2}} \|_{\prod_{j \in \overline{\mathbb{Z}_+/2}} \mathfrak{A}_j} := \sup_j \|\alpha_j\|_{\mathfrak{A}_j} < \infty$. Within this setting a **continuous bundle of C^* -algebras** over $\overline{\mathbb{Z}_+/2}$ (with fibers $\{\mathfrak{A}_j\}_{j \in \overline{\mathbb{Z}_+/2}}$) is a C^* -subalgebra $\mathfrak{A} \subset \prod_{j \in \overline{\mathbb{Z}_+/2}} \mathfrak{A}_j$ such that: (i) $(\|\alpha_j\|_{\mathfrak{A}_j})_{j \in \overline{\mathbb{Z}_+/2}} \in C(\overline{\mathbb{Z}_+/2})$ for all $(\alpha_j)_{j \in \overline{\mathbb{Z}_+/2}} \in \mathfrak{A}$, where $C(\overline{\mathbb{Z}_+/2})$ denotes the space of sequences $(\alpha_j)_{j \in \overline{\mathbb{Z}_+/2}}$, $\alpha_j \in \mathbb{C}$, such that $\alpha_\infty = \lim_{j \rightarrow \infty} \alpha_j$; (ii) $\alpha\alpha \in \mathfrak{A}$ for all $\alpha \in \mathfrak{A}$ and $\alpha \in C(\overline{\mathbb{Z}_+/2})$. A **strict deformation quantization (SDQ)** is then defined by the following data:

1. A commutative Poisson C^* -algebra \mathfrak{A}_∞ , with Poisson structure $\{, \}$: $\mathfrak{A}_\infty \times \mathfrak{A}_\infty \rightarrow \mathfrak{A}_\infty$;
2. A continuous bundle of C^* -algebras [18] $\tilde{\mathfrak{A}} \subset \prod_{j \in \overline{\mathbb{Z}_+/2}} \mathfrak{A}_j$;
3. A family of linear maps, called **quantization maps**, $Q_j: \mathfrak{A}_\infty \rightarrow \mathfrak{A}_j$, $j \in \overline{\mathbb{Z}_+/2}$, such that:
 - (a) $Q_\infty(\alpha) = \alpha$ for all $\alpha \in \mathfrak{A}_\infty$, moreover, $Q_j(\alpha)^* = Q_j(\alpha^*)$ and $(Q_j(\alpha))_{j \in \overline{\mathbb{Z}_+/2}} \in \tilde{\mathfrak{A}}$.
 - (b) For all $\alpha, \beta \in \mathfrak{A}_\infty$ the **Dirac-Groenewold-Rieffel (DGR) condition** holds:

$$\lim_{j \rightarrow \infty} \|Q_j(\{\alpha, \beta\}) - i(2j + 1)[Q_j(\alpha), Q_j(\beta)]\|_{\mathfrak{A}_j} = 0. \tag{3}$$

- (c) For all $j \in \overline{\mathbb{Z}_+/2}$, $Q_j(\mathfrak{A})$ is a dense $*$ -subalgebra of \mathfrak{A}_j .

In this framework $j \in \mathbb{Z}_+/2$ is interpreted as a semiclassical parameter—in fact, $h_j := 1/(2j + 1)$ is the proper semiclassical parameter—and $j \rightarrow \infty$ corresponds to the semiclassical limit. Given a sequence $(\omega_j)_{j \in \mathbb{Z}_+/2}$ of states such that $\omega_j \in S(\mathfrak{A}_j)$, the semiclassical limit is obtained considering the weak*-limit points of the sequence $(\omega_j \circ Q_j)_{j \in \mathbb{Z}_+/2}$ of functionals over \mathfrak{A}_∞ —each such weak*-limit point defines a state on \mathfrak{A}_∞ .

In this paper we provide a concrete, yet general SDQ model where the relation between classical and quantum KMS conditions (1)–(2) can be studied rigorously. In particular we will focus on S^2 -valued spin lattice systems on $\Gamma := \mathbb{Z}^d$, $d \in \mathbb{N}$, cf. [15,28], which are described by the renown quasi-local algebras $B_\infty^\Gamma, B_j^\Gamma$. The latter are C^* -inductive limits of corresponding C^* -inductive systems $\{B_\infty^\Lambda\}_{\Lambda \in \Gamma}, \{B_j^\Gamma\}_{\Lambda \in \Gamma}$ where, for any finite region $\Lambda \Subset \Gamma$, B_∞^Λ (resp. B_j^Λ) denotes the algebra of observables localized in Λ for the classical (resp. quantum) system, cf. Section 2.1. Within this setting classical and quantum KMS states have been investigated in detail [15,28]. Moreover, this setting fits within the framework of the Berezin quantization, which identifies a SDQ for the physical system associated with finite regions, cf. [10,44].

Within this framework we may summarize our results as follows:

- (I) In Theorem 2.5 we construct a SDQ for the spin lattice system associated to the infinite region Γ , extending the results of [10, 44]. This completes the framework in which we will subsequently investigate the properties of the semiclassical limits of KMS quantum states. It is worth to mention that the study of thermal equilibrium leads to physically relevant results only for infinitely extended systems: Thus, the construction of our SDQ is well-suited for the purposes of this study, *cf.* 1–1 below.
- (II) We study the properties of weak*-limit points of KMS quantum states, in particular, in Theorem 3.5 we prove that they all fulfil the KMS classical condition (2). This provides a rigorous derivation of the classical KMS condition from the quantum KMS condition along the line of [32].
- (III) We investigate further the relationship between classical and quantum thermal equilibrium with a specific focus on phase transitions. The latter describe the uniqueness/non-uniqueness of KMS states and are of utmost relevance for describing when a physical system undergoes an abrupt change in its macroscopic behaviour, *e.g.* gas-to-liquid condensation. It is common folklore that classical and quantum phase transition at non-vanishing temperature should be in bijection. Yet, to the best of our knowledge, no mathematically rigorous proof of this claim has been given. In Sect. 4 we prove that, within the model \mathbb{S}^2 -valued lattice spin system considered in Sect. 2, under a sufficiently mild assumption, *cf.* (44), classical and quantum KMS states are unique for temperatures higher than a common sufficiently high threshold temperature. This result is a consequence of Theorems 4.1, 4.7 which provides two new sufficient conditions for the uniqueness of KMS classical and quantum states. Thus, our result is in line with the claimed equivalence between classical and quantum phase transitions.

It is worth to point out that our results are companions of other existing works in this area. In particular, [44] already described an abstract framework which covers the Berezin SDQ for a lattice system in a finite region. Our result 1 generalizes this setting for a specific model but allowing to deal with a spin lattice system on an infinitely extended region: This is important because physically interesting results on thermal equilibrium, *e.g.* phase transitions, can only be described on infinitely extended system. Concerning 1, in [25] the semiclassical limit of KMS quantum states has been investigated under an assumption which is an abstract version of our Lemma 3.4. In [3] the classical and quantum KMS conditions were related for the case of the Bose-Hubbard system on a finite graph. Similarly, [58] deals with the semiclassical limit of Gibbs quantum and classical states, *i.e.* KMS states on a spin lattice system associated with a finite region: From this point of view, our result 1 can be seen as a generalization of [58] to a physically more interesting scenario. Finally, concerning 1, Theorems 4.1, 4.7 are inspired by [15, Prop. 6.2.45], which provides a sufficient condition for uniqueness of KMS quantum states with an argument based on a quantum version of the Kirkwood-Salzburg equations. Our result provides a classical analogous of [15, Prop. 6.2.45], moreover, it strengthens some of its conclusion—specifically the j -dependence of the inverse critical temperature, *cf.* Remark 4.9. It is also worth to point out that our results cover the regime of high temperatures. For low temperatures it was shown in [11] that, whenever chessboard estimates can be used to prove a phase transition in the classical model, the corresponding quantum model will have a similar phase transition provided $\beta^2 \ll j$.

The paper is organized as follows. Section 2 describes the model of interest and construct a SDQ suitable for our purposes. Section 3 deals with the semiclassical limit of KMS quantum states, proving that each weak*-limit point fulfils the classical KMS

condition (2). This requires a few technical results, *cf.* Lemmata 3.2–3.4. Finally Sect. 4 deals with the topic of classical and quantum phase transitions. In particular, Sect. 4.1 is devoted to the proof of a uniqueness result for KMS classical states, while Sect. 4.2 deals with an analogous result for KMS quantum states. Eventually the relation between these results is discussed, *cf.* Remark 4.9, leading to the proof that, under reasonably mild assumptions, classical and quantum phase transitions are absent for temperatures higher than a common threshold temperature.

2. Berezin SDQ on a Lattice System

The goal of this section is to provide a SDQ in the sense of Berezin for a classical lattice system. For definiteness we will consider the lattice \mathbb{Z}^d , $d \in \mathbb{N}$, where to each site $x \in \mathbb{Z}^d$ one associates the spin space \mathbb{S}^2 , in the classical case, or \mathbb{C}^{2j+1} , $j \in \mathbb{Z}_+/2$, in the quantum case. Here j will play the role of a semiclassical parameter, the semiclassical limit being $j \rightarrow \infty$. Within this setting the Berezin SDQ $Q_j : C(\mathbb{S}^2) \rightarrow M_{2j+1}(\mathbb{C})$ identifies a classical-to-quantum map between the algebras of observables associated to the system [10]. We will prove that such SDQ can be lifted to a quantization on the associated quasi local-algebra on the whole Γ , *cf.* Proposition 2.5, *i.e.* for the infinitely extended classical system. This generalizes [44] where the Berezin SDQ for a classical system localized in a finite region $\Lambda \Subset \mathbb{Z}^d$ was considered.

In Sect. 2.1 we briefly introduce the data for the classical and quantum system on a single site $x \in \mathbb{Z}^d$. Section 2.2 recollects the relevant properties of the standard Berezin deformation quantization, which will play a role also for the discussion in Sect. 4. Finally, Sect. 2.3 extends the result of [44] by constructing a Berezin SDQ for the infinitely extended classical system on Γ .

2.1. Classical and quantum lattice systems on Γ . In this section we will briefly summarize the data of the classical and quantum spin system we will consider for the rest of the paper, *cf.* [15,22,28].

At a classical level, we will consider the lattice $\boxed{\Gamma} := \mathbb{Z}^d$, $d \in \mathbb{N}$. The spin configuration space $\boxed{\mathbb{S}_x^2} := \mathbb{S}^2$ at each $x \in \mathbb{Z}^d$ is a closed symplectic manifold. By definition, the algebra of classical observables at $x \in \mathbb{Z}^d$ is the C^* -algebra $\boxed{B_\infty} := C(\mathbb{S}_x^2)$, the latter being also a Poisson C^* -algebra with Poisson bracket $\{, \}_{B_\infty}$ defined on $\boxed{\dot{B}_\infty} := C^\infty(\mathbb{S}_x^2)$.

For any finite region $\Lambda \Subset \mathbb{Z}^d$ the algebra of classical observables $\boxed{B_\infty^\Lambda}$ associated with Λ is defined by $B_\infty^\Lambda := \overline{\otimes_{x \in \Lambda} B_\infty} \simeq C(\mathbb{S}_\Lambda^2)$ where the spin configuration space is now $\boxed{\mathbb{S}_\Lambda^2} := \otimes_{x \in \Lambda} \mathbb{S}^2 \simeq (\mathbb{S}^2)^{|\Lambda|}$. Notice that B_∞^Λ is a Poisson C^* -algebra with Poisson bracket $\{, \}_{B_\infty^\Lambda} : \dot{B}_\infty^\Lambda \times \dot{B}_\infty^\Lambda \rightarrow \dot{B}_\infty^\Lambda$ defined on the dense $*$ -sub-algebra $\boxed{\dot{B}_\infty^\Lambda} := C^\infty(\mathbb{S}_\Lambda^2)$ and associated with the symplectic structure of \mathbb{S}_Λ^2 .

In the thermodynamic limit one identifies the C^* -algebra $\boxed{B_\infty^\Gamma}$ of quasi-local classical observables on Γ with $B_\infty^\Gamma := C(\mathbb{S}_\Gamma)$, where $\boxed{\mathbb{S}_\Gamma} := (\mathbb{S}^2)^\Gamma$ is compact in the product topology. It is worth to point out that B_∞^Γ is the C^* -direct limit of the C^* -direct system

$\{B_\infty^\Lambda\}_{\Lambda \in \mathbb{Z}^d}$. The latter is characterized by the C^* -injective maps

$$\iota_{\Lambda_1}^{\Lambda_0} : B_\infty^{\Lambda_0} \rightarrow B_\infty^{\Lambda_1}, \quad \iota_{\Lambda_1}^{\Lambda_0} a_{\Lambda_0} := a_{\Lambda_0} \otimes \bigotimes_{x \in \Lambda_1 \setminus \Lambda_0} I_\infty \quad \forall a_{\Lambda_0} \in B_\infty^{\Lambda_0}, \quad (4)$$

where $\Lambda_0 \subset \Lambda_1 \in \mathbb{Z}^d$ while $I_\infty \in B_\infty$ denotes the constant function $I_\infty \equiv 1$. Denoting by $\iota^\Lambda : B_\infty^\Lambda \rightarrow B_\infty^\Gamma$ the associated C^* -inclusion maps we observe that

$$\dot{B}_\infty^\Gamma := \bigcup_{\Lambda \in \mathbb{Z}^d} \iota^\Lambda \dot{B}_\infty^\Lambda, \quad (5)$$

is a dense $*$ -subalgebra of B_∞^Γ . With a standard slight abuse of notation in what follows we will identify B_∞^Λ and $\iota^\Lambda B_\infty^\Lambda$, therefore, we will drop the inclusions maps $\iota^\Lambda, \iota_{\Lambda_2}^{\Lambda_1}$.

As described in [23], B_∞^Γ is a Poisson C^* -algebra with Poisson structure defined on \dot{B}_∞^Γ . In particular one observes that the maps $\iota_{\Lambda_1}^{\Lambda_0} : B_\infty^{\Lambda_0} \rightarrow B_\infty^{\Lambda_1}$ are Poisson, namely

$$\iota_{\Lambda_1}^{\Lambda_0} \{a_{\Lambda_0}, \tilde{a}_{\Lambda_0}\}_{B_\infty^{\Lambda_0}} := \{\iota_{\Lambda_1}^{\Lambda_0} a_{\Lambda_0}, \iota_{\Lambda_1}^{\Lambda_0} \tilde{a}_{\Lambda_0}\}_{B_\infty^{\Lambda_1}}. \quad (6)$$

Out of Eq. (6) the Poisson structure $\{, \}_{B_\infty^\Gamma}$ is defined by

$$\{, \}_{B_\infty^\Gamma} : \dot{B}_\infty^\Gamma \times \dot{B}_\infty^\Gamma \rightarrow \dot{B}_\infty^\Gamma, \quad \{a_{\Lambda_1}, a_{\Lambda_2}\}_{B_\infty^\Gamma} := \{a_{\Lambda_1}, a_{\Lambda_2}\}_{B_\infty^{\Lambda_1 \cup \Lambda_2}}. \quad (7)$$

Notice that, in fact,

$$\{a_{\Lambda_1}, a_{\Lambda_2}\}_{B_\infty^{\Lambda_1 \cup \Lambda_2}} = \{a_{\Lambda_1}, a_{\Lambda_2}\}_{B_\infty^{\Lambda_1 \cap \Lambda_2}}. \quad (8)$$

On the quantum side, we will consider a spin lattice system over Γ , where each site $x \in \Gamma$ is associated with a finite dimensional algebra of non-commutative observables. Specifically, let $j \in \mathbb{Z}_+/2$ and let $\boxed{B_j} := M_{2j+1}(\mathbb{C})$: The latter will be considered the algebra of quantum observables at each site $x \in \Gamma$ —as we will see in Sect. 2.2 j will play the role of a semiclassical parameter. For any $\Lambda \in \Gamma$ we then set $\boxed{B_j^\Lambda} := \overline{\bigotimes_{x \in \Lambda} B_j}$. Then the collection $\{B_j^\Lambda\}_{\Lambda \in \Gamma}$ form a C^* -direct system, cf. [15, 54], with injective C^* -maps denoted by, with a slight abuse of notation,

$$\iota_{\Lambda_1}^{\Lambda_0} : B_j^{\Lambda_0} \rightarrow B_j^{\Lambda_1}, \quad \iota_{\Lambda_1}^{\Lambda_0} A_{\Lambda_0} := A_{\Lambda_0} \otimes \bigotimes_{x \in \Lambda_1 \setminus \Lambda_0} I_j \quad \forall A_{\Lambda_0} \in B_j^{\Lambda_0}, \quad (9)$$

where $\Lambda_0 \subset \Lambda_1 \in \Gamma$ while $I_j \in B_j$ is the identity matrix.

The algebra $\boxed{B_j^\Gamma}$ of quantum observables in the thermodynamic limit is the C^* -direct limit of the C^* -direct system $\{B_j^\Lambda\}_{\Lambda \in \Gamma}$. With a slight abuse of notation we will denote by $\iota^\Lambda : B_j^\Lambda \rightarrow B_j^\Gamma$ the associated C^* -inclusion maps. In particular $\boxed{\dot{B}_j^\Gamma} := \bigcup_{\Lambda \in \Gamma} \iota^\Lambda B_j^\Lambda$ is a dense $*$ -algebra of B_j^Γ , moreover, $\|\iota^\Lambda a_\Lambda\|_{B_j^\Gamma} = \|a_\Lambda\|_{B_j^\Lambda}$ for all $a_\Lambda \in B_j^\Lambda$. Similarly to the classical case we will identify B_j^Λ and $\iota^\Lambda B_j^\Lambda$ and drop the inclusion maps $\iota^\Lambda, \iota_{\Lambda_1}^{\Lambda_0}$ when not strictly necessary.

2.2. *Berezin SDQ for a single site system.* This section focuses on the standard Berezin quantization of the sphere \mathbb{S}^2 [10,13,17,27,46,53]. We will recall without proof the main results, pointing out useful consequences which we were not able to find in the existing literature, cf. Remark 2.2.

To begin with we consider the Lie group $SU(2)$ and denote by $\{J_i\}_{i=1}^3$ the generators of the corresponding Lie algebra $\mathfrak{su}(2)$ with commutation relations $[J_1, J_2] = iJ_3$ and extended cyclically. Adopting the standard physicist’s notation we denote by $\boxed{D^{(j)}}: SU(2) \rightarrow M_{2j+1}(\mathbb{C})$ the irreducible representation of $SU(2)$ of spin $j \in \mathbb{Z}_+/2$, cf. [29, Section 5.4]. We will denote by

$$|j, m\rangle \in \mathbb{C}^{2j+1} \quad m \in \{-j, \dots, j\}, \tag{10}$$

the orthonormal basis of \mathbb{C}^{2j+1} made by the eigenvectors of $D^{(j)}(J_3)$, the latter denoting the infinitesimal generator of $D^{(j)}(e^{-iJ})$ —we adopt the bra-ket notation, cf. [45]. In particular

$$D^{(j)}(J_3)|j, m\rangle = m|j, m\rangle \quad \langle j, m|j, m'\rangle = \delta_{m,m'}.$$

The **coherent state** associated with $\sigma \in \mathbb{S}^2 \simeq SU(2)/U(1)$ is defined by

$$|j, \sigma\rangle := D^{(j)}(\sigma)|j, j\rangle := D^{(j)}(e^{-i\phi(\sigma)J_z}e^{-i\theta(\sigma)J_y})|j, j\rangle, \tag{11}$$

where $(\phi(\sigma), \theta(\sigma)) \in (-\pi, \pi) \times (0, \pi)$ are the spherical coordinates associated with σ . By a standard argument, cf. [46], the family of coherent states $\{|j, \sigma\rangle\}_{\sigma \in \mathbb{S}^2}$ form an over-complete set in \mathbb{C}^{2j+1} in the sense that

$$\int_{\mathbb{S}^2} |j, \sigma\rangle\langle j, \sigma| d\mu_j(\sigma) = I, \tag{12}$$

where the integral in the left-hand side is computed in the weak sense while $|j, \sigma\rangle\langle j, \sigma|$ denotes the orthogonal projector along $|j, \sigma\rangle$ —here μ_j denotes the standard measure on \mathbb{S}^2 normalized so that $\mu_j(\mathbb{S}^2) = 2j + 1$.

