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Perazzo algebras, strongly Koszulness, Arithmetic Complexes, and Vasconcelos invariant

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In lovely memory of

Fiorindo Mauro (1968 - 2015) *Fiorindo Agostino (1945 - 2025)*
Cesaro Paola (1947 - 2025)

To my family

> *"Posso dire che la mia vita perderebbe gran parte del suo significato se rinunciassi alla speranza di ritrovare in qualche modo le persone che mi sono state più care, se non credessi alle parole del Credo: aspetto la risurrezione dei morti e la vita del mondo che verrà."*
>
> – Ennio De Giorgi

Abstract

The following doctorate thesis is based on four different projects. They lie in the intersection between Commutative Algebra, Algebraic Geometry, and Combinatorics.

The first project treats Perazzo 3-folds and the weak Lefschetz property. We characterise the Hilbert function of Perazzo algebras proving that it has a maximum and a minimum. We use this classification to fully display the weak Lefschetz property for those algebras. A classification on minimal Perazzo 3-folds is given together with a deep study on their geometry and their Jordan type.

The second project investigates strongly Koszul algebras. We examine the combinatorial structures underlying these algebras with the aim of describing them in families. Although the results obtained so far are preliminary, they offer new insights into the nature of these algebras.

The third regards the study of cohomology groups of line bundles on flag varieties. Using recent results of Raicu and VandeBogert, we study flag varieties on the projective space over the integers extending many known results. A key ingredient is the concept of arithmetic complexes, and a uniform identification formula for these complexes.

In the last project, we study the asymptotic properties of the Vasconcelos number. We prove that the (local) Vasconcelos number of $M/I_1^{n_1} \cdots I_r^{n_r} N$ is eventually a minimum of finitely many linear functions on $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$. In the particular case $r = 1$, we prove that the Vasconcelos number is eventually linear and the leading coefficient can be computed using the theory of Rees algebras.

Contents

Introduction

Algebra and geometry have always been closely connected. The Babylonians and Greeks were already translating geometric problems into algebraic ones. The formal development of Algebra came later, especially during the Islamic Golden Age through the work of mathematicians like Al-Khwarizmi. In more recent centuries, the contributions of Hilbert, Emmy Noether, Zariski, Grothendieck, and others shaped what we now know as Algebraic Geometry and Commutative Algebra. During the last century, Combinatorics emerged as an independent field with strong links to Geometry, Topology, Representation Theory, and Commutative Algebra. At the same time, many of the questions studied in Commutative Algebra have their origins in Algebraic Combinatorics and Algebraic Geometry. This exchange of ideas has not only provided new tools for addressing these questions, but has also led to the development of important concepts within Commutative Algebra itself. With this background, the present doctoral thesis explores several problems in Commutative Algebra that arise from geometric questions, conjectures, and results, as well as problems approached through combinatorial methods.

This thesis is based on four distinct projects that constitute the candidate's research activity during the doctoral period from 2022 to 2025. Each of these projects is presented in a separate chapter, arranged in chronological order according to their starting date. The four chapters are unified by a common goal, that is developing an algebraic framework to study problems arising from geometric contexts. The aim is to continue the study of homogeneous ideals (or equivalently projective varieties) by mean of their algebraic invariants in order to unravel their intrinsic nature. The thesis concludes with a brief appendix on the theory of minimal free resolutions and related invariants. There we consider modules on polynomial rings over a field or a general Noetherian ring, and their quotients.

We now proceed by presenting the four projects, summarising the results within.

Perazzo algebras and Lefschetz properties

The study of Lefschetz properties lies at the intersection of several mathematical areas, including Combinatorics, Algebra, Geometry, and Topology. In 1950, S. Lefschetz proved the so-called "Hard Lefschetz theorem" in the context of compact Kähler manifolds which imposes strict conditions on the behaviour of their cohomology rings. This seminal result motivated algebraists to seek purely algebraic analogues, leading to the modern notions of the *weak* and *strong Lefschetz properties* (*WLP* and *SLP*). In recent decades, this research area has gone through a remarkable expansion: This is shown not only by the constant increase in publications appearing each year, but also by the growing number of conferences and workshops explicitly devoted to this topic, and by the growing number of mathematicians interested in these properties. Particular attention has been devoted to the Lefschetz properties of *Gorenstein* algebras, whose rich homological structure makes them especially compatible with this line of study.

In Chapter 1, we investigate the weak Lefschetz property for a special family of Artinian Gorenstein $\mathbf{k}$-algebras. A classical theorem of Macaulay relates Artinian Gorenstein algebras A together with homogeneous polynomials f through the Macaulay's inverse system. We denote A_f the algebra associated with the form f. A homogeneous polynomial in $\mathbf{k}[x_0, \dots, x_n][u_1, \dots, u_m]$ is called a *Perazzo form* if it is of type $f = x_0p_0 + x_1p_1 + \cdots + x_np_n + g$, where $n, m \geqslant 2$, $p_i \in \mathbf{k}[u_1, \cdots, u_m]_{d-1}$ are algebraically dependent but linearly independent and $g \in \mathbf{k}[u_1, \cdots, u_m]_d$. These polynomials are the building blocks in studying forms with vanishing Hessian in five variables [66]. In particular, every

Perazzo form has zero Hessian since its partial derivatives are algebraically dependent. Using the theory of inverse systems, and the characterisation of Lefschetz properties with *higher Hessians* (Theorem 1.8), Perazzo forms give rise to a family of Artinian Gorenstein algebras failing the strong Lefschetz property.

We focus our attention to the first non-trivial case, Perazzo 3-folds $f = p_0x_0 + p_1x_1 + p_2x_2$ living inside $\mathbf{k}[x_0, x_1, x_2][u, v]$. The aim of this chapter is to study the Hilbert function and the WLP of the *Perazzo algebra* A_f. We provide an explicit formula for computing the Hilbert function involving the catalecticant matrices of the forms p_0, p_1, and p_2. As a consequence, Hilbert functions of Perazzo algebras are always unimodal. Furthermore, there exists an upper and a lower bound on their possible shapes depending only on the degree of f.

By proving general results on WLP over Gorenstein algebras, we are able to characterise Perazzo algebras having the WLP. We see that it is fully characterised by the Hilbert function, and A_f has WLP if and only if its Hilbert function does not reach the maximum value in "the centre". In particular, the generic Perazzo algebra, which has the maximum possible Hilbert function, fails the WLP, while Perazzo algebras with minimum Hilbert function always have WLP. The latter are exceptionally special: We are able to fully classify them into three classes up to a linear change of variables. Further, we also characterise them by mean of a geometrical argument involving the reciprocal position of the projective plane $\mathbb{P}(\langle p_0, p_1, p_2\rangle_{\mathbf{k}})$ with the Rational Normal Curve in $\mathbb{P}(\mathbf{k}[u, v]_{d-1})$.

The chapter ends with a deep study on Jordan type of Perazzo algebras with minimum Hilbert function. Since all the algebras have the WLP and fails SLP, we have strong conditions on the generic Jordan type. We are also interested in studying non-Lefschetz elements and the induced multiplication maps by mean of the Jordan type. We see that for Lefschetz elements there are only two possible cases and one of them is the generic Jordan type. We give conditions also on the Jordan type of non-Lefschetz elements. We also give explicit computations for one of the three families.

Strongly Koszul algebras

The study of minimal free resolutions is a central topic in modern Commutative Algebra and Algebraic Geometry. They were first studied by David Hilbert who was interested in syzygies in polynomial rings. One of the most important result on this topic is the Hilbert's Syzygy Theorem: every module over a polynomial ring S has a finite minimal S-free resolutions. This result opened a new research topic which resulted in the creation of many new invariants such as the Betti numbers, the projective dimension, and the Castelnuovo-Mumford regularity.

Set $R = S/I$ with I a non-zero homogeneous ideal contained in $\mathfrak{m}_S^2$, where $\mathfrak{m}_S$ is the maximal irrelevant ideal of S. A graded R-module admits an essentially unique minimal free resolution, although this resolution may be infinite. In particular, the minimal R-free resolution of the base field $\mathbf{k} = R/\mathfrak{m}_SR$ is always infinite. In this frame, Koszul algebras are a nice generalization of polynomial rings. By a theorem of Avramov, Eisenbud, and Peeva (Theorem 2.1), the relative regularity $\operatorname{reg}_R(\mathbf{k})$ is either infinity or zero. In the latter, we call the algebra *Koszul*. A graded module M over a Koszul algebra always has finite regularity $\operatorname{reg}_R(M)$ which is bounded by the Castelnuovo-Mumford regularity $\operatorname{reg}_S(M)$ of M seen as S-module.

The family of strong Koszul algebra is the subject of study in Chapter 2. We say that a standard graded $\mathbf{k}$-algebra R is strongly Koszul if there exists $\mathcal{B} = \{y_1, \ldots, y_n\}$ a $\mathbf{k}$-basis of R_1 such that every colon ideal $(y_a : a \in A) : y_b$ with $A \subset [n]$ and $b \notin A$ is generated by a subset of $\mathcal{B}$. The aim of this chapter is to construct a suitable framework with the aim of classifying and characterise all such algebras. We define the *strongly Koszul masks*, or simply masks, as a combinatorial object that encodes some (if not all) of the combinatorial structure underlying a strongly Koszul algebra. Given such a mark the first main question is whether there exists at least one algebra supporting it. If such an algebra exists the second main question is whether the set of all the algebras supported on the given mask can be given the structure of an algebraic variety or of a scheme. With these questions in mind, we prove several results regarding sub-families of strongly Koszul algebras like monomial quadratic algebras, and quadratic hypersurfaces. We prove that several invariants of a strongly Koszul algebra and of its defining ideal (such as dimension, Hilbert series and number of generators) can be obtained from the underlying mask.

We conclude the chapter by introducing a strategy which we believe can be used to construct all possible algebras satisfying a fixed mask. Given a graded quotient $R = S/I$, we prove that a set of generators for I can be constructed from the information contained in the colon ideals $(y_a : a \in [k]) : y_{k+1}$ for $k = 0, \ldots, n-1$. Moreover, these generators are constructed explicitly. In particular, if R is strongly Koszul, such generating set is minimal. Using this parametrization of the ideal defining R, we are working on a potential algorithm to get the equations of the variety of all the algebras supporting on the given mask. This is still work in progress.

Arithmetic complexes for stable sheaf cohomology

In Algebraic Geometry the study of coherent and quasi-coherent sheaves on projective varieties plays a prominent role. In our case, we are interested in a special family of varieties called flag varieties. Recall that given a vector space V over an algebraically closed field $\mathbf{k}$, the flag variety Fl_n is defined as

$$\mathrm{Fl}_n := \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V : \dim V_i = i\}.$$

By their very nature, these varieties can be studied and are interesting form several prospectives and point of view including Representation Theory and Algebraic Combinatorics. An important and widely developed line of research today focuses on the study of line bundles on these varieties. The celebrated Borel-Weil–Bott Theorem provides a complete characterisation of the cohomology of line bundles in the case where the base field $\mathbf{k}$ has characteristic zero. This is not the case in positive characteristic, where the behavior of line bundle cohomology is far more subtle and significantly less understood. However, some results about the (non) vanishing of the cohomology groups of the line bundles are known. We focus on the work of Raicu and VandeBogert [116]. They use a connection between the these line bundles of Fl_n and the skew-Schur functors $\mathbb{S}_{\lambda\backslash\mu}$ applied to the cotangent sheaf Ω of $\mathbb{P}(V)$ (Theorem 3.7). By letting the dimension of V tending to infinity, they prove that the cohomology groups stabilises to a group $H^j_{st}(\mathbb{S}_{\lambda\backslash\mu}\Omega)$ that they name *stable sheaf cohomology* groups. Furthermore, they define explicit *arithmetic complexes* $C^{\mathbf{k}}_\bullet(\underline{w})$ that computes the stable cohomology for specific skew-shapes. This allows them to prove connection formulae that translates to a relation formulae for the stable cohomology groups for the line bundles.

The content of Chapter 3 arose from a question posed by the two authors during the summer school *PRAGMATIC 2023 - Cohomology and Frobenius* and presented as a conjecture in [59]. They ask if it is possible to generalise these arithmetic complexes to the ring of integer valued polynomials as $C_\bullet(\underline{w})$ and find an explicit isomorphism

$$C_\bullet(x, 1^d) \cong C_\bullet(-x-2d, 1^d),$$

where x is a variable and d is a positive integer. Raicu and VandeBogert have already proven such isomorphism in their paper by mean of induction strategy and homological considerations, but no explicit formula has been found. In this chapter we give an explicit formula for this isomorphism. We see that the maps representing it depends only on integer values. This allows us to generalise some of the results within [59] to the projective space over the integers. In particular we prove the following.

Theorem. *Let Ω be the cotangent sheaf over $\mathbb{P}^n(\mathbb{Z})$ and $\mathbb{S}_{\lambda/\mu}$ be a skew Schur functor. The following is true for all i.*

i) $H^i(\mathbb{P}^n(\mathbb{Z}), \mathbb{S}_{\lambda/\mu}\Omega)$ *is independent of n for $n \gg 0$, and is denoted by* $H^i_{st}(\mathbb{S}_{\lambda/\mu}\Omega)$.

ii) If $\mathbb{S}_{\lambda/\mu}$ is a ribbon Schur functor corresponding to $(w_0, ..., w_d)$, then

$$H^i_{st}(\mathbb{S}_{\lambda/\mu}\Omega) = H_{|w|-i}(C_\bullet(\underline{w}) \otimes_R \mathbb{Z}).$$

iii) If λ is a two column partition, i.e. the conjugate partition for λ is of the form $\lambda' = (m, d)$ for integers $m, d \geqslant 0$, then

$$H^i_{st}(\mathbb{S}_\lambda \Omega) = H_{d+m-i}(\breve{C}_\bullet(-m-d-1, 1^d)[-d] \otimes_R \mathbb{Z}),$$

where $\breve{C}_\bullet(-m-d-1, 1^d)$ denotes the dual of the complex $C_\bullet(-m-d-1, 1^d)$.

iv) Consider λ as in iii) and let $d \geqslant 1$, then

$$H^i_{st}(\mathbb{S}_\lambda \Omega) = H^{2m+2-i}_{st}(\mathbb{S}_{(d+1, 1^{m-d})} \Omega).$$

Vasconcelos invariant for graded modules

In Coding Theory, one of the principal tools in evaluating the efficiency of a linear code is the minimum distance function. In a recent work [39], the authors introduced the so called Vasconcelos invariant in studying the regularity index of the minimum distance in the setting of projective Reed-Muller type codes. This study led to a series of papers and results in different mathematical areas like Commutative Algebra and Combinatorics, where the Vasconcelos invariant has been generalised and contextualised. In those papers, despite the connection to Coding Theory had faded, it became clear the mutual connection with this invariant and other classical algebraic invariants like the Castelnuovo-Mumford regularity, and the initial degree.

A broad definition can be stated as follows. Let R be a Noetherian $\mathbb{N}$-graded ring, and let M be a $\mathbb{Z}$-graded R-module. Then for every associated prime $\mathfrak{p} \in \operatorname{Ass}_R(M)$, we define the local Vasconcelos number as

$$v_{\mathfrak{p}}(M) := \inf\{u \in \mathbb{Z} : \text{there exists } x \in M_u \text{ such that } \mathfrak{p} = 0 :_R x\},$$

while the Vasconcelos number of M is defined as

$$v(M) := \inf\{v_{\mathfrak{p}}(M) : \mathfrak{p} \in \operatorname{Ass}_R(M)\}.$$

Many authors have applied the Vasconcelos invariant $v(R/I)$ for specific ideals I as monomial ideals, binomial edge ideals, and compared it with the Castelnuovo-Mumford regularity. In Chapter 4 we are interested in studying its asymptotic behaviour. That is, given homogeneous ideals $I_1, \dots, I_s \subset R$, and graded modules $N \subset M$, what can we say about the (local) Vasconcelos number of $M/\mathbf{I}^{\underline{n}}N$ and $\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}}N$ as a function of $\underline{n} \in \mathbb{N}^s$? Here $\mathbf{I}^{\underline{n}} = I_1^{n_1} \cdots I_s^{n_s}$ with $\underline{n} = (n_1, \dots, n_s)$. We must note that the local Vasconcelos number $v_{\mathfrak{p}}(M/\mathbf{I}^{\underline{n}}N)$ is well-defined for every $\underline{n} \gg \underline{0}$ since the set of associated primes of $M/\mathbf{I}^{\underline{n}}N$ eventually stabilises. The same can be said also for $\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}}N$.

We translate the computation of the local Vasconcelos number, to a problem of calculating the initial degree of a suitable graded R-module. By considering some mild assumptions, we are able to prove that every local Vasconcelos number of all the aforesaid modules is eventually the minimum of finitely many linear functions. The same statement holds true also for the Vasconcelos number. In particular, the leading coefficients of the linear functions corresponds to the degrees in which the ideals $I_1, \dots, I_s \subset R$ are generated. These results have been proved by considering a wider setting where the same behaviour is shown for the initial degree.

In the simple case where only one homogeneous ideal I is considered, i.e. $s = 1$, the minimum degenerates and the (local) Vasconcelos number is eventually a linear function. The leading coefficient is the degree of a generator of a reduction ideal of I. Moreover, we also prove that the leading coefficients of $\operatorname{indeg}\big(I^nM/I^{n+1}N\big)$, $v\big(I^nM/I^{n+1}N\big)$ and $v\big(M/I^{n+1}N\big)$ are all equal and a formula involving modules over the Rees algebra $\mathcal{R}(J)$ is given.

We then dedicate a section on (non-) examples completing the mentioned results. Finally we discuss the nature of the Vasconcelos invariant in the local case. We prove some results on the linearity of the Vasconcelos number in a special case. We see that in general, its behaviour diverges from the graded cases in unexpected ways.

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With deep gratitude,

Luca Fiorindo

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Notations and Conventions

Throughout the thesis we adopt the following notations and conventions.

- The end of proof will be signal with a tombstone symbol $\square$, the end of a remark by $\triangle$, while the end of an example by $\diamondsuit$.
- Given a finite set P, we indicate its cardinality as $\#P$.
- The set of natural numbers $\mathbb{N}$ includes the number 0.
- The set $\mathbb{N}^r$ of r-tuples of natural numbers $\underline{n} = (n_1, \ldots, n_r)$ is endowed with the component-wise addition. By $\underline{0}$ and $\underline{1}$, we denote the r-tuples $(0, \ldots, 0)$ and $(1, \ldots, 1)$ respectively.
- When a ring is considered, we always intend a unitary, commutative and Noetherian ring.
- We say that a ring R is $\mathbb{N}^r$-graded if there exists a decomposition $R = \oplus_{\underline{n} \in \mathbb{N}^r} R_{\underline{n}}$ such that $R_{\underline{n}} R_{\underline{m}} \subset R_{\underline{n}+\underline{m}}$ for every $\underline{n}, \underline{m} \in \mathbb{N}^r$. In particular, every homogeneous part $R_{\underline{n}}$ is a $R_{\underline{0}}$-module.
- If not stated otherwise, a polynomial ring $S = A[x_1, \ldots, x_n]$ over a ring/field A is considered standard graded i.e. $\deg(x_i) = 1$ for every $i = 1, \ldots, n$.
- Let $R = \oplus_{i \geqslant 0} R_i$ be a graded ring. A module M over R is say to be graded if M has a $\mathbb{Z}$-graded module structure: That is, there exists a decomposition $M = \oplus_{n \in \mathbb{Z}} M_n$ such that $R_a M_b \subset M_{a+b}$ for every $a \in \mathbb{N}$, and $b \in \mathbb{Z}$. In particular, every homogeneous component M_n is a module over R_0.

  A homomorhpism of R-modules $\phi : M \to N$ is graded of degree $d \in \mathbb{Z}$ if $\phi(M_a) \subset N_{a+d}$ for every $d \in \mathbb{Z}$. We say that a homomorphism is graded if it is graded of degree 0.
- Let R be a ring, and M be a R-module. A prime ideal $\mathfrak{p}$ is associated with M if $\mathfrak{p} = 0 :_R m$ for some $m \in M$. The set of associated prime ideals is indicated with $\operatorname{Ass}_R(M)$. If R is $\mathbb{N}$-graded, and M is a $\mathbb{Z}$-graded R-module, then every associated prime is homogeneous.

Chapter 1

Perazzo algebras and Lefschetz properties

> *"It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."*
>
> – Emil Artin

In Algebraic Geometry, a projective hypersurface $X = V(F) \subset \mathbb{P}^N := \mathbb{P}^N(\mathbf{k})$ is said to have vanishing hessian if $\mathrm{hess}(F) := \det \mathrm{Hess}(F)$ vanishes identically. Here, $\mathbf{k}$ is an algebraically closed field of characteristic zero, and $Hess(F)$ is the classic Hessian matrix. If F has degree d, then $\mathrm{hess}(F)$ is homogeneous of degree $(d-2)(N+1)$. Hypersurfaces with vanishing hessian have been studied very frequently in the last centuries due to their peculiar geometry. A projective hypersurfice $X = V(F)$ is a cone if in a suitable coordiante system, the polynomial F does not involve one of the variables. Obviously if X is a cone then $\mathrm{hess}(F) = 0$. In 1851 and later in 1859 O. Hesse claimed and proved that any hypersurface with vanishing hessian is a cone ([78], [79]). Later, in 1876 P. Gordan and M. Nöther disproved Hesse results proving that in every projective space $\mathbb{P}^N$ with $N \geqslant 4$ there exist classes of examples of hypersurfaces with vanishing hessian that are not cones. Their arguments were based on the facts that X is a cone if and only if the partial derivatives of F are $\mathbf{k}$-linearly dependent, while X has vanishing hessian if and only if they are $\mathbf{k}$-algebraically dependent. Finally, X is smooth if and only if the partial derivatives form a regular sequence. They also gave a complete description of the hypersurfaces in $\mathbb{P}^4$, not cones, with vanishing hessian. Subsequent contributions were given by several authors. The paper went largely unnoticed by the Italian mathematical community until Perazzo, in 1900, brought it to the attention of many algebraic geometers, including R. Permutti, and A. Franchetta. We refer to [118] for an exhaustive bibliography.

In recent years, the study of Hessians has gained new attention in Commutative Algebra, in particular in the study of Lefschetz properties. Recall that a standard graded Artinian $\mathbf{k}$-algebra A has the *weak Lefschetz property (WLP)* if multiplication by a generic linear form ℓ has maximal rank in each degree. Similarly A has the *strong Lefschetz property (SLP)* if multiplication by ℓ^s has maximal rank in each degree for every positive integer s (see Definition 1.3). Although these properties may seem easy to check, the general picture is far from being understood. The most known and long-standing conjecture is that every Artinian Complete Intersection should have the SLP, which first appeared in [117]. For monomial Complete Intersections the conjecture holds true due to a result in [123]. Several partial results on low codimension can be found for example in [8, 17, 73]. A very recent result proves the SLP for complete intersections whose dual generator is a binomial [43]. These are just to mention a few of the many partial results that has appeared from the 1990's till today. Despite the large literature on this topic, the community remains divided: some researchers believe the conjecture to be true, while others think otherwise. At moment, a definite solution seems to be out of reach.

In the Gorenstein case the Complete Intersection conjecture turns out to be false also for the WLP. Stanley in [122] gave an example of a Gorenstein algebra with Hilbert function $(1, 13, 12, 13, 1)$:

An Artinian algebras satisfying the WLP/SLP has a unimodal Hilbert function. More in general, a non-trivial characterisation in terms of vanishing of Hessians has been found. Indeed, the Artinian Gorenstein algebra A_f with Macaulay dual generator f fails the SLP if $\text{hess}_f = 0$. More precisely, A_f fails the SLP if and only if one of certain generalised Hessians, called higher Hessians, of f vanishes, see J. Watanabe's [129] and J. Watanabe and T. Maeno's [100]. This result has been generalised to the WLP using the so called mixed Hessians, see R. Gondim and G. Zappalà's [63]. In particular, hypersurfaces with vanishing hessian all fail the strong Lefschetz Property. A natural question is whether they have or they fail also the weak Lefschetz Property. This question was considered by R. Gondim in [62], where he constructed examples of hypersurfaces having and failing the WLP.

In this chapter, we consider the case of $\mathbb{P}^4$, where the classification of hypersurfaces with vanishing hessian is complete. Following the terminology introduced by Gondim in [62] and using homogeneous coordinates x_0, x_1, x_2, u, v for $\mathbb{P}^4$, we say $X = V(F) \subset \mathbb{P}^4$ is a Perazzo hypersurface if F has degree $d \geqslant 3$, and it can be written as $F = p_0x_0 + p_1x_1 + p_2x_2 + g$, where p_0, p_1, p_2 are linearly independent forms of degree $d-1$ in u, v, and g is a form in u, v of degree d. Perazzo hypersurfaces have vanishing hessian since the forms p_0, p_1, and p_2 are always algebraically dependent. On the other hand, according to [130] and [131], any hypersurface of degree d, with $3 \leqslant d \leqslant 5$ in $\mathbb{P}^4$, which is not cone and has vanishing hessian, is a Perazzo hypersurface. In general, as proved by Gordan and Nöther in [66], all forms with vanishing hessian, not cones, are elements of $K[u, v][\Delta]$ where Δ is a Perazzo polynomial of the form $p_0x_0 + p_1x_1 + p_2x_2$ (see also [131, Theorem 7.3]). This type of forms defines the "Franchetta hypersurfaces" (see [118] for all the details). If $d = 3$, for such an f, the $\mathbf{k}$-algebra A_f fails the weak Lefschetz Property by construction. For $d = 4$ Gondim proved that every Artinian Gorenstein algebra of codimension 5 has the weak Lefschetz Property [62, Theorem 3.5].

The aim of this chapter is to study Artinian Gorenstein algebras associated with Perazzo 3-folds of any degree $d \geqslant 3$. After giving some preliminary definitions and results in Section 1.1, in Section 1.2 we study Hilbert function of Perazzo algebras. After providing an explicit formula in Proposition 1.22 involving the catalecticant matrices of p_0, p_1, and p_2, we prove in Propositions 1.24 and 1.27 that the h-vector has a maximum and a minimum, coinciding only if $d = 3, 4$. Then, we consider the weak Lefschetz Property in Section 1.3. First, we focus on algebras A_f whose Hilbert function attains the upper or the lower bound, giving a ground base for the general case. Our main results, contained in Theorems 1.34, 1.35, and 1.46, say that A_f has the weak Lefschetz Property if and only if HF_{A_f} does not reach its maximal possible value in the centre.

In Section 1.4, the focus is given to Perazzo algebras A_f having minimum Hilbert function. Using the theory of Waring rank for forms in 2 variables, we are able to obtain in Theorem 1.51 a complete list of Perazzo forms with minimum Hilbert function. The classification is in terms of the position of the linear space π generated by p_0, p_1, p_2 in $\mathbb{P}(K[u, v]_{d-1})$, with respect to the secant varieties of the rational normal curve C_{d-1}. It follows that, to ensure the minimality of A_f's Hilbert function, π has to meet C_{d-1}, and there are three possibilities: either π is an osculating plane to C_{d-1}, or it is tangent to C_{d-1} and meets the curve in a second point, or the intersection $\pi \cap C_{d-1}$ consists of three distinct points. We conclude with a geometrical study of the polar and Gauss maps associated with these 3-folds.

In Section 1.5 we will also consider Perazzo algebras A_f with minimum Hilbert function, and we compute their Jordan type following the classification given in the previous section. This computation is done both for weak Lefschetz and non-weak Lefschetz elements.

The results presented in this chapter were published in [54] in collaboration with Emilia Mezzetti, and Rosa Maria Miró-Roig, [51], the candidate's Master thesis, and [2] in collaboration with Nancy Abdallah, Nasrin Altafi, Pietro De Poi, Anthony Iarrobino, Pedro Macias Marques, Emilia Mezzetti, Rosa Maria Miró-Roig, and Lisa Nicklasson.

