



High contrast biphasic mechanical metafilter for elastic wave mitigation

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ABSTRACT

The intrinsic composite microstructure of heterogeneous cellular media provides effective mechanisms for governing the propagation of elastic harmonic waves. The dispersion properties of biphasic two-dimensional phononic crystals, characterized by strong acoustic impedance contrast between the constituent phases, are investigated. Two microstructural configurations are considered: a *native* configuration with periodic inclusions embedded in a homogeneous matrix, and an *engineered* configuration obtained by introducing inclusion-connecting channels that enhance the intercellular connectivity. The biphasic medium is modeled as a classical non-dissipative solid continuum, governed by differential equations with phase-dependent piecewise coefficients. Methodologically, a hybrid semi-analytical finite element framework is employed to determine the Floquet–Bloch dispersion relations. Parametric analyses reveal how impedance contrast and geometrical connectivity influence the emergence and amplification of low-frequency bandgaps. Bandgap opening stems from different physical mechanisms: in the native configuration, large bandgaps emerge for increasing impedance contrast, whereas in the engineered configuration, large bandgaps arise for decreasing impedance contrast. The systematic investigation of the interplay between elastic properties and geometrical cell-shaping in tailoring the band structure of biphasic cellular media provides a unified framework for their spectral design. A design chart mapping the bandgap amplitude onto the principal mechanical parameters is provided to guide the development of high-performance metafilters. Finally, time- and frequency-domain analyses of finite periodic arrays under broadband excitation demonstrate the ability of the proposed configurations to attenuate elastic waves within the predicted bandgap ranges, which fall within the range of human sensitivity.

1. Introduction

Acoustic waves in homogeneous solid media can freely propagate without dispersion of their harmonic components, in accordance with the classical physical–mathematical formulation of first-order linear elastic continua [1–4]. The mechanical parameters of the medium determine the characteristic spectral properties of the spatio-temporal propagation, including the dispersion relations between the wavefrequencies and wavevectors, the polarization of waveforms, and the phase and group velocities of wave packets. Under unbounded conditions, the spectral properties of naturally homogeneous media can be significantly modified by introducing artificial heterogeneities that locally modulate the mechanical parameters. As a consequence, extraordinary research efforts in solid mechanics and materials science are currently devoted to the conception, development, design and fabrication of heterogeneous and composite media aimed at surpassing the intrinsic spectral capabilities of conventional homogeneous materials.

Within this framework, phononic crystals have emerged over the past few decades as a new class of advanced materials specifically engineered to achieve functional, extreme, or unconventional spectral performances [5–10]. Conceptually, the core idea stems from the actual possibility to control the transmission of electromagnetic waves (photons), by tailoring the repetitive molecular arrangement of crystalline structures (photonic materials). Drawing inspiration from this characteristic spectral behavior [11,12], the guiding principle behind phononic materials is that periodic modulations of the mechanical properties of heterogeneous composite solids can be analogously designed to govern the propagation of acoustic waves (phonons).

The synergy between the refinement of mechanical formulations, the increase in computational resources, and the advent of micro-engineering, high-precision manufacturing, and high-fidelity prototyping is currently opening the horizon to an entirely new generation of phononic materials. The most straightforward approach to implementing periodic modulations consists in introducing voids or holes,

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soft or rigid inclusions, breathing cracks or bucklable cavities, and various forms of multiphase composition or layering [13–16]. These types of periodic modulators alter the wave dynamics at wavelengths equal to or larger than the periodicity constants, primarily through the constructive interference phenomenon characteristic of the Bragg scattering mechanism. A more sophisticated approach entails the realization of complex or structured periodic modulations by embedding oscillators – such as pillars, cantilevers, pendula, strips, pantographs or twistors – elastically coupled to the host material [17–20]. The natural frequencies of these oscillating modulators can be tuned to influence the wave dynamics of the host material, primarily through the local resonance phenomenon associated with Fano-like interference mechanisms [21–23]. The inertia amplification and flexibility extremization required to drive local resonances toward the low-frequency band often tends to trigger or exalt significant nonlinear behaviors – such as frequency hardening/softening or superharmonic/subharmonic generation – which are nowadays regarded as valuable additional possibilities for amplitude-dependent tuning and multi-harmonic synthonization [24–30].

Regardless of the methodological approach, the primary effect of introducing periodic scatterers is the opening of stopbands in the dispersion spectrum. Engineering applications range from impact absorption to wave focusing, vibration shielding, noise silencing, energy localization, among others [31–33]. Within this fascinating and rapidly developing scenario, a key principle is that groundbreaking spectral performances can be achieved only by fine-tuning the phase compositeness, the micromechanical architecture, and the geometrical topology of these dynamic systems. In this respect, a fundamental finding in the recent scientific literature is that high-contrast biphasic phononic crystals – particularly those exhibiting a strong acoustic impedance mismatch between the constituent phases – can generate complete stopbands even at low frequencies [11,34]. While several behavioral patterns governing stopband opening have been extensively investigated under moderate-contrast conditions, the high-contrast condition in biphasic solid–solid compositions has received comparatively less attention than their solid–fluid counterparts [35–38].

Based on this motivating background, the present work aims to systematically analyze the combined effects of (i) interphasic high contrast, and (ii) local microstructural modifications of the periodic cell on the dynamic response of biphasic phononic crystals. Specifically, the novel idea is to combine geometrical reshaping (through the opening of intercellular intraphasic channels) with long-range internal interactions (through the introduction of channel-bridging elastic couplings). This dual adaptation shifts the periodic medium into the realm of mechanical metamaterials characterized by subwavelength cellular microstructures [39–42], and directs the objectives toward the challenging design of high-performing passive acoustic devices such as sound and vibration metafilters or metapropagators [43–45].

The spectral design problem for phononic crystals and mechanical metamaterials essentially consists in identifying the geometrical and elastodynamic properties of the periodic cell (the *design variables*) that produce desired features in the dispersion spectrum (the *design targets*). This task belongs to the class of inverse problems, as it requires reversing the typical roles of data and unknowns in the (direct) eigenproblem governing the dispersion relation (*inverse eigenproblem*). In practice, *topological designers* commonly adopt a fine pixelization of the periodic cell, treating each pixel as a freely designable unit [46–51]. A less common strategy governs the geometry of the periodic cell by selecting a limited number of independent descriptive parameters. Due to the strongly nonlinear nature of inverse eigenproblems, the parametric spectral design of phononic crystals and mechanical metamaterials is rarely tractable in fully analytical form [52,53], and typically requires advanced numerical approaches [54–56], recently complemented by powerful artificial intelligence techniques, such as machine learning and deep learning schemes [57–61]. Such approaches have proven particularly effective when extremely large or high-dimensional design

spaces must be explored, or when surrogate models are required to approximate complex input–output relationships in inverse problems or multi-objective optimization functions.

By adopting the methodological framework of parametric design, the present work investigates extreme regions of the parameter space associated with high acoustic-impedance contrast, with the aim of identifying robust and potentially optimal configurations for vibration filtering. Two distinct periodic solutions are considered: a native biphasic configuration with isolated circular inclusions periodically embedded in a homogeneous matrix, and an engineered configuration in which geometrical inclusion-to-inclusion connections (channels) and long-range matrix-to-matrix elastic couplings (long-range links) are introduced. By systematically scanning the high-contrast regime for both configurations, the parametric analyses disclose analogies or differences in spectral behaviors, and clarify the role of elastic and geometrical engineering in tailoring the frequency band structure and opening low-frequency stopbands. The results confirm that extreme-contrast regions not only offer promising perspectives for optimization, but also expose qualitative differences between native and engineered configurations, highlighting new design pathways for high-performing mechanical metafilters.

The novel contributions of the present study therefore lie in (i) elucidating how elastic properties (impedance contrast) and geometrical shaping (intercellular connectivity) interact in governing bandgap opening in biphasic periodic media, and (ii) proposing a design strategy that leverages the existence of isospectral solutions of the inverse problem in high-contrast regions of the parameter space. The study establishes a design-oriented framework for high-contrast biphasic metafilters, with particular emphasis on the parametrically controllable mechanisms of interphasic interactions governing bandgap formation, shifting and amplification in the dispersion spectrum. Computational analyses are employed to verify the consistency between the spectral design of complete low-frequency bandgaps in the infinite periodic medium and the wave attenuation observed in the undamped forced response of finite domains.

The paper is organized as follows. Section 2 introduces the biphasic solid–solid periodic media under investigation, describing both the native and engineered periodic cell configurations and outlining the governing elastodynamic formulation. The dispersion properties are derived through a hybrid semi-analytical finite element approach, and a systematic parametric analysis is conducted to assess the influence of acoustic impedance contrast and intraphase connectivity on bandgap formation. In Section 3, the insights gained from the spectral analysis are applied to finite mechanical metafilters, whose performance is evaluated under transient excitation through finite element simulations. Finally, concluding remarks are provided in Section 4.

2. Biphasic solid–solid phononic crystal

The class of infinite two-dimensional solid media characterized by double spatial periodicity can be considered a limitless platform for the microstructural design of functional metamaterials. An efficient design strategy may consist in (i) assuming an initial simple topology, identified as the *native* geometrical configuration of the periodic medium, and (ii) modifying it by exploring periodicity-preserving combinations of geometrical changes and microstructural enrichments, in order to obtain a versatile microstructured topology, referred to as the *engineered* configuration. The topological and microstructural modifications can be targeted at improving a particular mechanical performance, and the engineered medium can definitely be regarded as a mechanical metamaterial.

According to this methodological scheme, the native configuration can be inspired by two-dimensional phononic crystals (Fig. 1a), generated by periodically tessellating the infinite plane with a square cell of side length L ensuring full edge-to-edge tiling. The biphasic (solid–solid) cell is made up of a matrix (phase P_1) which embeds a centered

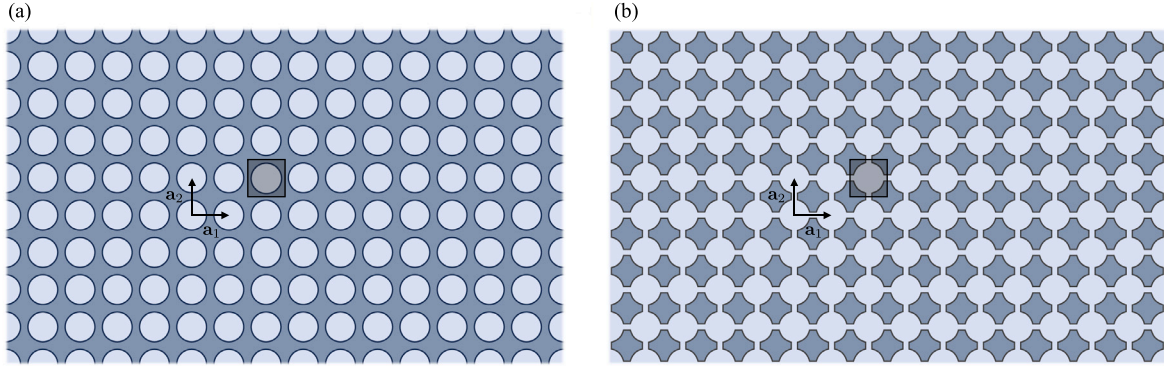


Fig. 1. Two-dimensional periodic media with double spatial periodicity, periodic square cell (gray region) and periodicity vectors $\mathbf{a}_1$ and $\mathbf{a}_2$: (a) native configuration of the phononic crystal, (b) engineered configuration of the mechanical metamaterial.

circular inclusion with radius R (phase $\mathcal{P}_I$). The nondimensional filling factor of the inclusions is $\alpha_f = \pi r^2$, where $r = R/L$ is the nondimensional radius (with $0 < r < 1/2$ for geometrical reasons). The relative amount of phase $\mathcal{P}_I$ with respect to phase $\mathcal{P}_I$ can also be measured by the volume ratio $\alpha_n = \pi r^2 / (1 - \pi r^2)$. The generic cell has mirror reflection symmetries relative to the principal inertia axes (Fig. 2a,b), so that the deriving periodic geometry belongs to the plane group $p4mm$ according to the crystallographic classification [62]. The geometric centroid of a particular cell C can be conveniently chosen as the origin of a global Cartesian reference frame, whose canonical unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ align with the periodicity directions.

Starting from the native configuration, the engineered configuration is obtained by opening straight channels connecting the circular inclusions of adjacent cells, and adding linear elastic links that bridge the channels and remotely connect the matrix islands (Fig. 1b). The channel width W and the link stiffness K can be regulated to increase or decrease the intraphasic connectivity characterizing the metamaterial (Fig. 2b). The volume ratio of the metamaterial is $\alpha_e = [r^2(\pi - 4S_e(w, r)) + w(2 - R_e(w, r))] / [1 + r^2(\pi - 4C_e(w, r)) - w(2 - R_e(w, r))]$, where $w = W/L$ is the nondimensional channel width (with $0 < w < 1$ for geometrical reasons) and the auxiliary quantities $S_e(w, r) = \arcsin(w/(2r))$, $C_e(w, r) = \arccos(w/(2r))$ and $R_e(w, r) = \sqrt{4r^2 - w^2}$ have been used. As long as r is fixed and w is strictly positive, the inequality $\alpha_e > \alpha_n$ holds. It can be noted that the engineerization does not alter the symmetry group of the periodic geometry and preserves the periodicity vectors $\mathbf{a}_1 = L\mathbf{e}_1$ and $\mathbf{a}_2 = L\mathbf{e}_2$.