At this stage the **Berezin quantization map** $Q_j: B_\infty \rightarrow B_j$ is defined by the weak integral

$$Q_j(a) := \int_{\mathbb{S}^2} a(\sigma)|j, \sigma\rangle\langle j, \sigma| d\mu_j(\sigma). \tag{13}$$

It is worth observing that $Q_j(a) \geq 0$ whenever $a \geq 0$, moreover, $\|Q_j(a)\|_{B_j} \leq \|a\|_{B_\infty}$. Furthermore, setting

$$\check{a}_j(\sigma) := \langle j, \sigma|Q_j(a)|j, \sigma\rangle = \int_{\mathbb{S}^2} a(\sigma')|\langle j, \sigma|j, \sigma'\rangle|^2 d\mu_j(\sigma'), \quad a \in C(\mathbb{S}^2), \tag{14}$$

one finds $\check{a}_j \in C(\mathbb{S}^2)$ and $\check{a}_j \rightarrow a$ in the sup-norm. Within this setting the data

$$B_j := \begin{cases} M_{2j+1}(\mathbb{C}) & j \in \mathbb{Z}_+/2 \\ C(\mathbb{S}^2) & j = \infty \end{cases}, \quad Q_j: B_\infty \rightarrow B_j \quad Q_j(a_j) := \begin{cases} Q_j(a_j) & j \in \mathbb{Z}_+/2 \\ a_\infty & j = \infty \end{cases},$$

identify a SDQ over a suitably defined bundle of C^* -algebras $\boxed{B_*} \subseteq \prod_{j \in \mathbb{Z}_+/2} B_j$, cf. [40, Thm. 8.1]. From a physical point of view, the semiclassical parameter is identified with $\boxed{h_j} := (2j + 1)^{-1}$.

Example 2.1. For later purposes, we report in this example the fairly explicit computation of the Berezin quantization of a generic spherical harmonic, cf. [46]. In more details, let $\boxed{Y_{\ell,m}} \in C^\infty(\mathbb{S}^2)$ be the spherical harmonic with parameter $\ell \in \mathbb{Z}_+$, $m \in [-\ell, \ell] \cap \mathbb{Z}$: Explicitly we set

$$Y_{\ell,m}(\sigma) := \sqrt{\frac{(\ell - m)!}{(\ell + m)!}} P_{\ell,m}[\cos \theta(\sigma)] e^{im\phi(\sigma)}, \tag{15}$$

where $P_{\ell,m}$ denotes the Legendre polynomial of order ℓ, m . Notice that this choice of normalization is such that $\|Y_{\ell,m}\|_{L^2(\mathbb{S}^2, \mu_\ell)} = 1$, moreover, $\|Y_{\ell,m}\|_{B_\infty} \leq 1$, cf. [55, Cor. 2.9]. The set $\{Y_{\ell,m}\}_{\ell,m}$ is a complete orthogonal system for $L^2(\mathbb{S}^2, \mu_0)$ made by orthogonal but not L^2 -normalized vectors. With this convention we also have [45, Section 3.6.2]

$$\overline{Y_{\ell,m}(\sigma)} = \langle \ell, m | D^{(j)}(\sigma) | \ell, 0 \rangle = D_{m,0}^{(\ell)}(\sigma), \tag{16}$$

where $\boxed{D_{m,k}^{(j)}(\sigma)} := \langle j, m | D^{(j)}(\sigma) | j, k \rangle$ denotes the **Wigner D-matrix**. By direct inspection we find

$$\begin{aligned} \langle j, m_1 | Q_j(Y_{\ell,m}) | j, m_2 \rangle &= \int_{\mathbb{S}^2} Y_{\ell,m}(\sigma) \langle j, m_1 | j, \sigma \rangle \langle j, \sigma | j, m_2 \rangle d\mu_j(\sigma) \\ &= \int_{\mathbb{S}^2} \overline{D_{m,0}^{(\ell)}(\sigma)} D_{m_1,j}^{(j)}(\sigma) \overline{D_{m_2,j}^{(j)}(\sigma)} d\mu_j(\sigma) \\ &= \text{CG}_{\ell,m;j,m_2}^{j,m_1} \text{CG}_{\ell,0;j,j}^{j,j}, \end{aligned}$$

where we used Schur orthogonality relations, cf. [37, Section 4.10], while $\boxed{\text{CG}_{j_1,m_1;j_2,m_2}^{j,m}}$ denotes the Clebsch-Gordan coefficients, cf. [45, Section 3]. We recall in particular that $\text{CG}_{j_1,m_1;j_2,m_2}^{j,m} \in \mathbb{R}$, moreover, the coefficient vanishes unless $m = m_1 + m_2$ and $|j_1 - j_2| \leq j \leq j_1 + j_2$. For later convenience it is also worth recalling that, for all $\sigma \in \mathbb{S}^2$,

$$D_{m_1,k_1}^{(j_1)}(\sigma) D_{m_2,k_2}^{(j_2)}(\sigma) = \sum_{j=|j_1-j_2|}^{j_1+j_2} \text{CG}_{j_1,m_1;j_2,m_2}^{j,m_1+m_2} \text{CG}_{j_1,k_1;j_2,k_2}^{j,k_1+k_2} D_{m_1+m_2,k_1+k_2}^{(j)}(\sigma),$$

which for the particular case of $k_1 = k_2 = 0$ and $j \in \mathbb{Z}_+$ reduces to

$$Y_{j_1,m_1} Y_{j_2,m_2} = \sum_{j=|j_1-j_2|}^{|j_1+j_2|} \text{CG}_{j_1,0;j_2,0}^{j,0} \text{CG}_{j_1,m_1;j_2,m_2}^{j,m} Y_{j,m_1+m_2}. \tag{17}$$

Overall we find

$$Q_j(Y_{\ell,m}) = \text{CG}_{\ell,0;j,j}^{j,j} \sum_{m'=-j}^j \text{CG}_{\ell,m;j,m'}^{j,m+m'} |j, m + m'\rangle \langle j, m'|. \tag{18}$$

In particular $Q_j(Y_{\ell,m}) = 0$ for $\ell > 2j$.

Remark 2.2. In Sect. 4, cf. Theorem 4.7, we will profit of the following well-known properties of the Berezin quantization map $Q_j : C(\mathbb{S}^2) \rightarrow M_{2j+1}(\mathbb{C})$. The latter have interesting and crucial consequences that we could not find in the existing literature and which we describe in the present remark. To begin with, let

$$\langle A|B \rangle_{\text{HS}} := \frac{1}{2j+1} \text{tr}(A^*B), \tag{19}$$

be the normalized Hilbert–Schmidt scalar product between $A, B \in M_{2j+1}(\mathbb{C})$. The irreducible representation $D^{(j)} : SU(2) \rightarrow M_{2j+1}(\mathbb{C})$, induces a new unitary representation $\tilde{D}^{(j)} : SU(2) \rightarrow \mathcal{B}(M_{2j+1}(\mathbb{C}))$, where $M_{2j+1}(\mathbb{C})$ is regarded as an Hilbert space with scalar product $\langle | \rangle_{\text{HS}}$, defined by

$$\tilde{D}^{(j)}(R)(A) := \text{Ad}_{D^{(j)}(R)}(A) = D^{(j)}(R)AD^{(j)}(R)^*. \tag{20}$$

By direct inspection $\tilde{D}^{(j)}(R)$ is unitary with respect to the Hilbert–Schmidt scalar product (19) on $M_{2j+1}(\mathbb{C})$, moreover $R \mapsto \tilde{D}^{(j)}(R)$ is a unitary representation of $SU(2)$. Though $D^{(j)}$ is irreducible, $\tilde{D}^{(j)}$ is not irreducible and by Peter–Weyl Theorem, cf. [29, §5.2], it decomposes into irreducible representations $\{D^{(\ell)}\}_{\ell \in \mathbb{Z}_+/2}$. Notably the Berezin quantization map $Q_j : C(\mathbb{S}^2) \rightarrow M_{2j+1}(\mathbb{C})$ of Equation (13) can be used to provide the explicit decomposition of $\tilde{D}^{(j)}$ in its irreducible components. To this avail, let

$$SU(2) \ni R \mapsto \hat{R} \in \mathcal{B}(L^2(\mathbb{S}^2, \mu_0)) \quad (\hat{R}a)(\sigma) := a(R^{-1}\sigma), \tag{21}$$

be the usual left-action unitary representation of $SU(2)$ —with a slight abuse of notation we dropped the isomorphism $SU(2)/\{\pm I\} \simeq SO(3)$. When necessary will denote by \hat{J}_k the infinitesimal generator of \hat{R} for $R = e^{-iJ_k}$. It is well known that the left-action representation decomposes into the irreducible representations of $SU(2)$ with integer spin by considering the L^2 -decomposition of $L^2(\mathbb{S}^2, \mu_0)$ in spherical harmonics. Actually, restricting the action of \hat{R} to the vector space spanned by $\{Y_{\ell,m}\}_{m \in [-\ell, \ell] \cap \mathbb{Z}}$ leads to a representation which is unitary equivalent to $D^{(\ell)}$.

At this stage we may observe that, by direct inspection,

$$\begin{aligned} Q_j(\hat{R}a) &= \int_{\mathbb{S}^2} a(R^{-1}\sigma)|j, \sigma\rangle\langle j, \sigma| d\mu_j(\sigma) = \int_{\mathbb{S}^2} a(\sigma)|j, R\sigma\rangle\langle j, R\sigma| d\mu_j(\sigma) \\ &= D^{(j)}(R) \int_{\mathbb{S}^2} a(\sigma)|j, \sigma\rangle\langle j, \sigma| d\mu_j(\sigma) D^{(j)}(R)^* = \tilde{D}^{(j)}(R)Q_j(a), \end{aligned} \tag{22}$$

where $a \in C(\mathbb{S}^2)$ while we used the rotation invariance of μ_j and the fact that $D^{(j)}(R)|j, \sigma\rangle = e^{i\alpha(\sigma, R)}|j, R\sigma\rangle$ for $\alpha(\sigma, R) \in \mathbb{R}$, cf. [46]. Thus, Q_j intertwines between the left-action representation and $\tilde{D}^{(j)}$. Moreover, it is well-known that

$$\begin{aligned} \langle Q_j(a)|Q_j(a') \rangle_{\text{HS}} &= \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \overline{a(\sigma)}a'(\sigma')|\langle j, \sigma|j, \sigma'\rangle|^2 d\mu_j(\sigma)d\mu_0(\sigma') \\ &= \langle a|\check{a}'_j \rangle_{L^2(\mathbb{S}^2, \mu_0)} = \langle \check{a}_j|a' \rangle_{L^2(\mathbb{S}^2, \mu_0)}, \end{aligned} \tag{23}$$

where we used both Eqs. (13)–(14).

Equations (22)–(23) have crucial consequences in the decomposition of $\tilde{D}^{(j)}$. In particular, Eq. (22) implies that $\hat{R}\check{a}_j = \widehat{[a \circ R^{-1}]_j}$, i.e. the left-action representation and the check-operator commute. At an infinitesimal level this implies

$$\Delta_{\mathbb{S}^2}\check{a}_j = \widehat{[\Delta_{\mathbb{S}^2}a]_j}, \tag{24}$$

where we observed that $\Delta_{\mathbb{S}^2} = \sum_{k=1}^3 \hat{J}_k^2$. Together with Eq. (23), Eq. (24) implies that

$$\widehat{[Y_{\ell,m}]_j} = c_{j,\ell} Y_{\ell,m}, \tag{25}$$

where $c_{j,\ell}$ is explicitly computed using Eq. (18), cf. Example 2.1:

$$\begin{aligned} c_{j,\ell} &= \langle Y_{\ell,m} | \widehat{[Y_{\ell,m}]_j} \rangle_{L^2(\mathbb{S}^2, \mu_\ell)} = \frac{2\ell+1}{2j+1} \langle Q_j(Y_{\ell,m}) | Q_j(Y_{\ell,m}) \rangle_{\text{HS}} \\ &= (\text{CG}_{\ell,0;j,j}^{j,j})^2 \frac{2\ell+1}{2j+1} \sum_{m'=-j}^j (\text{CG}_{\ell,m;j,m'}^{j,m+m'})^2 = (\text{CG}_{\ell,0;j,j}^{j,j})^2, \end{aligned}$$

where in the last line we used the symmetry property of the Clebsch-Gordan coefficients, cf. [37, Section 8].

Equations (23)–(25) imply that

$$\{Q_j(Y_{\ell,m}) \mid \ell \in \{0, \dots, 2j\}, m \in [-\ell, \ell] \cap \mathbb{Z}\},$$

form a complete orthogonal system in $M_{2j+1}(\mathbb{C})$ with respect to the Hilbert–Schmidt scalar product (19). (As an aside, we observe that this fact provides a quick proof of that $Q_j: \dot{B}_\infty \rightarrow B_j$ is surjective: Indeed, for any $A_j \in B_j$ we may consider $a_j := \sum_{\ell=0}^{2j} \sum_{m=-\ell}^{\ell} A_{\ell,m} Y_{\ell,m} \in \dot{B}_\infty$ where $A_{\ell,m} := \|Q_j(Y_{\ell,m})\|_{\text{HS}}^{-2} \langle Q_j(Y_{\ell,m}) | A_j \rangle_{\text{HS}}$ so that $A_j = Q_j(a_j)$.) Moreover, Eq. (22) ensures that Q_j intertwines between the left-action representation $R \mapsto \hat{R}$ and $\tilde{D}^{(j)}$. Since $R \mapsto \hat{R}$ is unitary equivalent to $D^{(\ell)}$ when restricted to the vector space spanned by $\{Y_{\ell,m}\}_{m \in [-\ell, \ell] \cap \mathbb{Z}}$, it follows that $\tilde{D}^{(j)}$ is unitary equivalent to $D^{(\ell)}$ when restricted to the vector space spanned by $\{Q_j(Y_{\ell,m})\}_{m \in [-\ell, \ell] \cap \mathbb{Z}}$. Thus, $\tilde{D}^{(j)}$ is not irreducible and decomposes into direct sum of the irreducible representations of $SU(2)$ with total spin $\ell \in \{0, \dots, 2j\}$, each of which taken with multiplicity one.

2.3. Berezin SDQ on Γ . The goal of this section is to prove that the Berezin SDQ $Q_j: \dot{B}_\infty \rightarrow B_j$, cf. Equation (13), lifts to a SDQ $Q_j^\Gamma: \dot{B}_\infty^\Gamma \rightarrow \dot{B}_j^\Gamma$ between the corresponding algebras of quasi-local observables for the corresponding infinitely extended systems.

To this avail we recall that in [44] the Berezin SDQ $Q_j: \dot{B}_\infty \rightarrow B_j$ has been lifted to a SDQ $\boxed{Q_j^\Lambda}^\Lambda: \dot{B}_\infty^\Lambda \rightarrow B_j^\Lambda$ for any finite region $\Lambda \Subset \Gamma$. In a nutshell, this boils down to define $\boxed{Q_j^\Lambda}$ by linear extension of the tensor product map $\otimes_{x \in \Lambda} Q_j$ and checking that the data $B_\infty^\Gamma, B_*^\Lambda, \{Q_j\}_{j \in \mathbb{Z}_+/2}$ fulfil the requirement of a SDQ—here $\boxed{B_*^\Lambda} := \otimes_{x \in \Lambda} B_*$. A non-trivial task in this setting is to prove that B_*^Λ is again a bundle of C^* -algebras over $\mathbb{Z}_+/2$: This is addressed in full generality in [44] by using the results of [36].

To extend the results of [10,44] to the case of an infinitely extended system over Γ we need to identify a suitable continuous bundle of C^* -algebras $B_*^\Gamma \subset \prod_{j \in \overline{\mathbb{Z}_+/2}} B_j^\Gamma$, where B_j^Γ are the quasi-local algebras introduced in Section 2.1. Eventually we will define suitable quantization maps $Q_j^\Gamma: \dot{B}_\infty^\Gamma \rightarrow B_j^\Gamma$ abiding by the requirements 3a-3c-3b of a SDQ.

Remark 2.3. Let $\{\mathfrak{A}_j\}_{j \in \overline{\mathbb{Z}_+/2}}$ be a collection of C^* -algebras. For later convenience we recall the following sufficient condition which identifies a continuous bundle of C^* -algebras $\mathfrak{A} \subset \prod_{j \in \overline{\mathbb{Z}_+/2}} \mathfrak{A}_j$ by defining a dense set of (a posteriori) elements of \mathfrak{A} —cf. [39, Prop. 1.2.3], [40, Prop. C.124]. In more details, let $\dot{\mathfrak{A}} \subseteq \prod_{j \in \overline{\mathbb{Z}_+/2}} \mathfrak{A}_j$ be such that:

1. For all $j \in \overline{\mathbb{Z}_+/2}$ the set $\{\mathfrak{a}_j \mid (\mathfrak{a}_j)_{j \in \overline{\mathbb{Z}_+/2}} \in \dot{\mathfrak{A}}\}$ is dense in \mathfrak{A}_j ;
2. $\dot{\mathfrak{A}}$ is a $*$ -algebra;
3. For all $(\mathfrak{a}_j)_{j \in \overline{\mathbb{Z}_+/2}} \in \dot{\mathfrak{A}}$, we have $(\|\mathfrak{a}_j\|_{\mathfrak{A}_j})_{j \in \overline{\mathbb{Z}_+/2}} \in C(\overline{\mathbb{Z}_+/2})$.

Then

$$\mathfrak{A} := \left\{ \mathfrak{a} \in \prod_{j \in \overline{\mathbb{Z}_+/2}} \mathfrak{A}_j \mid \forall \varepsilon > 0 \exists j_\varepsilon \in \mathbb{Z}_+/2, \exists \mathfrak{a}' \in \dot{\mathfrak{A}}: \|\mathfrak{a}_j - \mathfrak{a}'_j\|_{\mathfrak{A}_j} < \varepsilon \forall j \geq j_\varepsilon \right\}, \tag{26}$$

is the smallest continuous bundle of C^* -algebras over $\overline{\mathbb{Z}_+/2}$ which contains $\dot{\mathfrak{A}}$.

The following proposition identifies a continuous bundle of C^* -algebras B_*^Γ with fibers $\{B_j^\Gamma\}_{j \in \overline{\mathbb{Z}_+/2}}$.

Proposition 2.4. *Let $\dot{B}_*^\Gamma \subset \prod_{j \in \overline{\mathbb{Z}_+/2}} B_j^\Gamma$ be defined by*

$$\begin{aligned} \dot{B}_*^\Gamma &:= \text{Alg}(\dot{V}), \\ \dot{V} &:= \left\{ (\mathfrak{a}_j)_{j \in \overline{\mathbb{Z}_+/2}} \in \prod_{j \in \overline{\mathbb{Z}_+/2}} B_j^\Gamma \mid \exists \Lambda \in \Gamma, a_\Lambda \in \dot{B}_\infty^\Lambda: \mathfrak{a}_j = \begin{cases} Q_j^\Lambda(a_\Lambda) & j \in \mathbb{Z}_+/2 \\ a_\Lambda & j = \infty \end{cases} \right\}, \end{aligned} \tag{27}$$

where $\text{Alg}(\dot{V})$ denotes the algebra generated by the vector space \dot{V} .

Then \mathfrak{A} fulfils conditions 2.3–2.3-2.3 of Remark 2.3, thus, it identifies a continuous bundle of C^* -algebras B_*^Γ over $\overline{\mathbb{Z}_+/2}$ defined by

$$B_*^\Gamma := \{(\mathfrak{a}_j)_{j \in \overline{\mathbb{Z}_+/2}} \mid \forall \varepsilon > 0, \exists j_\varepsilon \in \mathbb{Z}_+/2, \exists \mathfrak{a}' \in \dot{B}_*^\Gamma: \|\mathfrak{a}_j - \mathfrak{a}'_j\|_{B_j^\Gamma} < \varepsilon \forall j \geq j_\varepsilon\}. \tag{28}$$

Proof. It suffices to prove that \dot{B}_*^Γ fulfils conditions 2.3–2.3-2.3.

1. Let $j \in \mathbb{Z}_+/2$ and consider $\{\mathfrak{a}_j \mid (\mathfrak{a}_j)_{j \in \overline{\mathbb{Z}_+/2}} \in \dot{B}_*^\Gamma\} \subset B_j^\Gamma$ —a similar argument applies for $j = \infty$. Since $Q_j: \dot{B}_\infty \rightarrow B_j$ is surjective, cf. Remark 2.2, the same holds for $Q_j^\Lambda: \dot{B}_\infty^\Lambda \rightarrow B_j^\Lambda$ for all $\Lambda \in \Gamma$. This implies that any $A_\Lambda \in B_j^\Lambda \subset \dot{B}_j^\Gamma$ can be written as $Q_j^\Lambda(a_{j,\Lambda})$ for some $a_{j,\Lambda} \in \dot{B}_\infty^\Lambda$. Thus, $A_j \in \{\mathfrak{a}_j \mid (\mathfrak{a}_{j'})_{j' \in \overline{\mathbb{Z}_+/2}} \in \dot{B}_*^\Gamma\}$ because $A_j = \mathfrak{a}_j$ for $(\mathfrak{a}_{j'})_{j' \in \overline{\mathbb{Z}_+/2}}$ defined by $\mathfrak{a}_{j'} := Q_{j'}^\Lambda(a_{j,\Lambda})$ for all $j' \in \overline{\mathbb{Z}_+/2}$. Condition 2.3 follows from the density of \dot{B}_j^Γ in B_j^Γ .