1.1 Preliminaries

Let $\mathbf{k}$ be a field of characteristic zero, and let A be a graded $\mathbf{k}$-algebra. In this chapter, we only consider standard graded $\mathbf{k}$-algebras if not specified otherwise. That is $A = \oplus_{i \geqslant 0} A_i$ with $A_0 = \mathbf{k}$, and $A = \mathbf{k}[A_1]$. Under this assumptions A is isomorphic to a quotient of a standard graded polynomial ring over a homogeneous ideal. If A is also Artinian, then we define the *Socle degree* to be the integer d such that the ring A can be written as $A = \oplus_{i=0}^{d} A_i$ with $A_d \neq 0$. The terminology of Socle degree is especially used when talking about Lefschetz properties. We leave [72, Remark 2.11] as example of basis reference.

Hilbert function

For a standard graded $\mathbf{k}$-algebra $A = \oplus_{i \geqslant 0} A_i$, its *Hilbert function* is a numerical function $\mathrm{HF}_A \colon \mathbb{N} \to \mathbb{N}$ defined as $\mathrm{HF}_A(t) = \dim_{\mathbf{k}} A_t$. Since the ring A is standard graded, the dimension $\dim_{\mathbf{k}} A_t$ is finite for all $t \geqslant 0$; therefore, the function HF_A is well-defined. When A is also Artinian with Socle degree d, then the information of the Hilbert function are captured by the *Hilbert vector*, h-vector for short, defined as $(1, \mathrm{HF}_A(1), \dots, \mathrm{HF}_A(d)) \in \mathbb{N}^{d+1}$. A Hilbert vector $h = (h_0, h_1, \dots, h_d)$ is said to be *unimodal* if, for some integer $s \in \mathbb{N}$,

$$h_0 \leqslant h_1 \leqslant \cdots \leqslant h_s \geqslant h_{s+1} \geqslant \cdots \geqslant h_d.$$

Analogously, it is called *symmetric* if

$$h_k = h_{d-k} \text{ for every } k = 0, 1, \dots, \left\lfloor \frac{d}{2} \right\rfloor.$$

The main and most known result about Hilbert functions is Macaulay's Theorem. Given integers $n, r \geqslant 1$, we define the *r-th binomial expansion of n* as

$$n = \binom{m_r}{r} + \binom{m_{r-1}}{r-1} + \cdots + \binom{m_e}{e}$$

where $m_r > m_{r-1} > \cdots > m_e \geqslant e \geqslant 1$ are uniquely determined integers (see W. Bruns and J. Herzog's [27, Lemma 4.2.6]). We write

$$n^{\langle r \rangle} = \binom{m_r + 1}{r + 1} + \binom{m_{r-1} + 1}{r} + \cdots + \binom{m_e + 1}{e + 1}, \text{ and}$$

$$n_{\langle r \rangle} = \binom{m_r - 1}{r} + \binom{m_{r-1} - 1}{r - 1} + \cdots + \binom{m_e - 1}{e}.$$

Example 1.1. We give some examples of binomial expansions in the following table.

(n, r)	$(5, 1)$	$(6, 2)$	$(7, 2)$	$(8, 3)$
r-th expansion	$\binom{5}{1}$	$\binom{4}{2}$	$\binom{4}{2} + \binom{1}{1}$	$\binom{4}{3} + \binom{3}{2} + \binom{1}{1}$
$n^{\langle r \rangle}$	$\binom{6}{2} = 15$	$\binom{5}{3} = 10$	$\binom{5}{3} + \binom{2}{2} = 11$	$\binom{5}{4} + \binom{4}{3} + \binom{2}{2} = 13$
$n_{\langle r \rangle}$	$\binom{4}{1} = 4$	$\binom{3}{2} = 3$	$\binom{3}{2} + \binom{0}{1} = 3$	$\binom{3}{3} + \binom{2}{2} + \binom{0}{1} = 2$

◇

The functions $H\colon \mathbb{N} \to \mathbb{N}$ that are Hilbert functions of graded standard **k**-algebras were characterised by Macaulay in [99] (see also [27]). Indeed, given a numerical function $H\colon \mathbb{N} \to \mathbb{N}$ the following conditions are equivalent:

(i) There exists a standard graded **k**-algebra A with H as Hilbert function,

(ii) H satisfies the so-called **Macaulay's inequality**, i. e.

$$H(0) = 1, \text{ and } H(t+1) \leqslant H(t)^{\langle t \rangle} \ \forall t \geqslant 1. \tag{1.1}$$

In some literature, a numerical function satisfying the Macaulay's inequality is also called a O-sequence. Notice that condition *(ii)* imposes strong restrictions on the Hilbert function of a standard graded **k**-algebra and, in particular, it bounds its growth. Another important restriction comes from the following **Green's theorem** which we recall for the sake of completeness.

Theorem 1.2. *Let A be an Artinian graded* **k***-algebra and let $\ell \in A_1$ be a general linear form. Denote by (h_i) the h-vector of A and by (h'_i) that of $A/(\ell)$. Then*

$$h'_t \leqslant (h_t)_{\langle t \rangle} \textit{ for all } t \geqslant 1.$$

Proof. See [67, Theorem 1]. ☐

Lefschetz properties

Definition 1.3. Let A be an Artinian graded **k**-algebra. We say that A has the *weak Lefschetz Property* (WLP, for short) if there exists a linear form $\ell \in A_1$ such that, for all integers $i \geqslant 0$, the multiplication map

$$\times \ell : A_i \longrightarrow A_{i+1}$$

has maximal rank, i.e. it is injective or surjective. In this case, the linear form ℓ is called a weak Lefschetz element of A. The set of weak Lefschetz elements forms a Zariski open subset of A_1. If A has the WLP, then this subset is dense in A_1, so the general element of A_1 is a weak Lefschetz element. We will often refer to this open set as the *Lefschetz locus* of A.

If for the general form $\ell \in A_1$ and for an integer j the map $\times \ell : A_{j-1} \longrightarrow A_j$ does not have maximal rank, we will say that the algebra A fails the WLP in degree j.

We say A has the *strong Lefschetz Property* (SLP, for short) if there exists a linear form $\ell \in A_1$ such that, for all integers $i \geqslant 0$ and $k \geqslant 1$, the multiplication map

$$\times \ell^k : A_i \longrightarrow A_{i+k}$$

has maximal rank. Such an element ℓ is called a strong Lefschetz element for A.

Whenever we present A as a quotient of a standard graded polynomial ring R over a homogeneous ideal I, we will often abuse notation and say that the ideal I has or fails the WLP, or the SLP.

The weak and strong Lefschetz properties were inspired by the work of S. Lefschetz about the cohomology ring of a complex manifold. In particular, the *Hard Lefschetz theorem* states that the cohomology ring associated with any compact Kähler manifold has the strong Lefschetz property (more details can be found in [97]).

Even if the problem of determining whether a given Artinian standard graded **k**-algebra has the WLP/SLP feels like a linear algebra problem, studying Lefschetz properties for families of algebras turned out to be extremely elusive. Part of the great interest in the WLP stems from the ubiquity of its presence and there are a long series of papers determining classes of Artinian algebras holding/failing the WLP but much more work remains to be done (see, for instance, [34] and [103]). The first result in this direction is due to Stanley [123] and Watanabe [128] and it asserts that the WLP holds for an Artinian complete intersection ideal generated by powers of linear forms.

Example 1.4. (1) The ideal $I = (x_1^3, x_2^3, x_3^3, x_1x_2x_3) \subset R = \mathbf{k}[x_1, x_2, x_3]$ fails to have the WLP. Set $A = R/I$. We have that for any linear form $\ell = ax_1 + bx_2 + cx_3 \in A_1$ the multiplication map

$$\times\ell : A_2 \cong \mathbf{k}^6 \longrightarrow A_3 \cong \mathbf{k}^6$$

does not have maximal rank. In fact, fix

$$\{x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_2\}, \text{ and } \{x_1^2x_2, x_1^2x_3, x_2^2x_1, x_2^2x_3, x_3^2x_1, x_3^2x_2\}$$

as a basis of A_2, and A_3 respectively. Then the multiplication map, associated with these bases, is given by the matrix

$$\begin{pmatrix} b & 0 & 0 & a & 0 & 0 \\ c & 0 & 0 & 0 & a & 0 \\ 0 & a & 0 & b & 0 & 0 \\ 0 & c & 0 & 0 & 0 & b \\ 0 & 0 & a & 0 & c & 0 \\ 0 & 0 & b & 0 & 0 & c \end{pmatrix},$$

which has zero determinant. More details on this example can be found in [22, Example 3.1].

(2) The ideal $I = (x_1^3, x_2^3, x_3^3, x_1^2x_2) \subset R = \mathbf{k}[x_1, x_2, x_3]$ has the WLP. We want to prove that $\ell = x_1 + x_2 + x_3$ is a weak Lefschetz element. Since the h-vector of $A = R/I$ is (1,3,6,6,4,1), we only need to check that the map $\times\ell : A_i \longrightarrow A_{i+1}$ is injective for $i = 1$ and surjective for $i = 2, 3, 4$. This is equivalent to check that $\dim_{\mathbf{k}}[A/(\ell)]_2 = 3$ and $[A/(\ell)]_i = 0$ for $i = 3, 4, 5$. We have

$$\begin{aligned} A/(\ell) &\cong R/(x_1^3, x_2^3, x_3^3, x_1^2x_2, x_1 + x_2 + x_3) \\ &\cong \mathbf{k}[x_1, x_2]/(x_1^3, x_2^3, x_1^3 + 3x_1^2x_2 + 3x_1x_2^2 + x_2^3, x_1^2x_2) \ , \\ &\cong \mathbf{k}[x_1, x_2]/(x_1^3, x_2^3, x_1^2x_2, x_1x_2^2) \end{aligned}$$

which proves what we want. With similar computations, one can compute the Hilbert functions of the algebras $A/(\ell^k)$ for $k \geqslant 1$, and check that I satisfies also the SLP. ♢

Remark 1.5. If an Artinian graded algebra A satisfy the WLP, then its h-vector is unimodal. Since $\times\ell : A_i \to A_{i+1}$ is surjective if and only if $[A/(\ell)]_{i+1} = 0$, we have that the surjectivity of $\times\ell : A_i \to A_{i+1}$ implies the surjectivity of $\times\ell : A_j \to A_{j+1}$ for every $j \geqslant i$. Therefore, once a map is surjective, all the following ones are as well, implying unimodality of the h-vector. △

Artinian Gorenstein ideals

In this subsection, we will characterise the Lefschetz elements for graded Artinian Gorenstein algebras A. Given $R = \mathbf{k}[x_0, \cdots, x_n]$, we denote by $S = \mathbf{k}[y_0, \cdots, y_n]$ the ring of differential operators on R, i.e., $y_i = \frac{\partial}{\partial x_i}$. We suppose that both polynomial rings are standard graded. For any homogeneous polynomial $f \in R_d$, we define

$$\operatorname{Ann}_S(f) := \{p \in S \mid p \cdot f = 0\} \subset S.$$

It is well known that $A_f := S/\operatorname{Ann}_S(f)$ is a standard graded Artinian Gorenstein $\mathbf{k}$-algebra. Conversely, the theory of inverse systems developed by Macaulay gives the following characterisation of standard graded Artinian Gorenstein $\mathbf{k}$-algebras.

Proposition 1.6. *Set $R = \mathbf{k}[x_0, \cdots, x_n]$ and let $S = \mathbf{k}[y_0, \cdots, y_n]$ be the ring of differential operators on R. Let $A = S/I$ be a standard graded Artinian $\mathbf{k}$-algebra. Then, A is Gorenstein if and only if there is $f \in R$ such that $A \cong S/Ann_S(f)$. We name f the* dual generator *of A. Moreover, isomorphic standard graded Gorenstein Artinian $\mathbf{k}$-algebras are defined by forms equal up to a linear change of variables in R.*

From now on, when we talk about Gorenstein algebras, we will always intend standard graded Gorenstein Artinian $\mathbf{k}$-algebras. Following the previous proposition, we write A_f to intend the Gorenstein algebra with f as dual generator. The degree of f coincides with the Socle degree of A. Moreover, one can prove that the h-vector of A_f is symmetric.

For Gorenstein algebras, due to the symmetry of their h-vector, we have that A has the strong Lefschetz property if and only if if there exists an element $\ell \in A_1$ such that the multiplication map

$$\times \ell^{e-2i} : A_i \longrightarrow A_{c-i}$$

is bijective for $i = 0, \cdots, \lfloor e/2 \rfloor$ being e the Socle degree of A. In literature, this condition is called *strong Lefschetz Property in the narrow sense.*

Definition 1.7. Let $f \in \mathbf{k}[x_0, \cdots, x_n]_d$ be a homogeneous polynomial and let $A = S/\operatorname{Ann}_S(f)$ be the associated Artinian Gorenstein algebra. Fix $\mathcal{B} = \{w_j \mid 1 \leqslant j \leqslant h_t := \dim A_t\} \subset A_t$ be an ordered $\mathbf{k}$-basis. The t-th (relative) *Hessian matrix* of f with respect to $\mathcal{B}$ is defined as the $h_t \times h_t$ matrix

$$\operatorname{Hess}^t_f = (w_i w_j(f))_{i,j},$$

whose entries are forms of degree $d - 2t$.

The t-th *Hessian of* f *with respect to* $\mathcal{B}$ is the homogeneous polynomial of degree $h_t(d-2t)$ defined as

$$\operatorname{hess}^t_f = \det(\operatorname{Hess}^t_f).$$

The 0-th Hessian is just the polynomial f and, in the case $\dim A_1 = n+1$ i.e. f is not a cone, the 1-st Hessian, with respect to the standard basis, is the classical Hessian. It is worthwhile to point out that the definition of Hessians and Hessian matrices of order t depends on the choice of a basis of A_t but the vanishing of the t-th Hessian is independent of this choice.

A result due to Watanabe establishes a useful link between the failure of Lefschetz properties and the vanishing of higher order Hessians. This result has been later generalised for weak Lefschetz elements in [63] by introducing the *mixed Hessians.*

Theorem 1.8. *Let* $f \in \mathbf{k}[x_0, \cdots, x_n]$ *be a homogeneous polynomial of degree* d *and let* $A = S/\operatorname{Ann}_S(f)$ *be the associated Artinian Gorenstein algebra. The linear form* $\ell = a_0 y_0 + \cdots + a_n y_n \in A_1$ *is a strong Lefschetz element of* A *if and only if* $\operatorname{hess}^t_f(a_0, \cdots, a_n) \neq 0$ *for* $t = 1, \cdots, [d/2]$. *More precisely, up to a nonzero multiplicative constant,* $\operatorname{hess}^t_f(a_0, \cdots, a_n)$ *is the determinant of the dual of the multiplication map* $\times \ell^{d-2t} : A_t \longrightarrow A_{d-t}$.

Proof. See [129, Theorem 4] and [100, Theorem 3.1]. □

Example 1.9. To illustrate Watanabe's theorem, we consider Ikeda's example of an Artinian Gorenstein algebra of codimension 4 failing WLP (see [86, Example 4.4]). We take

$$f = x_0 x_2^3 x_3 + x_1 x_2 x_3^3 + x_0^3 x_1^2 \in \mathbf{k}[x_0, x_1, x_2, x_3].$$

Let $S = \mathbf{k}[y_0, y_1, y_2, y_3]$ be the ring of differential operators on R. Using [M2], we compute $\operatorname{Ann}_S(f)$ and we get:

$$\operatorname{Ann}_S(f) = (y_0 y_3^2, y_1^2 y_3, y_0 y_1 y_3, y_0^2 y_3, y_1 y_2^2, y_0 y_2^2 - y_1 y_3^2, y_1^2 y_2, y_0 y_1 y_2, y_0^2 y_2, y_1^3,$$

$$y_3^4, y_2^2 y_3^2, y_2^4, y_0^2 y_1^2 - 2 y_2^3 y_3, y_0^3 y_1 - 2 y_2 y_3^3, y_0^4).$$

The h-vector of $A = S/\operatorname{Ann}_S(f)$ is (1, 4, 10, 10, 4, 1). We apply the above criterion to check that A fails the WLP in degree 3. To this end, we consider a $\mathbf{k}$-basis of A_2:

$$\{y_0^2, y_1^2, y_2^2, y_3^2, y_0 y_1, y_0 y_2, y_0 y_3, y_1 y_2, y_1 y_3, y_2 y_3\}.$$

The 2-nd Hessian of f with respect to this basis is

$$\mathrm{Hess}_f^2 = \begin{pmatrix} 0 & 12x_0 & 0 & 0 & 6x_1 & 0 & 0 & 0 & 0 & 0 \\ 12x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6x_3 & 6x_2 & 0 & 0 & 6x_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6x_3 & 6x_2 & 6x_1 \\ 6x_1 & 0 & 0 & 0 & 6x_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6x_3 & 0 & 0 & 0 & 0 & 0 & 0 & 6x_2 \\ 0 & 0 & 6x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6x_2 & 0 & 0 & 0 & 0 & 0 & 6x_3 \\ 0 & 0 & 6x_0 & 6x_1 & 0 & 6x_2 & 0 & 0 & 6x_3 & 0 \end{pmatrix}.$$

For any $(a_0, a_1, a_2, a_3) \in \mathbf{k}^4$, we have $\mathrm{hess}_f^2(a_0, a_1, a_2, a_3) = 0$. So, for any $\ell \in A_1$, the multiplication map $\times\ell : A_2 \longrightarrow A_3$ has zero determinant. Therefore, it is not bijective and we conclude that A fails the WLP. ♢

We now group together a series of results that are useful to check if an Artinian Gorenstein algebra has or not the weak Lefschetz property.

Proposition 1.10. *Let $A = \bigoplus_{i=0}^{d} A_i$ be a graded Artinian Gorenstein algebra and let $\ell \in A_1$. We denote $l_k : A_k \to A_{k+1}$ the restriction of the multiplication by ℓ at the k-th homogeneous part. Then the following statements hold true.*

1. *If l_{k_0} is injective for some $k_0 \leqslant d$, then l_k is injective for every $k \leqslant k_0$.*
2. *If l_{k_0} is surjective for some $k_0 \geqslant 0$, then l_k is surjective for every $k \geqslant k_0$.*
3. *If $\dim A_{k_0} = \dim A_{k_0+1}$ for some $0 \leqslant k_0 \leqslant d-1$, then ℓ is a weak Lefschetz element if and only if l_{k_0} is an isomorphism.*
4. *If $\dim A_{k_0-1} < \dim A_{k_0} > \dim A_{k_0+1}$ for some k_0, then ℓ is a weak Lefschetz element if and only if l_{k_0-1} is injective.*

Proof. For statements 1.,2., and 3. see [105, Proposition 2.1]. Statement 4. is a direct consequence of the the first two and the following fact: if l_{k_0-1} is injective then the map $l_{k_0} : A_{k_0} \to A_{k_0+1}$ is surjective. Since l_{k_0-1} is injective, its dual map $l^*_{k_0-1} : A^*_{k_0} \to A^*_{k_0-1}$ is surjective. Since A is a Gorenstein algebra, we can view this map as $l^*_{k_0-1} : A_{k_0} \to A_{k_0+1}$. This map coincide with the map $l_{k_0} : A_{k_0} \to A_{k_0+1}$, thus it is surjective. ☐

By Proposition 1.10 we also deduce the following property for A, an Artinian Gorenstein algebra: a necessary condition for A to have the weak Lefschetz properties is that

$$h_0 < h_1 < \cdots < h_s = h_{s+1} = \cdots = h_{\lceil \frac{d}{2} \rceil} = \cdots = h_{d-s} > h_{d-s+1} > \cdots > h_d$$

for some integer s. The quantity $h_s = \max\{h_i : i = 0, \ldots, d\}$ is called the *Sperner number* of A (cf. [72, Definition 2.39] and [72, Remark 3.7]). So, if we want to check the WLP, Conditions 3., and 4. of Proposition 1.10 exhaust all possible cases.

As a final note, the Hilbert vector of an Artinian Gorenstein algebra, which has the WLP/SLP, has restrictions on its value. In fact, a similar statement to Macaulay's Theorem can be proved for Gorenstein algebras involving the Lefschetz properties.

Theorem 1.11. *Let $h = (1, h_1, \ldots, h_d)$ be a tuple of non-negative integers. Then there exists an Artinian Gorenstein* $\mathbf{k}$*-algebra satisfying the WLP with h as Hilbert vector if and only if h is an SI-sequence, i.e.*

- *h is symmetric,*

- *h is unimodal,*
- *$\Delta h = (h_0, h_1 - h_0, \dots, h_t - h_{t-1})$ satisfies Macaulay's inequality, where $t = \min\{i | h_i \geqslant h_{i+1}\}$ (the index t makes sure the elements of Δh are strictly positive).*

The same statement holds true also by replacing WLP with SLP.

Proof. See [71, Theorem 1.2] and [7, Theorem 3.2]. □

Perazzo algebras

Definition 1.12. Fix $N \geqslant 4$. A *Perazzo* hypersurface $X \subset \mathbb{P}^N$ of degree d is a hypersurface defined by a form $f \in \mathbf{k}[x_0, \dots, x_n, u_1 \dots, u_m]$ of the following type:

$$f = x_0 p_0 + x_1 p_1 + \cdots + x_n p_n + g$$

where $n+m = N$, $n, m \geqslant 2$, $p_i \in \mathbf{k}[u_1, \cdots, u_m]_{d-1}$ are algebraically dependent but linearly independent and $g \in \mathbf{k}[u_1, \cdots, u_m]_d$.

It is worthwhile to point out that usually Perazzo hypersurfaces are assumed to be reduced and irreducible (see, for instance, [62, Definition 3.12]). We will insert these hypotheses if it is required.

Example 1.13. As a first example of Perazzo hypersurface we have the cubic 3-fold in $\mathbb{P}^4$ of equation:

$$f(x_0, x_1, x_2, u, v) = x_0 u^2 + x_1 uv + x_2 v^2.$$

It is a cubic hypersurface with vanishing hessian but not a cone. So, it provides the first counterexample to Hesse's claim: any hypersuface $X \subset \mathbb{P}^N$ with vanishing hessian is a cone ([78] and [79]). ◊

Hesse's claim, which is true for quadrics, was studied by Gordan and Nöther in [66] for hypersurfaces of degree $d \geqslant 3$. They proved it is true for $N \leqslant 3$ but it is false for any $N \geqslant 4$. More precisely they gave a complete classification of the hypersurfaces with vanishing hessian for $N = 4$ and a series of examples of hypersurfaces with vanishing hessian not cones for any $N \geqslant 5$. Subsequently Perazzo in [113] described all cubic hypersurfaces with vanishing hessian for $N = 4, 5, 6$. The results of Gordan-Nöther and of Perazzo have been recently considered and rewritten in modern language by many authors [18], [33], [56], [98], [60], [130] and [131].

Remark 1.14. We recall that the hypersurface defined by a polynomial f has vanishing hessian if and only if the partial derivatives of f are algebraically dependent, and it is a cone if and only if they are linearly dependent. It follows that the Perazzo hypersurfaces, introduced in Definition 1.12, have all vanishing first hessian and in general are not cones. △

In $\mathbb{P}^4$ the Gordan-Nöther classification states that, for degree $d \leqslant 5$, the hypersurfaces not cones with vanishing hessian are all Perazzo hypersurfaces, while for degree $d > 5$, a form of degree d with vanishing hessian, not cone, is an element of $\mathbf{k}[u, v][\Delta]$ where Δ is a Perazzo polynomial of the form $p_0 x_0 + p_1 x_1 + p_2 x_2$ (see [66] and [131, Theorem 7.3]).

In [100] Maeno and Watanabe found a connection between the vanishing of higher order hessians and Lefschetz properties, in particular with the SLP; then Gondim in [62] studied the WLP for some hypersurfaces with vanishing hessian.

In this note, we will concentrate our attention on Perazzo 3-folds X in $\mathbb{P}^4$ and our first goal will be to determine the maximum and minimum h-vector for the Gorenstein Artinian algebras associated with them. We will use the following notations: $R = \mathbf{k}[x_0, x_1, x_2, u, v]$ is the polynomial ring in 5 variables, $S = \mathbf{k}[y_0, y_1, y_2, U, V]$ is the ring of differential operators on R, and a Perazzo 3-fold $X \subset \mathbb{P}^4$ of degree d is defined by a form

$$f = x_0 p_0(u, v) + x_1 p_1(u, v) + x_2 p_2(u, v) + g(u, v) \in R_d. \tag{1.2}$$

If $d=3$, the corresponding algebras have all the same h-vector, and precisely $(1,5,5,1)$. In fact, by Remark 1.14, X not being a cone implies $h_1=h_2=5$. So, from now on, we will assume that $d\geqslant 4$ and we write

$$\begin{array}{rcl} p_0(u,v) & = & \sum_{i=0}^{d-1}\binom{d-1}{i}a_iu^{d-1-i}v^i, \\ p_1(u,v) & = & \sum_{i=0}^{d-1}\binom{d-1}{i}b_iu^{d-1-i}v^i, \\ p_2(u,v) & = & \sum_{i=0}^{d-1}\binom{d-1}{i}c_iu^{d-1-i}v^i, \text{ and} \\ g(u,v) & = & \sum_{i=0}^{d}\binom{d}{i}g_iu^{d-i}v^i. \end{array} \tag{1.3}$$

For any $2\leqslant k\leqslant\lfloor\frac{d+1}{2}\rfloor$, we define the matrices:

$$\begin{array}{ll} \mathcal{A}_k:=\begin{pmatrix} a_0 & a_1 & \cdots & a_{k-1} \\ a_1 & a_2 & \cdots & a_k \\ \vdots & \vdots & & \vdots \\ a_{d-k} & a_{d-k+1} & \cdots & a_{d-1} \end{pmatrix}, & \mathcal{B}_k:=\begin{pmatrix} b_0 & b_1 & \cdots & b_{k-1} \\ b_1 & b_2 & \cdots & b_k \\ \vdots & \vdots & & \vdots \\ b_{d-k} & b_{d-k+1} & \cdots & b_{d-1} \end{pmatrix}, \\ \mathcal{C}_k:=\begin{pmatrix} c_0 & c_1 & \cdots & c_{k-1} \\ c_1 & c_2 & \cdots & c_k \\ \vdots & \vdots & & \vdots \\ c_{d-k} & c_{d-k+1} & \cdots & c_{d-1} \end{pmatrix}, \text{ and} & \mathcal{G}_k:=\begin{pmatrix} g_0 & g_1 & \cdots & g_k \\ g_1 & g_2 & \cdots & g_{k+1} \\ \vdots & \vdots & & \vdots \\ g_{d-k} & g_{d-k+1} & \cdots & g_d \end{pmatrix}. \end{array} \tag{1.4}$$

The matrices $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$ and $\mathcal{G}_k$ are the building blocks of the matrices M_k, N_k and N'_k that will play an important role in the proof of our main results. They are defined as follows:

$$M_k:=\left(\mathcal{A}_k|\mathcal{B}_k|\mathcal{C}_k\right), \qquad N_k:=\begin{pmatrix} \mathcal{A}_{k+1} \\ \hline \mathcal{B}_{k+1} \\ \hline \mathcal{C}_{k+1} \end{pmatrix} \text{ and } \quad N'_k:=\begin{pmatrix} \mathcal{A}_{k+1} \\ \hline \mathcal{B}_{k+1} \\ \hline \mathcal{C}_{k+1} \\ \hline \mathcal{G}_k \end{pmatrix}. \tag{1.5}$$

Remark 1.15. (1) The matrices N_k and M_{k+1} contain the same 3 blocks of size $(d-k)\times(k+1)$.

(2) Since $M_k=N^t_{d-k}$, we have $\operatorname{rank} M_k=\operatorname{rank} N_{d-k}$.