2.1. Equation of motion

From the physical-mathematical viewpoint, the periodic medium can be described as an inhomogeneous (but piecewise homogeneous) isotropic material continuum $\mathcal{M}$ with non dissipative, linearly elastic behavior. According to the classical Cauchy formulation for the first-order dynamics and kinematics of deformable continua and assuming a plane state of strain, the in-plane stress tensor $\boldsymbol{\sigma}(\mathbf{x}, t)$ at the generic point $\mathbf{x} \in \mathcal{M}$ and time t can be related to the in-plane displacement field $\mathbf{u}(\mathbf{x}, t)$ by the constitutive relation $\boldsymbol{\sigma}(\mathbf{x}, t) = \lambda(\mathbf{x})(\nabla \cdot \mathbf{u}(\mathbf{x}, t))\mathbf{I} + \mu(\mathbf{x})(\nabla \mathbf{u}(\mathbf{x}, t) + \nabla \mathbf{u}(\mathbf{x}, t)^T)$, where $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are the Lamé parameters, characterizing the elastic behavior of the material. Consequently, by imposing the equilibrium of the generic infinitesimal volume of the continuum in the absence of external loads other than inertia forces, the partial differential equation of motion governing the free undamped oscillations reads

$$\nabla \cdot [\lambda(\mathbf{x})(\nabla \cdot \mathbf{u}(\mathbf{x}, t))\mathbf{I} + \mu(\mathbf{x})(\nabla \mathbf{u}(\mathbf{x}, t) + \nabla \mathbf{u}(\mathbf{x}, t)^T)] = \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}, t), \quad \forall \mathbf{x} \in \mathcal{M} \quad (1)$$

where $\rho(\mathbf{x})$ is the mass density. In the engineered configuration, the elastic links connect couples of points that belong to the parallel

borders of each channel. Within the linear range, the links/connection determine extra axial stresses $\mathbf{e}_1 \cdot \boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{e}_1 = k\Delta \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{e}_1$ (for pairs of $\mathbf{e}_2$ -aligned borders) and $\mathbf{e}_2 \cdot \boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{e}_2 = k\Delta \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{e}_2$ (for pairs of $\mathbf{e}_1$ -aligned borders), where $\Delta \mathbf{u}(\mathbf{x}, t)$ is the relative displacement between couples of interacting points across the channel and $k = K/(L - \sqrt{4R^2 - W^2})$ is the stiffness per unit length and unit depth. The elastic parameter K accounts for the matrix-to-matrix coupling across the engineered channels and can be regarded as an effective stiffness, which is conventionally assumed directly proportional to the matrix elastic modulus E_I and to the characteristic cell size L .

The piecewise functions associating the Lamé parameters $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ and the mass density $\rho(\mathbf{x})$ to each material point of the infinite continuum domain $\mathcal{M}$ describe and distinguish the different geometrical distributions of the two phases in the native and engineered configurations (Fig. 1). For both periodic configurations, by virtue of the sole periodicity, the functions $\lambda(\mathbf{x})$, $\mu(\mathbf{x})$ and $\rho(\mathbf{x})$ satisfy the conditions

$$\lambda(\mathbf{x} + \mathbf{a}_j) = \lambda(\mathbf{x}), \quad \mu(\mathbf{x} + \mathbf{a}_j) = \mu(\mathbf{x}), \quad \rho(\mathbf{x} + \mathbf{a}_j) = \rho(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{M} \quad (2)$$

for $j = 1, 2$. Therefore, by employing the time-Fourier transform, which is applied to the generic function $\mathbf{f}(\mathbf{x}, t)$ by the operation $\mathcal{F}[\mathbf{f}(\mathbf{x}, t)] = \int_{-\infty}^{\infty} \mathbf{f}(\mathbf{x}, t) \exp(-i\omega t) dt = \tilde{\mathbf{f}}(\mathbf{x}, \omega)$, Eq. (1) becomes

$$\nabla \cdot [\lambda(\mathbf{x})(\nabla \cdot \tilde{\mathbf{u}}(\mathbf{x}, \omega))\mathbf{I} + \mu(\mathbf{x})(\nabla \tilde{\mathbf{u}}(\mathbf{x}, \omega) + \nabla \tilde{\mathbf{u}}(\mathbf{x}, \omega)^T)] = -\omega^2 \rho(\mathbf{x}) \tilde{\mathbf{u}}(\mathbf{x}, \omega), \quad \forall \mathbf{x} \in \mathcal{M} \quad (3)$$

where $\tilde{\mathbf{u}}(\mathbf{x}, \omega)$ is the transformed displacement field and ω is the complex Fourier variable.

Mathematically, Eq. (3) is a linear system of second-order partial differential equations posed over an unbounded domain and endowed with periodic coefficients. According to the Floquet-Bloch Theorem [63], the problem admits spatially periodic solutions in the form $\tilde{\mathbf{u}}(\mathbf{x} + \mathbf{a}_j, \omega) = \exp(i\mathbf{k} \cdot \mathbf{a}_j) \tilde{\mathbf{u}}(\mathbf{x}, \omega)$ and $\nabla \tilde{\mathbf{u}}(\mathbf{x} + \mathbf{a}_j, \omega) = \exp(i\mathbf{k} \cdot \mathbf{a}_j) \nabla \tilde{\mathbf{u}}(\mathbf{x}, \omega)$, where $\mathbf{k}$ is the Floquet exponent. By virtue of their spatial periodicity, the unknown solutions can be determined by imposing the governing equations in the finite domain $\mathcal{A} = [-L/2, L/2] \times [-L/2, L/2]$ of the periodic cell C (Fig. 2c), without loss of generality. Therefore, a boundary value problem is stated, ruled by the field equation

$$\nabla \cdot [\lambda(\mathbf{x})(\nabla \cdot \tilde{\mathbf{u}}(\mathbf{x}, \omega))\mathbf{I} + \mu(\mathbf{x})(\nabla \tilde{\mathbf{u}}(\mathbf{x}, \omega) + \nabla \tilde{\mathbf{u}}(\mathbf{x}, \omega)^T)] = -\omega^2 \rho(\mathbf{x}) \tilde{\mathbf{u}}(\mathbf{x}, \omega), \quad \forall \mathbf{x} \in \mathcal{A} \quad (4)$$

and complemented by the Floquet-Bloch conditions applied to the cell boundary $\partial \mathcal{A}$. They express the quasi-periodicity of the solutions across the sub-boundaries $\partial \mathcal{A}_j^-$ and $\partial \mathcal{A}_j^+$ separated by the periodicity vector $\mathbf{a}_j$, in the form

$$\begin{aligned} \tilde{\mathbf{u}}(\mathbf{x}, \omega) \Big|_{\partial \mathcal{A}_1^+} &= \exp(i\mathbf{k} \cdot \mathbf{a}_1) \tilde{\mathbf{u}}(\mathbf{x}, \omega) \Big|_{\partial \mathcal{A}_1^-}, \\ \nabla \tilde{\mathbf{u}}(\mathbf{x}, \omega) \Big|_{\partial \mathcal{A}_1^+} &= \exp(i\mathbf{k} \cdot \mathbf{a}_1) \nabla \tilde{\mathbf{u}}(\mathbf{x}, \omega) \Big|_{\partial \mathcal{A}_1^-}, \end{aligned} \quad \forall \mathbf{x} \in \partial \mathcal{A}_1 \quad (5)$$

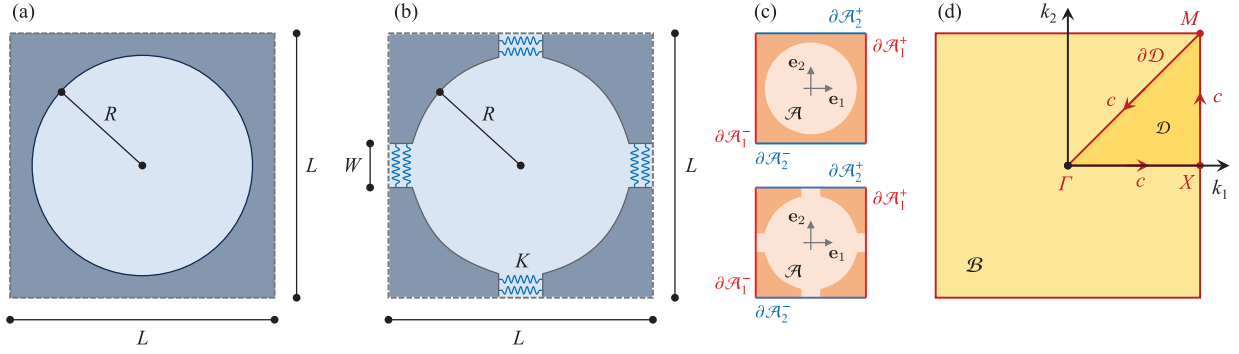


Fig. 2. Periodic cell with phase I (dark color) and phase II (light color): (a) native configuration, (b) engineered configuration, (c) domains of the eigenproblem, (d) first Brillouin zone.

$$\begin{aligned} \tilde{\mathbf{u}}(\mathbf{x}, \omega) \Big|_{\partial \mathcal{A}_2^+} &= \exp(i \mathbf{k} \cdot \mathbf{a}_2) \tilde{\mathbf{u}}(\mathbf{x}, \omega) \Big|_{\partial \mathcal{A}_2^-}, \\ \nabla \tilde{\mathbf{u}}(\mathbf{x}, \omega) \Big|_{\partial \mathcal{A}_2^+} &= \exp(i \mathbf{k} \cdot \mathbf{a}_2) \nabla \tilde{\mathbf{u}}(\mathbf{x}, \omega) \Big|_{\partial \mathcal{A}_2^-}, \quad \forall \mathbf{x} \in \partial \mathcal{A}_2 \end{aligned} \quad (6)$$

where $\partial \mathcal{A}_j^-$ and $\partial \mathcal{A}_j^+$ are distinguished by their outward normals $-\mathbf{e}_j$ and $\mathbf{e}_j$, respectively, while the joined sub-boundaries $\partial \mathcal{A}_j = \partial \mathcal{A}_j^- \cup \partial \mathcal{A}_j^+$. In particular, Eqs. (5)b and (6)b are equivalent to imposing the quasi-periodicity of the transformed stress tensor.

Expressing the boundary value problem in nondimensional terms can be useful to clarify the minimum number of independent factors governing the dynamics of the periodic materials under investigation. First, by adopting the length L and a known frequency Ω as reference quantities for the space and frequency domains, nondimensional forms of both the independent variables $\xi = \mathbf{x}/L$ and $\varpi = \omega/\Omega$ and the dependent variable $\tilde{\varphi} = \Omega/L\tilde{\mathbf{u}}$ can be introduced. Subsequently, it can be recalled that the Lamé parameters of phase P_J can be expressed in the form $\lambda_J = (c_{PJ}^2 - 2c_{SJ}^2)\rho_J = c_{PJ}(1 - 2\eta_J^2)Z_J$ and $\mu_J = c_{SJ}^2\rho_J = c_{PJ}\eta_J^2 Z_J$, where c_{SJ} and c_{PJ} are the speeds of pressure and shear waves, respectively, while $Z_J = c_{PJ}\rho_J$ is the acoustic impedance, with subscript $J = I, II$ related to the phase. Consequently, for the native cell, a minimal set π of four nondimensional independent parameters can be introduced

$$\begin{aligned} \eta_I^2 &= \frac{c_{SI}^2}{c_{PI}^2} = \frac{2\nu_I - 1}{2(\nu_I - 1)}, & \zeta &= \sqrt{\frac{Z_{II}}{Z_I}}, \\ \Gamma_\nu &= \frac{(\nu_I - 1)(2\nu_{II} - 1)}{(\nu_{II} - 1)(2\nu_I - 1)}, & \Gamma_\mu &= \frac{\mu_{II}}{\mu_I} = \frac{E_{II}(\nu_I + 1)}{E_I(\nu_{II} + 1)}, \end{aligned} \quad (7)$$

where ζ is denoted as the *Acoustic Impedance Ratio* (AIR) in the following. Therefore, by selecting the quantity $\Omega = c_{PI}/L$ as convenient reference frequency, the field Eq. (4) can be expressed in the nondimensional form