2. Condition 2.3 holds because $\dot{B}_*^\Gamma = \text{Alg}(\dot{V})$, moreover, \dot{V} is closed under $*$ -conjugation. Notice that \dot{V} is not an algebra because $Q_j^\Lambda(a_\Lambda \tilde{a}_\Lambda) \neq Q_j^\Lambda(a_\Lambda)Q_j^\Lambda(\tilde{a}_\Lambda)$, although this is true in the limit $j \rightarrow \infty$.
3. For any $(a_j)_{j \in \overline{\mathbb{Z}_+/2}} \in \dot{V}$ we have

$$\|a_j\|_{B_j^\Gamma} = \|Q_j^\Lambda(a_\Lambda)\|_{B_j^\Lambda} \xrightarrow{j \rightarrow \infty} \|a_\Lambda\|_{B_\infty^\Lambda} = \|a_\infty\|_{B_\infty^\Gamma},$$

where we used that $(Q_j^\Lambda(a_\Lambda))_{j \in \overline{\mathbb{Z}_+/2}} \in B_*^\Lambda$, therefore, $\overline{\mathbb{Z}_+/2} \ni j \mapsto \|Q_j^\Lambda(a_\Lambda)\|_{B_j^\Lambda}$ is continuous. Condition 2.3 follows from the latter observation together with the fact that $\|Q_j^\Lambda(a_\Lambda)Q_j^\Lambda(a'_\Lambda) - Q_j^\Lambda(a_\Lambda a'_\Lambda)\|_{B_j^\Lambda} \xrightarrow{j \rightarrow \infty} 0$ and $\dot{B}_*^\Gamma = \text{Alg}(\dot{V})$,

□

We now move to the definition of a SDQ associated with B_∞^Γ and B_*^Γ . For later convenience we observe that, by direct inspection,

$$Q_j^\Lambda \circ \iota_\Lambda^{\Lambda_0} = \iota_\Lambda^{\Lambda_0} \circ Q_j^{\Lambda_0} \quad \forall \Lambda_0 \subset \Lambda \in \Gamma. \tag{29}$$

Theorem 2.5. For $j \in \overline{\mathbb{Z}_+/2}$ let $Q_j^\Gamma: \dot{B}_\infty^\Gamma \rightarrow \dot{B}_j^\Gamma$ be the map defined by

$$Q_j^\Gamma(a_\Lambda) := \begin{cases} Q_j^\Lambda(a_\Lambda) & j \in \mathbb{Z}_+/2 \\ a_\Lambda & j = \infty \end{cases} \tag{30}$$

where $\Lambda \in \Gamma$, $a_\Lambda \in \dot{B}_\infty^\Lambda \subset \dot{B}_\infty^\Gamma$.

Then the data B_∞^Γ , B_*^Γ and $\{Q_j^\Gamma\}_{j \in \overline{\mathbb{Z}_+/2}}$ define a SDQ.

Proof. We will prove properties 3a-3b-3c.

3a By direct inspection $Q_j^\Gamma(a_\Lambda)^* = Q_j^\Gamma(a_\Lambda^*)$ for all $a_\Lambda \in B_\infty^\Lambda \subset \dot{B}_\infty^\Gamma$, moreover, $(Q_j^\Gamma(a_\Lambda))_{j \in \overline{\mathbb{Z}_+/2}} \in \dot{B}_*^\Gamma$ —cf. Equation (27)—thus, it defines a continuous section of B_*^Γ .

3b For all $a_\Lambda, \tilde{a}_\Lambda \in \dot{B}_\infty^\Lambda$ we have

$$Q_j^\Gamma(\{a_\Lambda, \tilde{a}_\Lambda\}_{B_\infty^\Gamma}) = Q_j^\Gamma(\{a_\Lambda, \tilde{a}_\Lambda\}_{B_\infty^\Lambda}) = Q_j^\Lambda(\{a_\Lambda, \tilde{a}_\Lambda\}_{B_\infty^\Lambda}),$$

therefore,

$$\begin{aligned} & \|Q_j^\Gamma(\{a_\Lambda, \tilde{a}_\Lambda\}_{B_\infty^\Gamma}) - ih_j^{-1}[Q_j^\Gamma(a_\Lambda), Q_j^\Gamma(\tilde{a}_\Lambda)]\|_{B_j^\Gamma} \\ &= \|Q_j^\Lambda(\{a_\Lambda, \tilde{a}_\Lambda\}_{B_\infty^\Lambda}) - ih_j^{-1}[Q_j^\Lambda(a_\Lambda), Q_j^\Lambda(\tilde{a}_\Lambda)]\|_{B_j^\Lambda} \xrightarrow{j \rightarrow \infty} 0, \end{aligned}$$

where in the last line we used property 3b for the quantization map $Q_j^\Lambda: \dot{B}_\infty^\Lambda \rightarrow B_j^\Lambda$.

3c For $j = \infty$ property 3c follows from the density of \dot{B}_∞^Γ in B_∞^Γ . For $j \in \mathbb{Z}_+/2$ it suffices to observe that

$$Q_j^\Gamma(\dot{B}_\infty^\Gamma) = \bigcup_{\Lambda \in \Gamma} Q_j^\Gamma(\dot{B}_\infty^\Lambda) = \bigcup_{\Lambda \in \Gamma} Q_j^\Lambda(\dot{B}_\infty^\Lambda) = \bigcup_{\Lambda \in \Gamma} B_j^\Lambda = \dot{B}_j^\Gamma,$$

where we used that $Q_j^\Lambda: \dot{B}_\infty^\Lambda \rightarrow B_j^\Lambda$ is surjective together with the definitions of \dot{B}_∞^Γ and \dot{B}_j^Γ . Since \dot{B}_j^Γ is dense in B_j^Γ , the claim follows.

□

3. The Semiclassical Limit of the Quantum KMS Condition

In this section we will study the notion of classical and quantum thermal equilibrium for the algebras B_j^Γ , $j \in \overline{\mathbb{Z}_+}/2$, of classical and quantum observables for the spin lattice systems over Γ introduced in Section 2.1. Thermal equilibrium is described by states $\omega_j^{\beta, \Gamma} \in S(B_j^\Gamma)$ fulfilling the KMS condition, cf. Equations (1)–(2): These conditions cover the notion of both classical and quantum thermal equilibrium.

Our main interest concerns the connection between classical and quantum thermal equilibrium. Specifically, while the physical justification of the quantum KMS condition has been the subject of many investigations, cf. [33, 50], the classical KMS condition is usually justified with a formal semi-classical limit of the quantum KMS condition, cf. [32]. Our main goal is to prove that this derivation can be proved rigorously within the framework of the SDQ introduced in Sect. 2.

In particular, using the quantization maps $Q_j^\Gamma : \dot{B}_\infty^\Gamma \rightarrow \dot{B}_j^\Gamma$ it is possible to analyse the semi-classical behaviour of sequences of quantum states $(\omega_j)_{j \in \overline{\mathbb{Z}_+}/2}$, $\omega_j \in S(B_j^\Gamma)$, by studying the weak*-limit points of the sequence $(\omega_j \circ Q_j^\Gamma)_{j \in \overline{\mathbb{Z}_+}/2}$ of classical states on $S(B_\infty^\Gamma)$ —it is worth noticing that Q_j^Γ preserves positivity because so does Q_j , cf. Equation (13). At this stage the natural question is whether a sequence of quantum KMS states $(\omega_j)_{j \in \overline{\mathbb{Z}_+}/2}$ leads to weak*-limit points $(\omega_j \circ Q_j^\Gamma)_{j \in \overline{\mathbb{Z}_+}/2}$ which fulfil the classical KMS condition. This is in fact what happens as we will prove in Theorem 3.5.

In Sect. 3.1 we will recall the notion of classical and quantum KMS condition within the framework introduced in Sect. 2.1, cf. [15, 22]. In passing, we will prove a characterization of the classical KMS condition, cf. Lemma 3.2, which is inspired by an analogous characterization of the quantum KMS condition known with the name of Roepstorff-Araki-Sewell auto-correlation lower bound, cf. [15, Thm. 5.3.15]. Section 3.2 is devoted to the proof of Theorem 3.5. The latter is based on Lemma 3.2 together with a result on the semiclassical limit of the quantum derivation associated with the quantum KMS condition, cf. Lemma 3.4.

3.1. Classical and quantum KMS condition on B_∞^Γ and B_j^Γ . In this section we briefly recall the notion of classical and quantum KMS conditions, cf. Equations (1)–(2), for the quasi-local algebras B_j^Γ , B_∞^Γ . Eventually we will move to the investigation of the semiclassical limit of quantum KMS states.

We will begin with the quantum KMS condition: Since the latter is very well-known, cf. [15], we will streamline its presentation by focusing on the main details which will be important for the forthcoming discussion.

The quantum KMS condition (1) relies on the choice of a strongly one-parameter group on the C^* -algebra of interest: For the particular case of B_j^Λ and B_j^Γ suitable one-parameter groups of automorphisms τ^Λ , τ^Γ are identified by considering a family $\boxed{\Phi_j} = \{\Phi_{j, \Lambda}\}_{\Lambda \in \Gamma}$ of self-adjoint elements $\Phi_{j, \Lambda} \in B_j^\Lambda \subset B_j^\Gamma$. We will refer to $\Phi_{j, \Lambda}$ as the **potential** associated with $\Lambda \in \Gamma$: For fixed $\Lambda \in \Gamma$ the **quantum Hamiltonian** $\boxed{H_{j, \Lambda}} \in B_j^\Lambda$ associated to Φ_j is defined by $H_{j, \Lambda} := \sum_{X \in \Lambda} \Phi_{j, X}$. The latter induces a strongly one-parameter group $t \mapsto \boxed{\tau_t^\Lambda}$ on B_j^Λ whose generator $\boxed{\delta_j^\Lambda}$ is given by $\delta_j^\Lambda := i[H_{j, \Lambda}, \cdot]$. Within this setting it can be shown that there exists a unique $(\beta, \delta_j^\Lambda)$ -

KMS quantum state $\boxed{\omega_j^{\beta,\Lambda}}$ on B_j^Λ called **quantum Gibbs state** and defined by

$$\omega_j^{\beta,\Lambda}(A_\Lambda) := \frac{\text{Tr}(e^{-\beta H_{j,\Lambda}} A_\Lambda)}{\text{Tr}(e^{-\beta H_{j,\Lambda}})} \quad \forall A_\Lambda \in B_j^\Lambda. \tag{31}$$

Uniqueness of $(\beta, \delta_j^\Lambda)$ -KMS quantum states is a manifestation of the relative simple nature of thermal equilibrium for systems of finite size: Needless to say, this does not hold any more for infinitely extended system.

Thermal equilibrium on B_j^Γ is described by a suitable limit procedure $\Lambda \uparrow \Gamma$. For that, further mild assumptions on Φ_j are required: In particular, we will assume that

$$\exists \lambda > 0: \|\Phi\|_\lambda := \sum_{m \geq 0} e^{\lambda m} \sup_{x \in \Gamma} \sum_{\substack{X \ni x \\ |X|=m+1}} \|\Phi_{j,X}\|_{B_j^x} < \infty. \tag{32}$$

Within assumption (32) it can be shown that δ_j^Λ converges strongly on \dot{B}_j^Γ as $\Lambda \uparrow \Gamma$ to a C^* -derivation $\boxed{\delta_j^\Gamma}: \dot{B}_j^\Gamma \rightarrow B_j^\Gamma$ which generates a strongly continuous one-parameter group $t \mapsto \tau_t^\Gamma$ on B_j^Γ , cf. [15, Thm. 6.2.4]. In particular, δ_j^Γ is explicitly given by

$$\delta_j^\Gamma(A_\Lambda) = i \sum_{X \in \Gamma} [\Phi_X, A_\Lambda] = i \sum_{\substack{X \in \Gamma \\ X \cap \Lambda \neq \emptyset}} [\Phi_X, A_\Lambda] \quad \forall A_\Lambda \in B_j^\Lambda \subset \dot{B}_j^\Gamma.$$

For later convenience it is worth to recall that for all $A_\Lambda \in B_j^\Gamma, \Lambda \in \Gamma$, we may compute

$$\tau_t^\Gamma(A) = \sum_{n \geq 0} \frac{t^n}{n!} (\delta_j^\Gamma)^n(A_\Lambda), \tag{33}$$

where the series converges in B_j^Γ for $|t| \leq \lambda/2 \|\Phi_j\|_\lambda$, cf. [15, Thm. 6.2.4].

Thus, assumption (32) ensures the existence of a time evolution on B_j^Γ and thermal equilibrium is then described by (β, δ_j^Γ) -KMS quantum states on B_j^Γ . Notably, the (β, δ_j^Γ) -KMS condition does not select a unique state in general: Whenever uniqueness fails a **quantum phase transition** is said to occur.

Concerning the classical KMS condition (2) for the C^* -algebras B_∞^Λ and B_∞^Γ , we will again consider a family $\varphi = \{\varphi_\Lambda\}_{\Lambda \in \Gamma}$ of self-adjoint potentials $\varphi_\Lambda \in \dot{B}_\infty^\Lambda$, cf. [23]. For the classical lattice system in a finite region $\Lambda \in \Gamma$ this suffices to identify a $*$ -derivation $\boxed{\delta_\infty^\Lambda}: \dot{B}_\infty^\Lambda \rightarrow \dot{B}_\infty^\Lambda$ defined by $\delta_\infty^\Lambda := \{h_\Lambda\}$, where $\boxed{h_\Lambda} := \sum_{X \subset \Lambda} \varphi_X$ is called **classical Hamiltonian**. Similarly to the quantum case, the $(\beta, \delta_\infty^\Lambda)$ -KMS condition select a unique state $\boxed{\omega_\infty^{\beta,\Lambda}} \in S(B_\infty^\Lambda)$ called **classical Gibbs state** and defined by

$$\omega_\infty^{\beta,\Lambda}(a_\Lambda) := \frac{\int_{\mathbb{S}_\Lambda^2} a_\Lambda e^{-\beta h_\Lambda} d\mu_0^\Lambda}{\int_{\mathbb{S}_\Lambda^2} e^{-\beta h_\Lambda} d\mu_0^\Lambda} \quad \forall a_\Lambda \in B_\infty^\Lambda. \tag{34}$$

The description of thermal equilibrium for B_∞^Γ , *i.e.* for a classical lattice system on the infinite region Γ , requires further assumptions on φ . A sufficiently mild condition is provided by

$$\sup_{x \in \Gamma} \sum_{\substack{X \in \Gamma \\ X \ni x}} \|\varphi_X\|_{C^1(\mathbb{S}_X^2)} < \infty, \tag{35}$$

where $\|\cdot\|_{C^1(\mathbb{S}_\Lambda^2)}$ denote the C^1 -norm on $C^1((\mathbb{S}^2)^{|\Lambda|})$.

For a family $\varphi = \{\varphi_\Lambda\}_{\Lambda \in \Gamma}$ of potentials fulfilling condition (35) we may introduce a derivation $\boxed{\delta_\infty^\Gamma}: \dot{B}_\infty^\Gamma \rightarrow B_\infty^\Gamma$ defined by

$$\delta_\infty^\Gamma(a_\Lambda) := \sum_{X \in \Gamma} \{a_\Lambda, \varphi_X\}_{B_\infty^\Gamma} = \sum_{\substack{X \in \Gamma \\ X \cap \Lambda \neq \emptyset}} \{a_\Lambda, \varphi_X\}_{B_\infty^\Lambda} \quad \forall a_\Lambda \in B_\infty^\Lambda \subset \dot{B}_\infty^\Lambda. \tag{36}$$

Notice that, on account of Eq. (8), the sum over $X \in \Gamma$ is restricted to $X \cap \Lambda \neq \emptyset$: This implies well-definiteness of δ_∞^Γ because of condition (35). Similarly to the quantum case, δ_∞^Γ can be approximated by δ_∞^Λ , that is, $\delta_\infty^\Gamma(a_\Lambda) = \lim_{X \uparrow \Gamma} \delta_\infty^X(a_\Lambda)$ for all $a_\Lambda \in B_\infty^\Lambda \subset \dot{B}_\infty^\Gamma$. Thermal equilibrium on B_∞^Γ is described by considering $(\beta, \delta_\infty^\Gamma)$ -KMS classical states. Once again, the $(\beta, \delta_\infty^\Gamma)$ -KMS classical condition does not identify a unique state on B_∞^Γ in general, leading to the notion of **classical phase transition**.

Remark 3.1. (i) We stress that conditions (32)–(35)—see also (37)—are minimal requirements for the discussion of this section. In the forthcoming Sect. 4 we will to specialize further our assumptions, *cf.* Theorems 4.1–4.7. In applications these conditions are usually met because the family of potentials Φ_j, φ turn out to be of **finite range**, namely there exists $m \in \mathbb{Z}_+$ and $d > 0$ such that $\varphi_X = 0$ (*resp.* $\Phi_{j,X} = 0$) if $|X| > m$ or $\text{diam}(X) > d$.

(ii) Remarkably, any weak*-limit point of the sequence $(\omega_j^{\beta,\Lambda})_{\Lambda \in \Gamma}$ of $(\beta, \delta_j^\Lambda)$ -KMS quantum states leads to a (β, δ_j^Γ) -KMS quantum state, *cf.* [15, Cor. 6.2.19]. The existence of these weak*-limit points follows from a standard Hahn-Banach and weak*-compactness argument. Similar considerations apply to the sequence $(\omega_\infty^{\beta,\Lambda})_{\Lambda \in \Gamma}$ of $(\beta, \delta_\infty^\Lambda)$ -KMS classical states, ensuring the existence of $(\beta, \delta_\infty^\Gamma)$ -KMS classical states.

At this stage, we are in position to set our investigation of the semiclassical limit of quantum thermal states. Specifically, we will consider a family $\varphi = \{\varphi_\Lambda\}_{\Lambda \in \Gamma}$ abiding by the condition

$$\exists \lambda > 0: \sum_{m \geq 0} e^{\lambda m} \sup_{x \in \Gamma} \sum_{\substack{X \ni x \\ |X|=m+1}} \|\varphi_X\|_{C^1(\mathbb{S}_X^2)} < \infty. \tag{37}$$

The latter implies condition (35), furthermore, it ensures that the family $\Phi_j := \{Q_j^\Gamma(\varphi_\Lambda)\}_{\Lambda \in \Gamma}$ of quantum potentials fulfils (32). Thus, we may consider both $(\beta, \delta_\infty^\Gamma)$ -KMS classical states as well as (β, δ_j^Γ) -KMS quantum states for such families of classical and quantum potentials.

We now consider a sequence $(\omega_j^{\beta,\Gamma})_{j \in \mathbb{Z}_+/2}$ where for each $j \in \mathbb{Z}_+/2$ the state $\omega_j^{\beta,\Gamma} \in S(B_j^\Gamma)$ is a (β, δ_j^Γ) -KMS quantum state. Since $Q_j^\Gamma: \dot{B}_\infty^\Gamma \rightarrow B_j^\Gamma$ preserves positive

elements, we find that $\omega_j^{\beta,\Gamma} \circ Q_j^\Gamma \in S(B_\infty^\Gamma)$ is well-defined—here we implicitly extended $\omega_j^{\beta,\Gamma} \circ Q_j^\Gamma : \dot{B}_\infty^\Gamma \rightarrow \mathbb{C}$ using the continuity of $\omega_j^{\beta,\Gamma}$ and the density of $Q_j^\Gamma(\dot{B}_\infty^\Gamma)$, cf. item 3c. By weak*-compactness of $S(B_\infty^\Gamma)$ the sequence $(\omega_j^{\beta,\Gamma} \circ Q_j^\Gamma)_{j \in \mathbb{Z}_{+}/2}$ has weak*-limit points: A natural question is whether these satisfy the $(\beta, \delta_\infty^\Gamma)$ -KMS classical condition.

This problem has already been investigated in [58] for the case of a lattice system in a finite region. Therein it can be shown that the sequence $(\omega_j^{\beta,\Lambda} \circ Q_j^\Lambda)_{j \in \mathbb{Z}_{+}/2}$ has a limit for $j \rightarrow \infty$, moreover, it holds

$$\lim_{j \rightarrow \infty} \omega_j^{\beta,\Lambda} \circ Q_j^\Lambda = \omega_\infty^{\beta,\Lambda}. \tag{38}$$

In other words the semiclassical limit of the sequence of quantum Gibbs state (31) is the classical Gibbs state (31). Theorem 3.5 generalizes this result to lattice systems on the infinite region Γ .

3.2. Semi-classical limit of quantum KMS condition. The goal of this section is to prove Theorem 3.5, which shows that weak*-limit points of KMS quantum states are KMS classical states.

The proof requires two main ingredients. The first one is a useful characterization of the $(\beta, \delta_\infty^\Gamma)$ -KMS classical condition, cf. Lemma 3.2: The latter is inspired by an analogous characterization of (β, δ_j^Γ) -KMS quantum states, cf. Remark 3.3. The second piece of information is a control of the semiclassical limit of the derivation δ_j^Γ . Specifically, for any finite region $\Lambda \Subset \Gamma$, the DGR condition (3) implies that $h_j^{-1} \delta_j^\Lambda \circ Q_j^\Lambda - Q_j^\Lambda \circ \delta_\infty^\Lambda \xrightarrow{j \rightarrow \infty} 0$ strongly on \dot{B}_∞^Λ . This is a key property which notably holds also for the lattice system on the entire Γ , cf. Lemma 3.4.