(3) As we will see in the proof of Propositions 1.24 and 1.27, the h-vector of $S/\operatorname{Ann}_S(f)$ is minimal if and only if for all k, $2\leqslant k\leqslant\lfloor\frac{d}{2}\rfloor$, $\operatorname{rank} M_k=\operatorname{rank} N'_k=3$. △

Jordan type

Definition 1.16. The *Jordan type* $P_\ell=P_{\ell,A}$ of a graded Artinian algebra A and a linear form ℓ of A_1 is the partition of $\dim_{\mathbf{k}} A$ determining the Jordan block decomposition for the multiplication map $\times\ell\colon A\to A$.

Example 1.17. Let $I=(x^3,xy,y^4)\subset\mathbf{k}[x,y]$ and set $A=\mathbf{k}[x,y]/I$. Consider the element $\ell=x+y\in A_1$. Since the algebra A is Artinian, the element ℓ is idempotent and the only eigenvalue of the multiplication map $\times\ell\colon A\to A$ is 0. It remains to determine the number and the dimension of the Jordan normal blocks. It is easy to see that $\{\ell^4,\ell^3,\ell^2,\ell,1,x\ell,x\}$ is $\mathbf{k}$-basis of A, and the matrix representing the multiplication map $\times\ell$ with respect to this basis is

$$\left(\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Thus, we obtain $P_{\ell,A}=(4,2)$. ♢

Remark 1.18. Let $P_{\ell,A}=(p_1,\dots,p_s)$ be the Jordan type of ℓ in a graded Artinian algebra A.

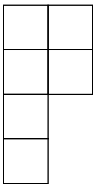
1. There are elements $z_1, \dots, z_s \in A$ such that $\{\ell^i z_j : 1 \leqslant j \leqslant s, 0 \leqslant p_j - 1\}$ is a $\mathbf{k}$-basis for A. We name this basis a *pre-Jordan basis* for the multiplication by ℓ in A. If in addition $\ell^{p_i} z_i = 0$ for each i, we call it a *Jordan basis*. The Jordan blocks are determined by the *strings* $S_j = \{z_i, \ell z_i, \dots, \ell^{p_i - 1} z_i\}$, $j = 1, \dots, s$, and A decomposes as
$$A = \langle S_1 \rangle \oplus \cdots \oplus \langle S_s \rangle.$$
2. Since A is standard graded, the elements $z_1, \dots, z_s \in A$ constructing a Jordan basis can be refined using $\mathbf{k}[\ell]$-linear operation to turn them homogeneous. The $\mathbf{k}[\ell]$-modules $\langle S_j \rangle$ become graded and the above decomposition is consistent with the graded structure of A. △

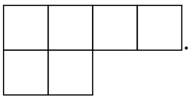
The Jordan type of a linear form ℓ for an Artinian algebra A determines whether or not ℓ is a weak or strong Lefschetz element of A. In order to explain the connection we need the following definition, and for more details on Artinian algebras and Jordan type the reader is invited to look at [85].

Definition 1.19 (Dominance order)**.** Given two partitions
$$P = (p_1, p_2, \dots, p_s), \quad p_1 \geqslant p_2 \geqslant \cdots \geqslant p_s$$
and
$$Q = (q_1, q_2, \dots, q_t), \quad q_1 \geqslant q_2 \geqslant \cdots \geqslant q_t$$
of an integer n, the *dominance partial order* is defined as
$$Q \leqslant P \text{ if } \sum_{j=1}^{i} q_j \leqslant \sum_{j=1}^{i} p_j, \text{ for all } \ i \leqslant \min\{s, t\}.$$

Definition 1.20. If $H = (h_0, \dots, h_d)$ is the h-vector of some Artinian graded algebra A, we can rearrange its values in a non-increasing order to form a partition of $\dim_{\mathbf{k}} A$. The conjugate partition $H^\vee$ of H is defined as
$$H^\vee = (h_0^\vee, \dots, h_r^\vee), \text{ where } h_i^\vee = \#\{k : h_k \geqslant i\}.$$

For example, following Example 1.17, the Hilbert function of $A = \mathbf{k}[x, y]/(x^3, xy, y^4)$ is $H = (1,\ 2,\ 2,\ 1)$. Viewing H as a partition, we obtain $H = (2,\ 2,\ 1,\ 1)$ and $H^\vee = (4,\ 2)$. The Young diagrams of H and $H^\vee$ are respectively





We can see that $H^\vee$ is obtained by switching rows and columns of H in its Young diagram.

Proposition 1.21. *Let A be a graded Artinian algebra with Hilbert function* HF_A*, then*

(i) a linear form ℓ is a weak Lefschetz element for A if and only if HF_A *is unimodal and the number of parts in $P_{\ell,A}$ is equal to the Sperner number of A (i. e. the maximum value of* HF_A*);*

(ii) a linear form ℓ is a strong Lefschetz element for A if and only if $P_{\ell,A}$ is the conjugate partition $\mathrm{HF}_A^\vee$*;*

(iii) there is a dense open set $U \subseteq A_1$ such that if ℓ is a linear form in U then, with respect to the dominance order, $P_{\ell,A} \geqslant P_{\ell_0,A}$, for any $\ell_0 \in A_1$.

Proof. See [72, Proposition 3.5] or [85, Proposition 2.10]. □

In Proposition 1.21 (iii), we call the Jordan type of an element in the open set U the *generic linear Jordan type of A* and denote it by P_A (see [85, Lemma 2.54, Definition 2.55]; there the notation for generic linear Jordan type is $P_{1,A}$, to distinguish this from the generic Jordan type, coming from a general element of the maximal ideal). We underline that $P_{\ell,A} \leqslant \mathrm{HF}_A^\vee$ for every $\ell \in A_1$.

Computational remarks

While using Macaulay2 [M2], in checking whenever a Perazzo algebra A has the WLP, we use the following procedure: After computing the hilbert vector (h_i) of A, we can consider a random linear form ℓ and compute the hilbert vector (h'_i) of $A/(\ell)$ and check whether the following equality holds

$$h'_i = \max\{h_i - h_{i-1}, 0\}, \text{ for } i \geqslant 1.$$

The Macaulay2 code is the following.

```
-- Check WLP for a single linear element
-- Assumes A = R/I is a Perazzo algebra

isWLP = (I) -> (
        if dim I != 0 then error "The␣quotient␣is␣not␣Artinian.";

        -- Hilbert function of A
        s := regularity I;
        HA := for i from 0 to s list hilbertFunction(i, I);

        -- Choose a random linear form
        ell := random(1,ring I); -- random linear form in R_1
        J := I + ideal(ell);

        -- Hilbert function of A/(ell)
        HB := for i from 0 to s list hilbertFunction(i, J);

        -- Check the equality:
        -- HB(i) = max{ HA(i) - HA(i-1), 0 }
        for i from 0 to s do (
                expected :=
                if i == 0 then HA#0
                else max(HA#i - HA#(i-1), 0);
                if HB#i != expected then return false;
        );
        return true;
);

--Check WLP for a given number of trials

isWLPgeneral = (I, trials) -> (
        for t from 1 to trials do (
                if not isWLP(I) then return false;
        );
        return true;
);
```

By generating a limited number of linear forms and check the relate h-vector, it is revealed whether the algebra satisfies WLP. If this is not the case, the probability of the algebra not having the property is quite high but a proper computation is needed to check that every linear element fails WLP. Usually, this is done for Gorenstein algebra with Theorem 1.8 and its generalizations. The above script can be used more in general for Artinian level algebras due to Proposition 1.10. We recall that an Artinian algebra A with maximal irrelevant ideal $\mathfrak{m}$ and Socle degree d is called *level* if $0 :_A \mathfrak{m} = A_d$. Every Gorenstein Artinian algebra is automatically level.

1.2 Hilbert function of Perazzo 3-folds

In this section we study the Hilbert function of Artinian Gorenstein algebras defined by a Perazzo form. We give explicit formulas to compute it obtaining a result on its minimality and maximality. Crucial is Theorem 1.31 that proves unimodality of the Hilbert function. We follow notations present in Section 1.1.

Proposition 1.22. *Let* $f = x_0p_0(u,v) + x_1p_1(u,v) + x_2p_2(u,v)$ *be a form of degree* d *defining a Perazzo* 3*-fold in* $\mathbb{P}^4$*. Let* $h = (h_0, h_1, \dots, h_d)$ *be its* h*-vector. Then* $h_0 = h_d = 1, h_1 = h_{d-1} = 5$ *and, for* $2 \leqslant i \leqslant d-2$*,* $h_i = 4i + 1 - m_i - n_i$*, where* $m_i = 3i - \operatorname{rank} M_i$ *and* $n_i = i + 1 - \operatorname{rank} N_i$*.*

Proof. Recall that the h-vector of an Artinian Gorenstein algebra is symmetric and, hence, we only have to compute h_i for $0 \leqslant i \leqslant \lfloor \frac{d}{2} \rfloor$. We have

$$h_i = \dim A_i = \dim S_i - \dim \operatorname{Ann}_S(f)_i = \binom{4+i}{i} - \dim \operatorname{Ann}_S(f)_i.$$

So, we have to compute $\dim \operatorname{Ann}_S(f)_i$ for any i, $0 \leqslant i \leqslant \lfloor \frac{d}{2} \rfloor$. Since $p_0(u,v)$, $p_1(u,v)$ and $p_2(u,v)$ are $\mathbf{k}$-linearly independent, we have $\dim \operatorname{Ann}_S(f)_1 = 0$ and, hence, $h_1 = 5$.

We observe that, for any $i \geqslant 2$, $\operatorname{Ann}_S(f)_i$ contains $(y_0, y_1, y_2)^{i-k}(U,V)^k$, for $0 \leqslant k \leqslant i-2$. Therefore

$$\dim A_i \leqslant \binom{4+i}{i} - \sum_{k=0}^{i-2}(k+1)\binom{i-k+2}{2} = 4i + 1.$$

We have to compute the numbers

$$m_i = \dim(\operatorname{Ann}_S(f)_i \cap (y_0, y_1, y_2)(U,V)^{i-1}),$$

$$n_i = \dim(\operatorname{Ann}_S(f)_i \cap (U,V)^i),$$

and we will get

$$\dim A_i = 4i + 1 - m_i - n_i. \tag{1.6}$$

This can be done because there are no linear dependence relations between the two parts, given the bi-homogeneous nature of f with respect to the two groups of variables x_0, x_1, x_2 and u, v.

To compute m_i we consider a general polynomial of degree i in $(y_0, y_1, y_2)(U,V)^{i-1}$:

$$(\alpha_0 U^{i-1} + \alpha_1 U^{i-2}V + \cdots + \alpha_{i-1}V^{i-1})y_0 + (\beta_0 U^{i-1} + \cdots + \beta_{i-1}V^{i-1})y_1 + (\gamma_0 U^{i-1} + \cdots)y_2.$$

It belongs to $\operatorname{Ann}_S(f)_i$ if and only if

$$\alpha_0 p_{0,u^{i-1}} + \alpha_1 p_{0,u^{i-2}v} + \cdots + \alpha_{i-1}p_{0,v^{i-1}} + \beta_0 p_{1,u^{i-1}} + \cdots + \gamma_0 p_{2,u^{i-1}} + \cdots + \gamma_{i-1}p_{2,v^i} = 0.$$

The partial derivatives of p_0, p_1, p_2 appearing in the above expression have degree $d-i$; setting equal to zero the coefficients of the $d-i+1$ monomials in u, v, we get a homogeneous linear system of $d-i+1$ equations in the $3i$ unknowns $\alpha_0, \dots, \alpha_{i-1}, \beta_0, \dots, \beta_{i-1}, \gamma_0, \dots, \gamma_{i-1}$. The matrix of the coefficients is M_i, therefore $m_i = 3i - \operatorname{rank} M_i$, and we are done.

To compute n_i we consider a general polynomial of degree i in U, V:

$$\delta_0 U^i + \delta_1 U^{i-1}V + \cdots + \delta_i V^i$$

and we impose that it belongs to $\operatorname{Ann}_S(f)_i$. We get

$$(\delta_0 p_{0,u^i} + \delta_1 p_{0,u^{i-1}v} + \cdots + \delta_i p_{0,v^i})x_0 + (\delta_0 p_{1,u^i} + \cdots)x_1 + (\delta_0 p_{2,u^i} + \cdots)x_2 = 0.$$

Looking at the coefficients of x_0, x_1, x_2 and then the coefficients of the monomials in u, v of degree $d-i-1$ and $d-i$, we get a homogeneous linear system of $3(d-i) + (d-i+1)$ equations in $i+1$ unknowns, whose matrix of the coefficients is N_i. We conclude that $n_i = i + 1 - \operatorname{rank} N_i$. The proof is complete. □

Remark 1.23. We observe that the expression for h_i can also be written in the form $h_i = \operatorname{rank} M_i + \operatorname{rank} N_i$. In fact, if we write a unique linear system to compute the dimension of the space $\operatorname{Ann}_S(f)_i \cap [(y_0, y_1, y_2)(U,V)^{i-1} + (U,V)^i]$, the matrix of this linear system results to be $\left(\begin{array}{c|c} 0 & N_i \\ \hline M_i & 0 \end{array}\right)$.

In the general case, when f is as in (1.2) with $g \neq 0$, equality (1.6) is not necessarily true, but only the inequality $\dim A_i \geqslant 4i + 1 - m_i - n_i$ holds true. An explicit example is provided by the form $f = x_0u^9 + x_1u^8v + x_2v^9 + u^5v^5$.

On the other hand, in this more general situation the matrix associated with the linear system to be considered to compute h_i is $\left(\begin{array}{c|c} 0 & N_i \\ \hline M_i & \mathcal{G}_i \end{array}\right)$. This implies, for every index i, the series of inequalities

$$\operatorname{rank} M_i + \operatorname{rank} N_i \leqslant h_i \leqslant \operatorname{rank} M_i + \operatorname{rank} N_i'.$$

Clearly, every time $\operatorname{rank} N_i = \operatorname{rank} N_i'$, we obtain a relation as in Proposition 1.22. This is obviously the case when $g = 0$. It is also the case if one of the polynomials p_0, p_1, p_2 is general enough. Indeed, we observe that N_i has maximal rank if and only if its columns are linearly independent; so if the rank of one of the matrices $\mathcal{A}_{i+1}$, $\mathcal{B}_{i+1}$, $\mathcal{C}_{i+1}$ is computed by the number of columns then N_i has maximal rank. This happens if one of the polynomials p_0, p_1, p_2 is general enough in view of [83, Proposition 3.4]. △

Proposition 1.24. *Let $d \geqslant 4$. The maximum h-vector of the Artinian Gorenstein algebras $S/Ann_S(f)$ associated with the Perazzo 3-folds of degree d in $\mathbb{P}^4$ is:*

(1) If $d = 4t$ then

$$h_i = \begin{cases} 4i+1 & \text{for } 0 \leqslant i \leqslant t \\ 4t+2 & \text{for } t+1 \leqslant i \leqslant 2t \\ h_{d-i} & \text{for } 2t+1 \leqslant i \leqslant 4t \end{cases};$$

(2) If $d = 4t+1$ then

$$h_i = \begin{cases} 4i+1 & \text{for } 0 \leqslant i \leqslant t \\ 4t+3 & \text{for } t+1 \leqslant i \leqslant 2t \\ h_{d-i} & \text{for } 2t+1 \leqslant i \leqslant 4t+1 \end{cases};$$

(3) If $d = 4t+2$ then

$$h_i = \begin{cases} 4i+1 & \text{for } 0 \leqslant i \leqslant t \\ 4t+4 & \text{for } t+1 \leqslant i \leqslant 2t+1 \\ h_{d-i} & \text{for } 2t+2 \leqslant i \leqslant 4t+2 \end{cases};$$

(4) If $d = 4t+3$ then

$$h_i = \begin{cases} 4i+1 & \text{for } 0 \leqslant i \leqslant t+1 \\ 4t+5 & \text{for } t+2 \leqslant i \leqslant 2t+1 \\ h_{d-i} & \text{for } 2t+2 \leqslant i \leqslant 4t+3 \end{cases}.$$

Example 1.25. For low values of d, the maximal Hilbert functions in Proposition 1.24 look like the following.

$$
\begin{aligned}
&d = 6 = 4\cdot 1+2 && h=(1,5,8,8,8,5,1)\\
&d = 7 = 4\cdot 1+3 && h=(1,5,9,9,9,9,5,1)\\
&d = 8 = 4\cdot 2+0 && h=(1,5,9,10,10,10,9,5,1)\\
&d = 9 = 4\cdot 2+1 && h=(1,5,9,11,11,11,11,9,5,1)\\
&d = 10 = 4\cdot 2+2 && h=(1,5,9,12,12,12,12,12,9,5,1)\\
&d = 11 = 4\cdot 2+3 && h=(1,5,9,13,13,13,13,13,13,9,5,1)\\
&d = 12 = 4\cdot 3+0 && h=(1,5,9,13,14,14,14,14,14,13,9,5,1)
\end{aligned}
$$

♢

Proof of Proposition 1.24. Let f be a form of degree d as in (1.2) with p_0, p_1, p_2, g as in (1.3). Being the h-vector symmetric, we only have to compute h_i for $0 \leqslant i \leqslant \frac{d}{2}$.

In view of Proposition 1.22 and Remark 1.23, the maximal Hilbert function is obtained when m_i, n_i are minimal for any i, i.e. when the ranks of the matrices M_i, N_i' are as large as possible.

Clearly $\operatorname{rank} M_i \leqslant \min\{3i, d-i+1\}$. Therefore

$$\operatorname{rank} M_i \leqslant \begin{cases} 3i \text{ for } i \leqslant \frac{d+1}{4};\\ d-i+1 \text{ for } i \geqslant \frac{d+1}{4}.\end{cases}$$

Regarding N_i', we observe that, in our situation, $i+1 \leqslant 3(d-i)+(d-i+1)$, so always $\operatorname{rank} N_i' \leqslant i+1$. This gives upper bounds on h_i depending on the class of congruence of the degree d modulo 4, that are precisely those in the statement of this Proposition.

To conclude the proof, we claim that these bounds are achieved. To this end, we observe that, in view of the expressions (1.3), the columns of the matrices $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{G}_i$ contain up to a constant the coefficients of the partial derivatives of order $i-1$ of p_0, p_1, p_2, g respectively. But, if p_0, p_1, p_2, g are general enough, then, by [83, Proposition 3.4], for any i their partial derivatives of order $i-1$ are as linearly independent as possible in $\mathbf{k}[u,v]_{d-1-i}$. This means that the ranks of the matrices $\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i, \mathcal{G}_i$ are as large as possible. This proves our claim. □

The family given in the proof of Proposition 1.24 is not much explicit because it simply states that the forms p_0, p_1, p_2 need to be general enough. The following example gives an explicit class of examples attending the upper bound.

Example 1.26. The following example shows that the upper bound for the h-vector given in Proposition 1.24 is achieved. For any integer $d \geqslant 5$, we write $d = 3r+\epsilon$ with $0 \leqslant \epsilon \leqslant 2$. We take

$$
\begin{aligned}
p_0(u,v) &= \sum_{i=0}^{r-1} \binom{d-1}{i}\frac{1}{i+1}u^{d-1-i}v^i,\\
p_1(u,v) &= \sum_{i=r}^{2r-1+\epsilon} \binom{d-1}{i}\frac{1}{i+1}u^{d-1-i}v^i, \text{ and}\\
p_2(u,v) &= \sum_{i=2r+\epsilon}^{d-1} \binom{d-1}{i}\frac{1}{i+1}u^{d-1-i}v^i.
\end{aligned}
$$

The Artinian Gorenstein algebra $S/\operatorname{Ann}_S(f)$ associated with the Perazzo 3-fold $X \subset \mathbb{P}^4$ with equation $f = x_0p_0 + x_1p_1 + x_2p_2$ has the maximum h-vector described in Proposition 1.24. Indeed, $h_0 = 1$ and $h_1 = 5$ (because p_0, p_1 and p_2 are $\mathbf{k}$-linearly independent). Moreover, for all $2 \leqslant k \leqslant \lfloor \frac{d}{2} \rfloor$, we have $\operatorname{rank} M_k = \min\{3k, d-k+1\}$ and $\operatorname{rank} N_k' = \min\{k+1, 4(d-k)+1\}$ and the result follows from the equality $h_k = 4k+1-m_k-n_k$ with $m_k = 3k - \operatorname{rank} M_k$ and $n_k = \operatorname{rank} N_k'$. More details can be found in [51, Example 3.20] where the rank of these matrices is explicitly computed. ♢

To determine the minimum h-vector for the Gorenstein Artinian algebra associated with a Perazzo 3-fold X in $\mathbb{P}^4$ we need first to recall some results about the growth of the Hilbert function of standard graded $\mathbf{k}$-algebras and to fix some additional notation. As an application of Macaulay's theorem, we have:

Proposition 1.27. *Let $d \geqslant 4$. Let $R = \mathbf{k}[x_0, x_1, x_2, u, v]$ and $S = \mathbf{k}[y_0, y_1, y_2, U, V]$ be the ring of differential operators on R. The minimum h-vector of the Artinian Gorenstein algebras $A = S/Ann_S(f)$ associated with the Perazzo 3-folds of degree d in $\mathbb{P}^4$ is:*

$$(1,\ 5,\ 6,\ 6,\ \cdots\ 6,\ 6,\ 5,\ 1).$$

Proof. The proof proceeds as follows: we first prove that the cited t-uple is less than any possible h-vector associated with a Perazzo 3-fold, with respect to the termwise order; then, we give examples of Perazzo forms that have this h-vector. Let

$$h_A = (h_0,\ h_1,\ h_2,\ h_3,\ \cdots\ h_{d-2},\ h_{d-1},\ h_d)$$

be the h-vector of A. First of all we observe that, arguing as in the proof of Proposition 1.24, we get that $6 \leqslant h_2 \leqslant 9$ which, together with the fact that the h-vector of any standard graded Artinian Gorenstein algebra is symmetric, gives us that a lower bound for h_A looks like

$$(1,\ 5,\ 6,\ h_3,\ \cdots\ h_{d-2},\ 6,\ 5,\ 1).$$

This concludes the first step for $d \leqslant 5$. We will now prove that if $d \geqslant 6$, for any i, $3 \leqslant i \leqslant d-3$, $h_i \geqslant 6$.

First we assume $d \geqslant 8$. If $h_j \leqslant 5$ for some $5 \leqslant j \leqslant d-3$, using Macaulay's inequality $h_{t+1} \leqslant h_t^{\langle t \rangle}$ for all $t \geqslant 1$, we get that $h_i \leqslant 5$ for all $i \geqslant j$ contradicting the fact that $h_{d-2} = 6$. Therefore, $h_j \geqslant 6$ for all $5 \leqslant j \leqslant d-2$ and, by symmetry, we also have $h_3, h_4 \geqslant 6$.

For $d = 6, 7$, we must show that $h_3 \geqslant 6$. This last equality follows after a straightforward computation which shows that $(y_0, y_1, y_2)^3 \oplus (y_0, y_1, y_2)^2(U, V) \subset \operatorname{Ann}_S(f)_3$, $\dim\{(\alpha_0 U^2 + \alpha_1 UV + \alpha_2 V^2)y_0 + (\beta_0 U^2 + \beta_1 UV + \beta_2 V^2)y_1 + (\gamma_0 U^2 + \gamma_1 UV + \gamma_2 V^2)y_2 \in \operatorname{Ann}_S(f)_3\} \leqslant 6$ and $\dim\{\delta_0 U^3 + \delta_1 U^2 V + \delta_2 UV^2 + \delta_3 V^3 \in \operatorname{Ann}_S(f)_3\} \leqslant 1$. Therefore, $h_3 = \dim S_3/\operatorname{Ann}_S(f)_3 \geqslant \binom{7}{4} - 29 = 6$.

Summarizing, we have got that for any $d \geqslant 4$ and for $2 \leqslant i \leqslant d-2$, it holds $h_i \geqslant 6$. To finish the proof it suffices to give an example with $h_i = 6$, for any i, $2 \leqslant i \leqslant d-2$. We take the homogeneous polynomial of degree d:

$$f(x_0, x_1, x_2, u, v) = u^d x_0 + u^{d-1} v x_1 + v^d x_2.$$

It is easy to check that it has the desired h-vector. □

Remark 1.28. From Proposition 1.22 it follows that $A = S/\operatorname{Ann}_S(f)$ has minimum h-vector if and only if $\operatorname{rank} M_i = \operatorname{rank} N_i' = 3$ for any i with $2 \leqslant i \leqslant \lfloor \frac{d}{2} \rfloor$. We note that none of these ranks can be strictly less than 3 due to the assumption that p_0, p_1, p_2 are linearly independent. △

Remark 1.29. From Propositions 1.24 and 1.27, it follows that for $d = 4$ the unique possible h-vector is $(1, 5, 6, 5, 1)$. Instead, for $d = 5$, we can obtain only the maximal h-vector $(1, 5, 7, 7, 5, 1)$, and the minimal h-vector $(1, 5, 6, 6, 5, 1)$. For bigger values of d, also some intermediate cases are a priori possible. △

Lemma 1.30. *Let $F = x_0 p_0 + x_1 p_1 + x_2 p_2 + g$ be a Perazzo form of degree $d \geqslant 4$ and let A_F be the associated Artinian Gorenstein algebra. Then, for a general linear form $\ell \in A_F$, the polynomial $\ell \circ F$ defines a Perazzo form of degree $d-1$.*

Proof. We can write $\ell = a_0 X_0 + a_1 X_1 + a_2 X_2 + b_0 U + b_1 V$ for some coefficients $a_i, b_j \in \mathbf{k}$ not all zero. Then we can exhibit the action of ℓ on F as

$$\ell \circ F = x_0 \tilde{p_0} + x_1 \tilde{p_1} + x_2 \tilde{p_2} + \left(a_0 p_0 + a_1 p_1 + a_2 p_2 + b_0 \frac{\partial g}{\partial u} + b_1 \frac{\partial g}{\partial v} \right)$$

with

$$\tilde{p_0} = b_0\frac{\partial p_0}{\partial u} + b_1\frac{\partial p_0}{\partial v}, \quad \tilde{p_1} = b_0\frac{\partial p_1}{\partial u} + b_1\frac{\partial p_1}{\partial v}, \text{ and } \quad \tilde{p_2} = b_0\frac{\partial p_2}{\partial u} + b_1\frac{\partial p_2}{\partial v}.$$

The form $\ell \circ F$ has degree $d-1 \geqslant 3$. It remains to prove that the polynomials $\tilde{p_0}, \tilde{p_1}, \tilde{p_2}$ are linearly independent, for a general choice of ℓ.

Let $\ell' = b_0u + b_1v$ and consider the map $\mathbf{k}[u,v]_{d-1} \to \mathbf{k}[u,v]_{d-2}$, given by $p \mapsto \ell' \circ p$. The kernel of this map is the one-dimensional space $\langle L^{d-1}\rangle$, where $L = b_1u - b_0v = (\ell')^{\perp}$ (a classical result of A. Terracini and others, see the Apolarity Lemma [84, Lemma 1.15], and a historical note in [84, Page 58]). Now let $W = \langle p_0, p_1, p_2\rangle$. Since the p_i are linearly independent, W has dimension 3. It does not fill the whole space $\mathbf{k}[u,v]_{d-1}$, because $d-1 \geqslant 3$. Since powers of linear forms span $\mathbf{k}[u,v]_{d-1}$, there is an open dense set B in $\mathbf{k}^2$ such that for any $(b_0, b_1) \in B$, the form L^{d-1} misses W. So, for any such pair (b_0, b_1), the map

$$\ell' : W \to \ell' \circ W$$

is an isomorphism, and therefore $\tilde{p_0}, \tilde{p_1}, \tilde{p_2}$ are linearly independent. □

Theorem 1.31. *The Hilbert function of an Artinian Gorenstein algebra associated with a Perazzo hypersurface of degree $d \geqslant 5$ in $\mathbb{P}^4$ is unimodal.*

Proof. We claim that $h_i < \frac{1}{2}(i+3)(2d-3i)$ for $i \leqslant \frac{d}{2} - 1$.