$$\begin{aligned} \nabla \cdot [\Psi_z(\xi)(1 - 2\eta_I^2\Psi_\nu(\xi))(\nabla \cdot \tilde{\varphi}(\xi, \varpi))\mathbf{I} + \eta_I^2\Psi_\mu(\xi)(\nabla \tilde{\varphi}(\xi, \varpi) + \nabla \tilde{\varphi}(\xi, \varpi)^\top)] &= \\ = -\varpi^2\Psi_\rho(\xi)\tilde{\varphi}(\xi, \varpi), & \quad \forall \xi \in \mathcal{H} \end{aligned} \quad (8)$$

where $\mathcal{H}$ is the nondimensionalization of the domain $\mathcal{A}$. The variable coefficients are nondimensional piecewise π -dependent functions, reading

$$\begin{aligned} \Psi_\rho(\xi) &= \begin{cases} 1 & \xi \in \mathcal{Y}_I \\ \zeta^4 \frac{\Gamma_\nu}{\Gamma_\mu} = \frac{Z_{II}^2 E_I(\nu_{II} + 1)(\nu_I - 1)(2\nu_{II} - 1)}{Z_I^2 E_{II}(\nu_I + 1)(\nu_{II} - 1)(2\nu_I - 1)} & \xi \in \mathcal{Y}_{II} \end{cases}, \\ \Psi_\nu(\xi) &= \begin{cases} 1 & \xi \in \mathcal{Y}_I \\ \Gamma_\nu = \frac{(\nu_I - 1)(2\nu_{II} - 1)}{(\nu_{II} - 1)(2\nu_I - 1)} & \xi \in \mathcal{Y}_{II} \end{cases}, \end{aligned} \quad (9)$$

$$\begin{aligned} \Psi_z(\xi) &= \begin{cases} 1 & \xi \in \mathcal{Y}_I \\ \frac{\Gamma_\mu}{\Gamma_\nu} = \frac{E_{II}(\nu_I + 1)(\nu_{II} - 1)(2\nu_I - 1)}{E_I(\nu_{II} + 1)(\nu_I - 1)(2\nu_{II} - 1)} & \xi \in \mathcal{Y}_{II} \end{cases}, \\ \Psi_\mu(\xi) &= \begin{cases} 1 & \xi \in \mathcal{Y}_I \\ \Gamma_\mu = \frac{E_{II}(\nu_I + 1)}{E_I(\nu_{II} + 1)} & \xi \in \mathcal{Y}_{II} \end{cases}, \end{aligned} \quad (10)$$

where $\mathcal{Y}_I$ and $\mathcal{Y}_{II}$ are the nondimensionalization of the phase domains $\mathcal{P}_I$ and $\mathcal{P}_{II}$. The equation is complemented by nondimensional Floquet-Bloch boundary conditions, expressing the quasi-periodicity of the solutions across the parallel sub-boundaries $\partial \mathcal{H}_j^-$ and $\partial \mathcal{H}_j^+$ separated by the nondimensional periodicity vector $\mathbf{b}_j = \mathbf{a}_j/L$ and reading

$$\begin{aligned} \tilde{\varphi}(\xi, \varpi) \Big|_{\partial \mathcal{H}_1^+} &= \exp(i \boldsymbol{\beta} \cdot \mathbf{b}_1) \tilde{\varphi}(\xi, \varpi) \Big|_{\partial \mathcal{H}_1^-}, \\ \nabla \tilde{\varphi}(\xi, \varpi) \Big|_{\partial \mathcal{H}_1^+} &= \exp(i \boldsymbol{\beta} \cdot \mathbf{b}_1) \nabla \tilde{\varphi}(\xi, \varpi) \Big|_{\partial \mathcal{H}_1^-}, \quad \forall \xi \in \partial \mathcal{H}_1 \end{aligned} \quad (11)$$

$$\begin{aligned} \tilde{\varphi}(\xi, \varpi) \Big|_{\partial \mathcal{H}_2^+} &= \exp(i \boldsymbol{\beta} \cdot \mathbf{b}_2) \tilde{\varphi}(\xi, \varpi) \Big|_{\partial \mathcal{H}_2^-}, \\ \nabla \tilde{\varphi}(\xi, \varpi) \Big|_{\partial \mathcal{H}_2^+} &= \exp(i \boldsymbol{\beta} \cdot \mathbf{b}_2) \nabla \tilde{\varphi}(\xi, \varpi) \Big|_{\partial \mathcal{H}_2^-}, \quad \forall \xi \in \partial \mathcal{H}_2 \end{aligned} \quad (12)$$

where the sub-boundary $\partial \mathcal{H}_j = \partial \mathcal{H}_j^- \cup \partial \mathcal{H}_j^+$ and $\boldsymbol{\beta} = \mathbf{k}L$ is the nondimensional Floquet exponent.

A strong simplification occurs if the materials constituting the two phases possess the same Poisson coefficient ($\nu_{II} = \nu_I =: \nu$). In this case, the piecewise functions $\Psi_\nu(\xi) = 1$ and $\Psi_\mu(\xi) = \Psi_z(\xi)$, so that Eq. (8) becomes

$$\begin{aligned} \nabla \cdot [\Psi_z(\xi)((1 - 2\eta^2)(\nabla \cdot \tilde{\varphi}(\xi, \varpi))\mathbf{I} + \eta^2(\nabla \tilde{\varphi}(\xi, \varpi) + \nabla \tilde{\varphi}(\xi, \varpi)^\top))] &= \\ = -\varpi^2\Psi_\rho(\xi)\tilde{\varphi}(\xi, \varpi), & \quad \forall \xi \in \mathcal{H} \end{aligned} \quad (13)$$

and depends on a reduced set π_r of only three nondimensional parameters

$$\eta^2 = \frac{2\nu - 1}{2(\nu - 1)}, \quad \zeta = \sqrt{\frac{Z_{II}}{Z_I}}, \quad \Gamma_\mu = \zeta^4 \frac{\rho_I}{\rho_{II}} = \frac{E_{II}}{E_I}, \quad (14)$$

because the non-unitary piecewise functions assume the simplified expression

$$\begin{aligned} \Psi_\rho(\xi) &= \begin{cases} 1 & \xi \in \mathcal{Y}_I \\ \zeta^4/\Gamma_\mu & \xi \in \mathcal{Y}_{II} \end{cases}, \\ \Psi_z(\xi) = \Psi_\mu(\xi) &= \begin{cases} 1 & \xi \in \mathcal{Y}_I \\ \Gamma_\mu & \xi \in \mathcal{Y}_{II} \end{cases}, \end{aligned} \quad (15)$$

and consequently – given a certain parameter η – two periodic media (superscripts a and b) are governed by the same nondimensional equation (same ζ and Γ_μ) if the respective phases satisfy $E_{II}^a/E_I^a = E_{II}^b/E_I^b$ and $\rho_{II}^a/\rho_I^a = \rho_{II}^b/\rho_I^b$. For the engineered cell, the set of independent parameters includes also the nondimensional stiffness $\kappa = K/(E_I W)$.

Table 1Elastodynamical properties of the materials selected for the phase P_I of the periodic cell.

Material	ρ_I	E_I	ν_I	c_{SI}	c_{PI}	K
Nylon	1000 kg/m ³	1.75 GPa	0.34	811 m/s	1610 m/s	4.44×10^7 N/m
Aluminum	2700 kg/m ³	69 GPa	0.34	3099 m/s	6153 m/s	1.74×10^9 N/m

2.2. Wave propagation

From the mathematical viewpoint, the field Eq. (8) and the complementary β -dependent boundary conditions (11) and (12) establish a parametric Sturm–Liouville eigenproblem. Solving the eigenproblem consists in determining the infinite eigenpairs $(\omega^2, \tilde{\varphi})$ providing nontrivial solutions, where the square Fourier variable ω^2 serves as unknown eigenvalue and $\tilde{\varphi}$ plays the role of unknown eigenfunction. According to the so-called *inverse method* (see [7]), each eigenpair $(\omega^2, \tilde{\varphi})$ is determined as a function of the Floquet exponent β , adopted as free vector variable. From a physical point of view, the eigenvalue $\omega^2(\beta)$ provides the dispersion relation $\omega(\beta)$ characterizing the plane wave harmonically oscillating in time with circular frequency ω while forward propagating (from $\partial\mathcal{H}_I^-$ to $\partial\mathcal{H}_I^+$) in space with wavevector β . The associated eigenvector $\tilde{\varphi}(\beta)$ describes the propagating waveform. For both the native and engineered configurations, the wavevector β spans the nondimensional square first Brillouin zone $\mathcal{B} = [-\pi, \pi] \times [-\pi, \pi]$, as illustrated in Fig. 2d. The set of all the dispersion functions $\omega^2(\beta)$ over the $\mathcal{B}$ -domain is the dispersion spectrum. For computational efficiency, the analysis of the dispersion relations can be restricted to the irreducible Brillouin contour $\partial\mathcal{D}$, which is the boundary of the triangular irreducible Brillouin zone $\mathcal{D} \subset \mathcal{B}$, defined by the vertices Γ (with $\beta = (0, 0)$), X (with $\beta = (\pi, 0)$), and M (with $\beta = (\pi, \pi)$). This restriction is allowed for the class of symmetry under consideration (p4 mm) based on statistical analyses [62]. The closed contour $\partial\mathcal{D}$ is spanned by the curvilinear abscissa known as the *reduced wavevector* $c \in [0, (2 + \sqrt{2})\pi]$. The curves representing the multivalued function $\varphi(c) = \omega(c)/(2\pi)$ are the branches of the dispersion diagram. The real-valued codomain and imaginary-valued codomain of the dispersion function $\omega(c)$ correspond to frequency passbands and stopbands in the spectral diagram, in which the free propagation of harmonic waves is allowed or inhibited, respectively.

By virtue of the nondimensionalization, all the periodic media described by the same set π of parameters are governed by the same nondimensional Eqs. (8) and boundary conditions (11)–(12). Consequently, they are characterized by the same nondimensional spectra and can be referred to as *isospectral media*. Therefore, under the assumption of all phases with the same Poisson coefficients, the relations $E_{II}^a/E_I^a = E_{II}^b/E_I^b$ and $\rho_{II}^a/\rho_I^a = \rho_{II}^b/\rho_I^b$ can be regarded as *isospectrality conditions*. An interesting consequence is that inverting the phases in two isospectral media preserves their isospectrality. Once the nondimensional dispersion relation $\omega(\beta)$ (or $\varphi(\beta)$) is obtained, its dimensional counterpart is determined as $\omega(\mathbf{k}) = \omega(\mathbf{k}L)\Omega = \omega(\mathbf{k}L)c_{PI}/L$. Accordingly, periodic media characterized by nondimensional isospectrality are also dimensionally isospectral if they have the same cell side ($L^a = L^b$) and the same wave speed $c_{PI}^a = c_{PI}^b$. Under the assumption of all phases with the same Poisson coefficients, the latter condition of dimensional isospectrality can be also expressed as $E_I^a/E_I^b = \rho_I^a/\rho_I^b$.

The eigenvalue problem has been solved numerically in discrete form by using the finite element method. Two-dimensional second-order triangular Lagrangian elements have been used to discretize the cell domain $\mathcal{A}$, where the maximum finite element size is 1/6 of the shortest nondimensional wavelength of interest. The nondimensional parameters that geometrically characterize the native cell and engineered cell have been initially selected to be $r = 2/5$ and $w = 1/5$, corresponding to volume ratios $\alpha_n = 1.011$ and $\alpha_e = 1.416$, respectively. Under the assumption of same Poisson coefficients, the nondimensional parameters that elastically characterize the cells have been selected

to be $\eta = 1/4$, $\kappa = 21/5$ and $\Gamma_\mu = 39$. Therefore, parametric analyses have been carried out by varying the acoustic impedance ratio ζ in the range $\zeta \in [1/10, 10]$ (for both the configurations) and the nondimensional channel width w in the range $w \in [1/100, 1/2]$ (for the engineered configuration). Once the nondimensional spectrum has been determined, the dimensional spectrum has been obtained for (i) the native configuration geometrically defined by a side length of $L = 30$ mm, and a circular inclusion with radius $R = 12$ mm, and (ii) the engineered configuration enriched by channels with width $W = 6$ mm. The wave velocities c_{PI} assigned to rescale the frequency axis of the dimensional spectrum correspond to selecting either a *softer matrix* (Nylon with $c_{PI} = 1610$ m/s) or a *harder matrix* (Aluminum with $c_{PI} = 6153$ m/s) for the phase P_I . A summary of the dimensional data is reported in Table 1.