We begin with the characterization of the classical KMS condition. The following result is essentially the classical version of the Roepstorff-Araki-Sewell auto-correlation lower bound, cf. [15, Thm. 5.3.15].

Lemma 3.2. *Let \mathfrak{A} be a commutative Poisson C^* -algebra, let $\delta : \mathfrak{A} \rightarrow \mathfrak{A}$ be a $*$ -derivation and consider $\beta \in [0, \infty)$. A state $\omega \in S(\mathfrak{A})$ is a (β, δ) -KMS classical state if and only if*

$$-i\beta\omega(\mathfrak{a}^*\delta(\mathfrak{a})) \geq -i\omega(\{\mathfrak{a}, \mathfrak{a}^*\}), \tag{39}$$

for all $\mathfrak{a} \in \mathfrak{A}$.

We will call inequality (39) the **classical auto-correlation lower bound**, cf. Remark 3.3.

Proof of Lemma 3.2. If $\omega \in S(\mathfrak{A})$ is a (β, δ) -KMS classical state then it also fulfils the classical auto-correlation lower bound (39): Indeed, by direct inspection

$$-i\omega(\{\mathfrak{a}, \mathfrak{a}^*\}) = -i\beta\omega(\mathfrak{a}^*\delta(\mathfrak{a})).$$

Conversely, let $\omega \in S(\mathfrak{A})$ be such that (39) is fulfilled: We will prove that ω fulfils the (β, δ) -KMS classical condition (2).

To begin with we observe that the classical auto-correlation lower bound implies in particular that $-i\omega(\mathfrak{a}^*\delta(\mathfrak{a})) \geq 0$ for all $\mathfrak{a} = \mathfrak{a}^* \in \mathfrak{A}$: In particular $-i\omega(\mathfrak{a}^*\delta(\mathfrak{a})) \in \mathbb{R}$ for

all $\mathfrak{a} = \mathfrak{a}^* \in \dot{\mathfrak{A}}$. This implies that $\omega(\delta(\mathfrak{a})) = 0$ for all $\mathfrak{a} \in \dot{\mathfrak{A}}$: The proof is a classical counterpart of [15, Lem. 5.3.16] and will be reviewed for the sake of clarity. In particular for $\mathfrak{a} = \mathfrak{a}^* \in \mathfrak{A}$ we find

$$\omega(\delta(\mathfrak{a}^2)) = \omega(\mathfrak{a}^* \delta(\mathfrak{a})) + \overline{\omega(\mathfrak{a}^* \delta(\mathfrak{a}))} = 0.$$

Thus $\omega(\delta(\mathfrak{a})) = 0$ on all positive elements $\mathfrak{a} \in \dot{\mathfrak{A}}$: Since finite linear combinations of the latter elements generate \mathfrak{A} we conclude that $\omega \circ \delta = 0$.

Thus, $\omega(\delta(\mathfrak{a})) = 0$ for all $\mathfrak{a} \in \mathfrak{A}$. Evaluating condition (39) for $\mathfrak{a}^* \in \dot{\mathfrak{A}}$ we find

$$-i\beta\omega(\mathfrak{a}\delta(\mathfrak{a}^*)) \geq -i\omega(\{\mathfrak{a}^*, \mathfrak{a}\}) = i\omega(\{\mathfrak{a}, \mathfrak{a}^*\}) \geq i\beta\omega(\mathfrak{a}^*\delta(\mathfrak{a})) = -i\beta\omega(\mathfrak{a}\delta(\mathfrak{a}^*)).$$

where we used that $\omega(\delta(\mathfrak{a}\mathfrak{a}^*)) = 0$ together with the fact that \mathfrak{A} is commutative. Thus, all inequalities must be equalities and we find

$$\omega(\{\mathfrak{a}, \mathfrak{a}^*\}) = \beta\omega(\mathfrak{a}^*\delta(\mathfrak{a})). \quad (40)$$

Let now $\mathfrak{a}, \mathfrak{b} \in \dot{\mathfrak{A}}$: Evaluation of (40) for $\mathfrak{a} + \mathfrak{b}$ leads to

$$\begin{aligned} & \omega(\{\mathfrak{a}, \mathfrak{a}^*\}) + \omega(\{\mathfrak{b}, \mathfrak{b}^*\}) + \omega(\{\mathfrak{a}, \mathfrak{b}^*\}) + \omega(\{\mathfrak{b}, \mathfrak{a}^*\}) \\ &= \omega(\{\mathfrak{a} + \mathfrak{b}, \mathfrak{a}^* + \mathfrak{b}^*\}) = \beta\omega((\mathfrak{a} + \mathfrak{b})^*\delta(\mathfrak{a} + \mathfrak{b})) \\ &= \beta\omega(\mathfrak{a}^*\delta(\mathfrak{a})) + \beta\omega(\mathfrak{b}^*\delta(\mathfrak{b})) + \beta\omega(\mathfrak{a}^*\delta(\mathfrak{b})) + \beta\omega(\mathfrak{b}^*\delta(\mathfrak{a})). \end{aligned}$$

Equating the terms linear in \mathfrak{a} we find the (β, δ) -KMS classical condition. \square

Remark 3.3. The classical auto-correlation lower bound (39) is a classical analogue of the quantum auto-correlation lower bound, cf. [15, Thm. 5.3.15]. To state the latter let \mathfrak{A} be a non-commutative C^* -algebra and let τ be a strongly continuous one-parameter group of $*$ -automorphisms on \mathfrak{A} with infinitesimal generator δ . Then $\omega \in \mathcal{S}(\mathfrak{A})$ is a (β, δ) -KMS quantum state if and only if

$$-i\beta\omega(\mathfrak{a}^*\delta(\mathfrak{a})) \geq \omega(\mathfrak{a}^*\mathfrak{a}) \log \left(\frac{\omega(\mathfrak{a}\mathfrak{a}^*)}{\omega(\mathfrak{a}^*\mathfrak{a})} \right), \quad (41)$$

for all \mathfrak{a} in the domain of δ . In the latter inequality the function $u, v \mapsto u \log(u/v)$ is defined by

$$u \log(u/v) := \begin{cases} u \log(u/v) & uv > 0 \\ 0 & u = 0, v > 0 \\ +\infty & u > 0, v = 0 \end{cases}$$

Notice that the quantum auto-correlation lower bound (41) trivialises to δ -invariance —i.e. $\omega \circ \delta = 0$ — if \mathfrak{A} is commutative. As we will see, cf. the proof of Theorem 3.5, the quantum auto-correlation lower bound (41) reduces to the classical auto-correlation lower bound (39) only for suitably scaled $*$ -derivations.

We now move to the discussion of the semiclassical limit of δ_j^Γ . To this avail we observe that the DGR condition (3) implies

$$\lim_{j \rightarrow \infty} \left\| h_j^{-1} \delta_j^\Lambda(Q_j^\Lambda(a_\Lambda)) - Q_j^\Lambda(\delta_\infty^\Lambda(a_\Lambda)) \right\|_{B_j^\Lambda} = 0 \quad \forall a_\Lambda \in \dot{B}_\infty^\Lambda. \quad (42)$$

The following lemma proves that the same property holds in the thermodynamical limit.

Lemma 3.4. For all $a_\Lambda \in \dot{B}_\infty^\Lambda \subset \dot{B}_\infty^\Gamma$ it holds

$$\lim_{j \rightarrow \infty} \left\| h_j^{-1} \delta_j^\Gamma(Q_j^\Gamma(a_\Lambda)) - Q_j^\Gamma(\delta_\infty^\Gamma(a_\Lambda)) \right\|_{B_j^\Gamma} = 0. \tag{43}$$

Proof. Let $a_\Lambda \in \dot{B}_\infty^\Lambda$. According to Definition (30) we have

$$\delta_j^\Gamma(Q_j^\Gamma(a_\Lambda)) = \delta_j^\Gamma(Q_j^\Lambda(a_\Lambda)) = \frac{1}{i} \sum_{X \cap \Lambda \neq \emptyset} [Q_j^\Lambda(a_\Lambda), Q_j^X(\varphi_X)].$$

Similarly, the continuity of $Q_j^\Gamma : \dot{B}_\infty^\Gamma \rightarrow \dot{B}_j^\Gamma$ implies

$$Q_j^\Gamma[\delta_\infty^\Gamma(a_\Lambda)] = Q_j^\Gamma\left(\sum_{X \cap \Lambda \neq \emptyset} \{a_\Lambda, \varphi_X\}_{B_\infty^\Lambda}\right) = \sum_{X \cap \Lambda \neq \emptyset} Q_j^{\Lambda \cup X}(\{a_\Lambda, \varphi_X\}_{B_\infty^\Lambda}).$$

Overall we have

$$\begin{aligned} & \left\| h_j^{-1} \delta_j^\Gamma(Q_j^\Gamma(a_\Lambda)) - Q_j^\Gamma(\delta_\infty^\Gamma(a_\Lambda)) \right\|_{B_j^\Gamma} \\ & \leq \sum_{X \cap \Lambda \neq \emptyset} \left\| \frac{1}{ih_j} [Q_j^\Lambda(a_\Lambda), Q_j^X(\varphi_X)] - Q_j^{\Lambda \cup X}(\{a_\Lambda, \varphi_X\}_{B_\infty^\Lambda}) \right\|_{B_j^{\Lambda \cup X}}. \end{aligned}$$

For each $j \in \mathbb{Z}_+/2$ the series

$$\sum_{X \cap \Lambda \neq \emptyset} \left\| \frac{1}{ih_j} [Q_j^\Lambda(a_\Lambda), Q_j^X(\varphi_X)] - Q_j^{\Lambda \cup X}(\{a_\Lambda, \varphi_X\}_{B_\infty^\Lambda}) \right\|_{B_j^{\Lambda \cup X}},$$

converges on account of condition (37). We will now prove that it vanishes in the limit $j \rightarrow \infty$. Notice that each term of the series vanishes as $j \rightarrow \infty$ on account of the DGR condition (3). We will now prove that each term can be bounded as

$$\left\| \frac{1}{ih_j} [Q_j^\Lambda(a_\Lambda), Q_j^X(\varphi_X)] - Q_j^{\Lambda \cup X}(\{a_\Lambda, \varphi_X\}_{B_\infty^\Lambda}) \right\|_{B_j^{\Lambda \cup X}} \leq c_\Lambda \|\varphi_X\|_{C^1(\mathbb{S}_X^2)},$$

for a j -independent constant $c_\Lambda > 0$. This will allow to apply dominated convergence, concluding the proof.

To this avail we consider the Banach space

$$C^0(\mathbb{S}_{\Lambda^c}^2, C^1(\mathbb{S}_\Lambda^2)) = C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c}, \quad \|f\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c}} := \sup_{\sigma_{\Lambda^c} \in \mathbb{S}_{\Lambda^c}^2} \|f(\sigma_{\Lambda^c})\|_{C^1(\mathbb{S}_\Lambda^2)}.$$

where \otimes_ε denotes the injective tensor product of Banach spaces, cf. [51, Section 3.2].

Let $T_j : \dot{B}_\infty^\Gamma \rightarrow \dot{B}_j^\Gamma$ be the linear operator defined by

$$T_j(b_X) := \frac{1}{ih_j} [Q_j^\Lambda(a_\Lambda), Q_j^X(b_X)] - Q_j^{\Lambda \cup X}(\{a_\Lambda, b_X\}_{B_\infty^\Lambda}),$$

for all $b_X \in \dot{B}_\infty^X \subset \dot{B}_\infty^\Gamma$, $X \in \Gamma$. By direct inspection we find

$$\|T_j(b_X)\|_{B_j^\Gamma} \leq c_\Lambda \|b_X\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c}},$$

which implies that T_j has a unique extension $T_j: C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow B_j^\Gamma$. Notice that, the DGR condition (3) entails $\|T_j(b_X)\|_{B_j^\Gamma} \xrightarrow{j \rightarrow \infty} 0$ for all $b_X \in \dot{B}_\infty^X$, $X \in \Gamma$. Moreover,

$$\|T_j(b_X)\|_{B_j^\Gamma} \leq \|T_j\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow B_j^\Gamma} \|b_X\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c}} \leq \|T_j\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow B_j^\Gamma} \|b_X\|_{C^1(\mathbb{S}_\Lambda^2)},$$

where $\|T_j\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow B_j^\Gamma}$ denotes the operator norm of T_j . Thus, if we were able to prove that $\sup_j \|T_j\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow B_j^\Gamma} < \infty$ then condition (37) would entail that

$$\sum_{X \cap \Lambda \neq \emptyset} \left\| \frac{1}{i\hbar_j} [Q_j^\Lambda(a_\Lambda), Q_j^X(\varphi_X)] - Q_j^{\Lambda \cup X}(\{a_\Lambda, \varphi_X\}_{B_\infty^\Lambda}) \right\|_{B_j^{\Lambda \cup X}} = \sum_{X \cap \Lambda \neq \emptyset} \|T_j(\varphi_X)\|_{B_j^\Lambda},$$

converges, moreover, it vanishes as $j \rightarrow \infty$ by dominated convergence, concluding the proof.

Thus, we prove that $\sup_j \|T_j\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow B_j^\Gamma} < \infty$. For that let us consider the linear map

$$T_j^\Lambda: C^1(\mathbb{S}_\Lambda^2) \rightarrow B_j^\Lambda \quad T_j^\Lambda(b_\Lambda) := \frac{1}{i\hbar_j} [Q_j^\Lambda(a_\Lambda), Q_j^\Lambda(b_\Lambda)] - Q_j^\Lambda(\{a_\Lambda, b_\Lambda\}_{B_\infty^\Lambda}).$$

We then "lift" T_j^Λ to

$$\hat{T}_j^\Lambda: C^1(\mathbb{S}_\Lambda^2) \rightarrow \prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda, \quad [\hat{T}_j^\Lambda(b_\Lambda)]_p := \begin{cases} T_j^\Lambda(b_\Lambda) & p = j \\ 0 & p \neq j \end{cases},$$

where $\prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda$ denotes the full C^* -direct product, cf. [12]. By direct inspection \hat{T}_j^Λ is linear and bounded, moreover, the DGR condition (3) implies that

$$\begin{aligned} \sup_{j \in \mathbb{Z}_+/2} \|\hat{T}_j^\Lambda(b_\Lambda)\|_{\prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda} &= \sup_{j \in \mathbb{Z}_+/2} \sup_{p \in \mathbb{Z}_+/2} \|[\hat{T}_j^\Lambda(b_\Lambda)]_p\|_{B_p^\Lambda} \\ &= \sup_{j \in \mathbb{Z}_+/2} \|T_j^\Lambda(b_\Lambda)\|_{B_j^\Lambda} < \infty, \end{aligned}$$

for all $b_\Lambda \in C^1(\mathbb{S}_\Lambda^2)$. By Banach-Steinhaus Theorem it follows that

$$\sup_{j \in \mathbb{Z}_+/2} \|\hat{T}_j^\Lambda\|_{C^1(\mathbb{S}_\Lambda^2) \rightarrow \prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda} < \infty.$$

We then consider the inductive tensor product of $\hat{T}_j^\Lambda: C^1(\mathbb{S}_\Lambda^2) \rightarrow \prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda$ with $Q_j^{\Lambda^c}: B_\infty^{\Lambda^c} \rightarrow B_j^{\Lambda^c}$, cf. [51, Prop. 3.2]. The latter is the unique bounded linear map

$$\hat{T}_j^\Lambda \otimes_\varepsilon Q_j^{\Lambda^c}: C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow \prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda \otimes B_j^{\Lambda^c},$$

which extends the algebraic tensor product $\hat{T}_j^\Lambda \otimes Q_j^{\Lambda^c}$, that is,

$$(\hat{T}_j^\Lambda \otimes_\varepsilon Q_j^{\Lambda^c})(b_\Lambda \otimes b_{\Lambda^c}) = \hat{T}_j^\Lambda(b_\Lambda) \otimes Q_j^{\Lambda^c}(b_{\Lambda^c}),$$

for all $b_\Lambda \in C^1(\mathbb{S}_\Lambda^2)$ and $b_{\Lambda^c} \in B_\infty^{\Lambda^c}$. Notice that $\prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda \otimes B_j^{\Lambda^c}$ is a C^* -algebra once completed with its unique C^* -cross norm. The uniqueness of such norm is due to the

fact that $B_j^{\Lambda^c}$ is the C^* -inductive limit of finite dimensional, hence nuclear, C^* -algebras, therefore, it is nuclear as well, cf. [12, Section II.9.4.5]. We recall that

$$\begin{aligned} \|\hat{T}_j^\Lambda \otimes_\varepsilon Q_j^{\Lambda^c}\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow \prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda \otimes B_j^{\Lambda^c}} &= \|\hat{T}_j^\Lambda\|_{C^1(\mathbb{S}_\Lambda^2) \rightarrow \prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda} \|Q_j^{\Lambda^c}\|_{B_\infty^{\Lambda^c} \rightarrow B_j^{\Lambda^c}} \\ &= \|\hat{T}_j^\Lambda\|_{C^1(\mathbb{S}_\Lambda^2) \rightarrow \prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda}, \end{aligned}$$

therefore, $\sup_j \|\hat{T}_j^\Lambda \otimes_\varepsilon Q_j^{\Lambda^c}\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow \prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda \otimes B_j^{\Lambda^c}} < \infty$. Finally, let

$$\hat{T}_j : C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow \prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda \otimes B_j^{\Lambda^c} \quad [\hat{T}_j(f)]_p := \begin{cases} T_j(f) & p = j \\ 0 & p \neq j \end{cases}.$$

Then \hat{T}_j is linear and continuous, moreover, by direct inspection

$$\hat{T}_j(b_\Lambda \otimes b_{\Lambda^c}) = \hat{T}_j^\Lambda(b_\Lambda) \otimes Q_j^{\Lambda^c}(b_{\Lambda^c}).$$

It follows that $\hat{T}_j = \hat{T}_j^\Lambda \otimes_\varepsilon Q_j^{\Lambda^c}$ and therefore

$$\begin{aligned} \|T_j\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow B_j^\Gamma} &= \|\hat{T}_j\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow \prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda \otimes B_j^{\Lambda^c}} \\ &= \|\hat{T}_j^\Lambda \otimes_\varepsilon Q_j^{\Lambda^c}\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c} \rightarrow \prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda \otimes B_j^{\Lambda^c}} \\ &= \|\hat{T}_j^\Lambda\|_{C^1(\mathbb{S}_\Lambda^2) \rightarrow \prod_{p \in \mathbb{Z}_+/2} B_p^\Lambda}, \end{aligned}$$

showing that $\sup_j \|T_j\|_{C^1(\mathbb{S}_\Lambda^2) \otimes_\varepsilon B_\infty^{\Lambda^c}} < \infty$. □

We are in position to prove Theorem 3.5, which guarantees that, for any sequence $(\omega_j^{\beta, \Gamma})_{j \in \mathbb{Z}_+/2}$ of (β, δ_j^Γ) -KMS quantum states on B_j^Γ , any weak*-limit point of the sequence $(\omega_j^{\beta, \Gamma} \circ Q_j^\Gamma)_{j \in \mathbb{Z}_+/2} \subset S(B_\infty^\Gamma)$ is a $(\beta, \delta_\infty^\Gamma)$ -KMS classical states. At its core, the proof is based on the observation that the quantum (β, δ_j^Γ) -KMS condition coincides with the quantum $(\beta/h, h\delta_j)$ -KMS condition, $h = 2j + 1$, together with the application of Lemma 3.4 and of the classical auto-correlation lower bound (39).