To prove the claim, we observe that Proposition 1.24 implies $h_i \leqslant d+2$ for any index i. Therefore it is enough to prove that, if $i \leqslant \frac{d}{2} - 1$, then

$$d + 2 < \tfrac{1}{2}(i+3)(2d-3i), \tag{1.7}$$

which is equivalent to

$$d > \frac{3i^2 + 9i + 4}{2(i+2)}.$$

Since our assumption ensures that $d \geqslant 2i + 2$, we only need to check that

$$2i + 2 > \frac{3i^2 + 9i + 4}{2(i+2)}.$$

It is immediate to check that this last inequality is satisfied for all $i \geqslant 1$. Applying [106, Proposition 2.6], we conclude that $h_{i+1} \geqslant h_i$ and hence we have the unimodality of the Hilbert function of A_F. □

Theorem 1.31 does not generalise to $\mathbb{P}^N$ with $N > 4$.

Example 1.32. Via computations on [M2], we constructed algebras A_F with Hilbert function

$$(1,\ 6,\ 15,\ 28,\ 43,\ 42,\ 43,\ 28,\ 15,\ 6,\ 1),$$

which is not unimodal. In these examples, $F = x_0p_0 + x_1p_1 + x_2p_2$ where p_0, p_1, p_2 are forms of degree 9 in the variables u, v, w, chosen randomly by the computer program. For more on Perazzo n-folds we refer to [104]. ♢

In the proof of Proposition 1.44 we need to exclude certain vectors as possible h-vectors of Perazzo algebras. It was pointed out by Mats Boij that the Hilbert function $(1,5,6,8,6,5,1)$ cannot occur for any Artinian Gorenstein algebra, and this can be proven by using Betti tables of lexicographic ideals and a cancellation argument. We state the result below. For more about lexicographic ideals we refer to [99].

Proposition 1.33. *There is no Artinian Gorenstein algebra with Hilbert function* $(1,5,6,8,6,5,1)$.

Proof. Let $A = R/I$ be an Artinian Gorenstein algebra with Hilbert function $(1,5,6,8,6,5,1)$. In [99], Macaulay proved that there exists a lexicographic ideal I_{lex} with the same Hilbert function. By [111, Theorem 1.1], the Betti numbers $\beta_{ij}(S/I)$ must arise from $\beta_{ij}(S/I_{\text{lex}})$ by consecutive cancellations. Using a computer algebra program, e.g. [M2], one can compute the minimal Betti table of S/I_{lex} which is

```
+-----------------------+
|       0  1  2  3  4  5|
|total: 1 22 67 84 49 11|
|    0: 1  -  -  -  -  -|
|    1: -  9 20 20 10  2|
|    2: -  2  5  4  1  -|
|    3: -  4 15 21 13  3|
|    4: -  2  7  9  5  1|
|    5: -  4 16 24 16  4|
|    6: -  1  4  6  4  1|
+-----------------------+
```

Here we use the convention that the entry at row j and column i is the Betti number $\beta_{i,i+j}$. To get an Artinian Gorenstein algebra A, the Betti number $\beta_{5,6}(S/I_{\text{lex}}) = 2$ should be cancelled to zero. This can only happen if the sum $\beta_{4,6}(S/I_{\text{lex}}) + \beta_{6,6}(S/I_{\text{lex}})$ is at least 2, a contradiction. □

1.3 Weak Lefschetz property

In this section we study weak Lefschetz property for Perazzo 3-folds. We start by considering Perazzo algebras with minimal and maximal Hilbert function. This study gives a base to study the general case. Between these two parts, it is present an intermezzo about Hilbert function and weak Lefschetz property for some Artinian Gorenstein algebras.

1.3.1 Perazzo 3-folds with minimal and maximal h-vector

From Theorem 1.8 and Remark 1.14, since the Perazzo 3-folds have vanishing first hessian, it follows that the associated algebras A fail the strong Lefschetz Property. In particular the map

$$\times \ell^{d-2} : A_1 \longrightarrow A_{d-1}$$

is not an isomorphism for every $\ell \in A_1$. The goal of this section is to analyze whether the Artinian Gorenstein algebra A associated with a Perazzo 3-fold $X \subset \mathbb{P}^4$ has the WLP. If $d = 3$, clearly A fails also WLP. But Gondim has proved that, for any Perazzo 3-fold of degree 4, A has the WLP (see [62, Theorem 3.5]). More precisely, we will see that, in any degree $d \geqslant 5$, WLP holds when A has minimum h-vector and fails when it has maximum h-vector.

Theorem 1.34. *Let $X \subset \mathbb{P}^4$ be a Perazzo 3-fold of degree $d \geqslant 5$ and equation*

$$f = x_0p_0(u,v) + x_1p_1(u,v) + x_2p_2(u,v) + g(u,v) \in R_d = \mathbf{k}[x_0,x_1,x_2,u,v]_d.$$

Let $S = \mathbf{k}[y_0,y_1,y_2,U,V]$ be the ring of differential operators on R. If $A = S/\operatorname{Ann}_S(f)$ has maximum h-vector, then A fails WLP.

Proof. According to the parity of the socle degree of A, we distinguish two cases.
Case 1: d is odd. Write $d = 2r+1$.To show that A fails WLP, we will prove that for any $L \in A_1$, the multiplication map

$$\times L : A_r \longrightarrow A_{r+1}$$

is not bijective. By Theorem 1.8, it is enough to see the vanishing of the r-th Hessian hess^r_f of $f = x_0p_0(u,v) + x_1p_1(u,v) + x_2p_2(u,v) + g(u,v)$ with respect to a suitable $\mathbf{k}$-basis $\mathcal{B}$ of A_r. First we can notice that a basis $\mathcal{B}$ made of classes with a monomial representative always exists. So, Hess^r_f is just a submatrix of dimension $h_r \times h_r$ of the following matrix:

$$\Big(\frac{\partial^{2r} f}{\partial u^\alpha \partial v^\beta \partial x_0^\gamma \partial x_1^\delta \partial x_2^\eta}\Big)_{\alpha+\beta+\gamma+\delta+\eta=2r}$$

where monomials are lexicographic ordered (for simplicity). Knowing that f is linear in the variables x_0, x_1, x_2, the above matrix can be partially computed as:

$$\left(\begin{array}{cccc|cccc|ccc}
\frac{\partial^{2r} f}{\partial u^{2r}} & \frac{\partial^{2r} f}{\partial u^{2r-1}\partial v} & \cdots & \frac{\partial^{2r} f}{\partial u^{r}\partial v^{r}} & \frac{\partial^{2r-1} p_0}{\partial u^{2r-1}} & \frac{\partial^{2r-1} p_0}{\partial u^{2r-2}\partial v} & \cdots & \frac{\partial^{2r-1} p_2}{\partial u^{r}\partial v^{r-1}} & 0 & \cdots & 0 \\
\frac{\partial^{2r} f}{\partial u^{2r-1}\partial v} & \frac{\partial^{2r} f}{\partial u^{2r-2}\partial v^2} & \cdots & \frac{\partial^{2r} f}{\partial u^{r-1}\partial v^{r+1}} & \frac{\partial^{2r-1} p_0}{\partial u^{2r-2}\partial v} & \frac{\partial^{2r-1} p_0}{\partial u^{2r-3}\partial v^2} & \cdots & \frac{\partial^{2r-1} p_3}{\partial u^{r-1}\partial v^{r}} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^{2r} f}{\partial u^{r}\partial v^{r}} & \frac{\partial^{2r} f}{\partial u^{r-1}\partial v^{r+1}} & \cdots & \frac{\partial^{2r} f}{\partial u^{r-1}\partial v^{2r}} & \frac{\partial^{2r-1} p_0}{\partial u^{r-1}\partial v^{r}} & \frac{\partial^{2r-1} p_0}{\partial u^{r-2}\partial v^{r+1}} & \cdots & \frac{\partial^{2r-1} p_2}{\partial u^{2r-1}} & 0 & \cdots & 0 \\
\hline
\frac{\partial^{2r-1} p_0}{\partial u^{r-1}\partial v^{r}} & \cdots & \cdots & \frac{\partial^{2r-1} p_0}{\partial u^{r-1}\partial v^{r}} & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial^{2r-1} p_2}{\partial u^{r}\partial v^{r-1}} & \cdots & \cdots & \frac{\partial^{2r-1} p_2}{\partial u^{2r-1}} & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
\hline
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0
\end{array}\right)$$

The three vertical (respectively, horizontal) blocks are composed respectively by $r+1$, $3r$, $\binom{r+4}{4} - (4r+1)$ columns (respectively, rows). Thus every possible choice of a $h_r \times h_r$ submatrix turns out to have at least an all zero sub-submatrix of size $(h_r - (r+1)) \times (h_r - (r+1))$. We now use the hypothesis of A to have maximum h-vector and Proposition 1.24 to obtain that $h_r = 2r+3$. We have just proved that Hess^r_f, matrix of dimension $(2r+3) \times (2r+3)$, has an all zero submatrix of dimension $(r+2) \times (r+2)$: this implies that hess^r_f identically vanishes.

Case 2: d is even. Write $d = 2r+2$. Note that, since the h-vector is maximum, then $h_r = h_{r+1} = h_{r+2}$. Using again the hessian criterion of Watanabe's (Theorem 1.8), we will check that for any $\ell \in A_1$, the multiplication map

$$\times \ell^2 \colon A_r \longrightarrow A_{r+2}$$

is not bijective. This implies that for any $\ell \in A_1$, the multiplication map

$$\times \ell \colon A_r \longrightarrow A_{r+1}$$

is not bijective and, hence, A fails the WLP.

Same adapted argument of the previous case can be used also here. In fact, the matrix to be considered is Hess^r_f which is now of size $(2r+4) \times (2r+4)$ which is even bigger than the previous case. Thus, as discussed above, its determinant is always zero. $\square$

In contrast with the last result we have that if an Artinian Gorenstein algebra A associated with a Perazzo 3-fold has minimum h-vector, then A has the WLP.

Theorem 1.35. *Let $X \subset \mathbb{P}^4$ be a Perazzo 3-fold of degree $d \geqslant 5$ and equation*

$$f = x_0 p_0(u,v) + x_1 p_1(u,v) + x_2 p_2(u,v) + g(u,v) \in R = \mathbf{k}[x_0, x_1, x_2, u, v]_d.$$

Let $S = \mathbf{k}[y_0, y_1, y_2, U, V]$ be the ring of differential operators on R. If $A = S/\operatorname{Ann}_S(f)$ has minimum h-vector, then A has WLP.

Proof. For $5 \leqslant d \leqslant 7$ see Section 1.4 where a full classification of Perazzo 3-folds with minimal h-vector is given. Assume $d \geqslant 8$. By the minimality assumption, $h_2 = h_3 = \cdots = h_{d-2} = 6$. By Proposition 1.10, if for a general linear form $\ell \in A_1$, the multiplication map

$$\times \ell \colon A_2 \longrightarrow A_3$$

is bijective, then

$$\times \ell \colon A_1 \longrightarrow A_2$$

is injective, and for all $j \geqslant 2$,

$$\times\ell\colon A_j \longrightarrow A_{j+1}$$

is surjective, therefore A has the WLP. By the symmetry property of Artinian Gorenstein algebras,

$$\times\ell\colon A_2 \longrightarrow A_3$$

is bijective if and only if

$$\times\ell\colon A_{d-3} \longrightarrow A_{d-2}$$

is bijective. So, let us prove the bijection of this last map. To this end, for a general linear form $\ell \in A_1$, we consider the exact sequence:

$$A_{d-3} \longrightarrow A_{d-2} \longrightarrow [S/(\operatorname{Ann}_S(f), \ell)]_{d-2} \longrightarrow 0.$$

It follows that $\times\ell\colon A_{d-3} \longrightarrow A_{d-2}$ is bijective if and only if $[S/(\operatorname{Ann}_S(f), \ell)]_{d-2} = 0$. Using the hypothesis $d-2 \geqslant 6$ (and, hence, $h_{d-2} \leqslant d-2$) and Theorem 1.2 we get

$$\dim[S/(\operatorname{Ann}_S(f), \ell)]_{d-2} \leqslant (h_{d-2})_{\langle d-2 \rangle} = 0$$

which proves what we want. □

Remark 1.36. As a consequence of Theorem 1.35, all forms of degree d which define a Perazzo 3-fold with minimum h-vector are examples of forms with zero first order hessian, and all hessians of order t different from zero, for $2 \leqslant t \leqslant \lfloor \frac{d}{2} \rfloor$. △

For Gorenstein Artinian algebras associated with Perazzo 3-folds X in $\mathbb{P}^4$ and with intermediate h-vector both possibilities occur: there are examples failing WLP and examples satisfying WLP as next example shows. We will see in the next sections that the crucial information is given by the h-vector.

Example 1.37. 1.- Let $X \subset \mathbb{P}^4$ be the Perazzo 3-fold of equation

$$f(x_0, x_1, x_2, u, v) = u^6 x_0 + (u^2v^4 + u^4v^2)x_1 + v^6 x_2 \in \mathbf{k}[x_0, x_1, x_2, u, v]_7.$$

Let $S = \mathbf{k}[y_0, y_1, y_2, U, V]$ be the ring of differential operators on R. We have

$$\operatorname{Ann}_S(f) = \langle y_0^2, y_1^2, y_2^2, y_0y_1, y_0y_2, y_1y_2, y_0V, y_2U, y_0U^2 + 15y_1U^2 - 15y_1V^2 - y_2V^2,$$
$$U^3V - UV^3, 15y_1U^4 - y_2V^4, UV^5, V^7, U^7\rangle.$$

Therefore, the Artinian Gorenstein algebra $A = S/Ann_S(f)$ has h-vector: (1, 5, 7, 8, 8, 7, 5, 1). Using Macaulay2 [M2] we check that for a general linear form $\ell \in A_1$, the multiplication map

$$\times\ell\colon A_3 \longrightarrow A_4$$

is bijective and, hence, A satisfies the WLP. It does not have the SLP because for any linear form $\ell \in A_1$

$$\times\ell^3\colon A_2 \longrightarrow A_5$$

is not surjective by Theorem 1.8.

2.- Let $X \subset \mathbb{P}^4$ be the Perazzo 3-fold of equation

$$f(x_0, x_1, x_2, u, v) = u^6 x_0 + u^3v^3 x_1 + v^6 x_2 \in \mathbf{k}[x_0, x_1, x_2, u, v]_7.$$

Let $S = \mathbf{k}[y_0, y_1, y_2, U, V]$ be the ring of differential operators on R. We have

$$\operatorname{Ann}_S(f) = \left\langle y_0^2, y_1^2, y_2^2, y_0y_1, y_0y_2, y_1y_2, y_0v, y_2u, 20y_1U^3 - y_2V^3, y_0U^3 - 20y_1V^3, UV^4, U^4V, V^7, U^7 \right\rangle.$$

Therefore, the Artinian Gorenstein algebra $A = S/Ann_S(f)$ has h-vector: (1, 5, 7, 9, 9, 7, 5, 1). Computing the third hessian, since it results to be zero, we get that for any linear form $\ell \in A_1$, the multiplication map

$$\times\ell\colon A_3 \longrightarrow A_4$$

is not bijective and, hence, A fails the WLP. ◇

1.3.2 Intermezzo on some Artinian Gorenstein algebras

Let $A = R/I$ be a graded Artinian Gorenstein algebra. In this part we prove that specific conditions on the Hilbert function of A ensure the WLP. To this aim we now prove the following lemma.

Lemma 1.38. *Let $A = A_F$ be an Artinian Gorenstein graded $\mathbf{k}$-algebra and set $I = \operatorname{Ann} F$. Then for every linear form $\ell \in A_1$ the sequence*

$$0 \longrightarrow \frac{R}{(I\colon \ell)}(-1) \longrightarrow \frac{R}{I} \longrightarrow \frac{R}{(I,\ell)} \longrightarrow 0 \tag{1.8}$$

is exact. Moreover $\frac{R}{(I\colon \ell)}$ is an Artinian Gorenstein graded algebra with $\ell \circ F$ as Macaulay dual generator.

Proof. We get the result cutting the exact sequence

$$0 \longrightarrow \frac{(I\colon \ell)}{I}(-1) \longrightarrow \frac{R}{I}(-1) \xrightarrow{\times \ell} \frac{R}{I} \longrightarrow \frac{R}{(I,\ell)} \longrightarrow 0$$

into two short exact sequences. The second fact is a straightforward computation. □

Proposition 1.39. *Let $A = R/I$ be an Artinian Gorenstein algebra of even socle degree $d = 2s$ with Hilbert function*

$$H = (1, h_1, h_2, \dots, h_{s-1}, h_s, h_{s-1}, \dots, h_2, h_1, 1)$$

with $h_1, \dots, h_s$ consecutive increasing integers and $h_k \leqslant k$ for some $\frac{d}{2} + 1 \leqslant k \leqslant d - 1$. Then A has the WLP.

Proof. Consider the exact sequence (1.8) in Lemma 1.38 where ℓ is a general linear form. Denote by $h'_k = \dim[R/(I,\ell)]_k$ and $\tilde{h}_k = \dim[R/(I\colon \ell)]_k$. Then $h_k - \tilde{h}_{k-1} = h'_k$ for all $k > 0$ and $h_0 = h'_0 = 1$. Let k_0 be the smallest $\mathbf{k}$ such that $h_k \leqslant k$. By Theorem 1.2 $h'_k = 0$ for all $k \geqslant k_0$, and therefore, $\tilde{h}_k = h_k$ for all $k \geqslant k_0$. Since R/I and $R/(I\colon \ell)$ are Gorenstein, by symmetry we have $h'_k = h_k - \tilde{h}_{k-1} = h_k - (h_k - 1) = 1$ for all $k \leqslant d - k_0 + 1$. Hence, Macaulay's inequality (1.1) forces $h'_{s+1} \leqslant 1$. If $(h'_{s-1}, h'_s, h'_{s+1}) = (1,1,1)$ then $\left(\tilde{h}_{s-2}, \tilde{h}_{s-1}, \tilde{h}_s\right) = (1,2,1)$ which again contradicts both Macaulay's theorem and the Gorenstein property of $R/(I\colon \ell)$. Hence, $h'_{s+1} = 0$ and R/I has the WLP. □

Example 1.40. Every Artinian Gorenstein algebra with h-vector

$$(1, 5, 6, 7, \dots, s-1, s, s-1, \dots, 7, 6, 5, 1)$$

and socle degree at least 6 has the WLP. ♢

As another example of Hilbert function forcing WLP we have:

Proposition 1.41. *Every Artinian Gorenstein algebra with h-vector*

$$h_0 < h_1 < \cdots < h_{t-1} < h_t = \cdots = h_s > h_{s+1} > \cdots > h_{d-1} > h_d$$

where $s \geqslant t + 2$ and $h_s \leqslant s$ has the WLP.

Proof. It is enough to check that for a general $\ell \in R_1$ the multiplication map $\times\ell\colon A_{d/2} \to A_{d/2+1}$ is surjective if d is even, resp. $\times\ell\colon A_{(d-1)/2} \to A_{(d+1)/2}$ if d is odd. So, it suffices to prove that $[R/(I,\ell)]_t = 0$ for $t \geqslant (d+1)/2$.

Since $h_s \leqslant s$, applying Green's theorem we get $[R/(I,\ell)]_s = 0$ i.e. the multiplication map $\times\ell\colon A_{s-1} \to A_s$ is an isomorphism. By duality the multiplication map $\times\ell\colon A_t \to A_{t+1}$ is also an isomorphism. This implies that $[R/(I,\ell)]_{t+1} = 0$. Therefore, $[R/(I,\ell)]_i = 0$ for $i \geqslant t+1$ and we are done. □

Remark 1.42. For a graded Artinian Gorenstein algebra A of codimension n and h-vector satisfying $h_t = h_{t+1} \leqslant t$, for some t, a result of Gotzmann shows that A is the quotient of the coordinate ring of a zero-dimensional scheme in $\mathbb{P}^{n-1}$ of degree h_t, see [62] or [84, Proposition C.32] for more details. Therefore, such an algebra A has the WLP.

The special case of Proposition 1.41 when $h_{s-1} = h_s \leqslant s-1$ follows from the result of Gotzmann.
$\triangle$

Example 1.43. Every Artinian Gorenstein algebra with h-vector

$$(1, 4, 6, 7^k, 6, 4, 1), \quad \text{for } k \geqslant 5$$

has the WLP. Gotzmann's result proves it for every $k \geqslant 6$ and the case $k = 5$ is implied by Proposition 1.41.
$\diamondsuit$

1.3.3 The General case

This part is devoted to the proof of Theorem 1.46. The following proposition serves as the basis for an induction in its proof.

Proposition 1.44. *Let A_f be an Artinian Gorenstein algebra of socle degree 6 associated with a Perazzo hypersurface $V(f) \subset \mathbb{P}^4$. The algebra A_f fails WLP if and only if the Hilbert function is maximal, i. e. $(1, 5, 8, 8, 8, 5, 1)$.*

Proof. By Theorem 1.31 and Lemma 1.33

$$(1, 5, 6, 6, 6, 5, 1), \quad (1, 5, 6, 7, 6, 5, 1), \quad (1, 5, 7, 7, 7, 5, 1),$$
$$(1, 5, 7, 8, 7, 5, 1), \quad \text{and} \quad (1, 5, 8, 8, 8, 5, 1).$$

are the possible h-vectors for A_f. We know that A_f has the WLP if its Hilbert function is the minimal $(1, 5, 6, 6, 6, 5, 1)$. The case of $(1, 5, 6, 7, 6, 5, 1)$ is covered by Example 1.40. We also know that A_f fails the WLP if it has the maximal Hilbert function $(1, 5, 8, 8, 8, 5, 1)$. Let us now see that the h-vector $(1, 5, 7, 7, 7, 5, 1)$ can be excluded.

Assume that $A_f = R/I$ has the Hilbert function $(1, 5, 7, 7, 7, 5, 1)$. Take ℓ a general linear form and denote by h' the Hilbert function of $R/(I, \ell)$. As in the previous case, the Hilbert function of $R/(I\colon \ell)$ is $(1, 5, a, a, 5, 1)$ with $a = 6$ or 7. We consider another general linear form $\ell' \in A_1$ and the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & {[R/(I : \ell)]_2} \arrow[r] \arrow[d] & A_3 = [R/I]_3 \arrow[r] \arrow[d] & {[R/(I, \ell)]_3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {[R/(I\colon \ell)]_3} \arrow[r] & A_4 = [R/I]_4 \arrow[r] & {[R/(I, \ell)]_4} \arrow[r] & 0
\end{tikzcd}$$

where the vertical morphisms are given by the multiplication by ℓ'. If $a = 6$, $R/(I\colon \ell)$ has minimal Hilbert function $(1, 5, 6, 6, 5, 1)$. Therefore, $R/(I\colon \ell)$ has the WLP and the first vertical map is an isomorphism. $R/(I, \ell)$ has Hilbert function $(1, 4, 2, 1, 1)$ and so the last vertical map is surjective while the middle map is not surjective because A_f fails the WLP in degree 4. If $a = 7$, $R/(I\colon \ell)$ has maximal Hilbert function $(1, 5, 7, 7, 5, 1)$. Therefore, $R/(I\colon \ell)$ fails the WLP while A_f has the WLP. Hence the middle map is an isomorphism which contradicts the fact that the first vertical map is not injective.

It only remains to prove WLP in the case that A_f has the Hilbert function $(1, 5, 7, 8, 7, 5, 1)$. In that case, let ℓ be a general linear form, and note that A_f has the WLP if and only if $\times\ell\colon [A_f]_3 \to [A_f]_4$ is surjective.

Write $A_f = R/I$ and consider the algebra $R/(I\colon \ell)$. By Lemma 1.30 this is a Gorenstein algebra associated with a Perazzo form of degree five. Recall that the Hilbert function of $R/(I, \ell)$ is determined by the Hilbert functions of R/I and $R/(I\colon \ell)$ as explained in Lemma 1.38. If $R/(I\colon \ell)$ has the Hilbert function $(1, 5, 7, 7, 5, 1)$ then $R/(I, \ell)$ has the Hilbert function $(1, 4, 2, 1)$. In particular $[R/(I, \ell)]_4 = 0$, and we are done. The only other possibility for the Hilbert function of $R/(I\colon \ell)$ is the minimal

$(1,5,6,6,5,1)$, in which case $R/(I,\ell)$ would have the Hilbert function $(1,4,2,2,1)$. We consider another general linear form $\ell' \in A_1$ and the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & {[R/(I\colon \ell)]_2} \arrow[r] \arrow[d] & A_3 = [R/I]_3 \arrow[r] \arrow[d] & {[R/(I,\ell)]_3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {[R/(I\colon \ell)]_3} \arrow[r] & A_4 = [R/I]_4 \arrow[r] & {[R/(I,\ell)]_4} \arrow[r] & 0
\end{tikzcd}$$

where the vertical morphisms are given by the multiplication by ℓ'. The algebra $R/(I\colon \ell)$ has the WLP and hence

$$\times \ell' \colon [R/(I\colon \ell)]_2 \longrightarrow [R/(I\colon \ell)]_3$$

is an isomorphism. Moreover, it is easy to see that

$$\times \ell' \colon [R/(I,\ell)]_3 \longrightarrow [R/(I,\ell)]_4$$

is surjective as $[R/(I,\ell)]_4$ is a one dimensional space. By the snake lemma

$$\times \ell' \colon [R/I]_3 \longrightarrow [R/I]_4$$

is surjective, which implies the WLP. □

Remark 1.45. Note that the Hilbert function $(1,5,7,7,7,5,1)$ cannot occur for a Perazzo hypersurface, but can still occur for some Artinian Gorenstein algebra. Indeed, it is the Hilbert function of the algebra defined by the Macaulay dual generator $F = x_0^6 + x_0x_1^5 + x_0x_2^5 + x_1^6 + x_2^6 + u^6 + v^6$. △

We can now give the classification of Artinian Gorenstein algebras associated with Perazzo hypersurfaces of degree $d \geqslant 5$ in $\mathbb{P}^4$ with the weak Lefschetz property. Recall that such algebras have Sperner number at most $d+2$.

Theorem 1.46. *Let A_f be an Artinian Gorenstein algebra associated with a Perazzo hypersurface $V(f) \subset \mathbb{P}^4$ of degree $d \geqslant 5$. Let $(h_0, h_1, \dots, h_d)$ be its h-vector. The algebra A_f has the WLP if and only if $\#\{i \mid h_i = d+2\} \leqslant 1$.*

Proof. Assume first that $\#\{i \mid h_i = d+2\} \geqslant 2$ (resp. $\geqslant 3$) if d is odd (resp. even). Consider the Hessian of f of order $(d-1)/2$ (resp. $(d-2)/2$). To compute it we fix a monomial basis $\mathcal{B}$ of the homogeneous component of A_f of degree $(d-1)/2$ (resp. $(d-2)/2$), formed by $d+2$ monomials. Since there are at most $(d+1)/2$ monomials involving only u,v (resp. $d/2$), the number of monomials containing some of the variables x,y,z is at least $(d+3)/2$ (resp. $(d+4)/2$). This implies that in the Hessian matrix there is a block of zeros of order at least $(d+3)/2$ (resp. $(d+4)/2$), which forces the determinant to vanish. Therefore the multiplication map by a general linear form does not have maximal rank in degree $(d+1)/2$ (resp. in degree $d/2$), and A_F fails the WLP.