A selection of significant cases extracted from the parametric analyses is reported in Fig. 3, in which the dispersion diagrams of the native and engineered configurations are compared for the softer and harder matrices. Since the selected dimensional data respect the condition of nondimensional isospectrality (see Table 1), the dispersion spectra coincide if represented in the nondimensional frequency φ (right-hand vertical coordinate), while differ if represented in the dimensional frequency f (left-hand vertical coordinate). Different acoustic impedance ratios ζ are considered, which can be obtained by modifying the material of the phase P_{II} (without changing the material of the phase P_I). The frequency passbands – where the free propagation of waves is permitted – and frequency stopbands (or *bandgaps*) – where the free propagation of waves is inhibited – are distinguished with green and red backgrounds, respectively. Generally, both configurations possess a highly dense spectrum, with prevalence of passbands over stopbands. Complete absence of stopbands (a condition known as *gapless dispersion*) is obtained for acoustic impedance ratio ζ close to unity. The physical reason is that nearly-unitary acoustic impedance ratio ($\zeta \simeq 1$) describe quasi-homogeneous media, whose dispersion spectra slightly differ from those of perfectly homogeneous non-dispersive Cauchy media ($\zeta = 1$). Considering the gapless dispersion as reference condition (obtained by setting $\zeta = 0.8$ and illustrated in Fig. 3a for the native configuration and Fig. 3d for the engineered configuration), a low-frequency full/complete bandgap, systematically located between the third and the fourth dispersion curves, can be opened by either

- *increasing* the acoustic impedance ratio ζ in the native configuration (*high-impedant inclusion*), or
- *decreasing* the acoustic impedance ratio ζ in the engineered configuration (*low-impedant inclusion*)

for both the softer and the harder matrices. The former mechanism of bandgap opening is testified by the comparison between Fig. 3a and Fig. 3b (corresponding to an increment of the acoustic impedance ratio ζ from 0.8 to 2.0), while the latter mechanism of bandgap opening is testified by the comparison between Fig. 3d and Fig. 3c (corresponding to a decrement of the acoustic impedance ratios ζ from 0.8 to 0.4). Inspection of the waveforms indicates that bandgap opening is governed by intra-cellular localization mechanisms. In the native configuration, increasing the acoustic impedance ratio lowers the frequency of a particular dispersion branch associated with waveforms strongly localized in the inclusion. The downward shift of this branch creates a separation between the lower-frequency and higher-frequency bundles of dispersion branches, giving rise to a frequency

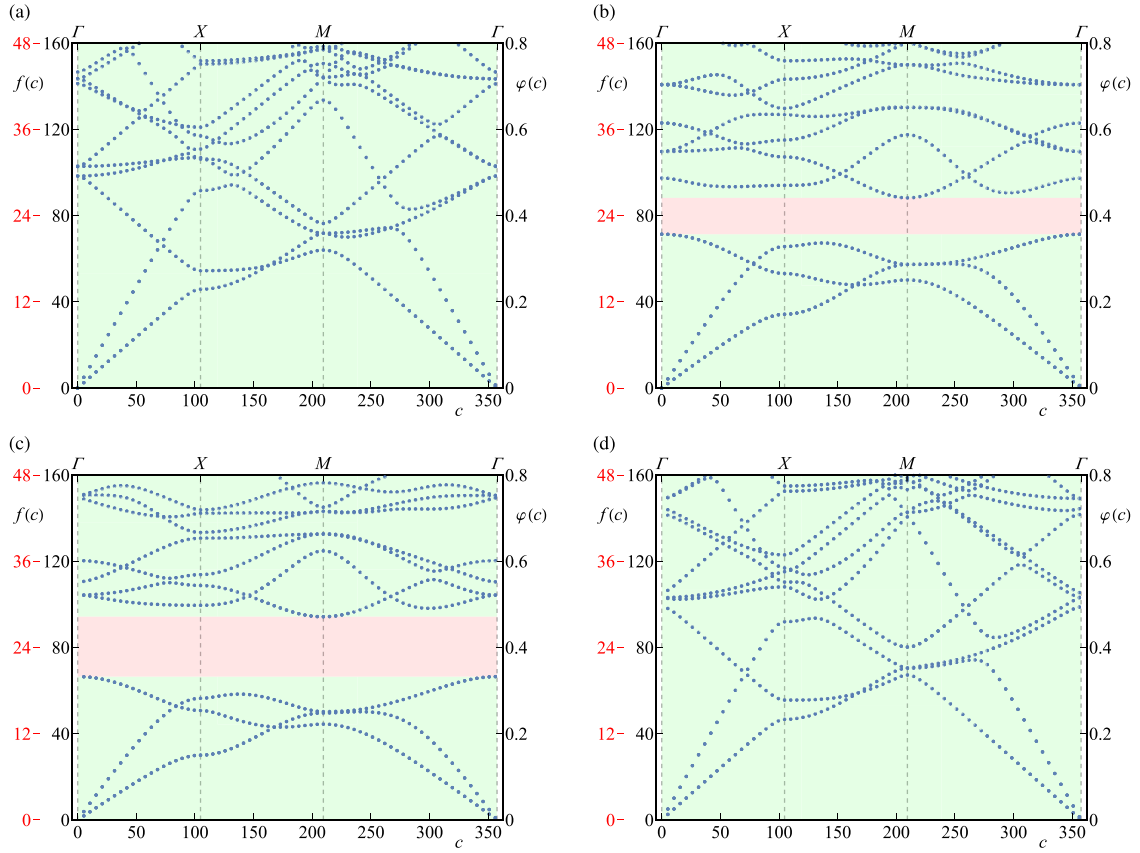


Fig. 3. Dispersion diagrams of nondimensional frequency $\varphi(c)$ and dimensional frequency $f(c)$ [kHz] versus reduced dimensional wavevector $c^* = c/L$ [rad/m] for: (top row) native configuration with: (a) $\zeta = 0.8$, (b) $\zeta = 2$; (bottom row) engineered configuration with: (c) $\zeta = 0.4$, (d) $\zeta = 0.8$. Black vertical axis: Aluminum, red vertical axis: Nylon. Asterisk is dropped.

bandgap. In the engineered configuration, the reduced connectivity of the matrix generates localization phenomena in the matrix islands at low impedance ratios, leading to bandgap opening related to a similar separation mechanism of the lower-frequency and higher-frequency bundles of dispersion branches. As the impedance ratio increases, the waveform localization in the matrix islands decreases, and the bandgap closes. At sufficiently high impedance ratios, waveform localization is activated again in the inclusions, and the bandgap reopens. The topological engineering therefore enables the localization mechanism to shift between the two phases, explaining the opposite trends observed in the two configurations. Recognizing these two opposite mechanisms of bandgap opening paves the way for designing mechanical metafilters based on the native or engineered configurations of biphasic periodic material to functionally inhibit the propagation of elastic waves.

Before outlining a methodological strategy for the design of acoustic metafilters, it may be worthwhile to briefly discuss the waveforms $\tilde{\varphi}(\beta)$. Mathematically, waveforms are complex-valued vector quantities (in the general case), providing a complete geometrical base to describe the harmonic motion of periodically propagating waves. Physically, waveforms are characteristic quantities that can be related to mechanical polarization and directional energy propagation in periodic materials [64–68]. For the cellular system under investigation, the waveforms of the native and engineered configurations characterized by acoustic impedance ratio $\zeta = 2$ are considered. The dimensional waveforms $\tilde{\mathbf{u}}_i(\mathbf{k}) = L/\Omega \tilde{\varphi}_i(\mathbf{k}L) = (\tilde{u}_1(\mathbf{k}), \tilde{u}_2(\mathbf{k}))_i^T$ associated with all the frequencies $\omega_i(\mathbf{k})$ of the dispersion spectrum in the limit of short wavelengths (point X of the reduced Brillouin zone, corresponding to wavevector $\mathbf{k} = (\pi/L, 0)$ or $c = \pi$) can be verified to be real-valued. As in any linear eigenproblems, the amplitude of the eigenfunctions (waveforms) is undetermined and a proper normalization criterion has

to be chosen. Here, unit-normalization with respect to the maximum norm of the displacement vector field over the periodic cell has been adopted. The two normalized components $\tilde{u}_1$ and $\tilde{u}_2$ of the waveforms $\tilde{\mathbf{u}}_i$ (wavecomponents) associated with the lowest frequencies ω_i ($i = 1, \dots, 5$) are shown in Fig. 4. From the qualitative viewpoint, the lower frequencies ω_i (for $i = 1, 2, 3$) tend to be associated with strongly polarized waveforms $\tilde{\mathbf{u}}_i$ (dominated by one wavecomponent and minimally participated by the other), whereas the higher frequencies ω_i ($i = 4, 5$) may correspond to hybrid waveforms $\tilde{\mathbf{u}}_i$ (significantly participated by both the wavecomponents). The comparison between the native and engineered configurations shows that introducing intercellular connectivity does not qualitatively change the waveforms (e.g., it does not invert the polarization), and only leads to minor quantitative modifications, at least at the lowest frequencies.

2.3. Parametric analysis for metafilter design

Once the presence of stopbands in the dispersion spectrum has been ascertained, a key aspect in evaluating the performance of mechanical metafilters is the quantification of the stopband amplitude, or stop bandwidth. Larger stop bandwidths imply stronger effects of wave attenuation over broader frequency ranges. From a functional perspective, stopbands are particularly desirable in the low-frequency range, which coincides with the interval of human sensitivity to sound and vibration (0–25 kHz). From a technical perspective, however, designing artificial media with large stopbands in this range remains inherently challenging, due to the dominant presence of pressure and shear passbands that – in natural and conventional materials – cover the low-frequency spectrum. Artificially opening a stopband at the lowest possible center frequency thus may represent a significant achievement

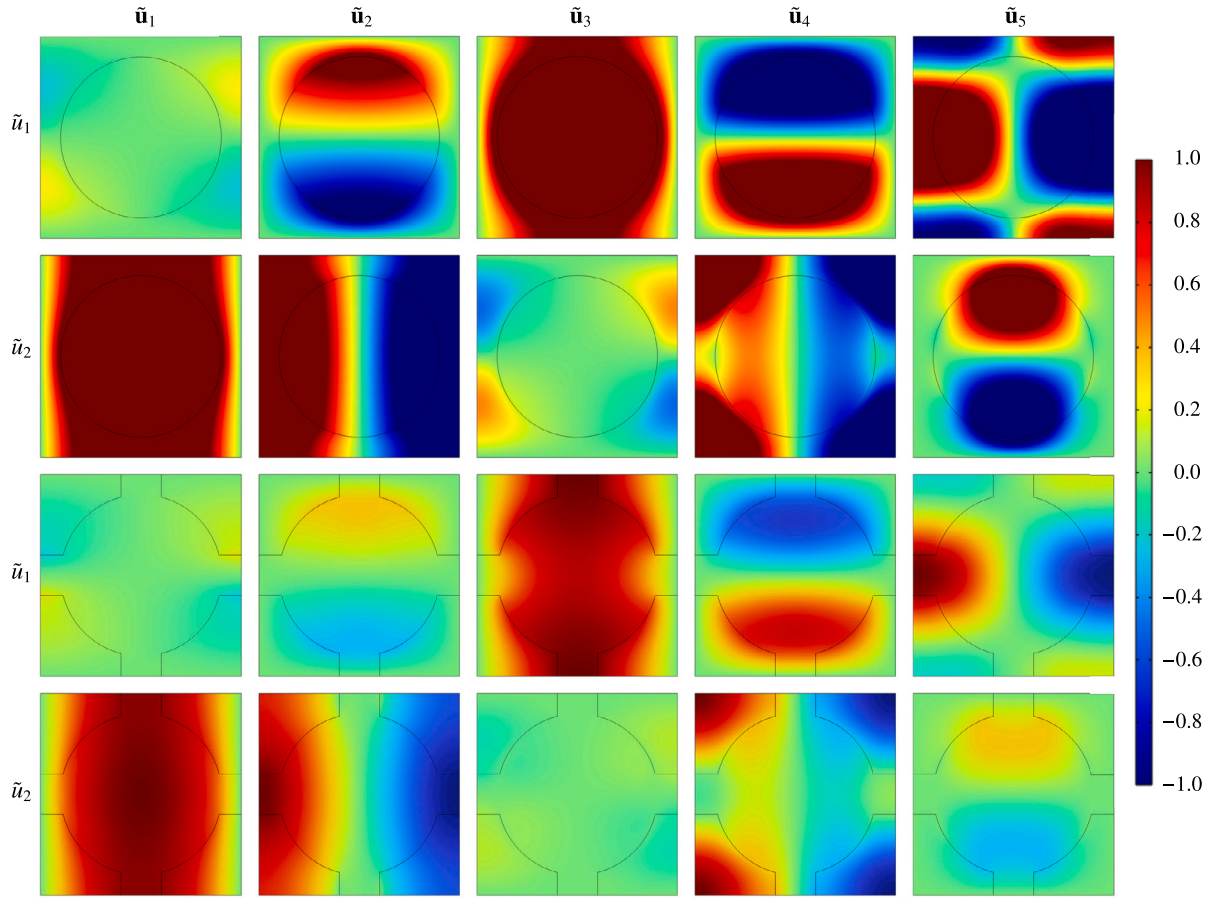


Fig. 4. Real-valued waveforms $\tilde{\mathbf{u}}_i = (\tilde{u}_1, \tilde{u}_2)^T$ associated with the lowest frequencies ω_i at the limit of short wavelengths ($c = \pi/L$) for the native cell (upper rows) and for the engineered cell (lower rows) with acoustic impedance ratio $\zeta = 2$.

in the design of mechanical metamaterials. To enable quantitative comparison across dissimilar cellular configurations and/or different parameter settings, the normalized nondimensional bandwidth $\Delta = \Omega_s/\omega_c$ can be considered as criterion of spectral performance, where Ω_s denotes the amplitude of the lowest-frequency stopband (*principal bandgap*), determined as the difference between the upper and lower boundaries of the stopband, while ω_c represents the center frequency, defined as the arithmetic mean (semisum) of these two boundaries. According to these definitions, larger normalized bandwidths correspond to wider stopbands located at lower centerfrequencies. Similar normalized metrics have been recurrently adopted as objective functions in topological and parametric optimization problems, targeted at maximizing the filtering effects for sound and vibration mitigation of phononic media [46,57,69,70].