Theorem 3.5. *For all $j \in \mathbb{Z}_+/2$ let $\omega_j^{\beta, \Gamma} \in S(B_j^\Gamma)$ be a (β, δ_j^Γ) -KMS quantum state on B_j^Γ . Then, any weak*-limit point $\omega_\infty^{\beta, \Gamma} \in S(B_\infty^\Gamma)$ of the sequence $(\omega_j^{\beta, \Gamma} \circ Q_j^\Gamma)_{j \in \mathbb{Z}_+/2}$ is a $(\beta, \delta_\infty^\Gamma)$ -KMS classical state.*

Proof. Up to moving to a subsequence in j we may assume

$$\omega_\infty^{\beta, \Gamma}(a_\Lambda) = \lim_{j \rightarrow \infty} \omega_j^{\beta, \Gamma}(Q_j^\Lambda(a_\Lambda)) \quad \forall a_\Lambda \in B_\infty^\Lambda \subset B_\infty^\Gamma.$$

We will prove that $\omega_\infty^{\beta, \Gamma}$ fulfils the classical $(\beta, \delta_\infty^\Gamma)$ -KMS conditions by proving condition (39), cf. Lemma 3.2. To this avail let $a_\Lambda \in \hat{B}_\infty^\Lambda$ and consider

$$\begin{aligned} -i\beta\omega_\infty^{\beta, \Gamma}(a_\Lambda^* \delta_\infty^\Gamma(a_\Lambda)) &= \lim_{j \rightarrow \infty} (-i\beta)\omega_j^{\beta, \Gamma}(Q_j^\Gamma(a_\Lambda^* \delta_\infty^\Gamma(a_\Lambda))) \\ &= \lim_{j \rightarrow \infty} (-i\beta)\omega_j^{\beta, \Gamma}(Q_j^\Lambda(a_\Lambda^*) Q_j^\Gamma(\delta_\infty^\Gamma(a_\Lambda))) \\ &= \lim_{j \rightarrow \infty} -i\beta h_j^{-1} \omega_j^{\beta, \Gamma}(Q_j^\Lambda(a_\Lambda^*) \delta_j^\Gamma(Q_j^\Lambda(a_\Lambda))) \quad \text{Lem. 3.4.} \end{aligned}$$

At this stage we may apply the quantum auto-correlation lower bound (41) for $\omega_j^{\beta,\Gamma}$, cf. Remark 3.3:

$$-i\beta\omega_j^{\beta,\Gamma}(Q_j^\Lambda(a_\Lambda^*)\delta_j^\Gamma(Q_j^\Lambda(a_\Lambda))) \geq \omega_j^{\beta,\Gamma}(Q_j^\Lambda(a_\Lambda^*)Q_j^\Lambda(a_\Lambda)) \log \left(\frac{\omega_j^{\beta,\Gamma}(Q_j^\Lambda(a_\Lambda^*)Q_j^\Lambda(a_\Lambda))}{\omega_j^{\beta,\Gamma}(Q_j^\Lambda(a_\Lambda)Q_j^\Lambda(a_\Lambda^*))} \right).$$

Thus, the DGR condition (3) leads to

$$\begin{aligned} & -i\beta\omega_\infty^{\beta,\Gamma}(a_\Lambda^*\delta_\infty(a_\Lambda)) \geq \\ & -\lim_{j \rightarrow \infty} \omega_j^{\beta,\Gamma}(Q_j^\Lambda(a_\Lambda^*)Q_j^\Lambda(a_\Lambda))h_j^{-1} \log \left(1 + \frac{\omega_j^{\beta,\Gamma}([\mathbb{Q}_j^\Lambda(a_\Lambda), \mathbb{Q}_j^\Lambda(a_\Lambda^*)])}{\omega_j^{\beta,\Gamma}(Q_j^\Lambda(a_\Lambda^*)Q_j^\Lambda(a_\Lambda))} \right) \\ & = -\lim_{j \rightarrow \infty} \omega_j^{\beta,\Gamma}(Q_j^\Lambda(a_\Lambda^*)Q_j^\Lambda(a_\Lambda))h_j^{-1} \log \left(1 + \frac{i}{2j+1} \frac{\omega_j^{\beta,\Gamma}(Q_j^\Lambda(\{a_\Lambda, a_\Lambda^*\}_{B_\infty^\Lambda}))}{\omega_j^{\beta,\Gamma}(Q_j^\Lambda(a_\Lambda^*)Q_j^\Lambda(a_\Lambda))} \right) \\ & = -i \lim_{j \rightarrow \infty} \omega_j^{\beta,\Gamma}(Q_j^\Lambda(\{a_\Lambda, a_\Lambda^*\}_{B_\infty^\Lambda})) = -i\omega_\infty^{\beta,\Gamma}(\{a_\Lambda, a_\Lambda^*\}_{B_\infty^\Lambda}). \end{aligned}$$

□

Remark 3.6. Proposition 3.5 has the following important consequence. Let $\beta \in [0, \infty)$ be such that there exists a unique $(\beta, \delta_\infty^\Gamma)$ -KMS classical state on B_∞^Γ , i.e., there are no classical phase transitions at β . In particular, there exists a unique limit point $\omega_\infty^{\beta,\Gamma} \in S(B_\infty^\Gamma)$ of $(\omega_\infty^{\beta,\Lambda})_{\Lambda \in \Gamma}$, cf. Remark 3.1. Let $(\omega_j^{\beta,\Gamma})_{j \in \mathbb{Z}_+/2}$ be any sequence such that $\omega_j^{\beta,\Gamma} \in S(B_j^\Gamma)$ is a (β, δ_j^Γ) -KMS quantum state on $S(B_j^\Gamma)$ for all $j \in \mathbb{Z}_+/2$. Then, any limit point of the sequence $(\omega_j^{\beta,\Gamma} \circ Q_j^\Gamma)_{j \in \mathbb{Z}_+/2}$ is a $(\beta, \delta_\infty^\Gamma)$ -KMS classical state by Proposition 3.5. However, by assumption there is only one of such states, thus, the whole sequence $(\omega_j^{\beta,\Gamma} \circ Q_j^\Gamma)_{j \in \mathbb{Z}_+/2}$ converges to the unique classical $(\beta, \delta_\infty^\Gamma)$ -KMS classical states $\omega_\infty^{\beta,\Gamma}$. In other words the absence of classical phase transitions implies that any sequence of (β, δ_j^Γ) -KMS quantum states has a semiclassical limit.

In the particular case where there is a unique (β, δ_j^Γ) -KMS quantum state $\omega_j^{\beta,\Gamma}$ on B_j^Γ —i.e., there are no quantum phase transitions for all $j \in \mathbb{Z}_+/2$ at β — the previous observation boils down to the equality

$$\lim_{j \rightarrow \infty} \left(\lim_{\Lambda \uparrow \Gamma} \omega_j^{\beta,\Lambda} \right) \circ Q_j^\Gamma = \omega_\infty^{\beta,\Gamma} = \lim_{\Lambda \uparrow \Gamma} \lim_{j \rightarrow \infty} \omega_j^{\beta,\Lambda} \circ Q_j^\Gamma,$$

where in the last equality we used the result of [58] together with Remark 3.1.

4. Common Absence of CPTs and of QPTs at High Temperatures

In this section we will explore the relation between classical and quantum phase transitions within the setting of the Berezin quantization on Γ , cf. Section 2. Our goal is to prove that, under mild conditions, above a common threshold critical temperature there is absence of both classical and quantum phase transitions, that is, uniqueness of both

$(\beta, \delta_\infty^\Gamma)$ -KMS classical states on B_∞^Γ and (β, δ_j^Γ) -KMS quantum states on B_j^Γ for all $j \in \mathbb{Z}_+/2$ holds for $\beta \leq \beta_c$ for a given $\beta_c \in (0, +\infty)$.

To this avail we will consider [15, Prop. 6.2.45], which provides a mild sufficient condition on a family of quantum potential $\Phi_j = \{\Phi_{j,\Lambda}\}_{\Lambda \in \Gamma}$ which ensures uniqueness of the corresponding (β, δ_j^Γ) -KMS quantum state on B_j^Γ for all $\beta \leq \beta_j$, where β_j is called **quantum critical inverse temperature**. This result does not suffice to our purposes because: (a) there is no clear classical counterpart for the assumption on the family Φ_j ; (b) the estimate for the critical inverse temperature β_j provided in [15, Prop. 6.2.45] is not uniform in $j \in \mathbb{Z}_+/2$, spoiling a comparison with the classical critical inverse temperature —cf. Remark 4.9.

To encompass these issues we revised the proof of [15, Prop. 6.2.45] in order to produce a new result which is uniform in $j \in \mathbb{Z}_+/2$. In particular, in Sect. 4.1 we prove a classical analogous of [15, Prop. 6.2.45], cf. Theorem 4.1. The latter result provides a sufficient condition, cf. (44), on the family of potential $\varphi = \{\varphi_\Lambda\}_{\Lambda \in \Gamma}$ which ensures uniqueness for the corresponding $(\beta, \delta_\infty^\Gamma)$ -KMS classical state on B_∞^Γ provided $\beta \leq \beta_\infty$, β_∞ being the **classical critical inverse temperature**. Condition (44) is essentially a classical version of the one required in [15], cf. (72), and essentially boils down to an estimate on the sup-norm of the derivatives of the potentials φ_Λ , $\Lambda \in \Gamma$.

Next, we provide a uniqueness result, cf. Theorem 4.7, for (β, δ_j^Γ) -KMS quantum states on B_j^Γ associated with the family of quantum potentials $\Phi_{j,\Lambda} = Q_j^\Gamma(\varphi_\Lambda)$ obtained by Berezin quantization of the classical potential. The novelty of this result relies on the bound on the critical inverse temperature, which is now uniform in $j \in \mathbb{Z}_+/2$, allowing for a better comparison with the classical critical inverse temperature. Finally, it is shown that the assumptions of Theorem 4.7 implies those of Theorem 4.1, cf. Remark 4.9. For this class of models, this establishes a sufficient criterion for the absence of both classical and quantum phase transitions.

4.1. Uniqueness result for classical KMS states. The goal of this section is to prove a uniqueness result for $(\beta, \delta_\infty^\Gamma)$ -KMS classical states, cf. Theorem 4.1. The latter identifies a sufficient condition on the classical interaction potentials $\varphi = \{\varphi_\Lambda\}_{\Lambda \in \Gamma}$ which ensures uniqueness of $(\beta, \delta_\infty^\Gamma)$ -KMS classical states on B_∞^Γ for sufficiently low β , that is, at sufficiently high temperature. This result is inspired from [15, Prop. 6.2.45] and adapted to the classical setting. In the forthcoming sections we will present an analogous uniqueness result in the quantum setting, cf. Theorem 4.7.

Theorem 4.1. *Let $\varphi = \{\varphi_\Lambda\}_{\Lambda \in \Gamma}$ with $\varphi \in C^{2s}(\mathbb{S}_\Lambda^2)$, $s > 7/4$, be such that*

$$\|\varphi\|_{0,s} := \sum_{m \geq 0} (C_\Delta^s K_s)^m \sup_{\substack{x \in \Gamma \\ |\Lambda|=m+1 \\ x \in \Lambda}} \|\varphi_\Lambda\|_{C^{2s}(\mathbb{S}_\Lambda^2)} < +\infty, \tag{44}$$

where $\|\cdot\|_{C^{2s}(\mathbb{S}_\Lambda^2)}$ is the C^{2s} -norm on $C^{2s}((\mathbb{S}^2)^{|\Lambda|})$ while $K_s > 1$ and $C_\Delta \geq 1$ are defined by

$$K_s := \sum_{\ell \in \mathbb{Z}_+} \frac{(2\ell + 1)^{5/2}}{[1 + \ell(\ell + 1)]^s}, \quad C_\Delta := \|1 - \Delta_{\mathbb{S}^2}\|_{C^2(\mathbb{S}^2) \rightarrow C^0(\mathbb{S}^2)}, \tag{45}$$

the latter being the operator norm of $1 - \Delta_{\mathbb{S}^2} : C^2(\mathbb{S}^2) \rightarrow C^0(\mathbb{S}^2)$. Then there exists a unique $(\beta, \delta_\infty^\Gamma)$ -KMS classical state on B_∞^Γ for all $\beta \in [0, \beta_{0,s})$ where

$$\beta_{0,s} := \frac{\log 2}{2C_\Delta^s K_s \|\varphi\|_{0,s}}. \tag{46}$$

The proof of Theorem 4.1 is rather long and will take the whole section. For the sake of clarity we start by collecting some observations in a series of Remarks 4.2–4.3–4.6. Together with Lemma 4.4 these will develop all technical tools need for the proof of Theorem 4.1.

Remark 4.2. For later convenience we will recollect in this remark a few useful observations concerning the Fourier-Laplace expansion on \mathbb{S}^2 and \mathbb{S}_Λ^2 , $\Lambda \Subset \Gamma$.

With reference to Example 2.1 we denote by $Y_{\ell,m} \in \dot{B}_\infty$ the spherical harmonic with parameter $\ell \in \mathbb{Z}_+$, $m \in [-\ell, \ell] \cap \mathbb{Z}$. With the choice of normalization of Eq. (15) we have $\|Y_{\ell,m}\|_{B_\infty} \leq 1$, moreover, the set $\{Y_{\ell,m}\}_{\ell,m}$ is a complete orthogonal system for $L^2(\mathbb{S}^2, \mu_0)$ made by orthogonal but not L^2 -normalized vectors. In particular for any $a \in L^2(\mathbb{S}^2, \mu_0)$ we have the following Fourier-Laplace expansion:

$$a = \sum_{\ell \in \mathbb{Z}_+} \sum_{m=-\ell}^{\ell} \hat{a}(\ell, m) Y_{\ell,m}, \quad \hat{a}(\ell, m) := (2\ell + 1) \langle Y_{\ell,m} | a \rangle_{L^2(\mathbb{S}^2, \mu_0)}. \tag{47}$$

It is worth recalling that the series in Eq. (47) converges in the L^2 -norm, moreover, the series converges uniformly with all derivatives if $a \in \dot{B}_\infty$. Furthermore, for $\ell, s \in \mathbb{Z}_+$

$$\begin{aligned} |\hat{a}(\ell, m)| &= (2\ell + 1) [1 + \ell(\ell + 1)]^{-s} |\langle Y_{\ell,m} | (1 - \Delta_{\mathbb{S}^2})^s a \rangle_{L^2(\mathbb{S}^2, \mu_0)}| \\ &\leq C_\Delta^s (2\ell + 1) [1 + \ell(\ell + 1)]^{-s} \|a\|_{C^{2s}(\mathbb{S}^2)} \|Y_{\ell,m}\|_{L^1(\mathbb{S}^2, \mu_0)} \\ &\leq C_\Delta^s (2\ell + 1)^{1/2} [1 + \ell(\ell + 1)]^{-s} \|a\|_{C^{2s}(\mathbb{S}^2)}, \end{aligned}$$

where C_Δ has been defined in Eq. (45) while we used the eigenvalues property $-\Delta_{\mathbb{S}^2} Y_{\ell,m} = \ell(\ell + 1) Y_{\ell,m}$ together with integration by parts and Cauchy-Schwarz inequality. The bound above ensures the uniform convergence of the series in Eq. (47) provided that $1/2 - 2s + 1 < -1$, the additional +1 factor arising from the summation over $m \in [-\ell, \ell]$.

The previous considerations generalize to the case of B_∞^Λ , $\Lambda \Subset \Gamma$. In particular, if $\ell_\Lambda \in \mathbb{Z}_+^\Lambda$ and $m_\Lambda \in \mathbb{Z}^\Lambda$, with $m_x \in [-\ell_x, \ell_x] \cap \mathbb{Z}$ for all $x \in \Lambda$, we set

$$Y_{\ell_\Lambda, m_\Lambda} := \bigotimes_{x \in \Lambda} Y_{\ell_x, m_x} \in \dot{B}_\infty^\Lambda. \tag{48}$$

The Fourier–Laplace expansion of $a_\Lambda \in \dot{B}_\infty^\Lambda$ is given by

$$a_\Lambda = \sum_{\ell_\Lambda \in \mathbb{Z}_+^\Lambda} \sum_{m_\Lambda} \hat{a}(\ell_\Lambda, m_\Lambda) Y_{\ell_\Lambda, m_\Lambda}, \quad \hat{a}(\ell_\Lambda, m_\Lambda) := \left(\prod_{x \in \Lambda} (2\ell_x + 1) \right) \langle Y_{\ell_\Lambda, m_\Lambda} | a_\Lambda \rangle_{L^2(\mathbb{S}_\Lambda^2, \mu_\Lambda^\Delta)}, \tag{49}$$

where we may estimate

$$|\hat{a}(\ell_\Lambda, m_\Lambda)| \leq C_\Delta^{s|\Lambda|} \prod_{x \in \Lambda} \left(\frac{(2\ell_x + 1)^{1/2}}{[1 + \ell_x(\ell_x + 1)]^s} \right) \|a_\Lambda\|_{C^{2s}(\mathbb{S}_\Lambda^2)}.$$

Remark 4.3. Let $\omega_\infty^\Gamma \in S(B_\infty^\Gamma)$ be an arbitrary state on B_∞^Γ . Then ω_∞^Γ is uniquely determined by its values on \hat{B}_∞^Λ for all $\Lambda \in \Gamma$. On account of Remark 4.2—cf. Equation (47)—it suffices to determine the values

$$\underline{\omega}_\infty^\Gamma(\ell_\Lambda, m_\Lambda) := \omega_\infty^\Gamma(Y_{\ell_\Lambda, m_\Lambda}), \quad \underline{\omega}_\infty^\Gamma(\ell_\emptyset, m_\emptyset) := \omega_\infty^\Gamma(1) = 1, \quad (50)$$

where $\ell_\Lambda \in \mathbb{N}^\Lambda$ while $m_\Lambda \in \mathbb{Z}^\Lambda$ are such that $m_x \in [-\ell_x, \ell_x] \cap \mathbb{Z}$ for all $x \in \Lambda$. Notice that we avoided to consider the case $\ell_x = 0$ for some $x \in \Lambda$ because of the following compatibility condition: If $\ell_x = 0$ then

$$\underline{\omega}_\infty^\Gamma(\ell_\Lambda, m_\Lambda) := \omega_\infty^\Gamma(Y_{\ell_\Lambda, m_\Lambda}) = \omega_\infty^\Gamma(Y_{\ell_{\Lambda \setminus \{x\}}, m_{\Lambda \setminus \{x\}}}) = \underline{\omega}_\infty^\Gamma(\ell_{\Lambda \setminus \{x\}}, m_{\Lambda \setminus \{x\}}).$$

At this stage we may regard $\underline{\omega}_\infty^\Gamma$ as an element of a suitable Banach space \underline{X} . The latter is defined by

$$\begin{aligned} \underline{X} := \{ \underline{f} = (f_\Lambda)_{\Lambda \in \Gamma} \mid f_\Lambda : \{(\ell_\Lambda, m_\Lambda) \in \mathbb{N}^\Lambda \times \mathbb{Z}^\Lambda \mid m_x \in [-\ell_x, \ell_x] \cap \mathbb{Z} \forall x \in \Lambda\} \rightarrow \mathbb{C} : \\ \| \underline{f} \|_{\underline{X}} := \sup_{\Lambda \in \Gamma} \sup_{\ell_\Lambda, m_\Lambda} |f_\Lambda(\ell_\Lambda, m_\Lambda)| < \infty \}, \end{aligned} \quad (51)$$

where $f_\emptyset \in \mathbb{C}$. It is worth observing that $\underline{\omega}_\infty^\Gamma \in \underline{X}$ because of our choice for the normalization of $Y_{\ell_\Lambda, m_\Lambda}$, cf. Example 2.1, which ensures that $\| \underline{\omega}_\infty^\Gamma \|_{\underline{X}} \leq 1$.

Summing up, a state $\omega_\infty^\Gamma \in S(B_\infty^\Gamma)$ is completely determined by the corresponding element $\underline{\omega}_\infty^\Gamma \in \underline{X}$.

The following Lemma is crucial and shows the relation between the classical $(\beta, \delta_\infty^\Gamma)$ -KMS condition and the action of $SU(2)$ (in fact, of $SO(3)$) on B_∞^Γ . In particular for all $x \in \Gamma$ let $R_x \in SU(2)$. Recalling Sect. 2 let $\hat{R}_x \in \mathcal{B}(L^2(\mathbb{S}^2, \mu_0))$ be the corresponding unitary operator on $L^2(\mathbb{S}^2, \mu_0)$ defined by $\hat{R}_x a_x := a_x \circ R_x^{-1}$. If $R_x = \exp[D_x]$, $D_x \in \mathfrak{su}(2)$, we will denote by \hat{D}_x the vector field on \mathbb{S}_x^2 corresponding to the infinitesimal generator of $\hat{R}_x(t)$, where $R_x(t) := \exp[tD_x]$ —for definiteness $\hat{R}_x(t) = \exp[-it\hat{D}_x]$. The action of \hat{R}_x on B_∞^x can be lifted to B_∞^Γ by setting, for all $a_\Lambda \in B_\infty^\Lambda$ and $\Lambda \in \Gamma$, $\hat{R}_x a_\Lambda(\sigma_\Lambda) := a_\Lambda(\sigma'_\Lambda)$ with $\sigma_\Lambda = (\sigma_y)_{y \in \Lambda}$, $\sigma'_\Lambda = (\sigma'_y)_{y \in \Lambda}$ being $\sigma'_y = \sigma_y$ for $y \neq x$ while $\sigma'_x := R_x^{-1} \sigma_x$.

Lemma 4.4. *Let $\omega_\infty^{\beta, \Gamma} \in S(B_\infty^\Gamma)$ be a $(\beta, \delta_\infty^\Gamma)$ -KMS classical states. Then, for any $x \in \Gamma$, $R_x \in SU(2)$ and $a_\Lambda \in \hat{B}_\infty^\Lambda$, $\Lambda \in \Gamma$, it holds*

$$\omega_\infty^{\beta, \Gamma}(a_\Lambda) = \omega_\infty^{\beta, \Gamma} \left(e^{\beta \sum_{X \ni x} (I - \hat{R}_x) \varphi_X} \hat{R}_x a_\Lambda \right). \quad (52)$$

Remark 4.5.