Let us now assume $\#\{i \mid h_i = d+2\} \leqslant 1$, and prove that A_F has the WLP. We have seen that the result holds for $d = 5,6$. We proceed by induction over d, treating the cases of d even and d odd separately.

CASE d **even**. Write $d = 2s$. The hypothesis $\#\{i \mid h_i = d+2\} \leqslant 1$ together with Theorem 1.31 implies that $h_{s+1} \leqslant d+1$. We let $I = \operatorname{Ann} F$ so that $A_F = R/I$. Take a general linear form $\ell \in A_F$ and consider the short exact sequence (1.8). By Lemma 1.30 we know that $R/(I\colon \ell)$ is an Artinian Gorenstein algebra associated with the Perazzo form $\ell \circ F$ of degree $d-1 = 2s-1$. Denote by h'_i and $\widetilde{h}_i$ the Hilbert functions of $R/(I,\ell)$ and $R/(I\colon \ell)$, and recall that $\widetilde{h}_i = h_{i+1} - h'_{i+1}$. Using Green's theorem (see Theorem 1.2) and the inequality $h_{s+1} < d+2$, we get that $h'_{s+1} \leqslant 1$. If $h'_{s+1} = 0$ the map

$$\times \ell \colon [R/I]_s \longrightarrow [R/I]_{s+1}$$

is surjective, and A_F has the WLP. Suppose instead $h'_{s+1} = 1$. Then, as $h_{s+1} \leqslant d+1$, we get $\widetilde{h_s} < d+1$. Therefore, we can apply our induction hypothesis to $R/(I\colon \ell)$ and conclude that $R/(I\colon \ell)$ satisfies the WLP. Consider now another general linear form $\ell' \in R/I$ and the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & {[R/(I\colon \ell)]_{i-1}} \arrow[r] \arrow[d] & {[R/I]_i} \arrow[r] \arrow[d] & {[R/(I,\ell)]_i} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {[R/(I\colon \ell)]_i} \arrow[r] & {[R/I]_{i+1}} \arrow[r] & {[R/(I,\ell)]_{i+1}} \arrow[r] & 0
\end{tikzcd} \tag{1.9}$$

for $i = s$, and where the vertical morphisms are given by the multiplication by ℓ'. Since $R/(I\colon \ell)$ satisfies the WLP, the leftmost vertical map is an isomorphism. On the other hand it is easy to check, as $h'_{s+1} = 1$, that the rightmost vertical map is an epimorphism. By the snake lemma, the middle vertical map is an epimorphism, which then gives A_f the WLP.

CASE d odd. Write $d = 2s+1$. By Theorem 1.31 we know that h is symmetric and unimodal. Since we assume $\#\{i \mid h_i = d+2\} \leqslant 1$ we have $h_s = h_{s+1} \leqslant d+1$ in this case. Applying Theorem 1.2 we get $h'_{s+1} \leqslant 1$. If $h'_{s+1} = 0$ we are already done, so suppose $h'_{s+1} = 1$. By Macaulay's inequality then $h'_i \leqslant 1$ for all $i > s$. Considering the commutative diagram (1.9) we see that the rightmost vertical map is surjective for $i > s$. Since $R/(I\colon \ell)$ is a Perazzo algebra of socle degree $2s$ it has the WLP, and therefore the leftmost vertical map in (1.9) is surjective for $i > s$. We can conclude that the middle map $\times\ell'\colon [R/I]_i \to [R/I]_{i+1}$ is surjective when $i > s$, and by duality also injective for $i < s$. This gives the equality $h'_i = h_i - h_{i-1}$ for $i \leqslant s$.

Next, we claim that $h'_i = 1$ for some $i \leqslant s$, or equivalently $h_i = h_{i-1} + 1$. To prove the claim, consider the possibility that $h_i \geqslant h_{i-1}+2$ for each $i = 2, \ldots, s$. Then we would have $h_s \geqslant h_1+2(s-1) = 5 + 2(s-1) = d+2$ which would contradict the assumption $h_s \leqslant d+1$.

Now Macaulay's inequality together with the fact that $h'_i = 1$ for some $i \leqslant s$ implies $h'_s = 1$. Finally, we consider the diagram (1.9) with $i = s$. In this case the leftmost vertical map is injective, and the rightmost is bijective as $h'_s = h'_{s+1} = 1$. We conclude that the middle map is injective, and therefore R/I has the WLP. ☐

Theorem 1.47. *Let A_f be an Artinian Gorenstein algebra associated with a Perazzo hypersurface $V(f) \subset \mathbb{P}^4$ of degree $d \geqslant 5$. Assume that A_F has the WLP and that ℓ is a weak Lefschetz element. Then the Hilbert function of $A_f/(\ell)$ is unimodal.*

Proof. Denote the h-vectors of A_f and $A_f/(\ell)$ by h and h', respectively. As ℓ is a weak Lefschetz element we have $h' = \Delta h$, meaning $h'_i = \max(h_i - h_{i-1}, 0)$. Using [54, Proposition 3.4] we get that $h' \leqslant 4$ for every i. Macaulay's inequality (1.1) on h', for every $i \geqslant 2$, implies that $(h')_{i+1} \leqslant h'_i$ unless $i = 2$, $h'_2 = 3$ and in that case $h'_3 \leqslant 4$. This means that h' is unimodal except when $h' = (1,4,3,4,\ldots)$. We will show that there is no Artinian Gorenstein algebra A_f associated with a Perazzo hypersurface $V(f) \subset \mathbb{P}^4$ such that $\Delta h = (1,4,3,4,\ldots)$. If such an algebra would exist, it would have Sperner number at least 12, which implies $d \geqslant 10$. But for $d = 10$ the vector h does not satisfy the necessary condition for WLP given in Theorem 1.46. Suppose $d \geqslant 11$. Then $h = (1,5,8,12,h_4,h_5,\ldots,h_5,h_4,12,8,5,1)$. We will show that h cannot occur as the Hilbert function of A_f by showing that $h_3 = 12$ forces $h_2 = 9$.

From Proposition 1.22 we obtain that

$$\operatorname{rank} M_2 + \operatorname{rank} N_2 \leqslant h_2 \leqslant \operatorname{rank} M_2 + \operatorname{rank} N'_2, \quad \text{and} \tag{1.10}$$

$$\operatorname{rank} M_3 + \operatorname{rank} N_3 \leqslant h_3 \leqslant \operatorname{rank} M_3 + \operatorname{rank} N'_3, \tag{1.11}$$

where M_2, M_3, N_2, N_3, N'_2, and N'_3 are the matrices defined in Equation (1.5).

We claim that $\operatorname{rank} M_3 \geqslant 8$ implies that $\operatorname{rank} M_2 = 6$. To show the claim notice that the matrix obtained by removing the first row (or the last row) from M_2 is a submatrix of M_3. Suppose $\operatorname{rank} M_2 < 6$ and assume that the last column of M_2 is a linear combination of the other columns. This implies that the last two columns of M_3 are in the span of the first 7 columns. So $\operatorname{rank} M_3 \leqslant 7$ and this proves the claim.

Observe that $\operatorname{rank} N_3' \leqslant 4$, so $h_3 = 12$ implies that $\operatorname{rank} M_3 \geqslant 8$ by Inequality (1.11). In this case the claim shows that M_2 has to have maximal rank, $\operatorname{rank} M_2 = 6$. On the other hand, $\operatorname{rank} M_3 \geqslant 8$ forces at least one of the three blocks $\mathcal{A}_3, \mathcal{B}_3$, or $\mathcal{C}_3$, defined in Equation (1.4), to have maximal rank, and therefore N_2' and N_2 both have maximal rank, that is equal to three. So $\operatorname{rank} M_2 + \operatorname{rank} N_2 = \operatorname{rank} M_2 + \operatorname{rank} N_2' = 6 + 3 = 9$ and by Inequality (1.10) we conclude that $h_2 = 9$ which completes the proof. □

Theorem 1.46 and Theorem 1.47 rule out some intermediate possibilities for h-vectors of Perazzo algebras in between the maximum and minimum h-vectors given by Proposition 1.24, and Proposition 1.27. In general, it is a difficult problem to determine whether a h-vector is the Hilbert function of an Artinian Gorenstein algebra, let alone one coming from a Perazzo hypersurface.

Example 1.48. There is no Artinian Gorenstein algebra associated with a Perazzo hypersurface in $\mathbb{P}^4$ with Hilbert function

$$H = (1, 5, 6, 7, 9, 9, 9, 7, 6, 5, 1).$$

If there were an algebra with this Hilbert function, then Theorem 1.46 would imply that it has WLP, and by Theorem 1.47 ΔH should be unimodal, which is a contradiction. ♢

1.4 Classification of Perazzo 3-folds with minimal h-vector

The goal of this section is to classify all Perazzo 3-folds X in $\mathbb{P}^4$ of degree $d \geqslant 5$ whose associated Artinian Gorenstein algebra $S/\operatorname{Ann}_S(f)$ has h-vector: $(1, 5, 6, 6, \cdots, 6, 6, 5, 1)$. As a corollary we will also classify all Perazzo 3-folds X in $\mathbb{P}^4$ of degree 5 whose associated Artinian Gorenstein algebra $S/\operatorname{Ann}_S(f)$ has the WLP.

We start the section with some technical lemmas and remarks.

Lemma 1.49. *Let* $f_1 = p_0(u,v)x_0 + p_1(u,v)x_1 + p_2(u,v)x_2$ *and* $f_2 = q_0(u,v)x_0 + q_1(u,v)x_1 + q_2(u,v)x_2$ *be two Perazzo 3-folds of degree* d *in* $\mathbb{P}^4$ *such that* $\langle p_0, p_1, p_2\rangle_{\mathbf{k}} = \langle q_0, q_1, q_2\rangle_{\mathbf{k}} \subset \mathbf{k}[u,v]_{d-1}$*. Then, the* h*-vectors of* $S/\operatorname{Ann}_S(f_1)$ *and* $S/\operatorname{Ann}_S(f_2)$ *coincide.*

Proof. By [84, Proposition A7] it is enough to prove that f_1 and f_2 define projectively equivalent 3-folds in $\mathbb{P}^4$. Write $q_0(u,v) = \lambda_0 p_0(u,v) + \lambda_1 p_1(u,v) + \lambda_2 p_2(u,v)$, $q_1 = \mu_0 p_0(u,v) + \mu_1 p_1(u,v) + \mu_2 p_2(u,v)$, $q_3 = \nu_0 p_0(u,v) + \nu_1 p_1(u,v) + \nu_2 p_2(u,v)$. We have

$$\begin{aligned} f_2 &= q_0(u,v)x_0 + q_1(u,v)x_1 + q_2(u,v)x_2 \\ &= (\lambda_0 p_0(u,v) + \lambda_1 p_1(u,v) + \lambda_2 p_2(u,v))x_0 + (\mu_0 p_0(u,v) + \mu_1 p_1(u,v) + \mu_2 p_2(u,v))x_1 + \\ &\quad + (\nu_0 p_0(u,v) + \nu_1 p_1(u,v) + \nu_2 p_2(u,v))x_2 \\ &= (\lambda_0 x_0 + \mu_0 x_1 + \nu_0 x_2)p_0(u,v) + (\lambda_1 x_0 + \mu_1 x_1 + \nu_1 x_2)p_1(u,v) + (\lambda_2 x_0 + \mu_2 x_1 + \nu_2 x_2)p_2(u,v). \end{aligned}$$

Therefore, f_1 and f_2 define projectively equivalent hypersurfaces in $\mathbb{P}^4$. □

We fix integers $d \geqslant 5$ and $2 \leqslant k \leqslant \lfloor \frac{d}{2} \rfloor$. We keep the notations introduced in Section 1.1. If $\operatorname{rank} M_k = 3$, then $\operatorname{rank} \mathcal{A}_k \leqslant 3$, $\operatorname{rank} \mathcal{B}_k \leqslant 3$ and $\operatorname{rank} \mathcal{C}_k \leqslant 3$. We will now explain the geometrical meaning of the rank of the matrices $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k$. To this end, we recall some basic facts about symmetric tensors in two variables. For more details see [15], [84], and [110].

Let us fix an integer $t \geqslant 3$ and consider the vector space $\mathbf{k}[u,v]_t$ of forms of degree t. Its elements can also be interpreted as symmetric tensors in two variables; by definition the Waring rank, or symmetric rank, of $p \in \mathbf{k}[u,v]_t$ is the minimum integer r such that there exist linear forms $l_1, \ldots, l_r \in \mathbf{k}[u,v]_1$ such that $p = l_1^t + \cdots + l_r^t$. In particular, a symmetric tensor p has Waring rank 1 if $p = l^t$ for a suitable linear form l, i.e. p is a pure power of degree t.

In the projective space $\mathbb{P}^t$, naturally identified with $\mathbb{P}(\mathbf{k}[u,v]_t)$, the set of (equivalence classes of) forms of Waring rank 1 is the image of the t-tuple Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^t$, that is the rational normal curve C_t of degree t. We recall that, for any $r \geqslant 1$, the r-secant variety of C_t is

$$\sigma_r(C_t) = \overline{\bigcup_{p_1,\dots,p_r \in C_t} \langle p_1, \dots, p_r\rangle}.$$

Clearly $C_t = \sigma_1(C_t) \subset \sigma_2(C_t) \subset \cdots$, and a general element of $\sigma_r(C_t)\backslash\sigma_{r-1}(C_t)$ is a symmetric tensor of Waring rank r, but if $r > 1$ $\sigma_r(C_t)$ contains also tensors of Waring rank $> r$. The dimension of $\sigma_r(C_t)$ is $\min\{2r-1, t\}$. Moreover, for any $r < \frac{t+1}{2}$, $\sigma_{r-1}(C_t)$ is the singular locus of $\sigma_r(C_t)$ (see [118, Proposition 1.2.2 and Corollary 1.2.3]).

We recall also that the tangential surface of C_t, TC_t, is the closure of the union of the embedded tangent lines to C_t. The tangent line at the point $l_1^t \in C_t$ is the set of tensors that can be written in the form $l_1^{t-1}l_2$, with l_2 a linear form. Similarly the osculating 3-fold of C_t, T^2C_t, is the closure of the union of the embedded osculating planes to C_t, and the osculating plane at l_1^t is the set of tensors that can be written in the form $l_1^{t-2}m$, with m a form of degree 2.

We are now ready to give the desired interpretation of the rank of the matrices introduced in Equation (1.4). We state and prove Proposition 1.50 for the form p_0 and the matrices $\mathcal{A}_k$; the analogous results hold true also for p_1, p_2, and their respectively catalecticant matrices $\mathcal{B}_k$, $\mathcal{C}_k$.

Proposition 1.50. *We fix an integer $d \geqslant 5$ and we keep the notations from Equation* (1.4). *It holds:*

(1) If $\operatorname{rank}\mathcal{A}_k = 1$ for some $2 \leqslant k \leqslant \lfloor\frac{d+1}{2}\rfloor$ (and, hence, for all $\mathbf{k}$), then $p_0 = \ell^{d-1}$ for some $\ell \in \mathbf{k}[u,v]_1$.

(2) If $\operatorname{rank}\mathcal{A}_k = 2$ for some $3 \leqslant k \leqslant \lfloor\frac{d+1}{2}\rfloor$ (and, hence, for all $\mathbf{k}$), then either $p_0 = \ell_1^{d-1} + \ell_2^{d-1}$ or $p_0 = \ell_1^{d-2}\ell_2$ for some $\ell_1, \ell_2 \in \mathbf{k}[u,v]_1$.

(3) If $\operatorname{rank}\mathcal{A}_k = 3$ for some $4 \leqslant k \leqslant \lfloor\frac{d+1}{2}\rfloor$ (and, hence, for all $\mathbf{k}$), then either $p_0 = \ell_1^{d-1} + \ell_2^{d-1} + (\lambda\ell_1 + \mu\ell_2)^{d-1}$ or $p_0 = \ell_1^{d-1} + \ell_2^{d-2}(\lambda\ell_1 + \mu\ell_2)$ for some $\ell_1, \ell_2 \in \mathbf{k}[u,v]_1$ and $\lambda, \mu \in \mathbf{k}^*$.

Proof. Let r be any integer such that $r+1 \leqslant k$. From [110, Theorem 1.3], it follows that all the minors of order $r+1$ of $\mathcal{A}_k$ vanish if and only if $[p_0] \in \sigma_r(C_{d-1})$. For $r = 1$, this gives (1). For $r = 2$, we get that if $\mathcal{A}_k$ has rank 2, then $p_0 \in \sigma_2(C_{d-1})$. From [15, Corollary 26], it follows that either p_0 has Waring rank 2 or $p_0 \in TC_{d-1}$; this proves (2). Similarly, for $r = 3$, $\operatorname{rank}\mathcal{A}_k = 3$ implies that $p_0 \in \sigma_3(C_{d-1})$. So, either the Waring rank of p_0 is 3, or p_0 belongs to the join of C_{d-1} and its tangential surface ([15, Corollary 26]). This proves (3). □

Theorem 1.51. *The Artinian Gorenstein algebra $S/\operatorname{Ann}_S(f)$ associated with a Perazzo 3-fold of degree $d \geqslant 5$ has h-vector:* (1, 5, 6, 6, ⋯ 6, 6, 5, 1) *if and only if, after a possible change of coordinates, one of the following cases holds:*

(i) $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + u^{d-3}v^2x_2$, or

(ii) $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + v^{d-1}x_2$, or

(iii) $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + (\lambda u + \mu v)^{d-1}x_1 + v^{d-1}x_2$ with $\lambda, \mu \in \mathbf{k}^*$.

Proof. As observed in Remark 1.28, the h-vector is minimal if and only if $\operatorname{rank} M_k = \operatorname{rank} N'_k = 3$ for any k. A straigthforward computation shows that for any f as in (i), (ii) or (iii) one has $\operatorname{rank} M_k = \operatorname{rank} N'_k = 3$ for any k and, therefore, $S/\operatorname{Ann}_S(f)$ has h-vector (1, 5, 6, ⋯ 6, 5, 1). To prove the converse, we first observe that if $\operatorname{rank} M_k = \operatorname{rank} N'_k = 3$ for any k, then the ranks of $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{G}_k$ are all bounded above by 3. We analyze first the various possibilities for p_0, p_1, p_2.

(I) $d \geqslant 7$ and $\operatorname{rank}\mathcal{A}_k = \operatorname{rank}\mathcal{B}_k = \operatorname{rank}\mathcal{C}_k = 3$ for $4 \leqslant k \leqslant \lfloor\frac{d+1}{2}\rfloor$ and p_0, p_1, p_2 all have Waring rank 3. We use [110, Corollary 1.2]: the spaces of the columns of $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k$ coincide, so there exist linear forms l_1, l_2, l_3 and suitable constants such that

$$p_0 = \lambda_0 l_1^{d-1} + \mu_0 l_2^{d-1} + \nu_0 l_3^{d-1}$$

$$p_1 = \lambda_1 l_1^{d-1} + \mu_1 l_2^{d-1} + \nu_1 l_3^{d-1}$$
$$p_2 = \lambda_2 l_1^{d-1} + \mu_2 l_2^{d-1} + \nu_2 l_3^{d-1}.$$

Since p_0, p_1, p_2 are linearly independent, the matrix $\begin{pmatrix} \lambda_0 & \mu_0 & \nu_0 \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu 2 \end{pmatrix}$ is invertible, so $\langle p_0, p_1, p_2 \rangle = \langle l_0^{d-1}, l_1^{d-1}, l_2^{d-1} \rangle$. In view of Lemma 1.49 in f we can replace p_0, p_1, p_2 with $l_0^{d-1}, l_1^{d-1}, l_2^{d-1}$.

(II) $d \geqslant 7$ and $\operatorname{rank} \mathcal{A}_k = 3$ for $4 \leqslant k \leqslant \lfloor \frac{d+1}{2} \rfloor$, but p_0 has Waring rank strictly > 3. So from Proposition 1.50 (3), p_0 is of the form $\ell_1^{d-1} + \ell_2^{d-2}(\alpha \ell_1 + \beta \ell_2)$ for some $\ell_1, \ell_2 \in \mathbf{k}[u,v]_1$ and $\alpha, \beta \in \mathbf{k}^*$. So up to the change of variables that sends l_1 into u, and l_2 into v, $p_0 = u^{d-1} + \alpha u v^{d-2} + \beta v^{d-1}$. Then

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & b_0 & b_1 & b_2 & c_0 & c_1 & c_2 \\ 0 & 0 & 0 & & & & & & \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & \alpha & b_{d-3} & b_{d-2} & b_{d-1} & c_{d-3} & c_{d-2} & c_{d-1} \\ 0 & \alpha & \beta & b_{d-2} & b_{d-1} & b_d & c_{d-2} & c_{d-1} & c_d \end{pmatrix}.$$

From $\operatorname{rank} M_3 < 4$ it follows $b_1 = \cdots = b_{d-3} = c_1 = \cdots = c_{d-3} = 0$. Therefore

$$p_1 = b_0 u^{d-1} + b_{d-2} u v^{d-2} + b_{d-1} v^{d-1}, \ p_2 = c_0 u^{d-1} + c_{d-2} u v^{d-2} + c_{d-1} v^{d-1},$$

and we can replace p_0, p_1, p_2 with $u^{d-1}, uv^{d-2}, v^{d-1}$.

(III) $\operatorname{rank} \mathcal{A}_k = 2$ for $3 \leqslant k \leqslant \lfloor \frac{d+1}{2} \rfloor$ and p_0 has Waring rank 2, so it can be written $p_0 = u^{d-1} + v^{d-1}$. Then M_3 is as in case (II) with $\alpha = 0$, $\beta = 1$ and

$$M_2 = \begin{pmatrix} 1 & 0 & b_0 & b_1 & c_0 & c_1 \\ 0 & 0 & b_1 & b_2 & c_1 & c_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & b_{d-2} & b_{d-1} & c_{d-2} & c_{d-1} \end{pmatrix}.$$

From $\operatorname{rank} M_2 < 4$ we deduce that

$$\operatorname{rank} \begin{pmatrix} b_1 & b_2 \\ \vdots & \vdots \\ b_{d-3} & b_{d-2} \end{pmatrix} < 2, \ \operatorname{rank} \begin{pmatrix} c_1 & c_2 \\ \vdots & \vdots \\ c_{d-3} & c_{d-2} \end{pmatrix} < 2, \ \operatorname{rank} \begin{pmatrix} b_1 & b_2 & \dots & b_{d-2} \\ c_1 & c_2 & \dots & c_{d-2} \end{pmatrix} < 2.$$

Therefore

$$(b_1, \dots, b_{d-2}) = (\lambda^{d-3}, \lambda^{d-4}\mu, \dots, \mu^{d-3}), \ (c_1, \dots, c_{d-2}) = (\sigma^{d-3}, \sigma^{d-4}\rho, \dots, \rho^{d-3}),$$

for suitable $\lambda, \mu, \sigma, \rho \in \mathbf{k}$. We get:

$$p_1 = b_0 u^{d-1} + uv((d-1)\lambda^{d-3} u^{d-3} + \binom{d-1}{2} \lambda^{d-4} \mu u^{d-4} v + \cdots) + b_{d-1} v^{d-1},$$

$$p_2 = c_0 u^{d-1} + uv((d-1)\sigma^{d-3} u^{d-3} + \binom{d-1}{2} \sigma^{d-4} \rho u^{d-4} v + \cdots) + c_{d-1} v^{d-1}.$$

We can also write

$$p_1 = b_0 u^{d-1} + uv\phi_{d-3} + b_{d-1} v^{d-1}, \ p_2 = c_0 u^{d-1} + kuv\phi_{d-3} + c_{d-1} v^{d-1}$$

where ϕ_{d-3} is a form of degree $d-3$ and $k \in \mathbf{k}$, because $(b_1, \dots, b_{d-2})$ and $(c_1, \dots, c_{d-2})$ are proportional. We can assume $b_0 = c_0 = 0$ and we get $v^{d-1} \in \langle p_0, p_1, p_2 \rangle$, hence $u^{d-1}, uv\phi_{d-3} \in \langle p_0, p_1, p_2 \rangle$. Finally, adding to $uv\phi_{d-3}$ suitable multiples of u^{d-1}, v^{d-1}, we get $(\lambda u + \mu v)^{d-1} \in \langle p_0, p_1, p_2 \rangle$.

(IV) $\operatorname{rank}\mathcal{A}_k = 2$ for $3 \leqslant k \leqslant \lfloor \frac{d+1}{2} \rfloor$ but p_0 has Waring rank > 2, so up to a change of variables $p_0 = u^{d-2}v$.

$$M_2 = \begin{pmatrix} 0 & 1 & b_0 & b_1 & c_0 & c_1 \\ 1 & 0 & b_1 & b_2 & c_1 & c_2 \\ \vdots & \vdots & & & & \\ 0 & 0 & b_{d-2} & b_{d-1} & c_{d-2} & c_{d-1} \end{pmatrix}$$

has rank 3, therefore

$$\operatorname{rank}\begin{pmatrix} b_2 & b_3 & c_2 & c_3 \\ \vdots & \vdots & \vdots & \vdots \\ b_{d-2} & b_{d-1} & c_{d-2} & c_{d-1} \end{pmatrix} < 2,$$

and arguing in a similar way to (III), we conclude that $\langle p_0, p_1, p_2\rangle$ is either of the form $\langle u^{d-1}, u^{d-2}v, (\lambda u + \mu v)^{d-1}\rangle$, or $\langle u^{d-1}, u^{d-2}v, u^{d-3}v^2\rangle$.

(V) $\operatorname{rank}\mathcal{A}_k = \operatorname{rank}\mathcal{B}_k = \operatorname{rank}\mathcal{C}_k = 1$, then p_0, p_1, p_2 are all pure powers of degree $d-1$.

(VI) Let π be the 2-plane generated by the polynomials p_0, p_1, p_2. If $d = 5$, $\pi \subset \mathbb{P}^4 = \mathbb{P}(\mathbf{k}[u,v]_4)$. The tangential variety TC_4 has codimension 2, so the intersection $\pi \cap TC_4 \neq \varnothing$. If π intersects TC_4 outside its singular locus C_4, up to a change of variables $u^3v \in \pi$ and we conclude as in (IV); otherwise, we are in the situation of (V). If $d = 6$, $\pi \subset \mathbb{P}^5 = \mathbb{P}(\mathbf{k}[u,v]_5)$. Now $\sigma_2(C_5)$ has codimension 2 and therefore $\pi \cap \sigma_2(C_5) \neq \varnothing$. Therefore we are either in the situation of (III) or of (IV).

We have proved that for any $d \geqslant 5$, if f defines a Perazzo 3-fold and $S/\operatorname{Ann}_S(f)$ has minimal h-vector, then the polynomials p_0, p_1, p_2 are as in (i), or (ii), or (iii).

It remains to find out how we can choose the polynomial g in each of the cases. From Proposition 1.22 we deduce that the only condition that g has to satisfy is $\operatorname{Ann}_S(p_0x_0 + p_1x_1 + p_2x_2)_3 = \operatorname{Ann}_S(f - g)_3 = \operatorname{Ann}_S(f)_3$. In other words, we impose that g is annihilated by a system of generators of $\operatorname{Ann}_S(f)_3$.

(i) If $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + u^{d-3}v^2x_2 + g$, we have that $\operatorname{Ann}_S(f)_3 = \langle V^3\rangle$ and so $g = g_0u^d + g_1u^{d-1}v + g_2u^{d-2}v^2$.

(ii) If $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + v^{d-1}x_2 + g$, we have that $\operatorname{Ann}_S(f)_3 = \langle UV^2\rangle$. This gives that $\sum_{i=2}^{d-1} g_i\binom{d}{i}(k-i)i(i-1)u^{k-i-1}v^{i-2} = 0$, so $g_2 = \cdots = g_{d-1} = 0$. Thus we get $g = g_0u^d + g_1u^{d-1}v + g_dv^d$.