Based on this background, the leading idea is to structure a design strategy according to a three-step procedure, consisting in (i) selecting the *interphase contrast* (described by the acoustic impedance ratio ζ) and the *intraphase connectivity* (described by the nondimensional channel width w) as the minimal set $\pi_d = (\zeta, w)$ of principal parameters, or design variables; (ii - Section 2.3.1) performing a parametric analysis to obtain a numerical relation $\Delta(\pi_d)$ over the admissible range of the design variables; (iii - Section 2.3.2) employing the relation $\Delta(\pi_d)$ to provide a characteristic behavioral chart to be used in a direct way (determine the normalized bandwidth Δ given a set of principal parameters π_d), as well as in an inverse way (identify the principal parameter set π_d producing a desired normalized bandwidth Δ). The strategy has some advantages. First, by exploiting the non-uniqueness of the inverse problem solution, families of iso-performant periodic media can be identified in the nondimensional parameter space (that is,

iso- Δ loci of nondimensional parameter pairs). Second, by virtue of the nondimensional formulation, a class of iso-performant periodic media can be associated with each solution of the inverse problem (that is, an iso- π_d group of different dimensional realizations corresponding to the same set of identified parameters), provided that the nondimensional isospectrality conditions are satisfied.

2.3.1. Stop bandwidth identification

The dispersion relation $\omega(c)$ is obtained as the numerical multi-valued solution of the governing eigenproblem under variation of the reduced wavevector c over a sufficiently fine discretization $\mathcal{C} = (c_1, c_2, \dots, c_n)$ of the Brillouin contour ∂D . The eigenvalues can be individually tracked by adopting suitable path-following techniques or, more commonly, a finite set S_i of N eigenvalues $\omega_j^2(c_i)$ can be computed for each wavenumber $c_i \in \mathcal{C}$ (with $j = 1, \dots, N$ and $i = 1, \dots, n$). As a consequence of the eigensolving process, the j th branch in the dispersion diagram is represented not as a continuous curve, but rather as a discrete locus $\mathcal{L}_j$ of n spectral points $\mathbf{p}_{ji} = (c_i, \omega_j(c_i))$. Therefore, two adjacent spectral points $\mathbf{p}_{hi}$ and $\mathbf{p}_{\ell(i+1)}$, resulting from consecutive and independent computations i and $i+1$, do not necessarily belong to the same locus $\mathcal{L}_j$. Indeed, the attribution $h = \ell = j$ cannot be taken for granted. This attribution ambiguity is particularly pronounced near curve intersections (crossings) or quasi-intersections (veerings), and may hinder the correct identification of stop bandwidths. The computational task of assigning each pair of adjacent spectral points to the correct locus $\mathcal{L}_j$, and therefore to the correct $\mathcal{L}_j$ -interpolating dispersion curve C_j , is referred to as *curve coloring*. This problem is recognized as a challenging issue in the spectral design of mechanical metamaterials, and can be formulated as an optimization problem aiming at maximizing the curve smoothness [71].

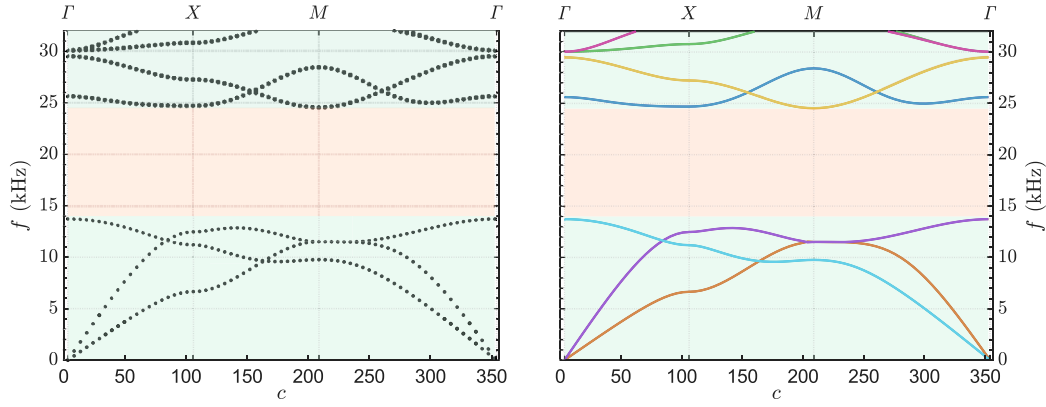


Fig. 5. Dispersion-branch reconstruction in the low-frequency spectrum: (left) unassigned spectral points obtained from the eigenvalue analyses; (right) the same spectral points assigned to continuous dispersion curves through the Hungarian algorithm.

Among the various techniques available in the literature, the curve coloring adopted in the following is based on the so-called Hungarian algorithm [72]. Basically, this algorithm is a general-purpose combinatorial method designed to optimize the solution of the assignment problem, which consists in assigning actions to agents so as to minimize (or maximize) a certain function of total cost. In extreme synthesis, for any two consecutive computations i and $i + 1$, the sets S_i and S_{i+1} of N eigenvalues are used to assemble an N -by- N cost matrix, which attributes a cost to each candidate pairing of eigenvalues (one for each set). The cost function is defined as a weighted linear combination of eigenvalue distance and forward difference mismatch between candidate pairings. The procedure for finding the minimum-cost matching consists of (i) row and column reduction, followed by (ii) iterative matrix updates, until a unique pairing for each row and column yields the optimal assignment. Extending this minimal-cost matching across the entire discretization $\mathcal{C}$ provides the optimal attribution of all spectral points to their respective loci $\mathcal{L}_j$. Finally, each locus is transformed into a dispersion curve via smooth interpolation, thereby reconstructing the full set of branches in the spectral diagram. A successful example of dispersion branch reconstruction is illustrated in Fig. 5: the left panel shows the colorless input data (spectral points), while the right panel displays the reconstructed spectrum (dispersion curves), with individual colors automatically assigned by the algorithm. A finer discretization $\mathcal{C}$ improves the accuracy of the results at the expense of computational speed. When accurate, the curve coloring enables the identification of local maxima and minima, the distinction between crossings and veerings, and ultimately the correct assessment of the complete stop bandwidth $\Delta(\pi_d)$ for the minimal set $\pi_d = (\zeta, w)$ of principal parameters. To guarantee proper computational reproducibility, the pseudocodes of the algorithms devised for data pre-processing (Algorithm 1), curve coloring (Algorithm 2), and stop-bandwidth evaluation (Algorithm 3) are reported in the Appendix.

2.3.2. Design chart

The numerical construction of the behavioral chart for the parametric design of mechanical metafilters requires the evaluation of the normalized stop bandwidth $\Delta(\pi_d)$ over a sufficiently wide and technically relevant range Π_d of the set $\pi_d = (\zeta, w)$ of design variables. To this end, the colored dispersion curves obtained from the parametric analyses described in Section 2.2 have been employed. Admissibility restrictions on the range Π_d might be related to the cell geometry (for w) or to the feasibility of attaining extreme values of material contrast (for ζ). In this respect, it may be worth recalling that the acoustic impedance ratio ζ , which governs the intracellular interphase contrast, is varied across two orders of magnitude, $\zeta \in [1/10, 10]$, while the nondimensional channel width w , which governs the intercellular intraphasic connectivity, is explored within the geometrically admissible

range $w \in [1/100, 1/2]$. As the channel width tends to zero, the behavior of the engineered cell is expected to converge to that of the native cell.

Based on these considerations, the behavior of the native configuration is analyzed first, by constructing the curve associated with the relation $\Delta(\zeta)$ and illustrated by yellow dots in Fig. 6a. From a qualitative perspective, the native configuration reveals that gapless dispersion (that is, $\Delta(\zeta) = 0$) persists over a broad interval of acoustic impedance ratios (approximately for $4/10 \leq \zeta \leq 15/10$), nearly centered on the no-contrast condition ($\zeta = 1$, indicated by the dotted-dashed line). This central interval is conventionally denoted as the *mid-AIR gapless range* $\mathcal{G}$ (green bar). Outside this range, small-amplitude bandgaps emerge in the left interval of lower acoustic impedance ratios (red bar), referred to as the *low-AIR gapped range* J_l (approximately for $\zeta < 4/10$), whereas large-amplitude bandgaps arise in the right interval of higher acoustic impedance ratios (red bar), referred to as the *high-AIR gapped range* J_h (approximately for $\zeta > 15/10$). This latter behavior corroborates the distinctive mechanism of bandgap formation induced by increasing the acoustic impedance ratio in the native configuration, as previously noted in Section 2.2. From a quantitative standpoint, the stop bandwidth systematically increases both with decreasing acoustic impedance ratios within J_l , and with increasing ratios within J_h . Comparatively, the stop bandwidths attainable in the high-AIR gapped range J_h largely exceed those observed in the low-AIR gapped range J_l . Consequently, the high-AIR gapped range J_h emerges as the most promising interval of parameters for the efficient design of mechanical metafilters based on the native configuration. This outcome is consistent with recurrent findings in the literature [11,31,35,73].

The behavior of the engineered configuration is first compared with that of the native configuration in the ζ -domain, by selecting a specific channel width ($w_o = 0.1$). Accordingly, the curve associated with the relation $\Delta(\zeta)$ is constructed and represented by blue dots in Fig. 6a. Similarly to the native configuration, the mid-AIR gapless range $\mathcal{G}$ (approximately for $6/10 \leq \zeta \leq 20/10$) coexists with the low-AIR gapped range J_l (approximately for $\zeta < 6/10$) and the high-AIR gapped range J_h (approximately for $\zeta > 20/10$). The former behavior confirms the distinctive mechanism of bandgap formation induced by decreasing the acoustic impedance ratio in the engineered configuration, as previously noted in Section 2.2. As a minor difference between the native and engineered configurations, the mid-AIR gapless range $\mathcal{G}$ of the engineered configuration is slightly right-shifted, while remaining nearly centered on the no-contrast condition ($\zeta = 1$). As a major difference, the largest stop-bandwidths are attainable in the low-AIR gapped range J_l rather than in the high-AIR gapped range J_h . The high-AIR gapped range J_h also disappears for larger specific channel width ($w_o = 0.3$, illustrated by sky blue dots in Fig. 6a). Consequently, the low-AIR gapped range J_l stands out as the most favorable parameter interval for the effective design of mechanical metafilters based on the engineered configuration.

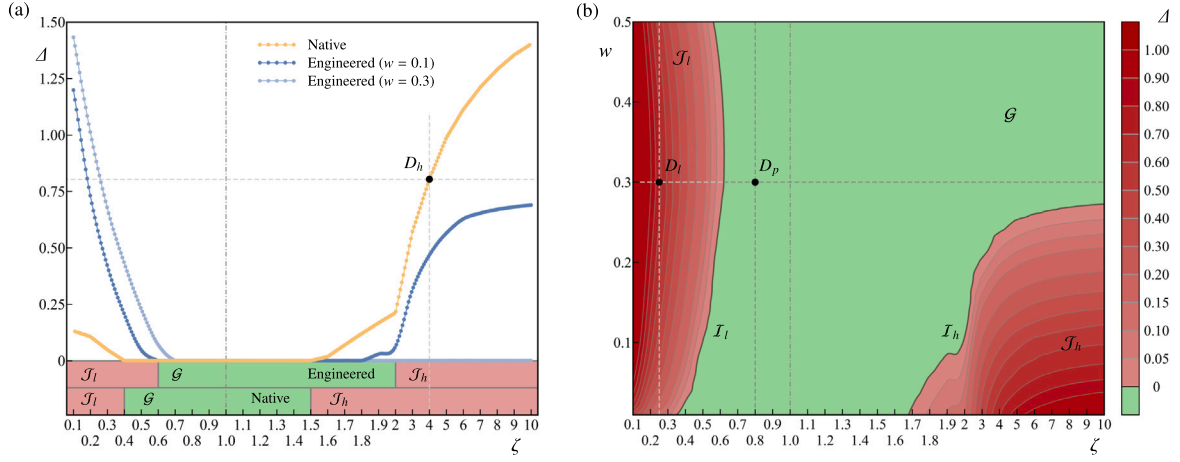


Fig. 6. Nondimensional stop bandwidth $\Delta(\pi_d)$ versus the design parameters: (a) behavior of the native and engineered configurations versus acoustic impedance ratio ζ , (b) behavioral chart of the engineered configuration in the range Π_d .