- (i) Lemma 4.4 is inspired by [24, Prop. 5], where Eq. (52) has been proved in the context of finite dimensional Poisson manifolds. Equation 52 can be understood as an adaptation to the present setting.
- (ii) Notice that the element $\sum_{X \ni x} (I - \hat{R}_x) \varphi_X \in B_\infty^\Gamma$ can be interpreted as the difference $h_\Gamma - \hat{R}_x h_\Gamma$ between the (ill-defined) classical Hamiltonian $h_\Gamma = \sum_{X \in \Gamma} \varphi_X$ and its R_x -rotated version: Loosely speaking, the locality of the involved rotation ensures that the above difference is finite.

Proof of Lemma 4.4. Let \hat{D}_x be the infinitesimal generator of \hat{R}_x . Since \mathbb{S}_x^2 is a symplectic manifold and \hat{D}_x is a Poisson vector field on \mathbb{S}_x^2 , there exists a closed 1-form $\alpha_x \in \Omega^1(\mathbb{S}_x^2)$ such that $\hat{D}_x a_x = \pi_x(\mathrm{d}a_x, \alpha_x)$ for all $a_x \in \dot{B}_\infty^x$, π_x being the Poisson bitensor associated with the symplectic 2-form on \mathbb{S}_x^2 . Let $\{U_x\}$ be an open cover of \mathbb{S}_x^2 such that $\alpha_x = \mathrm{d}g_{U_x}$, $g_{U_x} \in C^\infty(U_x)$, and let $\{\chi_{U_x}\}$ be a partition of unity associated with $\{U_x\}$. It follows that, for all $a_x \in \dot{B}_\infty^x$,

$$\hat{D}_x a_x = \sum_{U_x} \hat{D}_x (\chi_{U_x} a_x) = \sum_{U_x} \{\chi_{U_x} a_x, g_{U_x}\}_x.$$

The latter identity and the $(\beta, \delta_\infty^\Gamma)$ -KMS condition leads to

$$\omega_\infty^{\beta, \Gamma}(\hat{D}_x a_\Lambda) = -\beta \omega_\infty^{\beta, \Gamma} \left(a_\Lambda \sum_{U_x} \chi_{U_x} \sum_{X \ni x} \{g_{U_x}, \varphi_X\} \right) = \beta \omega_\infty^{\beta, \Gamma} \left[\left(\sum_{X \ni x} \hat{D}_x \varphi_X \right) a_\Lambda \right]. \quad (53)$$

Integration of the latter identity leads to Eq. (52). In more details we first observe that

$$\sum_{X \ni x} (I - \hat{R}_x) \varphi_X = \lim_{Y \uparrow \Gamma} (I - \hat{R}_x) h_Y, \quad h_Y = \sum_{X \subset Y} \varphi_X \in \dot{B}_\infty^Y,$$

where the limit converges in the B_∞^Γ -norm. We then consider the function

$$\omega_Y(t) := \omega_\infty^{\beta, \Gamma} \left[e^{\beta \sum_{X \subset Y}^{X \ni x} (I - \hat{R}_x(t)) \varphi_X} \hat{R}_x(t) a_\Lambda \right],$$

where $\hat{R}_x(t) = \exp[-it \hat{D}_x]$: Notice that

$$\omega_\infty^{\beta, \Gamma} \left(e^{\beta \sum_{X \ni x} (I - \hat{R}_x) \varphi_X} \hat{R}_x a_\Lambda \right) = \lim_{Y \uparrow \Gamma} \omega_Y(1).$$

By dominated convergence and on account of Eq. (53) we also have

$$\begin{aligned} -i \dot{\omega}_Y(t) &= \omega_\infty^{\beta, \Gamma} \left[e^{\beta \sum_{X \subset Y}^{X \ni x} (I - \hat{R}_x(t)) \varphi_X} \left([\hat{D}_x \hat{R}_x(t) a_\Lambda] - \beta (\hat{R}_x(t) a_\Lambda) \sum_{\substack{X \subset Y \\ X \ni x}} \hat{D}_x \hat{R}_x(t) \varphi_X \right) \right] \\ &= -\beta \omega_\infty^{\beta, \Gamma} \left[e^{\beta \sum_{X \subset Y}^{X \ni x} (I - \hat{R}_x(t)) \varphi_X} [\hat{R}_x(t) a_\Lambda] \sum_{\substack{X \subset Y \\ X \ni x}} \hat{D}_x \varphi_X \right] \\ &\quad + \omega_\infty^{\beta, \Gamma} \left[\hat{D}_x \left(e^{\beta \sum_{X \subset Y}^{X \ni x} (I - \hat{R}_x(t)) \varphi_X} [\hat{R}_x(t) a_\Lambda] \right) \right] \\ &= -\beta \omega_\infty^{\beta, \Gamma} \left[e^{\beta \sum_{X \subset Y}^{X \ni x} (I - \hat{R}_x(t)) \varphi_X} [\hat{R}_x(t) a_\Lambda] \sum_{\substack{X \subset Y \\ X \ni x}} \hat{D}_x \varphi_X \right] \\ &\quad + \beta \omega_\infty^{\beta, \Gamma} \left[e^{\beta \sum_{X \subset Y}^{X \ni x} (I - \hat{R}_x(t)) \varphi_X} [\hat{R}_x(t) a_\Lambda] \sum_{X \ni x} \hat{D}_x \varphi_X \right] \\ &= \beta \omega_\infty^{\beta, \Gamma} \left[e^{\beta \sum_{X \subset Y}^{X \ni x} (I - \hat{R}_x(t)) \varphi_X} [\hat{R}_x(t) a_\Lambda] \sum_{\substack{X \cap Y^c \neq \emptyset \\ X \ni x}} \hat{D}_x \varphi_X \right]. \end{aligned}$$

By direct inspection we also find

$$\lim_{Y \uparrow \Gamma} \sup_{t \in [0,1]} |\dot{\omega}_Y(t)| = 0,$$

thus, we have

$$\omega_\infty^{\beta, \Gamma} \left(e^{\beta \sum_{x \in \Lambda} (I - \hat{R}_x) \varphi_x} \hat{R}_x a_\Lambda \right) = \lim_{Y \uparrow \Gamma} \omega_Y(1) = \lim_{Y \uparrow \Gamma} \omega_Y(0) + \lim_{Y \uparrow \Gamma} \int_0^1 \dot{\omega}_Y(t) dt = \omega_\infty^{\beta, \Gamma} (a_\Lambda).$$

□

Remark 4.6. Lemma 4.4 suffices to prove Theorem 4.1 for the particular case $\beta = 0$, where the assumption (44) on the potential $\{\varphi_\Lambda\}_{\Lambda \in \Gamma}$ is not needed. In this case the unique $(0, \delta_\infty^\Gamma)$ -KMS classical states $\omega_\infty^{0, \Gamma}$ coincides with the normalized Poisson trace

$$\omega_\infty^{0, \Gamma} (a_\Lambda) = \int_{\mathbb{S}_\Lambda^2} a_\Lambda(\sigma_\Lambda) d\mu_0^\Lambda(\sigma_\Lambda). \tag{54}$$

Although this can be shown with several methods, it is instructive to prove it with the help of Lemma 4.4: This leads to a first intuition on the strategy we will employ in the proof of Theorem 4.1.

Let $\omega_\infty^{0, \Gamma} \in \mathcal{S}(B_\infty^\Gamma)$ be a $(0, \delta_\infty^\Gamma)$ -KMS classical states: We wish to prove that the associated element $\underline{\omega}_\infty^{0, \Gamma} \in \underline{\mathbf{X}}$ is necessarily of the form

$$\underline{\omega}_\infty^{0, \Gamma} (\ell_\Lambda, m_\Lambda) = \begin{cases} 1 & \Lambda = \emptyset \\ 0 & \Lambda \neq \emptyset \end{cases}. \tag{55}$$

This implies that $\omega_\infty^{0, \Gamma}$ abides by Eq. (54) on account of the Fourier-Laplace expansion (49). To prove Eq. (55) it suffices to observe that Lemma 4.4 implies $\omega_\infty^{0, \Gamma} \circ \hat{R}_x = \omega_\infty^{0, \Gamma}$.

Let $\Lambda \subseteq \Gamma$, $\ell_\Lambda \in \mathbb{N}^\Lambda$, $m_\Lambda \in \mathbb{Z}^\Lambda$, $m_x \in [-\ell_x, \ell_x] \cap \mathbb{Z}$ for all $x \in \Lambda$. For a fixed $x \in \Lambda$ we compute

$$\begin{aligned} \underline{\omega}_\infty^{0, \Gamma} (\ell_\Lambda, m_\Lambda) &= \omega_\infty^{0, \Gamma} (Y_{\ell_\Lambda, m_\Lambda}) = \int_{SU(2)} \omega_\infty^{0, \Gamma} (\hat{R}_x Y_{\ell_\Lambda, m_\Lambda}) dR_x \\ &= \omega_\infty^{0, \Gamma} \left(\int_{SU(2)} \hat{R}_x Y_{\ell_\Lambda, m_\Lambda} dR_x \right). \end{aligned}$$

where dR_x denotes the normalized Haar measure on $SU(2)$. At this stage we observe that the irreducibility of the left-representation of $SU(2)$ on the space generated by $\{Y_{\ell_x, m_x}\}_{m_x}$ entails

$$\int_{SU(2)} \hat{R}_x Y_{\ell_x, m_x} dR_x = \delta_{\ell_x}^0 \delta_{m_x}^0. \tag{56}$$

This implies

$$\int_{SU(2)} \hat{R}_x Y_{\ell_\Lambda, m_\Lambda} dR_x = \int_{SU(2)} \hat{R}_x Y_{\ell_x, m_x} dR_x \otimes Y_{\ell_{\Lambda \setminus \{x\}}, m_{\Lambda \setminus \{x\}}} = \delta_{\ell_x}^0 \delta_{m_x}^0 Y_{\ell_{\Lambda \setminus \{x\}}, m_{\Lambda \setminus \{x\}}} = 0,$$

where in the second equality we used that $\ell_x \in \mathbb{N}$.

Proof of Theorem 4.1. Let $\omega_\infty^{\beta,\Gamma} \in S(B_\infty^\Gamma)$ be a $(\beta, \delta_\infty^\Gamma)$ -KMS classical states. We will consider the associated element $\underline{\omega}_\infty^{\beta,\Gamma} \in \underline{X}$ as described in Remark 4.3: Our goal is to prove that $\underline{\omega}_\infty^{\beta,\Gamma}$ is the solution to a linear equation in \underline{X} which is unique under assumption (44).

To begin with, we choose an arbitrary but fixed bijection $\Gamma \simeq \mathbb{Z}$ and induce an ordering on Γ based on such map. Thus, for any $\Lambda \Subset \Gamma$ we may set $x := \min_{y \in \Lambda} y$ where the minimum is taken with respect to the chosen ordering. The choice of the ordering is only made to select a distinguished point $x \in \Lambda$ for any $\Lambda \Subset \Gamma$.

Let now $\ell_\Lambda \in \mathbb{N}^\Lambda$ and $m_\Lambda \in \mathbb{Z}^\Lambda$ be such that $m_y \in [-\ell_y, \ell_y] \cap \mathbb{Z}$ for all $y \in \Lambda$. Proceeding as in Remark 4.6 we compute

$$\begin{aligned} \underline{\omega}_\infty^{\beta,\Gamma}(\ell_\Lambda, m_\Lambda) &= \omega_\infty^{\beta,\Gamma}(Y_{\ell_\Lambda, m_\Lambda}) \\ &= \omega_\infty^{\beta,\Gamma}\left(\int_{SU(2)} (I - \hat{R}_x) Y_{\ell_\Lambda, m_\Lambda} dR_x\right) \\ &= \omega_\infty^{\beta,\Gamma}\left(\int_{SU(2)} \left(I - e^{\beta \sum_{X \ni x} (I - \hat{R}_x) \varphi_X}\right) Y_{\ell_\Lambda, m_\Lambda} dR_x\right), \end{aligned}$$

where in the second equality we used that $R_x \mapsto \hat{R}_x$ is a unitary irreducible representation when restricted on the vector space generated by $\{Y_{\ell_x, m_x}\}_{m_x}$ while $\ell_x \in \mathbb{N}$, cf. Remark 4.6. The third equality is nothing but Eq. (52). The exponential in the last term can be expanded in a series converging in B_∞^Γ , thus,

$$\begin{aligned} \underline{\omega}_\infty^{\beta,\Gamma}(\ell_\Lambda, m_\Lambda) &= - \sum_{n \geq 1} \frac{\beta^n}{n!} \sum_{\substack{X_1, \dots, X_n \\ x \in X_1 \cap \dots \cap X_n}} \omega_\infty^{\beta,\Gamma}\left(\int_{SU(2)} \prod_{k=1}^n (I - \hat{R}_x) \varphi_{X_k} Y_{\ell_\Lambda, m_\Lambda} dR_x\right) \\ &= - \sum_{n \geq 1} \frac{\beta^n}{n!} \sum_{\substack{X_1, \dots, X_n \\ x \in X_1 \cap \dots \cap X_n}} \sum_{\substack{\ell_{X_1}, \dots, \ell_{X_n} \\ m_{X_1}, \dots, m_{X_n}}} \int_{SU(2)} \prod_{k=1}^n C_{X_k, R_x}(\ell_{X_k}, m_{X_k}) dR_x \\ &\quad \omega_\infty^{\beta,\Gamma}\left(Y_{\ell_{X_1}, m_{X_1}} \cdots Y_{\ell_{X_n}, m_{X_n}} Y_{\ell_\Lambda, m_\Lambda}\right), \end{aligned} \tag{57}$$

where we used the Fourier-Laplace expansion discussed in Remark 4.2 and set

$$\begin{aligned} C_{X_k, R_x}(\ell_{X_k}, m_{X_k}) &:= \left(\prod_{y \in X_k} (2\ell_{X_k, y} + 1)\right) \langle Y_{\ell_{X_k}, m_{X_k}} | (I - \hat{R}_x) \varphi_{X_k} \rangle_{L^2(\mathbb{S}_{X_k}^2, \mu_0^{X_k})} \\ &= \left(\prod_{y \in X_k} \frac{(2\ell_{X_k, y} + 1)}{[1 + \ell_{X_k, y} (\ell_{X_k, y} + 1)]^s}\right) \langle Y_{\ell_{X_k}, m_{X_k}} | (I - \hat{R}_x) (I - \Delta_{\mathbb{S}^2, X_k})^s \varphi_{X_k} \rangle_{L^2(\mathbb{S}_{X_k}^2, \mu_0^{X_k})}, \end{aligned}$$

where $\Delta_{\mathbb{S}^2, X_k} := \bigotimes_{y \in X_k} \Delta_{\mathbb{S}^2, y}$ denotes the tensor product of the Laplacians $\Delta_{\mathbb{S}^2, y}$ acting on \mathbb{S}_y^2 .

At this stage it is important to carefully analyse the product of the spherical harmonics appearing in Eq. (57):

$$\begin{aligned} &Y_{\ell_{X_1}, m_{X_1}} \cdots Y_{\ell_{X_n}, m_{X_n}} Y_{\ell_\Lambda, m_\Lambda} \\ &= \left(\prod_{y \in \Lambda \cap S_n^c} Y_{\ell_{\Lambda, y}, m_{\Lambda, y}}\right) \prod_{y \in S_n} Y_{\tilde{\ell}_{\Lambda, y}, \tilde{m}_{\Lambda, y}} Y_{\tilde{\ell}_{X_1, y}, \tilde{m}_{X_1, y}} \cdots Y_{\tilde{\ell}_{X_n, y}, \tilde{m}_{X_n, y}}, \end{aligned}$$

where we set $\boxed{S_n} := X_1 \cup \dots \cup X_n$ while $\tilde{\ell}_{X_k} \in \mathbb{Z}_+^{S_n}$ denotes the extension of ℓ_{X_k} obtained by setting $\tilde{\ell}_{X_k,y} = 0$ for $y \notin X_k$ — \tilde{m}_{X_k} is defined similarly. In particular, by an iterated use of Eq. (17) we find, for all $y \in S_n$,

$$\begin{aligned} & Y_{\tilde{\ell}_{\Lambda,y}, \tilde{m}_{\Lambda,y}} \prod_{k=1}^n Y_{\tilde{\ell}_{X_k,y}, \tilde{m}_{X_k,y}} \\ &= \sum_{s_{y,1}=|\tilde{\ell}_{\Lambda,y}-\tilde{\ell}_{X_1,y}|}^{\tilde{\ell}_{\Lambda,y}+\tilde{\ell}_{X_1,y}} c_{s_{y,1}} Y_{s_{y,1}, \tilde{m}_{\Lambda,y}+\tilde{m}_{X_1,y}} \prod_{k=2}^n Y_{\tilde{\ell}_{X_k,y}, \tilde{m}_{X_k,y}} \\ &= \sum_{\substack{s_{y,1}, \dots, s_{y,n} \\ |s_{y,k-1}-\tilde{\ell}_{X_{k-1},y}| \leq s_{y,k} \leq |s_{y,k-1}+\tilde{\ell}_{X_{k-1},y}|}} \left(\prod_{k=1}^n c_{s_{y,k}} \right) Y_{s_{y,n}, \tilde{m}_{\Lambda,y}+\tilde{m}_{X_1,y}+\dots+\tilde{m}_{X_n,y}}, \end{aligned}$$

where $c_{s_{y,k}}, k \in \{1, \dots, n\}$ are defined in Eq. (17)—we omitted the m -dependence since it will not play any role. The particular values of $c_{s_{y,k}}$ are not important, however, we crucially observe that $|c_{s_{y,k}}| \leq 1$. For later convenience we also observe that

$$\begin{aligned} N(\ell_{X_1}, \dots, \ell_{X_n}) &:= \prod_{y \in S_n} \sum_{s_{y,1}=|\tilde{\ell}_{\Lambda,y}-\tilde{\ell}_{X_1,y}|}^{\tilde{\ell}_{\Lambda,y}+\tilde{\ell}_{X_1,y}} \dots \sum_{s_{y,n}=|s_{n-1,y}-\tilde{\ell}_{X_n,y}|}^{s_{n-1,y}+\tilde{\ell}_{X_n,y}} \\ &\leq \prod_{y \in S_n} \prod_{k=1}^n (2\tilde{\ell}_{X_k,y} + 1) = \prod_{k=1}^n \prod_{y \in X_k} (2\ell_{X_k,y} + 1), \end{aligned}$$

because $\sum_{s=|a-b|}^{a+b} = 2 \min\{a, b\} + 1$. This implies that, once considering the product over $y \in S_n$ and expanding the resulting sum, the product of spherical harmonics considered above can be written as a sum of at most $N(\ell_{X_1}, \dots, \ell_{X_n})$ terms of the form $Y_{\ell_{S_n}^k, m_{S_n}^k}$ where $\ell_{S_n}^k, m_{S_n}^k, k = 1, \dots, N(\ell_{X_1}, \dots, \ell_{X_n})$, are built out of $\ell_{\Lambda \cap S_n}, \ell_{X_1}, \dots, \ell_{X_n}$. Explicitly we have

$$\begin{aligned} & Y_{\ell_{X_1}, m_{X_1}} \dots Y_{\ell_{X_n}, m_{X_n}} Y_{\ell_{\Lambda}, m_{\Lambda}} \\ &= \left(\prod_{y \in \Lambda \cap S_n^c} Y_{\ell_{\Lambda,y}, m_{\Lambda,y}} \right) \prod_{y \in S_n} \sum_{s_{y,1}, \dots, s_{y,n}} \left(\prod_{k=1}^n c_{s_{y,k}} \right) Y_{s_{y,n}, \tilde{m}_{\Lambda,y}+\tilde{m}_{X_1,y}+\dots+\tilde{m}_{X_n,y}} \\ &= \left(\prod_{y \in \Lambda \cap S_n^c} Y_{\ell_{\Lambda,y}, m_{\Lambda,y}} \right) \sum_{k=1}^{N(\ell_{X_1}, \dots, \ell_{X_n})} C'(\ell_{S_n}^k, m_{S_n}^k) Y_{\ell_{S_n}^k, m_{S_n}^k}, \end{aligned}$$

where the explicit expression of the coefficients $C'(\ell_{S_n}^k, m_{S_n}^k)$ will not matter in the forthcoming discussion, however, it will be important to observe that $|C'(\ell_{S_n}^k, m_{S_n}^k)| \leq 1$.