(iii) If $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + (\lambda u + \mu v)^{d-1}x_1 + v^{d-1}x_2 + g$, we have that $\operatorname{Ann}_S(f)_3 = \langle \mu U^2V - \lambda UV^2\rangle$. Then we have the condition

$$\sum_{i=2}^{d-1}\binom{k-3}{i-1}(\mu g_i - \lambda g_{i+1})u^{d-k-2}v^{i-1} = 0 \iff \mu g_i - \lambda g_{i+1} = 0, \quad i = 1, \ldots, k-2.$$

So we can collect g_1 and complete the d-th power to obtain $g = au^d + b(\lambda u + \mu v)^d + cv^d$.

Therefore we obtain three different classes as follows:

(i) $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + u^{d-3}v^2x_2 + au^d + bu^{d-1}v + cu^{d-2}v^2$ with $a, b, c \in \mathbf{k}$, or

(ii) $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + v^{d-1}x_2 + au^d + bu^{d-1}v + cv^d$ with $a, b, c \in \mathbf{k}$, or

(iii) $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + (\lambda u + \mu v)^{d-1}x_1 + v^{d-1}x_2 + au^d + b(\lambda u + \mu v)^d + cv^d$ with $\lambda, \mu \in \mathbf{k}^*$ and $a, b, c \in \mathbf{k}$.

One last change of variables provides the desired classification. □

Remark 1.52. As we noticed in Lemma 1.49, the Hilbert function of the algebra $S/\operatorname{Ann}_S(f)$ depends only on the plane $\pi = \mathbb{P}(\langle p_0, p_1, p_2\rangle_{\mathbf{k}}) \subset \mathbb{P}^{d-1}$ and not on the choice of the three generators. In Theorem 1.51 we have proved that the h-vector is minimal if and only if the plane π is in one of the following positions: it is an osculating plane to the rational normal curve C_{d-1} (case (i)), or it contains the tangent line to C_{d-1} at a point and meets C_{d-1} also at a second point (case (ii)), or it intersects C_{d-1} at three distinct points (case (iii)). △

Remark 1.53. In Theorem 1.51, we have obtained a complete characterisation of the polynomials f such that $A = S/\operatorname{Ann}_S(f)$ has minimum h-vector for any $d \geqslant 5$. This allows us to conclude with a direct verification the proof of Theorem 1.35, proving the WLP of these algebras in the cases $5 \leqslant d \leqslant 7$. △

Corollary 1.54. *The Artinian Gorenstein algebra $S/\operatorname{Ann}_S(f)$ associated with a Perazzo 3-fold of degree 5 has the WLP if and only if, after a possible change of coordinates, one of the following cases holds:*

(i) $f(x_0, x_1, x_2, u, v) = u^4x_0 + u^3vx_1 + u^2v^2x_2 \in R_5$, *or*

(ii) $f(x_0, x_1, x_2, u, v) = u^4x_0 + u^3vx_1 + v^4x_2 \in R_5$, *or*

(iii) $f(x_0, x_1, x_2, u, v) = u^4x_0 + (\lambda u + \mu v)^4x_1 + v^4x_2 \in R_5$ *with* $\lambda, \mu \in \mathbf{k}^*$.

Proof. It follows from Theorems 1.51, 1.34, and 1.35. □

Note that, as consequence of the results of Gordan-Nöther, Corollary 1.54 gives also a complete classification of threefolds in $\mathbb{P}^4$ with vanishing hessian.

1.4.1 Geometrical considerations

In this section, we give a short geometrical description of the hypersurfaces in Theorem 1.51.

Case (i) corresponds to the union of the classic cubic Perazzo 3-fold in $\mathbb{P}^4$ of equation: $u^2x_0 + uvx_1 + v^2x_2 = 0$ with the non-reduced hyperplane of equation: $u^{d-3} = 0$. To describe the other two hypersurfaces, we first recall some known geometric properties of hypersurfaces with vanishing hessian. Let $X = V(f) \subset \mathbb{P}^N$ be such a hypersurface with $\operatorname{hess}_f = 0$.

We denote by

$$\nabla_f : \mathbb{P}^N \dashrightarrow (\mathbb{P}^N)^*$$

its *polar map* defined by

$$\nabla_f(p) = \left(\frac{\partial f}{\partial x_0}(p), \frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_N}(p)\right),$$

and by

$$\gamma : X \dashrightarrow (\mathbb{P}^N)^*$$

the restriction of ∇_f to X, i.e. the *Gauss map* of X, associating to each smooth point of X its embedded tangent space. The image of γ is the dual variety X^* of X. Let $Z = \overline{\nabla_f(\mathbb{P}^N)}$ be the closure of the image of the polar map. Then $X^* \subsetneq Z \subsetneq (\mathbb{P}^N)^*$ ([118, Corollary 7.2.8]). Moreover, if $N = 4$, Z is a cone with vertex a line over an irreducible plane curve, and its dual Z^* is a rational plane curve in $\mathbb{P}^4$, naturally identified with the bidual space $(\mathbb{P}^4)^{**}$ ([118, Lemma 7.4.13]).

Let X be a Perazzo hypersurface of degree d in $\mathbb{P}^4$ of equation (1.2). X contains the line $L : x_0 = x_1 = x_2 = 0$ and the plane $\Pi : u = v = 0$. From [118, Sections 7.3 and 7.4], it follows that Π is the singular locus of X with multiplicity $d-1$; moreover, X^* is a scroll surface of degree d, having the line Π^* as directrix. In particular, Π^* is also the vertex of Z, and the general plane ruling of the cone Z meets X^* along a line of the scroll. The curve Z^* is contained in Π and the hyperplanes containing Π cut on X, outside Π, a 1-dimensional family Σ of planes: they are all tangent to Z^* and meet L. If p is general in X, then the fibre of the Gauss map $\gamma^{-1}(\gamma(p))$ is the line $\langle p, p'\rangle$ where p' is the tangency point to Z^* of the plane of the family Σ passing through p.

We now see how this picture specialises if we consider the reduced, irreducible Perazzo 3-fold $X_1 \subset \mathbb{P}^4$ of equation

$$f_1(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + v^{d-1}x_2,$$

case (ii) in Theorem 1.51. We use coordinates $z_0, \ldots, z_4$ in $(\mathbb{P}^4)^*$. The equation of Z, which expresses the algebraic dependence of p_0, p_1, p_2, is $z_1^{d-1} - z_0^{d-2}z_2 = 0$; the one of Z^* is $(d-1)^{d-1}x_0^{d-2}x_2 + (d-2)^{d-2}x_1^{d-1} = 0$. They both represent rational curves of degree $d-1$ with a singular point of multiplicity $d-2$ with only one tangent line.

In case (iii) we have

$$f_2(x_0,x_1,x_2,u,v) = u^{d-1}x_0 + (\lambda u + \mu v)^{d-1}x_1 + v^{d-1}x_2 \text{ with } \lambda,\mu \in \mathbf{k}^*.$$

For low values of d we have checked with the help of Macaulay2 ([M2]) that Z is a cone over a rational curve of degree $d-1$ with $\frac{(d-2)(d-3)}{2}$ distinct nodes. Its dual Z^* results to be a rational curve of degree $2d-4$. If $d=5$, then Z^* has degree 6 and it has 3 cuspidal points of multiplicity 3 at the fundamental points $[1,0,0],[0,1,0],[0,0,1]$ and one node; if $d=6$, then Z^* has cuspidal points of multiplicity 4 at the fundamental points and 3 nodes; if $d=7$, then Z^* has cuspidal points of multiplicity 5 at the fundamental points and 6 nodes.

1.5 Jordan type for minimal Perazzo algebras

In this section, we study the Jordan types of an Artinian Gorenstein algebra corresponding to a Perazzo threefold of minimal Hilbert function, i. e. of the type $(1,5,6,\dots,6,5,1)$. An explicit classification of the possible dual generators f of degree $d \geqslant 5$ defining a Perazzo threefold with minimal Hilbert function is given in Theorem 1.51. Following a different and more concrete approach we are able to compute the Lefschetz locus for such cases.

Proposition 1.55. *For Perazzo threefolds with minimal Hilbert function, the Lefschetz elements are the linear forms $a_0X_0 + a_1X_1 + a_2X_2 + b_0U + b_1V$ satisfying the following conditions, for the three types of f listed in Theorem 1.51:*

(i) $b_0 \neq 0$,

(ii) $b_0b_1 \neq 0$,

(iii) $b_0b_1(b_0+\lambda b_1) \neq 0$.

Proof. Let $S = \mathbf{k}[y_0,y_1,y_2,U,V]$ be the ring of differential operators. By Theorem 1.35, we have $h_2 = h_3 = \cdots = h_{d-2} = 6$. By Proposition 1.10, A has the WLP if and only if the multiplication map

$$\times\ell\colon A_2 \longrightarrow A_3$$

is an isomorphism for some $L \in A_1$. To prove it, we instead check that the multiplication map

$$\times\ell^{d-4}\colon A_2 \longrightarrow A_{d-2}$$

is an isomorphism for some $L \in A_1$, i.e. the 2-nd Hessian of f does not vanish.

(i) $f(x_0,x_1,x_2,u,v) = u^{d-1}x_0 + u^{d-2}vx_1 + u^{d-3}v^2x_2$. A basis of A_2 is $\mathcal{B} = \{U^2, UV, V^2, y_0U, y_1U, y_2U\}$. So, the 2-nd Hessian is

$$\mathrm{hess}_f^2 = \det\begin{pmatrix} C & B \\ B^T & 0 \end{pmatrix} = -(\det B)^2,$$

where C, B are 3×3 matrices. In particular the matrix B has the form

$$B = (d-3)\begin{pmatrix} (d-1)(d-2)u^{d-4} & * & * \\ 0 & (d-2)u^{d-4} & * \\ 0 & 0 & u^{d-4} \end{pmatrix}.$$

Since B has non zero determinant, f does not have 2-nd vanishing Hessian. Moreover, the non Lefschetz elements have the form $\ell = k_0y_0 + k_1y_1 + k_2y_2 + k_3V$ with $k_i \in \mathbf{k}$.

(ii) $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + u^{d-2}vx_1 + v^{d-1}x_2$. A basis of A_2 is $\mathcal{B} = \{U^2, UV, V^2, y_0U, y_1U, y_2V\}$. So, the 2-nd Hessian is

$$\operatorname{hess}^2_f = \det \begin{pmatrix} C & B \\ B^T & 0 \end{pmatrix} = -(\det B)^2,$$

where C, B are 3×3 matrices. In particular the matrix B has the form

$$B = (d-2)(d-3) \begin{pmatrix} (d-1)u^{d-4} & * & * \\ 0 & u^{d-4} & * \\ 0 & 0 & (d-1)v^{d-4} \end{pmatrix}.$$

Since B has non zero determinant, f does not have 2-nd vanishing Hessian. Moreover, the non Lefschetz elements have the form $\ell = k_0y_0 + k_1y_1 + k_2y_2$ with $k_i \in \mathbf{k}$.

(iii) $f(x_0, x_1, x_2, u, v) = u^{d-1}x_0 + (\lambda u + \mu v)^{d-1}x_1 + v^{d-1}x_2$ with $\lambda, \mu \in \mathbf{k}^*$. A basis of A_2 is $\mathcal{B} = \{U^2, UV, V^2, y_0U, y_1U, y_2V\}$. So, the 2-nd Hessian is

$$\operatorname{hess}^2_f = \det \begin{pmatrix} C & B \\ B^T & 0 \end{pmatrix} = -(\det B)^2,$$

where C, B are 3×3 matrices. In particular the matrix B has the form

$$B = (d-1)(d-2)(d-3) \begin{pmatrix} u^{d-4} & * & 0 \\ 0 & \lambda^2\mu(\lambda u + \mu v)^{d-4} & 0 \\ 0 & * & v^{d-4} \end{pmatrix}.$$

Since B has non zero determinant, f does not have 2-nd vanishing Hessian. Moreover, the non Lefschetz elements have the form $\ell = k_0y_0 + k_1y_1 + k_2y_2$ with $k_i \in \mathbf{k}$. □

In our situation, the Jordan type of the algebra A_F is a partition of $6d - 6$. Let us start with the generic linear Jordan type $P_{A_F} = (p_1, \dots, p_t)$. We know that a general linear form $\ell \in A_F$ satisfies $\ell^d \neq 0$, so $p_1 = d + 1$. By Proposition 1.55 above, a general linear form $\ell \in A_F$ is a weak Lefschetz element. So, since the Sperner number is 6, this must be the number of parts in P_{A_F} (see Proposition 1.21). Since A_F does not have the SLP, P_{A_F} is strictly dominated by the conjugate of the Hilbert function: $H(A_F)^\vee = \big(d + 1, (d-1)^4, d-3\big)$. The only possibility is

$$P_{A_F} = \big(d + 1, (d-1)^3, (d-2)^2\big), \tag{1.12}$$

as any other partition of $6d - 6$ with $p_1 = d + 1$ and strictly dominated by $H(A_F)^\vee$ has more than 6 parts.

Remark 1.56. If we consider a general element ℓ' in the maximal ideal of A_F (not necessarily homogeneous), we could ask what its Jordan type is (this is the generic Jordan type of A_F, see [85, Definitions 2.1 and 2.55]). We know that this Jordan type is dominated by $H(A_F)^\vee$ and since a general element in the maximal ideal of A_F specialises to a general linear form in $(A_F)_1$, its Jordan type dominates the generic linear Jordan type (see the discussion before Lemma 2.54 in [85]). But if we compare the partition $H(A_F)^\vee$ with the generic linear Jordan type we have just obtained, we see that there is no partition between the two in the dominance order, so the generic Jordan type must be equal to one of these partitions. Since the Jordan type $H(A_F)^\vee$ implies that the vector space $(\ell')^{d-1}(A_F)_1$ has dimension 5 and this can only be attained if the same happens for $(\ell'_1)^{d-1}(A_F)_1$, where ℓ'_1 is the linear summand of ℓ', we see that the generic Jordan type of A_F is P_{A_F} as in (1.12). △

Let us now consider the possible Jordan types for multiplication by any linear form in A_F.

Theorem 1.57. *Let A_F be an Artinian Gorenstein algebra of a Perazzo threefold with minimal Hilbert function. The two Jordan types*

$$\big(d + 1, (d-1)^3, (d-2)^2\big) \quad \textit{and} \quad \big(d^2, (d-1)^2, (d-2)^2\big)$$

occur for multiplication by a weak Lefschetz element. For elements in the non-Lefschetz locus, the following holds regarding the Jordan basis, for the three types of F given in Theorem 1.51.

(i) All strings are of lengths at most 4*,*

(ii) The strings have lengths $\geqslant d-1$ *or* $\leqslant 3$*,*

(iii) The strings have lengths $\geqslant d-1$ *or* $\leqslant 2$*.*

Proof. We have already seen that the Jordan type of a general linear form is $\big(d+1,(d-1)^3,(d-2)^2\big)$, and that this is the only possible Jordan type for a weak Lefschetz element ℓ with $\ell^d \neq 0$. Let's consider a weak Lefschetz element ℓ such that $\ell^d = 0$. For instance, take $\ell = b_0u + b_1v$ with b_0, b_1 satisfying the conditions given in Proposition 1.55. In this case the Jordan strings have length at most d. We claim that the largest part of such a Jordan type is equal to d. By Proposition 1.21 every Jordan type partition has to be smaller than the generic Jordan type $\big(d+1,(d-1)^3,(d-2)^2\big)$ w. r. t. the dominance order, so there is no partition of $6d-6$ with exactly 6 parts that is dominated by the generic Jordan type and has parts of length at most $d-1$. Thus the largest part has length equal to d and by symmetry [40, Lemma 4.6] we must have at least 2 parts of length d. So, the Jordan type in this case is of the form (d^2,p_3,p_4,p_5,p_6) where $p_3+p_4+p_5+p_6 = 4d-6$, and since it has to be dominated by the generic Jordan type we must have $p_i \leqslant d-1$. The only possible partition in this case is $\big(d^2,(d-1)^2,(d-2)^2\big)$.

Let's now consider linear forms $\ell = a_0X_0+a_1X_1+a_2X_2+b_0U+b_1V$ which are not weak Lefschetz elements. Note that we always have $(x,y,z)^2 \subseteq \operatorname{Ann} F$. In the case of Theorem 1.51(i), the linear form ℓ not being a weak Lefschetz element means $b_0 = 0$, according to Proposition 1.55. As every monomial in x,y,z,v of degree four belongs to $\operatorname{Ann} F$ we then have $\ell^4 = 0$ in A_F. Therefore it's impossible to have a Jordan string of length more than four.

We move on to the case of Theorem 1.51(ii). By Proposition 1.55 we have $b_0 = 0$ or $b_1 = 0$. If $b_0 = b_1 = 0$ we have $\ell^2 = 0$ in A_F, and hence all strings have length at most two. Suppose $b_0 = 0$ and $b_1 \neq 0$. We may assume $b_1 = 1$. As $xv, yv^2 \in \operatorname{Ann} F$ we have

$$\ell^s = sa_2zv^{s-1}+v^s, \quad \text{and} \quad \ell^{s-1}z = zv^{s-1},$$

for any $s \geqslant 3$. This gives us two strings of length $d+1$ and $d-1$, and they span all polynomials in A_F in only the variables z and v. Using also the fact that $uv^2 \in \operatorname{Ann} F$ we get $\ell x = \ell^2 y = \ell^3 u = 0$ in A_F, and hence all remaining strings must have length at most three. Suppose instead $b_0 = 1$ and $b_1 = 0$. As $zu \in \operatorname{Ann} F$ we have

$$\ell^s = sa_0xu^{s-1}+sa_1yu^{s-1}+u^s$$

for $s \geqslant 2$ and

$$\ell^{d-1} = (d-1)a_0xu^{d-2}+u^{d-1} \neq 0.$$

Moreover

$$\ell^{d-2}x = xu^{d-2}, \quad \ell^{d-2}y = yu^{d-2}, \quad \ell^{d-2}v = (d-2)a_1yu^{d-3}v+u^{d-2}v.$$

It is straightforward to verify that ℓ^{d-1}, $\ell^{d-2}x$, $\ell^{d-2}y$, and $\ell^{d-2}v$ acting on F give four linearly independent elements, and therefore this gives us four Jordan strings of length at least $d-1$. We can also see that the one-dimensional space $[A_F]_d$ is covered, as $\ell^{d-1}x = xu^{d-1} \neq 0$. There is one more string starting in degree one, and as $\ell z = 0$ this string has length one. In degree $1 < s < d$ the monomials v^s and zv^{s-1} span a two dimensional subspace of $[A_F]_s$. Both monomials are in the kernel of multiplication by ℓ^2, therefore they cannot be in the span of ℓ^s, $\ell^{s-1}x$, $\ell^{s-1}y$, $\ell^{s-1}v$, except possibly when $s = d-2$. In any case this shows that the remaining strings have length at most two.

Finally we consider the last case of Theorem 1.51(iii). Here ℓ is non-Lefschetz if $b_0 = 0$, $b_1 = 0$, or $b_0+\lambda b_1 = 0$. Let's start by assuming $b_0 = 0$ and $b_1 = 1$. In this case

$$\ell^s = sa_1yv^{s-1}+sa_2zv^{s-1}+v^s \quad \text{when } s \geqslant 2.$$

The power ℓ^{d-1} together with

$$\ell^{d-2}y = v^{d-2}y, \quad \ell^{d-2}z = v^{d-2}z, \quad \ell^{d-2}u = (d-2)a_1yuv^{d-3}+uv^{d-2}$$

are linearly independent, and give us four strings of lengths at least $d-1$. As $\ell x = 0$ the remaining string starting in degree one will have length one. In degree $1 < s < d$ the elements $u^{s-1}(\lambda u - v)$ and xu^{s-1} are linearly independent, and both in the kernel of multiplication by ℓ^2. Hence all remaining strings have lengths at most two.

The case $b_0 = 1$ and $b_1 = 0$ is treated analogously. Suppose $b_0 + \lambda b_1 = 0$. We may assume $b_0 = \lambda$ and $b_1 = -1$. Note that $y(\lambda u - v) \in \operatorname{Ann} F$ in this case. We have

$$\ell^s = s\lambda a_0 x u^{s-1} - s a_2 z v^{s-1} + (\lambda u - v)^s \quad \text{for } s \geqslant 2.$$

It is straightforward to check that ℓ^{d-1} together with

$$\begin{aligned} &\ell^{d-2}x = x(\lambda u - v)^{d-2}, \quad \ell^{d-2}z = z(\lambda u - v)^{d-2}, \\ \text{and} \quad &\ell^{d-2}u = (d-2)\lambda a_0 x u^{d-2} + (\lambda u - v)^{d-2}u \end{aligned}$$

form a 4-dimensional space. Hence the Jordan basis has four strings of length at least $d-1$. As $\ell y = 0$ in A_F we get one string of length one starting in degree one. In higher degree the elements $y(u+\lambda v)^s$ and $u^s v$ form a two-dimensional space contained in the kernel of multiplication by ℓ^2. It follows that all remaining strings will have length at most two. ☐

Jordan type of a non-Lefschetz element

We do a more thorough study of the possible Jordan types in the case of Theorem 1.51 (i). As the possible Jordan types of Lefschetz elements were discussed in detail above, let now $\ell = a_0X_0 + a_1X_1 + a_2X_2 + b_0U + b_1V$ be a non-Lefschetz element. We present the Jordan types, as well as the strings in a pre-Jordan basis, in each case.

Conditions	P_{ℓ,A_f}	pre-Jordan basis
$a_0 \in \mathbf{k}$, $a_1 \in \mathbf{k}$, $a_2 \neq 0$, $b_1 \neq 0$	$(4^{d-2}, 2^d, 1^2)$	$U^i \mapsto \ell U^i \mapsto \ell^2 U^i \mapsto \ell^3 U^i$, $0 \leqslant i \leqslant d-3$ $U^iV \mapsto \ell U^iV$, $0 \leqslant i \leqslant d-3$ $X_1 \mapsto \ell X_1$, $U^{d-2} \mapsto \ell U^{d-2}$ X_0, U^{d-1}
$a_0 \in \mathbf{k}$, $a_1 \in \mathbf{k}$, $a_2 = 0$, $b_1 \neq 0$	$(3^{2d-4}, 2^2, 1^2)$	$U^i \mapsto \ell U^i \mapsto \ell^2 U^i$, $0 \leqslant i \leqslant d-3$ $X_2U^i \mapsto \ell X_2U^i \mapsto \ell^2 X_2 U^i$, $0 \leqslant i \leqslant d-3$ $X_1 \mapsto \ell X_1$, $U^{d-2} \mapsto \ell U^{d-2}$ X_0, U^{d-1}
$a_0 \in \mathbf{k}$, $a_1 \in \mathbf{k}$, $a_2 \neq 0$, $b_1 = 0$	$(2^{3d-6}, 1^6)$	$U^i \mapsto \ell U^i$, $0 \leqslant i \leqslant d-3$ $U^iV \mapsto \ell U^iV$, $0 \leqslant i \leqslant d-3$ $U^iV^2 \mapsto \ell U^iV^2$, $0 \leqslant i \leqslant d-3$ $X_0, X_1, X_0U, U^{d-2}, U^{d-2}V, U^{d-1}$
$a_0 \in \mathbf{k}$, $a_1 \neq 0$, $a_2 = 0$, $b_1 = 0$	$(2^{2d-2}, 1^{2d-2})$	$U^i \mapsto \ell U^i$, $0 \leqslant i \leqslant d-2$ $U^iV \mapsto \ell U^iV$, $0 \leqslant i \leqslant d-2$ X_2U^i, U^iV^2, $0 \leqslant i \leqslant d-3$ X_0, U^{d-1}
$a_0 \neq 0$, $a_1 = 0$, $a_2 = 0$, $b_1 = 0$	$(2^d, 1^{4d-6})$	$U^i \mapsto \ell U^i$, $0 \leqslant i \leqslant d-1$ U^iV^2, X_2U^i, $0 \leqslant i \leqslant d-3$ U^iV, X_1U^i, $0 \leqslant i \leqslant d-2$

We notice that the computations agree with Theorem 1.57. We also remark that these Jordan types are in a chain with respect to the dominance order:

$$(4^{d-2}, 2^d, 1^2) > (3^{2d-4}, 2^2, 1^2) > (2^{3d-6}, 1^6) > (2^{2d-2}, 1^{2d-2}) > (2^d, 1^{4d-6}).$$

Chapter 2

Strongly Koszul algebras

> *"If people do not believe that mathematics is simple, it is only because they do not realise how complicated life is."*
>
> – John von Neumann

The theory of minimal free resolutions lies at the heart of modern Commutative Algebra and Algebraic Geometry. It provides a systematic tool in studying modules over polynomials rings, or quasi-coherent sheaves over projective spaces. A classic theorem of Hilbert states that modules over a polynomial ring $S = \mathbf{k}[x_1, \ldots, x_n]$ have a finite minimal free resolution. A generalization of this theorem due to Auslander, Buchsbaum, and Serre characterise polynomial rings as the only standard graded rings in which every module has a finite minimal free resolution. Such a resolution is controlled by the minimal free resolution of the residue field seen as quotient of S by $m_S = (x_1, \ldots, x_n)$. When we consider a quotient $R = S/I$ of S by a homogeneous ideal I, then if I does not contain linear forms, we lose in general the finiteness of the minimal free resolution. In particular, this is always the case for $\mathbf{k} = R/\mathfrak{m}_R$, where $\mathfrak{m}_R = \mathfrak{m}_S R$. However, by imposing conditions on the minimal free resolution of $\mathbf{k}$, we are able to investigate minimal resolutions of every R-module. This is the basic idea underneath the definition of *Koszul algebra* (see Definition 2.2).

The authors of [77] defined a particular sub-class of Koszul algebras called *strongly Koszul algebras*. They followed the first experimentation of Fröberg in [57] in finding family of Koszul algebras, in a non-commutative setting, and they were inspired by the observations made in [28] about some special colon ideals in algebras defined by quadratic monomial relations.

Some recent results on strongly Koszul algebras include:

- A characterisation for edge algebras [80].
- A study of monomial ideals' free resolutions in strongly Koszul algebras based on iterated mapping cones [127].
- A sufficient condition for strongly Koszulness based on "tidy" quadratic RevLex-universal Gröbner basis [42].
- An equivalent condition for toric algebras still based on universal Gröbner bases [101].

In this chapter we introduce the so called strongly Koszul masks (cf. Definition 2.13) to give a general tool in studying strongly Koszul algebras. For a standard graded $\mathbf{k}$-algebra, we see that the conditions imposed on the behaviour of colon ideals $(x_{j_1}, \ldots, x_{j_s}) :_R x_{j_{s+1}}$ imply strong conditions on the Hilbert function of R. In particular, a mask generalises this type of relations showing that algebras fitting the given mask all have the same Hilbert function. The results within this chapter are based on many discussion with Aldo Conca.

2.1 Preliminaries

Let $\mathbf{k}$ be a field, $S = \mathbf{k}[x_1, \dots, x_n]$ be the polynomial ring in n variables over $\mathbf{k}$, and $R = S/I$ be a graded quotient with $I_1 = 0$. Set $\mathfrak{m}_S := (x_1, \dots, x_n)$ be the maximal irrelevant ideal of S, and $\mathfrak{m}_R := \mathfrak{m}_S R$. Whenever there is no confusion, we will name x_i both the variables in S and their cosets inside R. The following is a brief introduction to Koszul algebras, and in particular to strongly Koszul algebras.

Koszul algebras

The Auslander-Buchsbaum-Serre Theorem (Theorem A.9) characterises polynomial rings through the finiteness of the minimal free resolutions. Therefore, if we consider a graded quotient $R = S/I$ and I is not generated in degree one, then the minimal free resolution $K_\bullet$ of $\mathbf{k} = R/\mathfrak{m}_R$ is infinite. On the other hand, the DG-algebra structure on $K_\bullet$ is preserved. In fact, a (usually infinite) procedure called Tate complex allows to expand the Koszul complex $K_\bullet(\mathfrak{m}_R)$ into a minimal free resolution of $\mathbf{k}$ (see [11]).