The analysis of the stop-bandwidth is therefore extended to the entire admissible ranges spanned by the design parameters. Accordingly, the two-dimensional map of the relation $\Delta(\pi_d)$, represented by the contour plot in Fig. 6b, provides a comprehensive behavioral chart of the engineered configuration within the domain Π_d . From a qualitative perspective, the gapless dispersion occupies a broad two-dimensional neighborhood around the no-contrast locus (black dotted line with equation $\zeta = 1$). For nomenclature consistency, this central zone is referred to as the *mid-AIR gapless region* $\mathcal{G}$ (green area). On the left, the region $\mathcal{G}$ borders the *low-AIR gapped region* $\mathcal{I}_l$ (red-shaded area), separated by a behavior-changing boundary (black curve $\mathcal{I}_l$) that persists for all w -values. On the lower-right side, the region $\mathcal{G}$ instead borders the *high-AIR gapped region* $\mathcal{I}_h$ (red-shaded area), separated by a behavior-changing boundary (black curve $\mathcal{I}_h$) that exists only for small w -values. It is worth noting that the curves $\mathcal{I}_l$ and $\mathcal{I}_h$ cannot be determined analytically; however, within the domain Π_d they are accurately fitted by branches of the ellipses $\zeta^2 = 4225/10000[1 - 100/16(w - 35/100)^2]$ and $w^2 = 784/10000[1 - 100/7056(\zeta - 10)^2]$, respectively.

In summary, the parametric analyses of the band structure for both the native and engineered configurations clearly indicate that the largest stop bandwidth can be achieved through *high-contrast configurations*, obtained by maximizing or minimizing the acoustic impedance ratio ζ for the native configuration (within its high-AIR gapped region $\mathcal{I}_h$) and the engineered configuration (within its low-AIR gapped region $\mathcal{I}_l$), respectively. Methodologically, the behavioral chart can be employed to address either the direct or the inverse problem:

- in the *direct problem*, the chart simply predicts the nondimensional stop-bandwidth Δ_o (output) for the configuration described by a given parameter set π_d° (input);
- in the *inverse problem*, the chart identifies one or more parameter sets $\pi_d^\#$ (output) corresponding to isogapped configurations characterized by a target nondimensional stop-bandwidth $\Delta_\#$ (input).

Focusing on the engineered configuration, given a strictly positive target bandwidth ($\Delta_\# > 0$), the solution of the inverse problem may (i) not exist within the domain Π_d , or (ii) be non-unique, corresponding to a family of parameter sets $\pi_d^\#$ located in the low-AIR gapped region (gray iso- Δ contour lines in $\mathcal{I}_l$) and/or in the high-AIR gapped region (gray iso- Δ contour lines in $\mathcal{I}_h$). Since the geometrical and mechanical constraints are included in the definition of the admissible domain Π_d , no additional acceptance or convergence criteria are required.

Naturally, except for the most expedient applications, constructing a behavioral chart does not imply that the spectral design should be carried out through graphical procedures. Rather, the chart has

to be intended as illustration of a bidirectional (but non-bijective) map that can be built to relate the respective input and output of the direct and inverse eigenproblems. Specifically, its fundamental role is threefold (i) to reveal that the inverse problem governing the spectral design is solvable, albeit only through numerical methods, (ii) to show that solutions exist within the physically admissible parameter domain, despite the presence of geometrical and mechanical constraints, and (iii) to identify multiple equivalent, or *isogapped* (but not necessarily isospectral) solutions, that can coexist as a consequence of the intrinsic nonlinearity of the inverse eigenproblem. From the technical viewpoint, the multiplicity of isogapped solutions highlights the versatility of the engineered configuration, which offers a variety of equivalent design alternatives that can subsequently be selected according to additional functional requirements, such as lightweight construction, stiffness enhancement, technological manufacturability, or other application-specific objectives.

3. High contrast finite metafilter

From a technological perspective, a mechanical filter is conceived as a passive device designed to protect vulnerable or sensitive environments from vibration pollution. Mechanical metafilters – derived from periodic metamaterials – are inherently constrained in geometric size due to technical and technological considerations, such as fabrication feasibility and functional requirements. According to these considerations, the biphasic metamaterial provides an effective and flexible platform for designing a mechanical metafilter aimed at shielding critical assets from vibrations. The proposed mechanical metafilter consists of a rectangular array of 20-by- N periodic cells, where the number of columns N (metafilter thickness) is treated as a design variable.

The nondimensional formulation of the spectral analysis reveals that different constituent materials can exhibit the same normalized band structure, provided that they satisfy the isospectrality conditions. However, their dynamic behavior in terms of dimensional bandwidths and center frequencies may differ significantly. Specifically, when Nylon is employed as matrix (phase P_l), the lowest stopband typically appears between 10 and 25 kHz, whereas an Aluminum matrix tends to generate stopbands at higher frequencies. Consequently, since the metafilter design focuses on vibration attenuation within the human sensitivity range (0–20 kHz), Nylon represents the most suitable solution for validating the shielding performance. The selected frequency range is not an intrinsic limitation of the proposed design. Indeed, according to the scaling relation $f = \varphi c_{P_l}/L$, the same normalized band structure can be shifted toward lower or higher frequencies by appropriately scaling the cell size and/or wave velocities of constituent

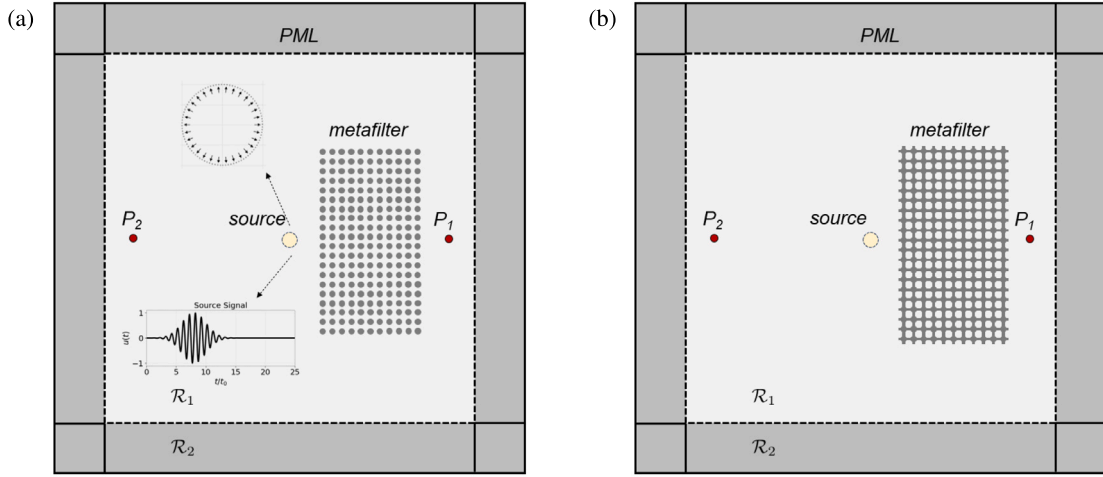


Fig. 7. Sketch of the computational models for virtual experiments: (a) native metafilter and (b) engineered metafilter.

materials. The validation is carried out by developing a computational model and analyzing its dynamic response to broadband excitation in both time and frequency domains during virtual experiments.

3.1. Virtual experiment

The virtual experiment simulates the forced propagation of elastic waves in a two-dimensional square domain (Fig. 7), adopting a geometric configuration consistent with benchmark testbeds reported in the literature [19,74–76]. The mechanical metafilter (shield) is embedded in the right-hand side of the domain (environment). Two variants of the virtual experiment are considered, by employing native cells (Experiment A), and engineered cells (Experiment B) for the cellular metafilter. The environmental domain is uniformly filled with the same constituent material as the cell matrix. The forced dynamics is assessed by extracting the environmental response (output) at two observation points P_1 and P_2 at positions $\mathbf{x}_1$ and $\mathbf{x}_2$, symmetrically located with respect to the central excitation source (input). This Single-Input-Multiple-Output configuration enables a one-run evaluation of the metafilter performance by comparing simultaneous shielded output (in P_1) and unshielded output (in P_2) under identical input conditions.

The forced oscillation problem is solved by the Finite Element method. Second-order quadrilateral and triangular finite elements of the Lagrange type are used. Time and space resolutions, as well as the geometrical dimensions of the environmental domain are adapted to the frequency range under investigation, which coincides with the first stopband (between 10 and 25 kHz). Accordingly, the time step is chosen as 1/20th of the minimum period contained in the forcing function, while the largest element side is 1/6th of the minimum wavelength involved in the analysis. Moreover, the side length of the square domain occupied by the base matrix corresponds to 80 wavelengths.

The external force is introduced through a broadband excitation applied at the boundary surrounding the central source, which occupies a small circular region. Specifically, a time-dependent Dirichlet condition is imposed by prescribing the boundary displacement $\mathbf{u}(\mathbf{x}, t)$ to oscillate radially according to a predefined time law, reading

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{cases} h(t) \sin(x_1) \cos(\arctan(x_2/x_1)) \\ h(t) \sin(x_1) \sin(\arctan(x_2/x_1)) \end{cases} \quad (16)$$

where the function

$$h(t) = \sin(\omega t) \exp\left(-\frac{(t-m)^2}{2n^2}\right), \quad (17)$$

describes an amplitude-modulated sinusoidal signal enveloped by a Gaussian function, with $m = 0.8$ ms and $n = 0.2$ ms defining the

envelope width. A sample of the time history and spatial distribution is shown in Fig. 7a. Since the existence and frequency bounds of the stopbands are intrinsic properties of the periodic native and engineered configurations, the dynamic performance of the mechanical metafilter is expected to be rather independent of the particular time law of the excitation. The broadband excitation pulse (16) allows transmission/attenuation of the passband and stopband components to be assessed within a single simulation.

3.1.1. Perfectly matched layer

The environmental domain is fully surrounded by absorbing boundary layers to prevent disturbing reflections. Implementing an effective decay layer capable of suppressing reflections for all incidence angles requires a dedicated treatment. A suitable approach consists in adopting a mixed displacement–stress unsplit-field *Perfectly Matched Layer* (PML) formulation [77,78]. The PML is constructed through a complex coordinate stretching defined by the real and imaginary stretching tensors $\mathbf{A}_{\text{re}}(\mathbf{x})$ and $\mathbf{A}_{\text{im}}(\mathbf{x})$, represented by diagonal matrices

$$\mathbf{A}_{\text{re}}(\mathbf{x}) = \begin{bmatrix} \alpha_2(x_2) & 0 \\ 0 & \alpha_1(x_1) \end{bmatrix}, \quad \mathbf{A}_{\text{im}}(\mathbf{x}) = \begin{bmatrix} \beta_2(x_2) & 0 \\ 0 & \beta_1(x_1) \end{bmatrix}, \quad (18)$$

depending on the stretching functions $\alpha_j(x_j)$ and $\beta_j(x_j)$ with $j = 1, 2$. This formulation introduces additional unknown fields: besides the displacement vector $\mathbf{u}(\mathbf{x}, t)$, an auxiliary stress-memory tensor $\mathbf{S}(\mathbf{x}, t)$ is incorporated to capture the time-domain effects of the complex stretching. The equations governing the free dynamics of the PML read

$$\nabla \cdot (\dot{\mathbf{S}}_{\text{re}} + \mathbf{S}_{\text{im}}) = \rho(a\ddot{\mathbf{u}} + b\dot{\mathbf{u}} + c\mathbf{u}), \quad (19)$$

$$a\ddot{\mathbf{S}} + b\dot{\mathbf{S}} + c\mathbf{S} = \mu [(\nabla\dot{\mathbf{u}})\mathbf{A}_{\text{re}} + \mathbf{A}_{\text{re}}(\nabla\dot{\mathbf{u}})^T + (\nabla\mathbf{u})\mathbf{A}_{\text{im}} + \mathbf{A}_{\text{im}}(\nabla\mathbf{u})^T] + \lambda \nabla \cdot (\mathbf{A}_{\text{re}}\dot{\mathbf{u}} + \mathbf{A}_{\text{im}}\mathbf{u}) \mathbf{I}, \quad (20)$$

where the spatially varying coefficients are given by

$$a = \alpha_1\alpha_2, \quad b = \alpha_1\beta_2 + \alpha_2\beta_1, \quad c = \beta_1\beta_2 \quad (21)$$

with dependence on $\mathbf{x}$ omitted for brevity. The auxiliary tensor $\mathbf{S}(\mathbf{x}, t)$ evolves according to the integral rule

$$\mathbf{S}(\mathbf{x}, t) = \int_0^t \sigma(\mathbf{x}, \tau) d\tau, \quad (22)$$

where $\sigma(\mathbf{x}, \tau)$ is the Cauchy stress tensor. The attenuation mechanism relies on the stretching functions $\alpha_j(x_j)$ and $\beta_j(x_j)$ working along the j th spatial direction x_j .