Summing up, any $(\beta, \delta_\infty^\Gamma)$ -KMS $\omega_\infty^{\beta, \Gamma} \in S(B_\infty^\Gamma)$ fulfils

$$\begin{aligned} \underline{\omega}_\infty^{\beta, \Gamma}(\ell_\Lambda, m_\Lambda) = & \\ & - \sum_{n \geq 1} \frac{\beta^n}{n!} \sum_{\substack{X_1, \dots, X_n \\ x \in X_1 \cap \dots \cap X_n}} \sum_{\substack{\ell_{X_1}, \dots, \ell_{X_n} \\ m_{X_1}, \dots, m_{X_n}}} \int_{SU(2)} \prod_{k=1}^n C_{X_k, R_x}(\ell_{X_k}, \\ & m_{X_k}) dR_x \sum_{k=1}^{N(\ell_{X_1}, \dots, \ell_{X_n})} C'(\ell_{S_n}^k, m_{S_n}^k) \omega_\infty^{\beta, \Gamma} \left(Y_{\ell_{S_n}^k, m_{S_n}^k} \prod_{y \in \Lambda \cap S_n^c} Y_{\ell_{\Lambda, y}, m_{\Lambda, y}} \right). \end{aligned}$$

Let $\boxed{X(n, \Lambda)} := \Lambda \cup S_n$ and set $\ell_{X(n, \Lambda)}^k := \ell_{S_n}^k \ell_{\Lambda \cap S_n^c}$ (i.e. $\ell_{X(n, \Lambda), y}^k = \ell_{\Lambda, y}$ if $y \in \Lambda \cap S_n^c$ and $\ell_{X(n, \Lambda), y}^k = \ell_{S_n, y}^k$ if $y \in S_n$) and similarly $m_{X(n, \Lambda)} = m_{S_n} m_{\Lambda \cap S_n^c}$. Then the above equality can be written as a linear equation in \underline{X} , in particular

$$(I - L_\infty^\beta) \underline{\omega}_\infty^{\beta, \Gamma} = \underline{\delta}. \tag{58}$$

Here $\underline{\delta} \in \underline{X}$ and $L_\infty^\beta \in \mathcal{B}(\underline{X})$ are defined by

$$\underline{\delta}_\Lambda(\ell_\Lambda, m_\Lambda) := \begin{cases} 1 & \Lambda = \emptyset \\ 0 & \Lambda \neq \emptyset \end{cases}, \tag{59}$$

while for all $\underline{f} \in \underline{X}$ we set $(L_\infty^\beta \underline{f})_\emptyset = 0$ and

$$\begin{aligned} (L_\infty^\beta \underline{f})_\Lambda(\ell_\Lambda, m_\Lambda) := & - \sum_{n \geq 1} \frac{\beta^n}{n!} \sum_{\substack{X_1, \dots, X_n \\ x \in X_1 \cap \dots \cap X_n}} \sum_{\substack{\ell_{X_1}, \dots, \ell_{X_n} \\ m_{X_1}, \dots, m_{X_n}}} \int_{SU(2)} \prod_{k=1}^n C_{X_k, R_x}(\ell_{X_k}, m_{X_k}) dR_x \\ & \sum_{k=1}^{N(\ell_{X_1}, \dots, \ell_{X_n})} C'(\ell_{S_n}^k, m_{S_n}^k) f_{X(n, \Lambda)}(\ell_{X(n, \Lambda)}^k, m_{X(n, \Lambda)}^k), \end{aligned} \tag{60}$$

for all non-empty $\Lambda \in \Gamma$ —we recall that we set $x := \min_{y \in \Lambda} y$.

At this stage we may bound $\|L_\infty^\beta\|_{\mathcal{B}(\underline{X})}$ in such a way that $\|L_\infty^\beta\|_{\mathcal{B}(\underline{X})} < 1$ if β is small enough. This will ensure that (58) has a unique solution $\underline{\omega}_\infty^{\beta, \Gamma} \in \underline{X}$, therefore, its associated state $\omega_\infty^{\beta, \Gamma}$ will be the unique $(\beta, \delta_\infty^\Gamma)$ -KMS classical states on B_∞^Γ . (In passing, the forthcoming estimates will also prove that L_∞^β is bounded on \underline{X} .) To this avail we observe that,

$$\begin{aligned} \sup_{\ell_\Lambda, m_\Lambda} |(L_\infty^\beta \underline{f})_\Lambda(\ell_\Lambda, m_\Lambda)| \leq & \|\underline{f}\|_{\underline{X}} \sum_{n \geq 1} \frac{\beta^n}{n!} \sum_{\substack{X_1, \dots, X_n \\ x \in X_1 \cap \dots \cap X_n}} \sum_{\substack{\ell_{X_1}, \dots, \ell_{X_n} \\ m_{X_1}, \dots, m_{X_n}}} \int_{SU(2)} \prod_{k=1}^n |C_{X_k, R_x} \\ & (\ell_{X_k}, m_{X_k})| \prod_{y \in X_k} (2\ell_{X_k, y} + 1) dR_x, \end{aligned}$$

where we used the bound on $|C'(\ell_{S_n}^k, m_{S_n}^k)|$ and on $N(\ell_{X_1}, \dots, \ell_{X_n})$. Moreover, proceeding as in Remark 4.2 we have

$$\begin{aligned} & \sum_{\substack{\ell_{X_1}, \dots, \ell_{X_n} \\ m_{X_1}, \dots, m_{X_n}}} \int_{SU(2)} \prod_{k=1}^n |C_{X_k, R_x}(\ell_{X_k}, m_{X_k})| \prod_{y \in X_k} (2\ell_{X_k, y} + 1) dR_x \\ & \leq \sum_{\ell_{X_1}, \dots, \ell_{X_n}} 2^n \prod_{k=1}^n \left(\prod_{y \in X_k} \frac{(\ell_{X_k, y} + 1)^{5/2}}{[1 + \ell_{X_k, y}(\ell_{X_k, y} + 1)]^s} \right) (C_\Delta^s)^{|X_k|} \|\varphi_{X_k}\|_{C^{2s}(\mathbb{S}_{X_k}^2)} \\ & = 2^n \prod_{k=1}^n (C_\Delta^s K_s)^{|X_k|} \|\varphi_{X_k}\|_{C^{2s}(\mathbb{S}_{X_k}^2)}, \end{aligned}$$

where K_s has been defined in Eq. (45). It follows that

$$\begin{aligned} \sup_{\ell_\Lambda, m_\Lambda} |(L_\infty^\beta \underline{f})_\Lambda(\ell_\Lambda, m_\Lambda)| & \leq \|\underline{f}\|_{\underline{X}} \sum_{n \geq 1} \frac{(2\beta)^n}{n!} \sum_{\substack{X_1, \dots, X_n \\ x \in X_1 \cap \dots \cap X_n}} \prod_{k=1}^n (C_\Delta^s K_s)^{|X_k|} \|\varphi_{X_k}\|_{C^{2s}(\mathbb{S}_{X_k}^2)} \\ & = \|\underline{f}\|_{\underline{X}} \sum_{n \geq 1} \frac{1}{n!} \left(2\beta \sum_{\substack{X \subseteq \Gamma \\ x \in X}} (C_\Delta^s K_s)^{|X|} \|\varphi_X\|_{C^{2s}(\mathbb{S}_X^2)} \right)^n. \end{aligned}$$

Finally, we have

$$\begin{aligned} \sum_{\substack{X \subseteq \Gamma \\ x \in X}} (C_\Delta^s K_s)^{|X|} \|\varphi_X\|_{C^{2s}(\mathbb{S}_X^2)} & = \sum_{m \geq 0} (C_\Delta^s K_s)^{m+1} \sum_{\substack{|X|=m+1 \\ x \in X}} \|\varphi_X\|_{C^{2s}(\mathbb{S}_X^2)} \\ & \leq \sum_{m \geq 0} (C_\Delta^s K_s)^{m+1} \sup_{x \in \Gamma} \sum_{\substack{|X|=m+1 \\ x \in X}} \|\varphi_X\|_{C^{2s}(\mathbb{S}_X^2)} = C_\Delta^s K_s \|\varphi\|_{0,s}, \end{aligned}$$

where $\|\varphi\|_{0,s}$ has been defined in Eq. (44). Thus, $L_\infty^\beta \in \mathcal{B}(\underline{X})$ with

$$\|L_\infty^\beta\|_{\mathcal{B}(\underline{X})} \leq \exp \left[2C_\Delta^s K_s \beta \|\varphi\|_{0,s} \right] - 1,$$

which implies $\|L_\infty^\beta\|_{\mathcal{B}(\underline{X})} < 1$ provided $\beta < \beta_{0,s}$, where $\beta_{0,s}$ has been defined in Eq. (46). □

4.2. Uniqueness result for quantum KMS state. The goal of this section is to prove a uniqueness result for (β, δ_j^1) -KMS quantum states on B_j^Γ , cf. Theorem 4.7 The latter applies under hypothesis very similar to those of Theorem 4.1. In fact, Theorems 4.1, 4.7 will imply that, under suitably mild assumptions, for high enough temperatures there is absence of both classical and quantum phase transitions, cf. Remark 4.9. The proof of Theorem 4.7 is inspired by [15, Prop. 6.2.45], see also [30], although it requires a different argument to ensure a uniform bound on $j \in \mathbb{Z}_+/2$.

Theorem 4.7. *Let $\varphi := \{\varphi_\Lambda\}_{\Lambda \in \Gamma}$ with $\varphi \in C^{2s}(\mathbb{S}_\Lambda^2)$, $s > 7/4$. Let assume that there exists $\varepsilon > 0$ such that*

$$\|\varphi\|_{\varepsilon,s} := \sum_{m \geq 0} (e^\varepsilon K_s C_\Delta^s)^m \sup_{y \in \Gamma} \sum_{\substack{|X|=m+1 \\ X \ni y}} \|\varphi_X\|_{C^{2s}(\mathbb{S}_X^2)} < +\infty, \quad (61)$$

where $C_s > 1$ and C_Δ have been defined in Eq. (45). Then there exists a unique (β, δ_j^Γ) -KMS quantum states on B_j^Γ for all $\beta \in [0, \beta_{\varepsilon,s})$ where

$$\beta_{\varepsilon,s} := \frac{\varepsilon}{1 + e^\varepsilon} \frac{1}{2K_s C_\Delta^s \|\varphi\|_{\varepsilon,s}}. \quad (62)$$

The proof of Theorem 4.7 is similar in spirit to the one of Theorem 4.1. As such, it requires a few technical observations which we will recollect in the following remark.

Remark 4.8.

(i) Recalling Remark 2.2, the set $\{Q_j(Y_{\ell,m}) \mid \ell \in \mathbb{Z}_+, m \in [-\ell, \ell] \cap \mathbb{Z}\}$ is an orthogonal basis of $B_j = M_{2j+1}(\mathbb{C})$ with respect to the Hilbert-Schmidt scalar product (19)—notice that $Q_j(Y_{\ell,m}) = 0$ if $\ell > 2j$, cf. Equation (18). In what follows we will normalize $Q_j(Y_{\ell,m})$ by setting

$$\mathfrak{Y}_{j|\ell,m} := \frac{1}{\sqrt{c_{j,\ell}}} Q_j(Y_{\ell,m}), \quad (63)$$

where $c_{j,\ell} > 0$ has been defined in Eq. (25): With this choice we find

$$\|\mathfrak{Y}_{j|\ell,m}\|_{\text{HS}} = \frac{1}{\sqrt{2\ell+1}}, \quad \|\mathfrak{Y}_{j|\ell,m}\|_{B_j} \leq 1. \quad (64)$$

Indeed by direct inspection we have

$$\|\mathfrak{Y}_{j|\ell,m}\|_{\text{HS}}^2 = \frac{1}{c_{j,\ell}} \langle Q_j(Y_{\ell,m}) | Q_j(Y_{\ell,m}) \rangle_{\text{HS}} = \frac{1}{c_{j,\ell}} \langle Y_{\ell,m} | \check{Y}_{\ell,m} \rangle_{L^2(\mathbb{S}^2, \mu_0)} = \frac{1}{2\ell+1},$$

where we used Eqs. (23) and (15). Moreover, for all $\psi \in \mathbb{C}^{2j+1}$,

$$\begin{aligned} \|\mathfrak{Y}_{j|\ell,m}\psi\|^2 &= \frac{1}{c_{j,\ell}} \sum_{m'=-j}^j (\text{CG}_{\ell,m;j,m'}^{j,m+m'})^2 (\text{CG}_{\ell,0;j,j}^{j,j})^2 |\langle j, m' | \psi \rangle|^2 \\ &= \sum_{m'=-j}^j (\text{CG}_{\ell,m;j,m'}^{j,m+m'})^2 |\langle j, m' | \psi \rangle|^2 \leq \|\psi\|^2, \end{aligned}$$

where we used the explicit expression obtained in Example 2.1. It is worth to mention that further properties of the matrices $\mathfrak{Y}_{j|\ell,m}$ have been studied in [34].

For any finite region $\Lambda \Subset \Gamma$ we define $\boxed{\mathfrak{Y}_{j|\ell_\Lambda, m_\Lambda}} \in B_j^\Lambda$ by setting

$$\mathfrak{Y}_{j|\ell_\Lambda, m_\Lambda} := \bigotimes_{x \in \Lambda} \mathfrak{Y}_{j|\ell_x, m_x} \in B_j^\Lambda \subset B_j^\Gamma,$$

where $\ell_\Lambda \in \mathbb{Z}_+^\Lambda$ and $m_\Lambda \in \mathbb{Z}^\Lambda$ are such that $m_x \in [-\ell_x, \ell_x]$ for all $x \in \Lambda$. Then $\{\mathcal{Y}_{j|\ell_\Lambda, m_\Lambda} \mid \ell_\Lambda \in \mathbb{Z}_+^\Lambda, m_\Lambda \in \mathbb{Z}^\Lambda, m_x \in [-\ell_x, \ell_x] \forall x \in \Lambda\}$ is a orthogonal basis of B_j^Λ with respect to the Hilbert-Schmidt scalar product (19). In particular, if $A_\Lambda \in B_j^\Lambda$ then

$$A_\Lambda = \sum_{\ell_\Lambda, m_\Lambda} \left(\prod_{x \in \Lambda} (2\ell_x + 1) \right) \langle \mathcal{Y}_{j|\ell_\Lambda, m_\Lambda} \mid A_\Lambda \rangle_{\text{HS}} \mathcal{Y}_{j|\ell_\Lambda, m_\Lambda}, \tag{65}$$

See Eq. (49) for comparison.

- (ii) A crucial step in the proof of Theorem 4.1 is the use of Eq. (56), cf. Remark 4.6. Thanks to Eq. (22) an analogous property holds in the quantum setting. Specifically, by proceeding as in Remark 4.6 we find

$$\int_{SU(2)} \tilde{D}^{(j)}(R) \mathcal{Y}_{j|\ell, m} dR = \frac{1}{\sqrt{c_{j, \ell}}} \mathcal{Q}_j \left(\int_{SU(2)} \hat{R} Y_{\ell, m} dR \right) = 0. \tag{66}$$

- (iii) Similarly to the classical case, cf. Remark 4.3, any state $\omega_j^\Gamma \in S(B_j^\Gamma)$ is uniquely determined by its associated element $\underline{\omega}_j^\Gamma \in \mathbf{X}$ defined by

$$\underline{\omega}_j^\Gamma(\ell_\Lambda, m_\Lambda) := \begin{cases} 1 & \Lambda = \emptyset \\ \omega_j^\Gamma(\mathcal{Y}_{j|\ell_\Lambda, m_\Lambda}) & \Lambda \neq \emptyset \end{cases}.$$

Notice that $\underline{\omega}_j$ is an element of \mathbf{X} , cf. Remark 4.3, because of the bound

$$|\underline{\omega}_j(\ell_\Lambda, m_\Lambda)| = |\omega_j(\mathcal{Y}_{j|\ell_\Lambda, m_\Lambda})| \leq \|\mathcal{Y}_{j|\ell_\Lambda, m_\Lambda}\|_{B_j^\Lambda} \leq 1,$$

where we used the second inequality in (64).

- (iv) For later convenience we also discuss a quantum version of Eq. (17). Indeed, once again thanks to Eq. (23), an analogous identity holds for the \mathcal{Y} 's. This can be either argued by observing that the $\mathcal{Y}_{j|\ell, m}$ are spherical tensors of order ℓ with respect to the representation $\tilde{D}^{(j)}$, cf. [45, Section 3.11], or by direct inspection. In more details, let $\ell_1, \ell_2 \in \mathbb{Z}_+$ and $m_1, m_2 \in \mathbb{Z}$ with $m_k \in [-\ell_k, \ell_k], k \in \{1, 2\}$. Equation (65) leads to

$$\mathcal{Y}_{j|\ell_1, m_1} \mathcal{Y}_{j|\ell_2, m_2} = \sum_{\ell} (2\ell + 1) \langle \mathcal{Y}_{j|\ell, m_1+m_2} \mid \mathcal{Y}_{j|\ell_1, m_1} \mathcal{Y}_{j|\ell_2, m_2} \rangle_{\text{HS}} \mathcal{Y}_{j|\ell, m_1+m_2},$$

where the restriction to $m = m_1 + m_2$ is obtained by acting on both side of the equality with $\tilde{D}^{(j)}(e^{iJ_z})$ or by computing the coefficients directly.

Equation (67) can be seen as a quantum version of Eq. (17). Moreover, the coefficient appearing in Eq. (67) can be computed in a fairly explicit fashion. In particular we find

$$\begin{aligned} & (2\ell + 1) |\langle \mathcal{Y}_{j|\ell, m_1+m_2} \mid \mathcal{Y}_{j|\ell_1, m_1} \mathcal{Y}_{j|\ell_2, m_2} \rangle_{\text{HS}}| \\ &= \frac{2\ell + 1}{2j + 1} \left| \sum_{M, M_1, M_2 = -j}^j \text{CG}_{\ell, m; j, M}^{j, M_1} \text{CG}_{\ell_1, m_1; j, M_2}^{j, M_1} \text{CG}_{\ell_2, m_2; j, M}^{j, M_2} \right| \\ &= \sqrt{(2\ell + 1)(2j + 1)} \left| \text{CG}_{\ell_1, m_1; \ell_2, m_2}^{\ell, m} \begin{Bmatrix} j & j & \ell_2 \\ \ell & \ell_1 & j \end{Bmatrix} \right|, \end{aligned}$$

where we used a few properties of the Clebsch-Gordan coefficients and the definition of the $6j$ -symbols, cf. [37, Sections 8.7 and 9.1]. Notice that the appearance of the Clebsch-Gordan coefficient $CG_{\ell_1, m_1; \ell_2, m_2}^{\ell, m}$ ensures that the coefficients vanish unless $|\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2$. Moreover, since it is known that the matrix

$$C_{pq} := \sqrt{2p+1}\sqrt{2q+1} \begin{Bmatrix} j_1 & j_2 & p \\ \ell_1 & \ell_2 & q \end{Bmatrix},$$

is orthogonal, cf. [37, 9.1.1], we have

$$(2\ell + 1)|\langle \mathcal{Y}_{j|\ell, m_1+m_2} | \mathcal{Y}_{j|\ell_1, m_1} \mathcal{Y}_{j|\ell_2, m_2} \rangle_{\text{HS}}| \leq 1.$$

Overall we have

$$\mathcal{Y}_{j|\ell_1, m_1} \mathcal{Y}_{j|\ell_2, m_2} = \sum_{\ell=|\ell_1-\ell_2}^{\ell_1+\ell_2} (2\ell + 1)c_{j|\ell, m_1, m_2} \mathcal{Y}_{j|\ell, m_1+m_2}, \quad |c_{j|\ell, m}| \leq 1, \quad (67)$$

which will play the same role in the proof of Theorem 4.7 of Eq. (17) in the proof of Theorem 4.1.