Another important side that we can control is the regularity. We have seen in Construction A.5 that over a polynomial ring, the minimal free resolution of $\mathbf{k}$ is linear, i.e. $\beta_{i,j}(\mathbf{k}) = 0$ whenever $i \neq j$. This is equivalent to say $\operatorname{reg}_S(\mathbf{k}) = 0$. An important result in this direction is the Avramov-Eisenbud-Peeva Theorem.

Theorem 2.1 (Avramov-Eisenbud-Peeva). *Let R be a finitely generated standard graded $\mathbf{k}$-algebra. Then the following are equivalent.*

1. $\operatorname{reg}_R(\mathbf{k}) = 0$*;*
2. $\operatorname{reg}_R(\mathbf{k})$ *is finite;*
3. $\operatorname{reg}_R(M)$ *is finite for every graded R-module M.*

If one of the above equivalent conditions holds, then $\operatorname{reg}_R(M) \leqslant \operatorname{reg}_S(M)$ *for every graded R-module M.*

Proof. Avramov and Eisenbud proved the equivalence of (1) and (3) [12], while Avramov and Peeva proved the equivalence between (1) and (2) [13]. □

Definition 2.2. Let R be a finitely generated standard graded $\mathbf{k}$-algebra. Then if one of the equivalent conditions in Theorem 2.1 is satisfied, we say that R is Koszul.

Koszul algebras were introduced by Priddy in 1970 in the classical paper [115]. Here he wanted to formalise a recurring behaviour occurring in many natural examples. He was interested in non-commutative $\mathbf{k}$-algebras where the base field $\mathbf{k}$ has a linear minimal resolution $K_\bullet$. In this more general setting, he introduced the concept of Koszul duality and proved that the Koszul dual algebra defines a minimal free resolution of $\mathbf{k}$. In this sense, the exterior algebra and the polynomial ring are one dual of the other.

Later, Fröberg in [57] constructed many class of examples of Koszul algebras. For instance, all algebras with defining ideal generated by quadratic monomials is Koszul. Another important class of examples is quadratic complete intersection which are always Koszul by theory of Tate complexes (see [11]). A general discussion on Koszul algebras can be found in [35, 37].

Two important tools in checking whenever a given algebra is Koszul are the Gröbner bases, Koszul filtrations, and the so called strongly Koszul algebras. The last is studied and discussed in Chapter 2, while the definition of the former are given below.

Definition 2.3. A $\mathbf{k}$-algebra $R = S/I$ is called G-quadratic if I is generated by a Gröbner basis of quadrics for some term order. Instead, it is called LG-quadratic if $R \cong R'/(y_1, \dots, y_t)$ where R' is G-quadratic and $y_1, \dots, y_t$ is a regular sequence in R' of linear elements. Finally, R is quadratic if I is generated in degree two.

Definition 2.4. A Koszul filtration $\mathfrak{F}$ of a $\mathbf{k}$-algebra R is a family of ideals in R satisfying the conditions:

(1) Every ideal in $\mathfrak{F}$ is generated by linear forms;

(2) The zero ideal and $\mathfrak{m}_R$ are in $\mathfrak{F}$;

(3) For every ideal $I \in \mathfrak{F}$, there exists $J \in \mathfrak{F}$, $J \subset I$ such that $J : I \in \mathfrak{F}$, and I/J is cyclic.

Moreover, if a set $\mathfrak{F} = \{0 \subset I_1 \subset \cdots \subset I_{n-1} \subset \mathfrak{m}_R\}$ is a Koszul filtration of R, we name it a Gröbner flag.

Remark 2.5. All these mathematical notions are strictly connected to each other.

$$\begin{tikzcd}
\text{G-quadratic} \arrow[r, Rightarrow] & \text{LG-quadratic} \arrow[d, Rightarrow] & \begin{array}{c}\text{complete intersection}\\ \text{of quadrics}\end{array} \arrow[l, Rightarrow] \\
 & \text{Koszul} \arrow[r, Rightarrow] & \text{quadratic} \\
\text{Gröbner flag} \arrow[uu, Rightarrow] \arrow[r, Rightarrow] & \text{Koszul filtration} \arrow[u, Rightarrow] &
\end{tikzcd}$$

In the above diagram any of the arrows cannot be reversed and explicit court-examples have been found. By Gröbner flag, and Koszul filtration, we mean that the algebra admits such an object. △

Strongly Koszul algebras

Definition 2.6. Let R be a $\mathbf{k}$-algebra. For every $A \subseteq [n]$, we set the ideal $J_A := (x_i : i \in A) \subseteq R$. We say that a R is *strongly Koszul*, or SK for short, if there exist $\{y_1, \dots, y_n\}$ a $\mathbf{k}$-basis of R_1 with the following property: For every $A \subseteq [n]$ and $b \in [n]\backslash A$, there exists $B \subseteq [n]$ such that $J_A :_R y_b = J_B$. Here, $[n] := \{1, 2, \dots, n\}$ for every $n \in \mathbb{Z}_{>0}$.

Remark 2.7.

1. Every strongly Koszul algebra is also Koszul. In fact, the family
$$\mathfrak{F} := \{(y_i : i \in A) : A \subset [n]\}$$
is a Koszul filtration. In particular, the Koszul filtration's Condition (3) is satisfied in a stronger way: For every given $(y_i : i \in A) \subset (y_j : j \in B)$ such that $(y_j : j \in B)/(y_i : i \in A)$ is cyclic then $(y_i : i \in A) :_R (y_j : j \in B) \in \mathfrak{F}$.
2. By a linear change of coordinates, we can always suppose that $y_i = x_i$. Therefore, from now on when we talk about strongly Koszul algebras, we intend with respect to the standard basis. △

From now on, we fix coordinates on S, and we are interested only in SK algebras with respect to those fixed coordinates. Moreover, use the frequent notation $J_A := (x_i : i \in A) \subseteq R$ for every $A \subseteq [n]$.

Example 2.8. Typical examples of strongly Koszul algebras are: algebras defined by quadratic monomial ideals and Veronese algebras $S^{(c)} = \oplus_{i \geqslant 0} S_{ci}$ of the polynomial ring S. Both cases relies in computing explicit colon ideals which can be done using the formula for monomial ideals in the first case, and the fact that $S^{(c)}$ is a direct summand of S in the latter. More in general, if $I \subset S$ is generated by monomials of degree $\leqslant d$, then $R^{(c)}$ is strongly Koszul for every $c \geqslant d - 1$ [37, Thm 13]. ♢

Remark 2.9. If R is a strongly Koszul algebra, then also $R_A = R/J_A$ is strongly Koszul for every $A \subseteq [n]$. In fact for every $B \subset [n]$ with $A \subset B$ we have the relation

$$J_B R_A :_{R_A} x_b = \frac{J_{A \cup B} :_R x_b}{J_A}.$$

Therefore, the information contained in these colon ideals in R_A is inherit by the one in R. △

Remark 2.10. If $A \subseteq [n]$ and $b \in [n] \backslash A$ then there is a short exact sequence of graded R-modules

$$0 \longrightarrow \frac{R}{J_A :_R x_b}(-1) \xrightarrow{\cdot x_b} \frac{R}{J_A} \longrightarrow \frac{R}{J_A + (x_b)} \longrightarrow 0.$$

In particular if R is SK, then all the rings in this short exact sequences are SK as well by Remark 2.9. Moreover, the rings on the sides will involve in principle less variables than the ring in the centre except when $J_A :_R x_b = J_A$. △

In the following discussion, we consider ways to construct new strongly Koszul algebras A starting from two known SK algebras A_1 and A_2. In all constructions, we have that A_1 is a retract of A, that is $A_1 \hookrightarrow A \twoheadrightarrow A_1$ and the composition is the identity of A_1. This implies the equality $I :_{A_1} b = (IA :_A b) \cap A_1$ for every ideal I of A_1, and $b \in A_1$.

Let $A_1 = \mathbf{k}[x_1, \ldots, x_n]/I$ and $A_2 = \mathbf{k}[y_1, \ldots, y_m]/J$ be two $\mathbf{k}$-algebras with I and J containing no linear forms.

The fiber product $A = A_1 \times_K A_2$ can be presented by a quotient $\mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]/U$ where $U = I + J + (x_i y_j : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$. Straightforward computations show that for $B \subseteq [n]$ and $C \subseteq [m]$ we have

$$\begin{aligned} [(x_i : i \in B) + (y_j : j \in C)] :_A x_k &= [(x_i : i \in B) :_{A_1} x_k] A + (y_1, \ldots, y_m) \quad \text{for } k \notin B, \\ [(x_i : i \in B) + (y_j : j \in C)] :_A y_l &= (x_1, \ldots, x_n) + [(y_j : j \in C) :_{A_2} y_l] A \quad \text{for } l \notin C. \end{aligned}$$

In particular, using Remark 2.9, we are able to state the following.

Proposition 2.11. *Let A_1 and A_2 be as above. If A_1 and A_2 are strongly Koszul then the fiber product $A_1 \times_K A_2$ is strongly Koszul. Furthermore, fix a coordinate system on $A_1 \times_K A_2$ that is the union of two coordinate systems on A_1 and A_2. If $A_1 \times_K A_2$ is strongly Koszul w.r.t. that coordinate system then both A_1 and A_2 are strongly Koszul.*

In the same manner, one can consider the tensor product $A = A_1 \otimes_K A_2$ which is isomorphic to $\mathbf{k}[x_1, \ldots, x_n, y_1, \ldots, y_m]/(I + J)$. Moreover, given $B \subseteq [n]$ and $C \subseteq [m]$ we get

$$\begin{aligned} [(x_i : i \in B) + (y_j : j \in C)] :_A x_k &= [(x_i : i \in B) :_{A_1} x_k] A + (y_j : j \in C) \quad \text{for } k \notin B, \\ [(x_i : i \in B) + (y_j : j \in C)] :_A y_l &= (x_i : i \in B) + [(y_j : j \in C) :_{A_2} y_l] A \quad \text{for } l \notin C. \end{aligned}$$

Together with Remark 2.9, a result similar to Proposition 2.11 is stated below.

Proposition 2.12. *Let $A_1 = \mathbf{k}[x_1, \ldots, x_n]/I$ and $A_2 = \mathbf{k}[y_1, \ldots, y_m]/J$ be two $\mathbf{k}$-algebras. If A_1 and A_2 are strongly Koszul then the tensor product $A_1 \otimes_K A_2$ is strongly Koszul. Furthermore, fix a coordinate system on $A_1 \otimes_K A_2$ that is the union of two coordinate systems on A_1 and A_2. If $A_1 \otimes_K A_2$ is strongly Koszul w.r.t. that coordinate system then both A_1 and A_2 are strongly Koszul.*

2.2 Strongly Koszul mask

Let $G = (V, E)$ be the oriented graph where $V = 2^{[n]}$ and $(A, B) \in E$ if $A \subseteq B$ and $\#(B \backslash A) = 1$. Equivalently, we can write the elements of E as pairs (A, k) where $A \subseteq [n]$ and $k \notin A$.

Definition 2.13. Given a function $M : E \longrightarrow 2^{[n]}$, we say that M is a *strongly Koszul mask*, or simply mask, if:

(1) $A \subseteq M(A, k)$ for every $(A, k) \in E$;

(2) if $A \subseteq B$, then $M(A, k) \subseteq M(B, k)$ for every $k \notin B$;

(3) $j \in M(A, k) \iff k \in M(A, j)$ for every $(A, k), (A, j) \in E$;

(4) $M(M(A,k),j) = M(M(A,j),k)$ for every $(A,k) \in E$, and $j \notin M(A,k)$;

(5) there exists a map $F : 2^{[n]} \longrightarrow \mathbb{Z}[\![z]\!]$ satisfying

- $F([n]) = 1$;
- $F(A) = \begin{cases} F(A \cup \{k\}) + zF(M(A,k)) & \text{if } A \subsetneq M(A,k) \\ \frac{1}{1-z}F(A \cup \{k\}) & \text{otherwise} \end{cases}$, given $(A,k) \in E$.

Notation 2.14. In the remaining parts of this chapter, we will refer many times to Property (5) of Definition 2.13 and the function F defined in it. We will simply refer to this property as Property (5); while, we will refer to F as the Hilbert series of M.

Proposition 2.15. *Let $M : E \longrightarrow 2^{[n]}$ be a strongly Koszul mask. Then the Hilbert series of M is uniquely defined.*

Proof. Let F and G be two functions both satisfying the two conditions imposed in Property (5). The aim is to prove that $F(A) = G(A)$ for every $A \subset [n]$. We argue by decreasing induction on $\#A$, the cardinality of A. Surely, $F([n]) = 1 = G([n])$. Suppose now that the functions F and G coincide for every subset of cardinality greater or equal than $s+1$. Then, for every set A of cardinality s and $k \notin A$ we have

$$F(A) = \frac{1}{1-z}F(A \cup \{k\}) = \frac{1}{1-z}G(A \cup \{k\}) = G(A)$$

if $A = M(A,k)$, and

$$F(A) = F(A \cup \{k\}) + zF(M(A,k)) = G(A \cup \{k\}) + zG(M(A,k)) = G(A)$$

if $A \subsetneq M(A,k)$. This concludes the proof. □

The non-trivial part of Property (5) lies in the existence of the function F, which depends only on M: this gives a restriction on all possible functions M that are a mask as we will see later.

The main idea is to mimic the behaviour of a strongly Koszul algebra R. From the definition of SK algebra, every time relations $J_A :_R x_b = J_B$ are established, we can construct a map $M_R : E \longrightarrow 2^{[n]}$ by seting $M_R(A,b) := B$. Properties (1)-(4) follows by the general properties of colon ideals. Property (5) wants to involve the Hilbert Series of R by defining $F_R(A) := \mathrm{HS}_{R/J_A}$. One has $F_R([n]) = \mathrm{HS}_{R/J_{[n]}} = \mathrm{HS}_K = 1$, and the recursive formula is a direct consequence of Remark 2.9.

Question 2.16. *For every mask M, is there a strongly Koszul algebra R such that $M = M_R$ and $F = F_R$? If this is not the case, which other properties should one add?*

Given a strongly Koszul mask M, if such algebra exists, then we say that M is *realisable*. The algebra realising a mask is not unique in general.

Example 2.17. Let $I = (x^2 + \lambda xy + y^2) \subseteq \mathbf{k}[x,y]$ with $\lambda \in \mathbf{k}$. Then we have

$$0 :_R x = 0 \quad 0 :_R y = 0 \quad (x) :_R y = (x,y) \quad (y) :_R x = (x,y).$$ ♢

Question 2.18. *Is it possible to describe algebras having the same mask? That is, given a mask, can one parametrise all algebras having the same given mask? What type of relations could arise?*

In considering both questions, it comes to help the action of $\mathcal{S}_n$ on the set of all possible masks. In fact, a permutation σ of the set $[n]$ induces an action on the values of a mask $M : E \to 2^{[n]}$, creating a new mask σM. This corresponds to a permutation of the set of all masks. In particular, the realisable property is invariant under this action.

Numerical considerations

Let $M : E \longrightarrow 2^{[n]}$ be a strongly Koszul mask. Which numerical invariants of a SK algebra realising M one can deduce from M itself? By Proposition 2.15, the Hilbert series of M is exactly the Hilbert series of every SK-algebra realising it. Therefore, all algebras realising M have the same dimension, and their defining ideals have the same minimal number of generators. Given the Hilbert series of R, $\mathrm{HS}_R(z) = \sum_{i\geqslant 0} \mathrm{HF}(R,i)z^i = h(z)/(1-z)^d$ with $h(1) \neq 0$, these invariants can be computed as

$$\mu(I) = \binom{n+1}{2} - \mathrm{HF}(R,2), \qquad \text{and} \qquad \dim R = d. \tag{2.1}$$

We want to show how to combinatorically compute such numbers from M. To this end, given $(A,k) \in E$ we define

$$r(A,k) := \text{cardinality of } M(A,k)\backslash A.$$

Lemma 2.19. *Let M be a mask with Hilbert series $F : 2^{[n]} \to \mathbb{Z}[\![z]\!]$. For every $A \subset [n]$, we define $F(A;i)$ as the integer coefficient of z^i in the series $F(A)(z)$.*

Then, for every $A \subset [n]$ there exists $d_A \in \mathbb{N}$, and a polynomial $h_A \in \mathbb{Z}[z]$ such that

$$F(A)(z) = \frac{h_A(z)}{(1-z)^{d_A}}, \text{ and } h_A(1) > 0.$$

Moreover, it holds $F(A;0) = 1$, and $F(A,1) = \#([n]\backslash A)$.

Proof. We proceed by strong induction on $m = \#([n]\backslash A)$ the cardinality of the complement of A. For $A = [n]$ we obtain $h_{[n]}(z) = 1 > 0$, and $d_{[n]} = 0$. In particular, $F([n];0) = 1$, and $F([n];1) = 0 = \#([n]\backslash[n])$.

Suppose now that the result is true for every set A such that $\#([n]\backslash A) \leqslant m$. Consider $B \subset [n]$ with $\#([n]\backslash B) = m+1$, and consider any $k \notin B$. By Property (5), there are two cases:

- If $B = M(B,k)$, then

$$F(B)(z) = \frac{F(B\cup\{k\})}{1-z} = F(B\cup\{k\}) + zF(B\cup\{k\}) + \text{ terms in } z \text{ of degree } \geqslant 2.$$

  In this case $F(B;0) = F(B\cup\{k\};0) = 1$, and

$$F(B;1) = F(B\cup\{k\};1) + F(B\cup\{k\};0) = \#([n]\backslash(B\cup\{k\})) + 1 = \#([n]\backslash B).$$

  Moreover, we set

$$h_B := h_{B\cup\{k\}}, \text{ and } d_B; = d_{B\cup\{k\}} + 1.$$

- If $B \subsetneq M(B,k)$, then

$$F(B)(z) = F(B\cup\{k\})(z) + zF(M(B,k))(z).$$

  In this case $F(B;0) = F(B\cup\{k\};0) = 1$, and

$$F(B;1) = F(B\cup\{k\};1) + F(M(B,k);0) = \#([n]\backslash(B\cup\{k\})) + 1 = \#([n]\backslash B).$$

  Moreover, we set

$$h_B := h_{B\cup\{k\}}(1-z)^{c-d_{B\cup\{k\}}} + zh_{M(B,k)}(1-z)^{c-d_{M(B,k)}}, \text{ and } d_B := c,$$

  where $c = \max\{d_{B\cup\{k\}}, d_{M(B,k)}\}$.

In both cases we have that $F(B)(z) = h_B(z)/(1-z)^{d_B}$, and $h_B(1) > 0$, together with the correct formulae for $F(B;0)$ and $F(B;1)$. □

Notation 2.20. With $\mathcal{S}(A)$ we intend the group of permutations of a given finite set A. If $A = [n]$, we use the notation $\mathcal{S}_n := \mathcal{S}([n])$.

Theorem 2.21. *Let $M : E \longrightarrow 2^{[n]}$ be a mask, and let F be its Hilbert series. For every $A \subset [n]$, we define $F(A; i)$ as the integer coefficient of z^i in the series $F(A)(z)$. Then the quantity*

$$\mu_\sigma(M) := r(\varnothing, \sigma(1)) + r(\{\sigma(1)\}, \sigma(2)) + \cdots + r(\{\sigma(1), \ldots, \sigma(n-1)\}, \sigma(n))$$

is independent on the permutation $\sigma \in \mathcal{S}_n$. We name this invariant $\mu(M)$. In particular, we have the formula $\mu(M) = \binom{n+1}{2} - F(\varnothing; 2)$. Suppose further that M is realised by $R = S/I$. Then the equality $\mu(M) = \mu(I)$ holds true.

Proof. The proof starts by checking the following claim.

Claim. $F(A; 2) = F(A \cup \{k\}; 2) + F(A; 1) - r(A, k)$ for any $(A, k) \in E$.

Let $(A, k) \in E$, then by Property (5) there are two possible cases:

- If $A = M(A, k)$, then $r(A, k) = 0$ and Lemma 2.19 implies
$$\begin{aligned} F(A;2) &= F(A \cup \{k\}; 2) + F(A \cup \{k\}; 1) + F(A \cup \{k\}; 0) = F(A \cup \{k\}; 2) + \#([n] \backslash (A \cup \{k\})) + 1 \\ &= F(A \cup \{k\}; 2) + \#([n] \backslash A) = F(A \cup \{k\}; 2) + F(A; 1) - r(A, k). \end{aligned}$$

- If $A \subsetneq M(A, k)$, then Lemma 2.19 implies
$$\begin{aligned} F(A;2) &= F(A \cup \{i\}; 2) + F(M(A, i); 1) = F(A \cup \{k\}; 2) + \#([n] \backslash M(A, k)) \\ &= F(A \cup \{k\}; 2) + \#([n] \backslash A) - \#(M(A, i) \backslash A) = F(A \cup \{k\}; 2) + F(A; 1) - r(A, i). \end{aligned}$$

This concludes the proof of the claim. We now use it repeatedly to compute $F(\varnothing; 2)$. Let $\sigma \in \mathcal{S}_n$ be any permutation. Then we have the following series of equalities.

$$\begin{aligned} F(\varnothing; 2) &= F(\{\sigma(1)\}; 2) + F(\varnothing; 1) - r(\varnothing, \sigma(1)) = F(\{\sigma(1)\}; 2) + n - r(\varnothing, \sigma(1)) \\ &= F(\{\sigma(1), \sigma(2)\}; 2) + F(\{\sigma(1)\}; 1) - r(\{\sigma(1)\}, \sigma(2)) + n - r(\varnothing, \sigma(1)) \\ &= F(\{\sigma(1), \sigma(2)\}; 2) + n + (n-1) - r(\varnothing, \sigma(1)) - r(\{\sigma(1)\}, \sigma(2)) \\ &\ \vdots \\ &= F(\{\sigma(1), \ldots, \sigma(n)\}; 2) + \sum_{j=1}^{n} j - \mu_\sigma(M) \\ &= \binom{n+1}{2} - \mu_\sigma(M). \end{aligned}$$

Here $F(\{\sigma(1), \ldots, \sigma(n)\}; 2) = F([n]; 2) = 0$ by Property (5). Since $F(\varnothing; 2)$ does not depend on the permutation σ, so does $\mu_\sigma(M)$. In particular, we obtain the equality $\mu(M) = \binom{n+1}{2} - F(\varnothing; 2)$.

If M is realised by a SK algebra $R = S/I$, then Equation (2.1) implies the relation $\mu(M) = \mu(I)$ since $F(\varnothing)$ is the Hilbert series of R. ☐

Theorem 2.22. *Let $M : E \longrightarrow 2^{[n]}$ be a mask. For any, $A \subset [n]$, let d_A be defined as in Lemma 2.19. Then*

$$d_A = \max\left\{ d \in \mathbb{N} : \begin{array}{c} \exists \sigma \in \mathcal{S}(B),\ \text{with } B := \{j_1, \ldots, j_c\} = [n] \backslash A,\ \text{such that for } d \text{ different indices} \\ m_1, \ldots, m_d \in [c] \text{ it holds } r(A \cup \{\sigma(j_1), \ldots, \sigma(j_{m_s})\}, \sigma(j_{m_s+1})) = 0,\ \text{for } s = 1, \ldots, d \end{array} \right\}.$$

Suppose further that M is realised by R. Then we get the equality

$$\dim R = \max\left\{ d \in \mathbb{N} : \begin{array}{c} \exists \sigma \in \mathcal{S}_n \text{ such that for } d \text{ different indices } m_1, \ldots, m_d \in [n] \\ \text{it holds } r(A \cup \{\sigma(1), \ldots, \sigma(m_s)\}, \sigma(m_{s+1})) = 0,\ \text{for } s = 1, \ldots, d \end{array} \right\}.$$

Proof. By Lemma 2.19, the series $F(A)$ can be written as $F(A)(z) = h_A(z)/(1-z)^{d_A}$, where d_A is a non-negative integer, and $h_A(1) > 0$. Following the proof of Lemma 2.19, given $(A,k) \in E$, there are two possible cases.

- If $A = M(A,k)$ then $r(A,k) = 0$, and $d_A = d_{A\cup\{k\}} + 1$, or;
- If $A \subsetneq M(A,k)$ then $r(A,k) > 0$, and $d_A = \max\{d_{A\cup\{k\}}, d_{M(A,k)}\}$.

The first part is now completed arguing by strong induction on $m = \#([n]\backslash A)$. The key point is how to construct a maximizing sequence of A from the maximizing sequences of $A \cup C$ with $C \subset [n]\backslash A$. When $M(A,k) = A$ then d_A increases by one as well as the number of times when $r(A,k) = 0$. Instead, if $A \subsetneq M(A,k)$ then in the maximizing sequence the number of times when

$$r(A \cup \{\sigma(j_1), \ldots, \sigma(j_{m_s})\}, \sigma(j_{m_s+1})) = 0$$

has not changed.

The second part follows from Equation 2.1, Lemma 2.19, and the fact that $F(\varnothing)$ is the Hilbert series of R. □

Example 2.23. We give an explicit example on how to use Theorems 2.21 and 2.22.

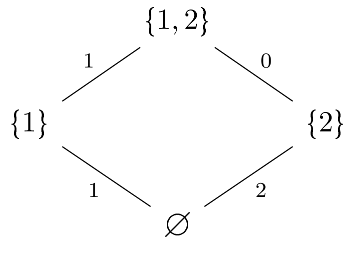
Set $n = 2$, and consider the mask defined as

$$\begin{array}{cc} M(\varnothing,1) = \{2\} & M(\varnothing,2) = \{1,2\} \\ M(\{1\},2) = \{1,2\} & M(\{2\},1) = \{2\} \end{array}$$

It is not hard to check that the Hilbert series of M is defined as

$$F(\varnothing) = \frac{1+z-z^2}{1-z} = 1 + 2z + z^2 + \ldots, \quad F(\{1\}) = 1 + z, \quad F(\{2\}) = \frac{1}{1-z}, \quad F([2]) = 1.$$

From M, we can add weights on the edges of the graph G. Given $(A,k) \in E$ we give to this edge the weight $r(A,k)$. In this example we obtain



On the two possible paths joining $\varnothing$ and $[2]$, we can see that the sum of the weight is constant and equal to $2 = \binom{n+1}{2} - F(\varnothing; 2)$ as stated in Theorem 2.21. Secondly, we can see that for every $A \subset [n]$ the maximum of the number of times that a zero appears in a path joining A to $[2]$ is equal to d_A. In particular, $d_\varnothing = 1$. This is guaranteed by Theorem 2.22.

The mask M is also realisable and it is realised by the algebra $R = \mathbf{k}[x,y]/(y^2, xy)$ for any field $\mathbf{k}$. One should notice that $\dim R = 1$, and $\mu(y^2, xy) = 2$. ♢

Remark 2.24. Theorem 2.21, together with the weight given to the graph G gives a visual hint in checking if a given map $M : E \to 2^{[n]}$ is a mask. This is clearly weaken than checking Property (5), but it is computationally faster and it gives a easier way to eliminate maps that are not strongly Koszul correspondences. △

Monomial ideals

As already mentioned, all monomial ideals define a strongly Koszul algebra. In particular, one notices that for a **k**-algebra R defined by a monomial ideal it follows:

$$J_A :_R x_b = J_A + 0 :_R x_b, \text{ for all } (A, b) \in E.$$

We say that a mask is *determined by atoms* if $M(A, k) = A \cup M(\varnothing, k)$ for all $(A, k) \in E$. It is clear that a mask of a monomial algebra is defined by atoms. The following result proves the opposite implication.