Within the square computational domain under investigation, with side length $2L_E$, the rectilinear interfaces separating the environmental domain from the PML (dashed lines in Fig. 7) are defined by the

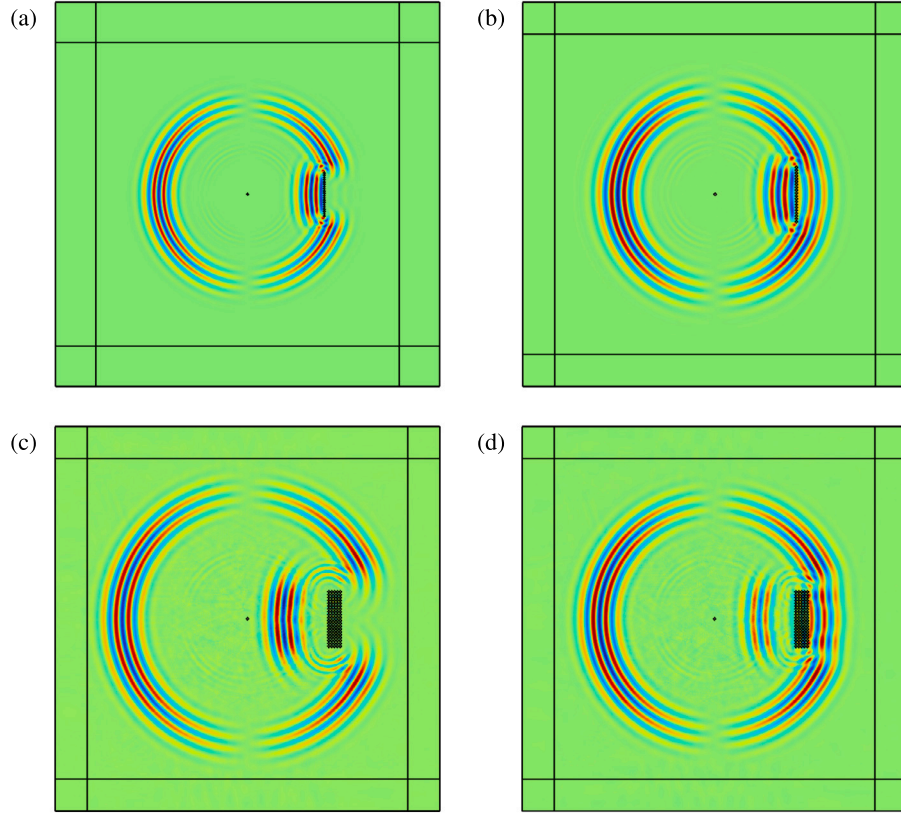


Fig. 8. Displacement component u_1 obtained for engineered metafilters with $w = 0.3$ for different thicknesses N and AIR values ζ : (top) thickness $N = 1$ and AIR (a) $\zeta = 0.25$ (stop) or (b) $\zeta = 0.80$ (pass); (bottom) thickness $N = 5$ and AIR (c) $\zeta = 0.25$ (stop) or (d) $\zeta = 0.80$ (pass).

equations $x_j = x_j^{\text{int}}$. Therefore, the environmental domain (light gray) occupies the region $\mathcal{R}_1 = \{\mathbf{x} : |x_j| \leq x_j^{\text{int}}\}$, whereas the PML (dark gray) occupies the region $\mathcal{R}_2 = \{\mathbf{x} : x_j^{\text{int}} \leq |x_j| \leq L_E\}$. Based on these definitions, the stretching functions can be defined as

$$\alpha_j(x_j) = 1 + \frac{3L_j^C}{2L_j^P} \log\left(\frac{1}{R}\right) \left(\frac{|x_j| - x_j^{\text{int}}}{L_j^P}\right)^{n_j}, \quad (23)$$

$$\beta_j(x_j) = \frac{3L_j^C}{2L_j^P} \log\left(\frac{1}{R}\right) \left(\frac{|x_j| - x_j^{\text{int}}}{L_j^P}\right)^{n_j}$$

where L_j^P and L_j^C denote the layer thickness and characteristic length, respectively, n_j is the polynomial order of the stretching profile, and R plays the role of reflection coefficient. For the virtual experiment under consideration, the layer thickness is set to ten times the maximum wavelength of interest, the characteristic length is chosen as half the layer thickness ($L_j^C = 1/2 L_j^P$), while the polynomial order and reflection coefficient are tuned to $n = 2$ and $R = 10^{-5}$. It is worth noting that, by setting the stretching functions to $\alpha_j = 1$ and $\beta_j = 0$, since $\dot{\mathbf{S}} = \boldsymbol{\sigma}$, Eqs. (19) and (20) reduce to the governing equations of the classical Cauchy continuum, which describe the free dynamics of both the environment and the biphasic shield. Consequently, the interfacial continuity of displacement and stress fields can be straightforwardly enforced by imposing the conditions

$$\mathbf{u}^-(\mathbf{x}, t) = \mathbf{u}^+(\mathbf{x}, t), \quad \boldsymbol{\sigma}^-(\mathbf{x}, t) = \dot{\mathbf{S}}^+(\mathbf{x}, t) \quad (24)$$

at all interfaces (i.e. at $|x_j| = x_j^{\text{int}}$). Here, superscripts $(-)$ and $(+)$ denote the fields approaching the interface from regions $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Finally, homogeneous Dirichlet boundary conditions $\mathbf{u}(\mathbf{x}, t) = \mathbf{0}$ are prescribed at the external limits of the computational domain (i.e. at $|x_j| = L_E$).

3.1.2. Metafilter performance

The metafilters based on the native cellular configuration (*native metafilter*) and engineered cellular configurations (*engineered metafilter*) are designed following the procedure outlined in Section 2.3. Accordingly, the configurations examined below are selected from the behavioral chart in Fig. 6 to assess the consistency between the stopbands predicted for these design points and the corresponding transmission response. Specifically, the native metafilter is obtained by selecting the *design point* D_h within the high-AIR gapped range J_h (black dot in Fig. 6 (left), corresponding to the parameter $\zeta = 4$). This choice determines the normalized amplitude $\Delta = 0.82$ and center frequency $\omega_c = 17.7$ kHz of the principal bandgap. The engineered metafilter is realized instead by selecting the *design point* D_l within the low-AIR gapped range J_l (black dot in Fig. 6 (right), corresponding to the design parameters $\zeta = 0.25$ and $w = 0.3$). This selection corresponds to normalized amplitude $\Delta = 0.34$ and center frequency $\omega_c = 19.9$ kHz of the principal bandgap. The performance of both metafilters is assessed in the time and frequency domains.

From a phenomenological perspective, the broadband excitation generates a multi-frequency localized packet of elastic waves that propagates radially from the source circle. The wave packet reaches the unshielded point P_2 undisturbed, on the left side, and impinges on the metafilter, on the right side. Depending on its configuration and design, the metafilter may either impede or allow wave propagation toward the shielded point P_1 . The engineered metafilter is considered first, by comparing the performance of a thin device ($N = 1$) and a thick device ($N = 5$) to qualitatively explore the effectiveness of the filtering properties in realistic cases where infinite repetitiveness of cells is not attainable. The initial comparison can be carried out by analyzing the displacement field in the entire environmental domain. In

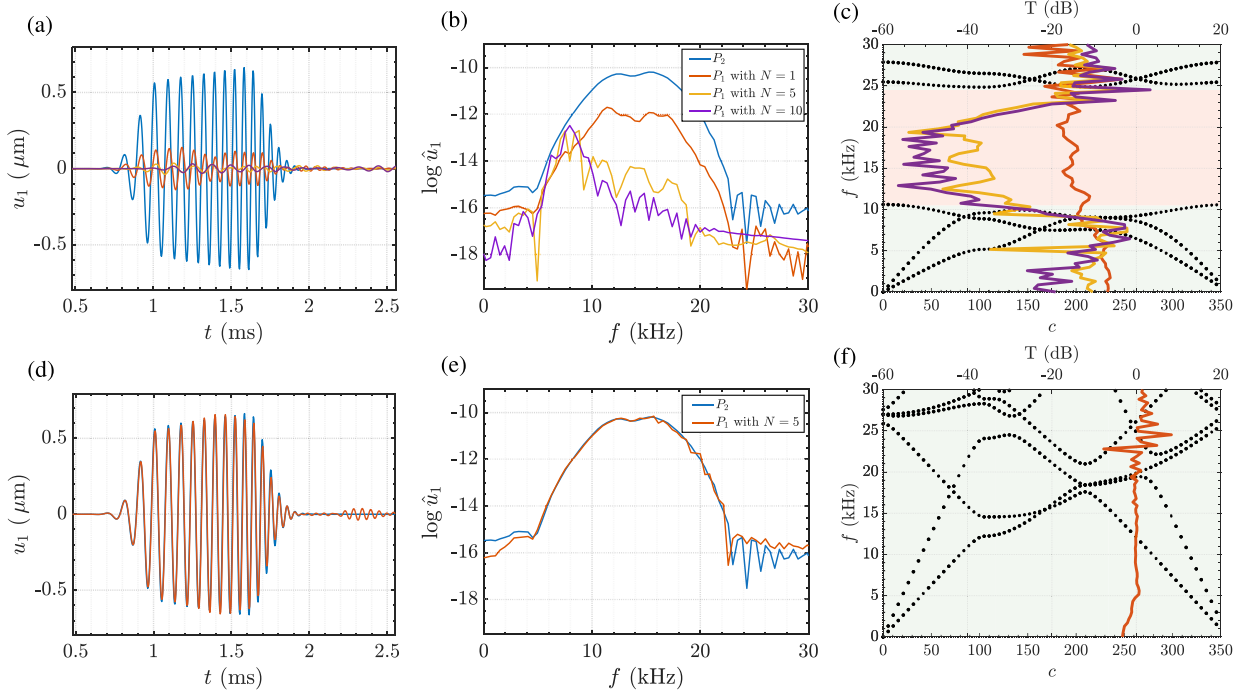


Fig. 9. Performance of the native metafilters with different thicknesses and (a,b,c) AIR $\zeta = 4$, and (d,e, f) $\zeta = 0.8$. Comparison of (a,d) time–history responses, (b,e) frequency-domain spectra, (c,f) transmission coefficients.

particular, analyzing the displacement component $u_1(\mathbf{x})$ confirms that the engineered metafilter is highly effective in shielding the output point P_1 (Fig. 8). The protective efficiency is appreciable for both thin and thick engineered metafilters (Fig. 8a,c), confirming that the filtering effect remains significant even for low-thickness devices. To strengthen the analysis outcomes by comparison, a counterexample is provided by designing a *metapropagator*, namely a device based on the engineered cellular configuration described by the design point D_p within the gapless range $\mathcal{G}$ (black dot in Fig. 6, corresponding to the design parameters $\zeta = 0.8$ and $w = 0.3$). The analysis of the displacement component $u_1(\mathbf{x})$ confirms that – outside the gapped regions – designing an engineered metapropagator allows most of the wave packet to pass and reach the output point P_1 with only minor disturbances, in both low- and high-thickness cases (Fig. 8b,d). Qualitatively similar results, here not reported for the sake of brevity, can be achieved for native metafilters based on the design curve in Fig. 6a.

The general findings achieved in the time domain can be quantitatively validated and strengthened by integration with results in the frequency domain, where transmission coefficients can provide a consistent and robust performance index for vibration reduction. Determining transmission coefficients requires a comparative evaluation of response intensities in the presence and absence of the metafilter, obtained by extracting characteristic variables of the response at the shielded and unshielded output points P_1 and P_2 . Specifically, the real-valued square of the transmission coefficient $T(\omega)$ can be computed according to a quadratic definition [37], as the ratio

$$T(\omega)^2 = \frac{\hat{v}(\mathbf{x}_1, \omega) \hat{v}^*(\mathbf{x}_1, \omega)}{\hat{v}(\mathbf{x}_2, \omega) \hat{v}^*(\mathbf{x}_2, \omega)} \quad (25)$$

where $\hat{v}(\mathbf{x}_j, \omega) = \mathcal{F}_n[v(\mathbf{x}_j, t)]$ stands for the numerical Fourier Transform of the response variable $v(\mathbf{x}_j, t)$ at the output point P_j , while the asterisk indicates the complex conjugate.

An integrated analysis of time-domain and frequency domain results for the native and engineered metafilters can be performed by selecting the displacement component u_1 as the characteristic variable of the response (Fig. 9). The native metafilter is considered first.