Proof of Thm. 4.7. Let $\omega_j^{\beta, \Gamma} \in S(B_j^\Gamma)$ be a (β, δ_j^Γ) -KMS quantum states and let $\underline{\omega}_j^{\beta, \Gamma} \in \underline{X}$ be its associated element of \underline{X} . Proceeding as in the proof of Theorem 4.1, we will prove that $\underline{\omega}_j^{\beta, \Gamma}$ is a solution to a linear equation in \underline{X} which admits a unique solution under assumption (61)

To this avail we consider the ordering on Γ induced by an arbitrary but fixed bijection $\Gamma \simeq \mathbb{Z}$ and set $x := \min_{y \in \Lambda} y$ for $\Lambda \in \Gamma$. Let $\ell_\Lambda \in \mathbb{N}^\Lambda$ and $m_\Lambda \in \mathbb{Z}^\Lambda$ be such that $m_y \in [-\ell_y, \ell_y] \cap \mathbb{Z}$ for all $y \in \Lambda$. Equation (66) leads to

$$\begin{aligned} \underline{\omega}_j^{\beta, \Gamma}(\ell_\Lambda, m_\Lambda) &= \omega_j^{\beta, \Gamma}(\mathcal{Y}_{j|\ell_\Lambda, m_\Lambda}) \\ &= \omega_j^{\beta, \Gamma} \left(\int_{SU(2)} \left[\mathcal{Y}_{j|\ell_\Lambda, m_\Lambda} - D^{(j)}(R_x) (\mathcal{Y}_{j|\ell_\Lambda, m_\Lambda}) D^{(j)}(R_x)^* \right] dR_x \right) \\ &= \omega_j^{\beta, \Gamma} \left(\int_{SU(2)} \mathcal{Y}_{j|\ell_\Lambda, m_\Lambda} D^{(j)}(R_x)^* (I - \tau_{i\beta}^\Gamma) D^{(j)}(R_x) dR_x \right), \end{aligned}$$

where we used Eq. (66) and the (β, δ_j^Γ) -KMS condition. We observe that $D^{(j)}(R_x) \in B_j^x$, thus, $\tau_{i\beta}^\Gamma(D^{(j)}(R_x))$ can be computed using Eq. (33) for $\beta < \lambda/2 \|\Phi_j\|_\lambda$, where $\Phi_{j, \Lambda} = Q_j^\Lambda(\varphi_\Lambda)$. We find

$$\begin{aligned} \underline{\omega}_j^{\beta, \Gamma}(\ell_\Lambda, m_\Lambda) &= - \sum_{n \geq 1} \frac{(-\beta)^n}{n!} \sum_{\substack{X_1, \dots, X_n \\ X_q \cap S_{q-1} \neq \emptyset}} \\ &\omega_j^{\beta, \Gamma} \left(\int_{SU(2)} \mathcal{Y}_{j|\ell_\Lambda, m_\Lambda} D^{(j)}(R_x)^* (\text{ad}_{Q_j^{X_n}(\varphi_{X_n})} \cdots \text{ad}_{Q_j^{X_1}(\varphi_{X_1})}) D^{(j)}(R_x) dR_x \right), \end{aligned}$$

where $S_0 := \{x\}$ and $S_q := S_{q-1} \cup X_q$ for $q \geq 1$, $\text{ad}_A(A') := [A, A']$. Denoting by

$$W_n := D^{(j)}(R_x)^* (\text{ad}_{Q_j^{X_n}(\varphi_{X_n})} \cdots \text{ad}_{Q_j^{X_1}(\varphi_{X_1})}) D^{(j)}(R_x),$$

we find

$$W_1 = D^{(j)}(R_x) * Q_j^{X_1}(\varphi_{X_1}) D^{(j)}(R_x) - Q_j^{X_1}(\varphi_{X_1}) = Q_j^{X_1}(\hat{R}_x \varphi_{X_1}) - Q_j^{X_1}(\varphi_{X_1}), \tag{68}$$

and by induction

$$W_q = Q_j^{X_q}(\hat{R}_x \varphi_{X_q}) W_{q-1} - W_{q-1} Q_j^{X_q}(\varphi_{X_q}) \quad q = 2, \dots, n. \tag{69}$$

Out of Eqs. (68)–(69) one may find a reasonably explicit expression for W_n . To this avail let

$$\psi_X^p := \begin{cases} \hat{R}_x \varphi_X & p = + \\ -\varphi_X & p = - \end{cases}.$$

We consider the set $\Psi := \{Q_j^{X_1}(\psi_{X_1}^\pm), \dots, Q_j^{X_n}(\psi_{X_n}^\pm)\}$ with an order relation \succ defined by

$$\begin{aligned} Q_j^{X_n}(\psi_{X_n}^+) &\succ Q_j^{X_{n-1}}(\psi_{X_{n-1}}^+) \succ \dots \succ Q_j^{X_1}(\psi_{X_1}^+) \succ Q_j^{X_1}(\psi_{X_1}^-) \succ Q_j^{X_2}(\psi_{X_2}^-) \\ &\succ \dots \succ Q_j^{X_n}(\psi_{X_n}^-). \end{aligned}$$

For two elements in $A, B \in \Psi$ we set $A \cdot_\succ B := AB$ if $A \succ B$ and $A \cdot_\succ B := BA$ if $B \succ A$. Then

$$W_n = \sum_{p \in \{\pm 1\}^n} Q_j^{X_1}(\psi_{X_1}^{p(1)}) \cdot_\succ \dots \cdot_\succ Q_j^{X_n}(\psi_{X_n}^{p(n)}).$$

At this stage we proceed similarly to Theorem 4.1 by expanding each ψ_X -term in its Fourier-Laplace expansion, cf. Remark 4.2. In particular we have for all $k \in \{1, \dots, n\}$ and $p \in \{+, -\}$,

$$\begin{aligned} Q_j^{X_k}(\psi_{X_k}^p) &= \sum_{\ell_{X_k,p}, m_{X_k,p}} C_{j|X_k,p}(\ell_{X_k,p}, m_{X_k,p}) \mathfrak{Y}_{j|\ell_{X_k,p}, m_{X_k,p}} C_{j|X_k,p}(\ell_{X_k,p}, m_{X_k,p}) := \\ &\left(\prod_{y \in X_k} c_{j, \ell_{X_k,p,y}}^{1/2} (2\ell_{X_k,p,y} + 1) \right) \langle Y_{\ell_{X_k,p}, m_{X_k,p}} | \psi_{X_k}^p \rangle_{L^2(\mathbb{S}_{X_j}^2, \mu_0^{X_j})}. \end{aligned}$$

Thus, we find

$$\begin{aligned} \omega_j^{\beta, \Gamma}(\ell_\Lambda, m_\Lambda) &= - \sum_{n \geq 1} \frac{(-\beta)^n}{n!} \sum_{\substack{X_1, \dots, X_n \\ X_q \cap X_{q-1} \neq \emptyset}} \sum_{p \in \{\pm 1\}^n} \sum_{\substack{\ell_{X_1,p}, \dots, \ell_{X_n,p} \\ m_{X_1,p}, \dots, m_{X_n,p}}} \int_{SU(2)} \prod_{k=1}^n C_{X_k,p}(\ell_{X_k,p}, \\ &m_{X_k,p}) dR_X \omega_j^{\beta, \Gamma} \left(\mathfrak{Y}_{j|\ell_\Lambda, m_\Lambda} \mathfrak{Y}_{j|\ell_{X_1,p(1)}, m_{X_1,p(1)}} \cdot_\succ \dots \cdot_\succ \mathfrak{Y}_{j|\ell_{X_n,p(n)}, m_{X_n,p(n)}} \right). \end{aligned}$$

We then expand the product of the \mathcal{Y} 's factors by means of Eq. (67) in exactly the same way we did for the classical spherical harmonics. Setting $S_n := X_1 \cup \dots \cup X_n$ we find

$$\begin{aligned} & \mathcal{Y}_{j|\ell_\Lambda, m_\Lambda} \mathcal{Y}_{j|\ell_{X_1, p(1)}, m_{X_1, p(1)}} \cdots \mathcal{Y}_{j|\ell_{X_n, p(n)}, m_{X_1, p(n)}} \\ &= \left(\prod_{y \in \Lambda \cap S_n^c} \mathcal{Y}_{j|\ell_{\Lambda, y}, m_{\Lambda, y}} \right) \prod_{y \in S_n} \sum_{s_{y,1}, \dots, s_{y,n}} \left(\prod_{k=1}^n c_{j, s_{y,k}} \right) \mathcal{Y}_{s_{y,n}, \tilde{m}_{\Lambda, y} + \tilde{m}_{X_1, p(1), y} + \dots + \tilde{m}_{X_n, p(n), y}} \\ &= \left(\prod_{y \in \Lambda \cap S_n^c} \mathcal{Y}_{j|\ell_{\Lambda, y}, m_{\Lambda, y}} \right) \sum_{k=1}^{N(\ell_{X_1, p}, \dots, \ell_{X_n, p})} C'_{j,p}(\ell_{S_n}^k, m_{S_n}^k) \mathcal{Y}_{j|\ell_{S_n}^k, m_{S_n}^k}, \end{aligned}$$

where $|C'_{j,p}(\ell_{S_n}^k, m_{S_n}^k)| \leq 1$ while

$$N(\ell_{X_1, p}, \dots, \ell_{X_n, p}) \leq \prod_{k=1}^n \prod_{y \in X_k} (2\ell_{X_k, p, y} + 1).$$

Setting again $X(n, \Lambda) := \Lambda \cup S_n$ we have that for any (β, δ_j^Γ) -KMS quantum states $\omega_j^{\beta, \Gamma} \in S(B_j^\Gamma)$ the corresponding element $\underline{\omega}_j^{\beta, \Gamma} \in \underline{X}$ solves the linear equation

$$(I - L_j^\beta) \underline{\omega} = \underline{\delta}. \quad (70)$$

where $\underline{\delta} \in \underline{X}$ has been defined in Eq. (59). The operator $L_j^\beta: \underline{X} \rightarrow \underline{X}$ is defined by setting, for all $\underline{f} \in \underline{X}$, $(L_j^\beta \underline{f})_\emptyset = 0$ and

$$\begin{aligned} (L_j^\beta \underline{f})_\Lambda(\ell_\Lambda, m_\Lambda) &= - \sum_{n \geq 1} \frac{(-\beta)^n}{n!} \sum_{\substack{X_1, \dots, X_n \\ X_q \cap S_{q-1} \neq \emptyset}} \sum_{p \in \{\pm 1\}^n} \sum_{\substack{\ell_{X_1, p}, \dots, \ell_{X_n, p} \\ m_{X_1, p}, \dots, m_{X_n, p}}} \int_{SU(2)} \prod_{k=1}^n C_{j|X_k, p}(\ell_{X_k, p}, \\ & m_{X_k, p}) dR_x \sum_{k=1}^{N(\ell_{X_1}, \dots, \ell_{X_n})} C'_{j,p}(\ell_{S_n}^k, m_{S_n}^k) f_{X(n, \Lambda)}(\ell_{X(n, \Lambda), p}^k, m_{X(n, \Lambda), p}^k), \end{aligned} \quad (71)$$

for all non-empty $\Lambda \Subset \Gamma$ where $x := \min_{y \in \Lambda} y$ while $\ell_{X(n, \Lambda), p, y} = \ell_{\Lambda, y}$ for all $y \in \Lambda \cap S_n^c$ and $\ell_{X(n, \Lambda), p, y} = \ell_{S_n, p, y}$ for $y \in S_n$.

It remains to prove that Eq. (70) has a unique solution under assumption (61). To this avail we observe that

$$\begin{aligned} |(L_j^\beta \underline{f})_\Lambda(\ell_\Lambda, m_\Lambda)| &\leq \|\underline{f}\|_{\underline{X}} \sum_{n \geq 1} \frac{\beta^n}{n!} \sum_{\substack{X_1, \dots, X_n \\ X_q \cap S_{q-1} \neq \emptyset}} \sum_{p \in \{\pm 1\}^n} \\ &\times \sum_{\substack{\ell_{X_1, p}, \dots, \ell_{X_n, p} \\ m_{X_1, p}, \dots, m_{X_n, p}}} \int_{SU(2)} \prod_{k=1}^n |C_{j|X_k, p}(\ell_{X_k, p}, m_{X_k, p})| \prod_{y \in X_k} (2\ell_{X_k, p, y} + 1) dR_x, \end{aligned}$$

where we used the bound on $C'_{j,p}$ and $N(\ell_{X_1,p}, \dots, \ell_{X_n,p})$. Moreover, again as in the proof of Theorem 4.1, we find

$$\begin{aligned} & \sum_{\substack{\ell_{X_1,p}, \dots, \ell_{X_n,p} \\ m_{X_1,p}, \dots, m_{X_n,p}}} \int_{SU(2)} \prod_{k=1}^n |C_{j|X_k,p}(\ell_{X_k,p}, m_{X_k,p})| \prod_{y \in X_k} (2\ell_{X_k,p,y} + 1) dR_x \\ & \leq \sum_{\ell_{X_1,p}, \dots, \ell_{X_n,p}} \prod_{k=1}^n \left(\prod_{y \in X_k} \frac{(2\ell_{X_k,p,y} + 1)^{5/2}}{[1 + \ell_{X_k,p,y}(\ell_{X_k,p,y} + 1)]^s} \right) C_\Delta^{s|X_k|} \|\varphi_{X_k}\|_{C^{2s}(\mathbb{S}_{X_k}^2)} \\ & = \prod_{k=1}^n (K_s C_\Delta^s)^{|X_k|} \|\varphi_{X_k}\|_{C^{2s}(\mathbb{S}_{X_k}^2)}, \end{aligned}$$

where C_Δ and K_s have been defined in Eq. (45). The above estimate is uniform over $p \in \{\pm 1\}^n$, therefore,

$$|(L_j^\beta \underline{f})_\Lambda(\ell_\Lambda, m_\Lambda)| \leq \|\underline{f}\|_{\mathbb{X}} \sum_{n \geq 1} \frac{(2\beta)^n}{n!} \sum_{\substack{X_1, \dots, X_n \\ X_q \cap S_{q-1} \neq \emptyset}} \prod_{k=1}^n (K_s C_\Delta^s)^{|X_k|} \|\varphi_{X_k}\|_{C^{2s}(\mathbb{S}_{X_k}^2)}.$$

Finally we apply the following estimate, cf. [15, Prop. 6.2.45]: If $\alpha_X \in \mathbb{R}_+$ for all $X \in \Gamma$ then for all $S \in \Gamma$

$$\sum_{X \cap S \neq \emptyset} \alpha_X \leq \sum_{x \in S} \sum_{X \ni x} \alpha_X = \sum_{x \in S} \sum_{m \geq 0} \sum_{\substack{|X|=m+1 \\ X \ni x}} \alpha_X \leq |S| \sum_{m \geq 0} \sup_{x \in \Gamma} \sum_{\substack{|X|=m+1 \\ X \ni x}} \alpha_X,$$

and by iteration, for $S_1 = \{x\}$ and $S_q := X_q \cup S_{q-1}$,

$$\begin{aligned} \sum_{\substack{X_1, \dots, X_n \\ X_q \cap S_{q-1} \neq \emptyset}} \alpha_{X_1} \cdots \alpha_{X_n} & \leq \sum_{m_1 \geq 0} \sup_{y \in \Gamma} \sum_{\substack{|X_1|=m_1+1 \\ X_1 \ni y}} \alpha_{X_1} \sum_{\substack{X_2, \dots, X_n \\ X_q \cap S_{q-1} \neq \emptyset}} \alpha_{X_2} \cdots \alpha_{X_n} \\ & \leq \sum_{m_1, \dots, m_n \geq 0} \prod_{k=1}^n (1 + m_1 + \dots + m_{k-1}) \sup_{y \in \Gamma} \sum_{\substack{|X_k|=m_k+1 \\ X_k \ni y}} \alpha_{X_k} \\ & \leq n! \varepsilon^{-n} e^\varepsilon \left(\sum_{m \geq 0} e^{\varepsilon m} \sup_{y \in \Gamma} \sum_{\substack{|X|=m+1 \\ X \ni y}} \alpha_X \right)^n, \end{aligned}$$

where in the last line we observe that, for all $\lambda > 0$,

$$\prod_{k=1}^n (1 + m_1 + \dots + m_{k-1}) \leq (1 + m_1 + \dots + m_n)^n \leq n! \varepsilon^{-n} e^{\varepsilon(1+m_1+\dots+m_n)}.$$

Therefore, setting $\alpha_X := (K_s C_\Delta^s)^{|X|} \|\varphi_X\|_{C^{2s}(\mathbb{S}_X^2)}$ we find

$$\begin{aligned} & |(L_j^\beta \underline{f})_\Lambda(\ell_\Lambda, m_\Lambda)| \\ & \leq \|\underline{f}\|_{\underline{X}} e^\varepsilon \sum_{n \geq 1} (2\varepsilon^{-1} \beta K_s C_\Delta^s)^n \left(\sum_{m \geq 0} (e^\varepsilon K_s C_\Delta^s)^m \sup_{\substack{y \in \Gamma \\ |X|=m+1 \\ X \ni y}} \|\varphi_X\|_{C^{2s}(\mathbb{S}_X^2)} \right)^n \\ & = \|\underline{f}\|_{\underline{X}} e^\varepsilon \sum_{n \geq 1} (2\varepsilon^{-1} \beta K_s C_\Delta^s \|\varphi\|_{\varepsilon,s})^n \\ & = \|\underline{f}\|_{\underline{X}} e^\varepsilon \frac{2\varepsilon^{-1} \beta K_s C_\Delta^s \|\varphi\|_{\varepsilon,s}}{1 - 2\varepsilon^{-1} \beta K_s C_\Delta^s \|\varphi\|_{\varepsilon,s}}, \end{aligned}$$

where we used assumption (61) and considered $\beta < \varepsilon/2 K_s C_\Delta^s \|\varphi\|_{\varepsilon,s}$ —notice that the latter value is lower than the previous bound $\beta < \varepsilon/2 \|\Phi_j\|_\varepsilon$ necessary to ensure the expansion of $\tau_{i\beta}^\Gamma(D^{(j)}(R_x))$ according to Eq. (33). It follows that $L_j^\beta \in \mathcal{B}(\underline{X})$, moreover,

$$\|L_j^\beta\|_{\underline{X}} \leq e^\varepsilon \frac{2\varepsilon^{-1} \beta K_s C_\Delta^s \|\varphi\|_{\varepsilon,s}}{1 - 2\varepsilon^{-1} \beta K_s C_\Delta^s \|\varphi\|_{\varepsilon,s}} < 1,$$

provided that $\beta < \beta_{\varepsilon,s}, \beta_{\varepsilon,s}$ being defined by Eq. (62). □

Remark 4.9. (i) By direct inspection assumption (61) implies (45). Thus, condition (61) is a sufficient condition which guarantees uniqueness of both $(\beta, \delta_\infty^\Gamma)$ -KMS classical states and (β, δ_j^Γ) -KMS quantum states for all $j \in \mathbb{Z}_+/2$ and for $\beta \leq \beta_{\varepsilon,s}$. In other words, (61) ensures the absence of both classical and quantum phase transitions at sufficiently high temperature. Furthermore, since $\|\varphi\|_{0,s} \leq \|\varphi\|_{\varepsilon,s}$ and $\varepsilon(1 + e^\varepsilon)^{-1} < \log 2$, it follows that $\beta_{\varepsilon,s} < \beta_{0,s}$, that is, the corresponding quantum inverse critical temperature is slightly lower with respect to the corresponding classical one—This ensures absence of phase transition starting from a common critical inverse temperature.

Moreover, on account of Proposition 3.5, cf. Remark 3.6, in this situation the classical limit $\lim_{j \rightarrow \infty} \omega_j^{\beta,\Gamma} \circ Q_j^\Gamma$ of the unique (β, δ_j^Γ) -KMS quantum states $\omega_j^\Gamma \in S(B_j^\Gamma)$ coincides with the unique $(\beta, \delta_\infty^\Gamma)$ -KMS classical states $\omega_\infty^{\beta,\Gamma} \in S(B_\infty^\Gamma)$.

(ii) It is worth to compare Theorem 4.7 with [15, Prop. 6.2.45]. The latter provide a sufficient condition for the uniqueness of (β, δ_j^Γ) -KMS for fixed $j \in \mathbb{Z}_+/2$ which is similar in spirit to (61)—in fact, Theorem 4.7 has been inspired by this latter result—namely

$$\|\varphi\|_{\text{BR},\varepsilon,j} := \sum_{m \geq 0} e^{\varepsilon m} (2j + 1)^m \sum_{\substack{|X|=m+1 \\ X \ni x}} \|Q_j^X(\varphi_X)\|_{B_j^X} < \infty. \tag{72}$$

Condition (72) is stronger than (61) because it only uses the B_j^X -norm of $Q_j^X(\varphi_X)$. However, it is not uniform of $j \in \mathbb{Z}_+/2$, in particular, it requires a faster and faster decay behaviour of the potential $\{\varphi_\Lambda\}_{\Lambda \in \mathbb{Z}^d}$ as $j \rightarrow \infty$. Moreover, the critical quantum inverse temperature $\beta_{\text{BR}}(j, \lambda)$ predicted by [15, Prop. 6.2.45] vanishes as $j \rightarrow \infty$. Finally, there is no classical version of condition (72). For all these reasons [15, Prop. 6.2.45] is not suitable for the comparison with the classical setting we are interested in.

Instead, Theorem 4.7 leads to a result which is uniform in j , allowing for a simpler comparison with Theorem 4.1. The latter theorem can be understood as a classical counterpart of the uniqueness result presented in [15]. From a technical point of view the uniform behaviour in j is obtained by trading the B_∞ -norm with the C^{2s} -norm for a suitably high s .

Acknowledgement We are indebted with V. Moretti for many helpful discussions on this project. N.D. and L. P. acknowledge the support of the GNFM group of INdAM. C. J. F. van de Ven is supported by a postdoctoral fellowship granted by the Alexander von Humboldt Foundation (Germany). He also thanks the support of S. Teufel und A. Capel.

Funding Open access funding provided by Università degli Studi di Genova within the CRUI-CARE Agreement.

Data availability Data sharing is not applicable to this article as no new data were created or analysed in this study.

Declarations

Conflict of interest statement. The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter discussed in this manuscript.

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Communicated by Y. Ogata