Comparison of the time histories at the shielded point P_1 (red, orange and purple curves) and the unshielded point P_2 (blue curve) reveals a marked reduction in response amplitudes throughout the entire oscillation interval (Fig. 9a). Broadband vibration attenuation is further confirmed by the comparison of the Fast Fourier Transforms of the displacement response (Fig. 9b), where the beneficial effect of increasing metafilter thickness (from $N = 1$ to $N = 5$ and $N = 10$) is also evident, particularly in the higher-frequency range. Lastly, the evaluation of the transmission coefficient shows that the filtering effect is effectively localized within the principal bandgap (red band in Fig. 9c), as expected, with frequency-dependent transmission coefficients indicating a substantial response reduction, even exceeding 40 dB for large-thickness devices ($N = 5$ and $N = 10$). Time and frequency domain comparison with the native metapropagator (designed with $\zeta = 0.8$) shows that the filtering effect almost completely disappears under gapless conditions (green band only in Fig. 9f), with negligible reduction of the displacement response even for large-thickness devices ($N = 5$ in Fig. 9d,e,f).

The engineered metafilter exhibits qualitatively and quantitatively similar results, confirming that the performance achieved by the native configuration in the high-AIR range can also be attained by the engineered configuration in the low-AIR range. Specifically, time and frequency domain analyses of the engineered metafiltered response confirm a significant attenuation of the displacement response at the shielded point P_1 , with broadband vibration reduction evident throughout the oscillation interval (Fig. 10a). The Fast Fourier Transform comparison highlights the persistence of the filtering effect (Fig. 10b), while transmission coefficients indicate substantial reductions of response amplitudes within the principal bandgap, in the range of 10–20 dB for large-thickness devices ($N = 5$ and $N = 10$ in Fig. 10c). Similarly, the engineered metapropagator behaves consistently with its native counterpart, showing negligible attenuation under gapless conditions, even for large-thickness devices.

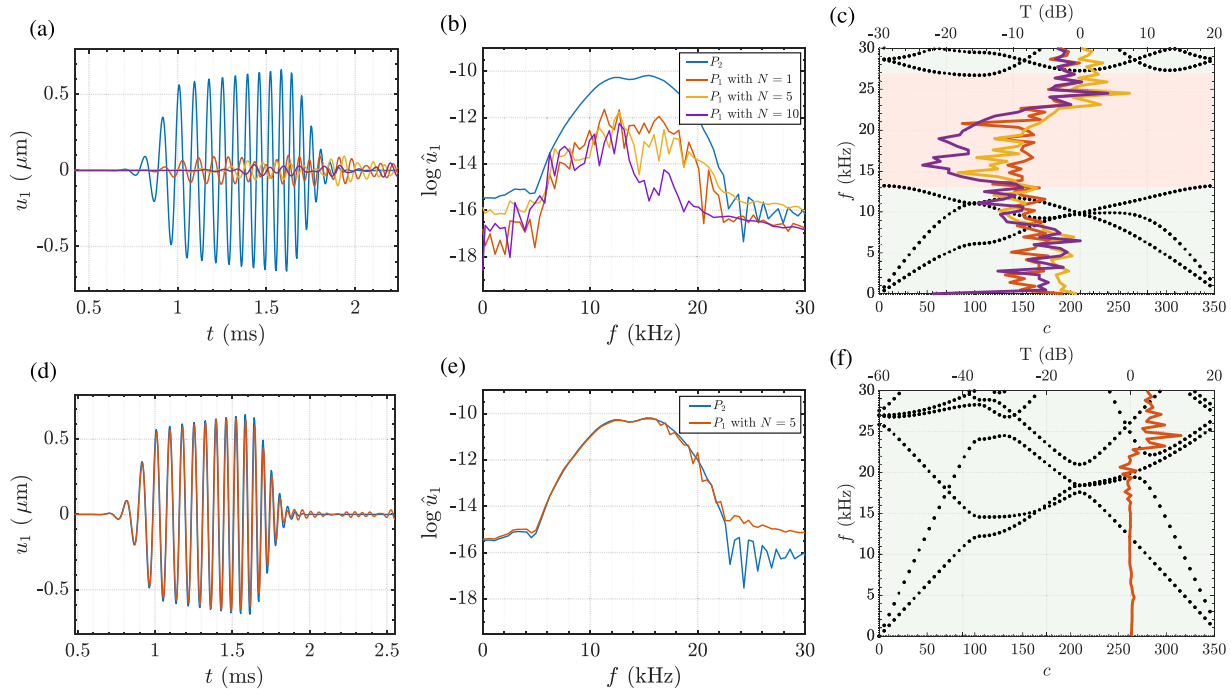


Fig. 10. Performance of the engineered metafilters (with $w = 0.3$) with different thicknesses and (a,b,c) AIR $\zeta = 0.25$, and (d,e, f) $\zeta = 0.8$. Comparison of (a,d) time-history responses, (b,e) frequency-domain spectra, (c,f) transmission coefficients.

4. Conclusions

Elastic wave propagation in biphasic solid–solid phononic crystals has been investigated with the primary objective of realizing compact mechanical metafilters for passive mitigation of sound and vibration. Each of the two solid phases has been modeled as a classical, non-dissipative Cauchy continuum. Two cellular architectures have been analyzed: (i) a native configuration and (ii) an engineered configuration. The native configuration consists of circular inclusions (Phase II) periodically distributed within a base matrix (Phase I). The engineered configuration introduces intraphasic connectivity between adjacent inclusions through intercellular channels, replacing matrix continuity with long-range, channel-bridging elastic couplings. This microstructural architecture enables the engineered biphasic medium to be regarded as a mechanical metamaterial.

The linear partial differential equations governing free oscillations have been formulated in a nondimensional form characterized by piecewise phase-dependent coefficients. Dispersion properties of harmonic wave solutions have been determined through a hybrid semi-analytical finite element approach, which reduces the domain of the associated eigenproblem to the periodic cell, by invoking Floquet–Bloch theory. The formulation reveals the existence of isospectral systems and establishes the conditions for isospectrality within the minimal sufficient space of independent parameters. Parametric analyses of dispersion spectra show that the opening of low-frequency stopbands is regulated by the combined effects of elastodynamic properties (the intracellular interphasic contrast described by the phase-to-phase acoustic impedance ratio) and geometrical features (the intercellular intraphasic connectivity described by the inclusion-to-inclusion channel width). In the native configuration, complete stopbands occur primarily at high impedance ratios, while gapless behavior persists near the no-contrast condition. In the engineered configuration, the introduction of intercellular connectivity qualitatively modifies the bandgap-opening mechanism, enabling bandgaps to appear not only at high impedance ratios but also at sufficiently low impedance ratios. The widest normalized stopbands occur for low-impedance ratios, while high-impedance

ratios produce narrower bandgaps for low connectivity levels. The common behavioral trait is that, regardless of configuration, high-contrast conditions yield the widest low-frequency bandgaps and represent the most promising framework for sound and vibration shielding.

All the parametric analyses have been synthesized into a nondimensional behavioral chart that relates the normalized amplitude of the lowest-frequency bandgap to the key parameters of interphasic contrast and intercellular connectivity (i.e., acoustic impedance ratio and channel width, respectively). This chart serves as a practical tool for both direct and inverse design, enabling the identification of iso-performant configurations and highlighting the role of engineered connectivity in expanding the design space of biphasic phononic crystals. Based on these findings, finite arrays of native and engineered cells have been employed to realize passive mechanical metafilters. Their performance in vibration shielding has been assessed through virtual experiments under transient broadband excitation. Analyses of the forced response in both time and frequency domains confirm a marked attenuation of transmitted waves within the predicted stopband frequencies, with decreasing transmission coefficients for growing numbers of cells. Comparisons between native and engineered metafilters show consistent agreement with predictions derived from dispersion properties. Overall, the study demonstrates that engineered biphasic phononic crystals provide an efficient and versatile platform for low-frequency vibration mitigation. By combining high interphasic contrast with tunable intercellular connectivity, compact passive metafilters with tailored spectral properties can be systematically designed, offering promising perspectives for practical and versatile vibration and noise control applications.

CRediT authorship contribution statement

Meng Wang: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Marco Lepidi:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Methodology, Investigation, Formal

analysis, Data curation, Conceptualization. **Annamaria Pau:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Luigi Martirano:** Writing – review & editing, Supervision, Project administration, Funding acquisition, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

Appendix. Computational methods

The computational pseudocode provided in this Appendix summarizes the programming workflow adopted for curve coloring in the dispersion spectrum and for the subsequent evaluation of stopband widths. Specifically, Algorithm 1 outlines the preprocessing steps applied to the discrete eigenfrequency dataset associated with a sufficiently fine discretization $\mathcal{D}_d$ of the parameter domain Π_d . Algorithm 2 details the reconstruction of dispersion branches using a cost-minimization strategy based on the Hungarian algorithm, ensuring optimal pairings of spectral points across two successive values of reduced wavevector in the discretization $\mathcal{C}$. Finally, Algorithm 3 presents the procedure for stopband width evaluation over the discretization $\mathcal{D}_d$. Together, these algorithms provide an explicit and reproducible scheme of how spectral data are processed to produce continuous dispersion curves and quantify bandgap properties.

Algorithm 1 Data preprocessing (for each point $p \in \mathcal{D}_d$)

Require: The dataset $\mathcal{Q}_i = \{(c_i, S_i^u), c_i \in \mathcal{C}\}$, with unsorted eigenfrequency sets $S_i^u = \{\omega_j^u(c_i)\}$

Ensure: Sorted eigenfrequency sets $S_i = \{\omega_j(c_i)\}$ for all $c_i \in \mathcal{C}$

- 1: Sort all eigenfrequency sets S_i in ascending order of $c_i \triangleright i = 1, \dots, n$
- 2: **for** each c_i **do**
- 3: Sort S_i in ascending frequency order and return sorted $S_i^s = \{\omega_j(c_i)\}$ $\triangleright j = 1, \dots, N$
- 4: **end for**

Algorithm 2 Dispersion branch reconstruction via Hungarian algorithm (for each point $p \in \mathcal{D}_d$)

Require: The sorted eigenfrequency sets S_i for all $c_i \in \mathcal{C}$

Ensure: Complete reconstruction of each locus $\mathcal{L}_j$ of spectral points

$$p_{ji} = (c_i, \omega_j(c_i))$$

- 1: Initialise each locus $\mathcal{L}_j$ from S_1 by assigning $p_{j1} = (c_1, \omega_j(c_1))$ and each eigendifference by setting $d_j(c_1) = 0$
- 2: Fix the real-valued cost weights W_1 (for the eigenfrequencies) and W_2 (for the eigendifferences)
- 3: **for** $i = 1$ to $(n - 1)$ **do**
- 4: Predict next spectral point $p_{ji}^+ = (c_{i+1}, \omega_j^+(c_i))$ for each locus $\mathcal{L}_j$, by setting $\omega_j^+(c_i) = \omega_j(c_i) + d_j(c_i)$
- 5: Use the eigenfrequency sets S_i and S_{i+1} to construct cost matrix $C(c_i)$ with jk -th entry

$$C_{jk}(c_i) = W_1 \left| \omega_j^+(c_i) - \omega_k(c_{i+1}) \right| + W_2 \left| (\omega_k(c_{i+1}) - \omega_j(c_i)) - d_j(c_i) \right|$$

- 6: Apply Hungarian algorithm to the cost matrix $C(c_i)$:
 - i. Subtract row minima and column minima to create initial zero positions
 - ii. Cover all zeros with minimum lines
 - iii. Adjust uncovered elements until an optimal (i.e. minimal-cost) matching is found by pairing each eigenfrequency $\omega_j(c_i)$ of S_i with a unique eigenfrequency $\omega_k(c_{i+1})$ of S_{i+1}
- 7: Re-sort the eigenfrequency set S_{i+1} according to the optimal matching and recalculate $d_j(c_i) = \omega_j(c_{i+1}) - \omega_j(c_i)$
- 8: Assign the optimally matching spectral point $p_{j(i+1)} = (c_{i+1}, \omega_j(c_{i+1}))$ to each locus $\mathcal{L}_j$
- 9: **end for**

Algorithm 3 Principal stopband identification and stopband width assessment.

Require: Completely reconstructed loci $\mathcal{L}_j$ of spectral points $p_{ji} = (c_i, \omega_j(c_i))$ for each point $p \in \mathcal{D}_d$

Ensure: Normalized stopband width $\Delta(p)$ for each point $p \in \mathcal{D}_d$

- 1: **for** each point $p \in \mathcal{D}_d$ **do**
- 2: Identify the eigenfrequency $\omega_b(p)$ of the spectral point p_b bounding the principal stopband from the bottom
- 3: Identify the eigenfrequency $\omega_t(p)$ of the spectral point p_t bounding the principal stopband from the top
- 4: Determine the normalized stopband width $\Delta(p) = 2(\omega_t(p) - \omega_b(p)) / (\omega_t(p) + \omega_b(p))$
- 5: **end for**

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