Proposition 2.25. *Every strongly Koszul mask determined by atoms is realisable, and it is realised by a unique algebra which is defined by monomial relations.*

Proof. Let M be a mask determined by atoms. Then consider the following monomial ideal

$$I = (x_i x_j : j \in M(\varnothing, i), 1 \leqslant i \leqslant n).$$

The condition $j \in M(\varnothing, i)$ implies that inside the algebra one has $x_i x_j = 0$ or equivalently $x_i x_j$ must lie in the defining ideal. A elementary computation shows that $M_{S/I} = M$.

Let J be another quadratic ideal, non necessarily monomial, such that S/J is SK, and $M_{R/J} = M$. We want to prove that $J = I$. From the previous argument we have $I \subseteq J$. On the other hand, since $M_{S/I} = M = M_{R/J}$, we have that they share the same Hilbert Series $F(\varnothing)$, where F depends only on M. Therefore, I and J have the same number of minimal generators which all lie in degree two; then, they must be equal. □

Hypersurfaces

We now focus on the case $I = (f)$ with $f \in S = \mathbf{k}[x_1, \dots, x_n]$ with $\deg f = 2$. We would like to characterise SK hypersurfaces.

Remark 2.26. By Theorems 2.21 and 2.22, if M is a realisable mask, we have the following necessary and sufficient condition for M to be realised by a hypersurface: For every permutation $\sigma \in \mathcal{S}_n$, there exists a unique index $c_\sigma \in \mathbb{N}$ such that

$$r(\{\sigma(1), \dots, \sigma(m)\}, \sigma(m+1)) = \begin{cases} 1 & \text{if } m = c_\sigma \\ 0 & \text{otherwise} \end{cases}.$$

In particular, if $R = S/(f)$ realises M, then for every $j \in M(A, k)\backslash A$ we must have $f - \lambda x_j x_k \in (x_i : i \in A)_2 \subset S_2$ for some $\lambda \in \mathbf{k}\backslash\{0\}$. This implies that $x_j x_k \in \sup(f)$. △

The following lemma shows that the support of f is the crucial information about the strongly kosulness of $S/(f)$.

Lemma 2.27. *Consider $x_a x_b \in S_2$ and let $f = \lambda x_a x_b + f_0$ and $g = x_a x_b + f_0$ with $f_0 \in S_2$ such that $x_a x_b \notin \sup(f_0)$ and $\lambda \in \mathbf{k}\backslash\{0\}$. Then the algebra $S/(f)$ is strongly Koszul if and only if the algebra $S/(g)$ is. In particular, $M_{S/(f)} = M_{S/(g)}$.*

Proof. We may assume without loss of generality that $f_0 \neq 0$, otherwise $(f) = (g)$, and there is anything to prove.

First, we underline that the support of f and the support of g by definition. Suppose that $S/(f)$ is SK. Consider $(A, k) \in E$. We want to compute $C := (J_A + (g)) :_S x_k$, where $J_A = (x_i : i \in A) \subset S$. There are several cases to consider.

- If $a, b \notin A \cup \{k\}$, then by Remark 2.26 we have $(J_A + (f)) :_S x_c = J_A + (f)$. We want to prove that $C = J_A + (g)$. Let $\ell \in C$ and $\ell \notin J_A + (g)$. Without loss of generality, we can suppose that ℓ does not depend on $\{x_i : i \in A\}$. From $\ell \in C$ one has $\ell x_k = \sum_{i \in A} h_i x_i + hg$. Since ℓ does not depend on $\{x_i : i \in A\}$, the equality reduces to $\ell x_k = hg$. If $h = 0$, then we have $\ell = 0$ which concludes the proof. Otherwise, suppose by contradiction that $h \neq 0$. From the equality $\ell x_k = hg$, one obtains that $x_k | gh$. Since x_k is a prime element it must divide one of the two elements. Since $k \neq a, b$ we have $x_b \nmid g$. Therefore $x_a | h$, but this also gives a contradiction because we would have that $\ell \in (g)$ which is false by assumption. Therefore, $h = 0$ implying $\ell = 0$.

- If $a = k$ and $b \notin A$, then

$$(J_A + (f)) :_S x_a = (J_A + (f)) + \begin{cases} (x_b) & \text{if } f_0 \in J_A \\ 0 & \text{if } f_0 \notin J_A \end{cases}.$$

We now prove that

$$C = (J_A + (g)) + \begin{cases} (x_b) & \text{if } f_0 \in J_A \\ 0 & \text{if } f_0 \notin J_A \end{cases}.$$

There are two possible cases.

  - If $f_0 \in J_A$, then $J_A + (f) = J_A + (x_a x_b) = J_A + (g)$ and therefore

$$C = (J_A + (f)) :_S x_a = J_A + (f) + (x_b) = J_A + (f_0) + (x_b) = J_A + (g) + (x_b).$$

  - If $f_0 \notin J_A$, given $\ell \in C$ and $\ell \notin J_A + (g)$, we want to prove that $\ell = 0$. The argument will be similar to the one given above. Without loss of generality, we can suppose that ℓ does not depend on $\{x_i : i \in A\}$. From $\ell \in C$ one has $\ell x_a = \sum_{i \in A} h_i x_i + hg$. Since ℓ does not depend on $\{x_i : i \in A\}$, the equality reduces to $\ell x_a = hg$. If $h = 0$, then we have $\ell = 0$ which concludes the proof. Otherwise, suppose by contradiction that $h \neq 0$. From the equality $\ell x_a = hg$, one obtains that $x_a | gh$. Since x_a is a prime element it must divide one of the two elements. If $x_a | g$ then $x_a | f$, or in other words $f = x_a * u$. Since (f) is strongly Koszul, and $u \in (f) : x_a$ the only option is that u is a constant multiple of variable, and $f_0 = 0$ giving a contradiction. Therefore $x_a | h$, but this also gives a contradiction because we would have that $\ell \in (g)$ which is false by assumption. Therefore, $h = 0$ implying $\ell = 0$.

- If $a \notin A$ and $b = c$, then by symmetry one can argue as in the previous case.

- If $a \in A$ or $b \in A$ then

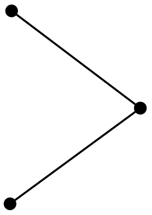
$$C = (J_A + (f_0)) :_S x_c = (J_A + I) :_S x_c$$

and the property is verified.

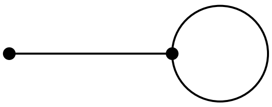
The opposite implication can be proved in the same manner by studying each case separately. □

Definition 2.28. Given a quadratic form f, we define the graph $\mathcal{G}_f = ([n], \mathcal{E}_f)$ as follows: $\{i, j\} \in \mathcal{E}_f$ if and only if $x_i x_j \in \sup(f)$.

Theorem 2.29. *The algebra $R = S/(f)$ is strongly Koszul with respect to the standard system of coordinates if and only if the graph $\mathcal{G}_f$ does not admit as induced subgraph one the following two connected graphs*



(1)

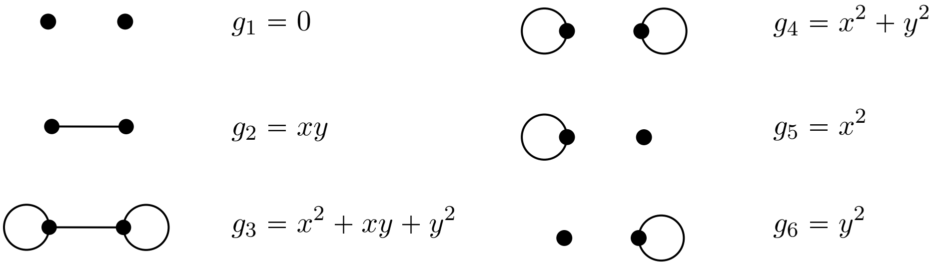


(2)

Proof. By iterating Lemma 2.27, we may assume that the coefficients of f are all 1.

Let $R = S/(f)$ be a SK algebra. Suppose by contradiction that $\mathcal{G}_f$ contains one of the above two subgraphs as an induced subgraph. It means that after quotienting R by the variables not involved in the subgraph, one obtains an algebra isomorphic to $\mathbf{k}[x_1, x_2, x_3]/(x_1(x_2 + x_3))$ in the first case, or to $\mathbf{k}[x_1, x_2]/(x_1(x_1 + x_2))$ in the second case. In both cases, the resulting algebra is not SK giving a contradiction by Remark 2.9.

On the other hand, suppose $\mathcal{G}_f$ does not contain the above sub-graphs. We want to prove that R is SK by strong induction on n the number of variables of the polynomial ring S. For $n = 2$ all possible graphs are listed below with the corresponding form. By explicit computations, one concludes that the corresponding algebras are all SK.



In the general case, we start by considering the colons $(f) :_S x_i$. We have that

$$(f) :_S x_i = \begin{cases} (f) & \text{if } x_i \nmid f \\ (\ell) & \text{if } f = \ell x_i \end{cases}$$

where ℓ is the sum of some variables. There are three possible cases for ℓ:

1. if ℓ is a variable, then we checked the property for this case;
2. if $\ell = x_j + x_k + \dots$ with $j \neq k$, and $j, k \neq i$, then $\mathcal{G}_f$ contains the left sub-graph giving a contradiction;
3. if $\ell = x_i + x_j + \dots$ with $i \neq j$, then $\mathcal{G}_f$ contains the right sub-graph giving again a contradiction.

In the other column ideals one has $((f) + J_A) :_S x_b = ((\tilde{f}) + J_A) :_S x_b$, where $\tilde{f}$ is obtained from f by evaluating the variables $\{x_i : i \in A\}$ in 0. Now $\mathcal{G}_{\tilde{f}}$ is obtained from $\mathcal{G}_f$ by restriction on $[n] \backslash A$. Therefore the algebra associated with $\tilde{f}$ satisfies the induction hypothesis and the thesis follows. □

Low number of variables

For $n = 2$ it is possible to explicitly compute all masks and verify that they all are realisable.

Theorem 2.30. *For $n = 2$, up to permutation, there are 8 masks which are all realisable. These correspond to all possible quadratic monomial ideals of $S = \mathbf{k}[x, y]$ together with $J_1 = (xy, ax^2 + by^2)$ as $a, b \in \mathbf{k} \backslash \{0\}$, and $J_2 = (\lambda_1 x^2 + \lambda_2 xy + \lambda_3 y^2)$ as $\lambda_1, \lambda_3 \in \mathbf{k} \backslash \{0\}$ and $\lambda_2 \in \mathbf{k}$.*

Proof. The proof proceeds by an elimination process by considering all possible values that a map $M \colon E \to 2^{[n]}$ could assume. Then using these information we compute all possible SK algebras $R = S/I$ each connected to the given mask.

We consider the case $M(\varnothing, 1) = \{1, 2\}$. This implies that $M(\{2\}, 1) = \{1, 2\}$. For the set $M(\varnothing, 2)$ there are only two possibilities.

- If $M(\varnothing, 2) = \{1, 2\}$, then automatically $M(\{1\}, 2) = \{1, 2\}$. In this case M is generated by atoms and the ideal $I = (x^2, xy, y^2)$ is the only ideal realising M.
- If $M(\varnothing, 2) = \{1\}$, then there are again two cases.

  - If $M(\{1\},2) = \{1,2\}$, then M is generated by atoms and there is only one ideal $I = (x^2, xy)$.
  - If $M(\{1\},2) = \{1\}$, then Theorem 2.21 implies that M cannot be a mask.

The other cases can be treated as the first case above. There are four possibilities: The map M is generated by atoms which implies a unique monomial ideal class; The map M is generated by a hypersurface which is treated by Theorem 2.29; For the ideal J_2 case one sees that x^2, y^2 can not lie in the generating ideal and the conditions on the generator $ax^2 + by^2$ are automatic from the conditions imposed by M. Finally, Theorem 2.21 implies that the remaining cases are not masks. We give a quick summarise of these algebras below, except for the trivial case $I = 0$.

- $M(\varnothing, 1) = \{1,2\}$
  - $M(\varnothing, 2) = \{1,2\} \implies I = (x^2, xy, y^2)$;
  - $M(\varnothing, 2) = \{1\} \implies I = (x^2, xy)$.
- $M(\varnothing, 1) = \{1\}$
  - $M(\varnothing, 2) = \{2\} \implies I = (x^2, y^2)$;
  - $M(\varnothing, 2) = \varnothing \implies I = (x^2)$.
- $M(\varnothing, 1) = \{2\}$
  - $M(\varnothing, 2) = \{1,2\} \implies I = (xy, y^2)$;
  - $M(\varnothing, 2) = \{1\} \implies \begin{cases} I = (xy) & \text{if } M(\{2\},1) = \{2\}, \\ I = J_1 & \text{if } M(\{2\},1) = \{1,2\}. \end{cases}$
- $M(\varnothing, 1) = \varnothing$
  - $M(\varnothing, 2) = \{2\} \implies I = (y^2)$;
  - $M(\varnothing, 2) = \varnothing \implies I = J_2$. ☐

Concerning the case $n = 3$, we can use CoCoA [1] to sample on Strongly Koszul algebras. We discover 120 possible different realisable masks. In particular, there are $64 = 2^6 = 2^{\binom{n+1}{2}}$ masks determined by atoms, and 56 non-monomial masks. By considering the action of $\mathcal{S}_3$ on these masks, we obtain 28 non-monomial masks. These computations have been done over the ring $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/101\mathbb{Z}$. We note that these computations are incomplete since we do not have the exact number of masks in the case $n = 3$ and further.

On the generating set

Let $R = S/I$ be a graded quotient of a polynomial ring $S = \mathbf{k}[x_1, \ldots, x_n]$ by a general homogeneous ideal I. We put ourselves in the following situation. Suppose that we don't know the ideal I, but we would like to understand a set of generators for I. Since we are only allow to search information inside the ring R, we look at the following colon ideals

$$(x_j : j \in [i]) :_R x_{i+1} = (x_j : j \in [i]) + (f_{i,k} : 1 \leqslant k \leqslant a_i) \subset R.$$

We shall suppose that the forms $f_{i,k}$ do not depend on the variables $x_1, \ldots, x_i$. We do not suppose that they are minimal set of generators. This information is equivalent to know the generators of

$$((x_j : j \in [i]) + I) :_S x_{i+1} = (x_j : j \in [i]) + I + (F_{i,k} : 1 \leqslant k \leqslant a_i) \subset S$$

for every $i = 0, \ldots, n-1$. Then, also the elements $F_{i,k}$ do not depend on $\{x_j : j \in [i]\}$. From this point, some elements of I are determined by the given information. In fact, we have that $G_{i,k} := F_{i,k}x_{i+1} - \sum_{j \in [i]} H_{i,k,j}x_j \in I$ for some suitable polynomials $H_{i,k,j} \in S$.

Theorem 2.31. *Consider the polynomials $F_{i,k}, H_{i,k,j}$, and $G_{i,k}$ as described above. Then $I = (G_{i,k} : 0 \leqslant i \leqslant n-1, 1 \leqslant k \leqslant a_i)$.*

Proof. Since every polynomial $G_{i,k}$ lies inside I, one containment is obvious.

Set $U := (G_{i,k} : 0 \leqslant i \leqslant n-1, 1 \leqslant k \leqslant a_i)$. Suppose by contradiction that $U \subsetneq I$. Let $<$ be the GRevLex term order on S such that $x_1 < x_2 < \cdots < x_n$. Consider the following set

$$\mathcal{A} := \{in_<(h) : h \in I \backslash U\}.$$

Since $\mathcal{A} \neq \varnothing$, by [96, Thm 1.4.19], there exists an element $g \in I \backslash U$ such that $in_<(g)$ is a minimum of $\mathcal{A}$. Without loss of generality we can choose g being homogeneous of degree $\deg g = d$. The monomial $in_<(g)$ can be written as $in_<(g) = mx_s$ where m involves only the variables $x_s, \ldots, x_n$. Then we can write $g = g_0 x_s + g_1$ with g_1 not involving the variable x_s. In particular, $in_<(g_0) = m$. Since mx_s is the leading term of g, we must have that $g_1 \in (x_1, \ldots, x_{s-1})$. This implies that $g_0 \in (I + (x_1, \ldots, x_{s-1})) :_S x_s$, but by the minimality of $in_<(g)$ and the construction of m we have that g_0 is not in I neither in $(x_1, \ldots, x_{s-1})$. Therefore there exists forms L_i and L'_j such that

$$H_g := g_0 - \sum_{i=1}^{s-1} L_i x_i - \sum_{j=1}^{a_{s-1}} L'_j F_{s-1,j} \in I.$$

Since $\deg H_g < \deg g$ we have that $in_< H_g < in_< g$ and therefore $H_g \in U \subset I$. Consider the element $g - H_g x_s \in I$. We want to prove that $g - H_g x_s \in U$ which gives a contradiction since $g \notin U$. Let's write down the expression of $g - H_g x_s$

$$\begin{aligned} g - H_g x_s &= g_0 x_s + g_1 - g_0 x_s + \sum_{i=1}^{s-1} L_i x_i x_s + \sum L'_j F_{s-1,j} x_s \\ &= g_1 + \sum_{i=1}^{s-1} L_i x_i x_s + \sum_{j=1}^{a_{s-1}} L'_j G_{s-1,j} x_s - \sum_{j=1}^{a_{s-1}} \sum_{t=1}^{s-1} L'_j H_{i,k,t} x_t \end{aligned}$$

Then $g - H_g x_s \in U$ if and only if $g_1 + \sum_{i=1}^{s-1} L_i x_i x_s - \sum_{j=1}^{a_{s-1}} \sum_{t=1}^{s-1} L'_j H_{i,k,t} x_t \in U$. But, this last element, which is in I, lies inside $(x_1, \ldots, x_{s-1})$. Therefore, its leading term is surely smaller than $in_<(g)$, implying that it must lie in U by the minimality of $in_<(g)$ in $\mathcal{A}$. The contradiction comes from assuming $\mathcal{A}$ is not empty, and the proof is completed. □

As shown in the following example, even if we suppose that $f_{i,k}$ is a minimal generating set inside the colon ideal, we can not prove that $\{G_{i,k} : 0 \leqslant i \leqslant n-1, 1 \leqslant k \leqslant a_i\}$ is a minimal set of generators for I.

Example 2.32. Let $I = (x^2 + y^2, xz) \subseteq S = \mathbf{k}[x, y, z]$, and set $R = S/I$. Then $0 :_R y = (yz)$, $(y) :_R x = (y, x, z)$, and $(y, x) :_R z = (y, x)$. In this case $a_0 = 1, a_1 = 2, a_2 = 0$, and we obtain the polynomials $G_{0,1} = y^2 z$, $G_{1,1} = x^2 + y^2$, $G_{1,2} = xz$. Surely $I = (G_{0,1}, G_{1,1}, G_{1,2})$, but they are not minimal since $G_{0,1} = zG_{1,1} - xG_{1,2}$. ♢

If R is strongly Koszul, then the elements $F_{i,k}$ can be taken as variables. Since $G_{i,k}$ are quadratic, linearly independent, in number exactly $\mu(I)$ by Theorem 2.21, and since I is quadratic by assumption, we obtain the following result.

Corollary 2.33. *If $R = S/I$ is strongly Koszul, then the ideal I is minimally generated by $\{G_{i,k} : 0 \leqslant i \leqslant n-1, 1 \leqslant k \leqslant a_i)\}$.*

Regarding Question 2.18, we consider the following construction. Let M be a realisable mask. The generators of algebras realising M can be found using Corollary 2.33: We know that these generators are of the form $G_{i,k} = X_k X_{i+1} - \sum_{j \in [i]} H_{i,j,k} X_j$ where the linear polynomials $H_{i,j,k}$ are unknown. Therefore we can replace every $H_{i,j,k}$ with polynomials of the type $a^1_{i,j,k} X_1 + \cdots + a^n_{i,j,k} X_n \in S[a^s_{i,j,k}]$.

At this point we want to find relations that these coefficients have to satisfy to ensure the restrictions given by M. That is, we want to find equations and inequalities in the variables $a^s_{i,j,k}$ such that the evaluation map $ev\colon S[a^s_{i,j,k}] \to S$ associated with a solution turns $S/ev(I)$ into a strongly Koszul algebra with mask M. This method is still work in progress, but we believe that with the right adjustments it could have good potentials.

This idea is better explained in the following example.

Example 2.34. Set $n = 3$, and $S = \mathbf{k}[x, y, z]$. Consider the following realisable mask

$$M(\varnothing, x) = \varnothing, M(\varnothing, y) = \varnothing, M(\varnothing, z) = \varnothing,$$
$$M(A, k) = [3] \text{ for } (A, k) \in E, \text{ and } A \neq \varnothing.$$

One possible algebra that realise it is given by $I = (y^2 - xz, yz - x^2, z^2 - xy)$. All algebras realising M have dimension 1 and Hilbert series $(1 + 2z)/(1 - z)$.

Step 1. We define the polynomials $G_{i,k}$, using polynomial parameters

$$G_1 = y^2 - (a_1x + a_2y + a_3z)x, \quad G_2 = yz - (b_1x + b_2y + b_3z)x, \quad G_3 = z^2 - (c_1x + c_2y + c_3z)x.$$

They lie inside $\bar{S} = S[a_i, b_i, c_i]$. Let $J = (G_1, G_2, G_3) \subset \bar{S}$. Given an evaluation map $ev : \bar{S} \to S$ that maps the parameters to coefficients in $\mathbf{k}$, we may consider the ideal $ev(J)$, and the quotient $R(J) := S/ev(J)$. We start by finding equations that the evaluation map must satisfy in such a way that $0 :_{R(J)} x = 0$, $(y) :_{R(J)} x = (x, y, z)$, $(x, y) :_{R(J)} z = (x, y, z)$.

First, we want to compute $J : x$ inside $\bar{S}$. This ideal is exactly the projection on the last coordinate of the syzygy module $\operatorname{syz}(G_1, G_2, G_3, x)$ (cf. [96, Lemma 3.2.13]). Since $\{G_1, G_2, G_3, x\}$ is a Gröbner basis, we get $J : x = J + (f_1, f_2)$ where

$$f_1 := (b_1x + b_2y + b_3z)y - (a_1x + a_2y + a_3z)z, \quad f_2 := (c_1x + c_2y + c_3z)y - (b_1x + b_2y + b_3z)z.$$

Therefore, $J : x = J$ if and only if $f_1, f_2 \in J$. Since all the polynomials are quadratic in the variables x, y, z we have that $J : x = J$ if and only if $\operatorname{rk}\mathcal{M} \leqslant 3$ where $\mathcal{M}$ is the matrix

$$\mathcal{M} = \begin{pmatrix} -a_1 & -a_2 & -a_3 & 1 & & \\ -b_1 & -b_2 & -b_3 & & 1 & \\ -c_1 & -c_2 & -c_3 & & & 1 \\ & b_1 & -a_1 & b_2 & b_3 - a_2 & -a_3 \\ & c_1 & -b_1 & c_2 & c_3 - b_2 & -b_3 \end{pmatrix}.$$

Instead, the conditions $(J + (x)) : y = (x, y, z)$ and $(J + (x, y)) : z = (x, y, z)$ are always satisfied since $J + (x) = (y^2, yz, z^2) + (x)$.

The affine variety describing the algebras satisfying $0 :_R (x) = 0, (x) :_R (y) = (x, y) :_R (z) = (x, y, z)$ is defined by the system of equations

$$\begin{cases} b_2b_3 - a_3c_2 + b_1 = 0 \\ b_2^2 - a_2c_2 + b_3c_2 - b_2c_3 - c_1 = 0 \\ b_1b_2 + b_3c_1 - a_1c_2 - b_1c_3 = 0 \\ a_3b_2 - a_2b_3 + b_3^2 - a_3c_3 - a_1 = 0 \\ a_2b_1 - a_1b_2 - b_1b_3 + a_3c_1 = 0 \end{cases} . \tag{2.2}$$

Step 2: Now we have to impose the other colon conditions. Using Macaulay2 [M2], one can compute these colons and obtain the following results.

$$\begin{aligned} J :_S y = J + (&f_1, yf_2, -b_2f_1 - b_3f_2 - z^2(a_3b_2 - a_2b_3 + b_3^2 - a_3c_3 - a_1) - yz(b_2b_3 - a_3c_2 + b_1) \\ &+ xz(a_2b_1 - a_1b_2 - b_1b_3 + a_3c_1)) \\ J :_S z = J + (&f_2, -b_2f_1 + xy(b_1b_2 + b_3c_1 - a_1c_2 - b_1c_3) + \\ &+ y^2(b_2^2 - a_2c_2 + b_3c_2 - b_2c_3 - c_1) + yz(b_2b_3 - a_3c_2 + b_1)) \end{aligned}$$

$$\begin{aligned}
(J+(y)) :_S x &= (y, xb_1 + zb_3, xa_1 + za_3, x^2c_1 + xzc_3 - z^2)\\
(J+(z)) :_S x &= (z, xc_1 + yc_2, xb_1 + yb_2, x^2a_1 + xya_2 - y^2)\\
(J+(y)) :_S z &= J + (y) + (xb_3c_1 - xb_1c_3 + zb_1, xa_3c_1 - xa_1c_3 + za_1, za_3b_1 - za_1b_3,\\
&\quad xa_3b_1 - xa_1b_3, xzb_1 + z^2b_3, xza_1 + z^2a_3)\\
(J+(z)) :_S y &= J + (z) + (yb_2c_1 - yb_1c_2, xb_2c_1 - xb_1c_2, xa_2c_1 - xa_1c_2 - yc_1,\\
&\quad xyc_1 + y^2c_2, xa_2b_1 - xa_1b_2 - yb_1, xyb_1 + y^2b_2)\\
(J+(x)) :_S z &= (x, y, z)\\
(J+(x,y)) :_S z &= (x, y, z)\\
(J+(x,z)) :_S y &= (x, y, z)
\end{aligned}$$

We can see that the only new conditions to obtain the desired algebra are:

$$\begin{cases} c_1b_2 - c_2b_1 \neq 0 \\ b_1a_3 - b_3a_1 \neq 0 \end{cases} . \tag{2.3}$$

These are not equalities because we need some elements to appear in the colon ideal. For example $(J+(z)) : x$ is generated by $\{z, xc_1+yc_2, xb_1+yb_2, x^2a_1+xya_2-y^2\}$. After evaluating the parameters, this ideal is equal to (x, y, z) if and only if $xc_1 + yc_2$ and $xb_1 + yb_2$ are linearly independent, that is $c_1b_2 - c_2b_1 \neq 0$.

Therefore

For example, another algebra with the same mask is generated by

$$y^2 - (3x + 4y + z)x, \quad yz - (x + 2z)x, \quad z^2 - (-2x + y - 7z)x. \qquad \diamondsuit$$

Chapter 3

Arithmetic complexes for stable sheaf cohomology

> *"A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies."*
>
> – Stefan Banach

A central problem connecting representation theory and Algebraic Geometry is the calculation of sheaf cohomology of line bundles on complete flag varieties over an algebraically closed field $\mathbf{k}$. Recall that for every natural number n, the flag variety Fl_n parametrises complete flags of a n-dimensional vector space V, where a complete flag of V is a filtration $0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V$ with $\dim_{\mathbf{k}} V_i = i$. In characteristic zero, the Borel–Weil–Bott theorem (Theorem 3.5) provides a complete description of these cohomology groups: Each line bundle admits at most one nonzero cohomology group, which itself realises an irreducible representation of the general linear group. In positive characteristic, however, the situation is markedly more subtle. Raicu and VandeBogert [116], for example, demonstrated that the number of irreducible constituents in a filtration of such cohomology groups cannot be bounded by any polynomial in the cohomological degree, the dimension of the flag variety, and the combinatorial data defining the line bundle. Despite these complications, several significant results are known in positive characteristic: Kempf's vanishing theorem (Theorem 3.6), Andersen's characterisation of the nonvanishing of the first cohomology group [9], and Donkin's complete classification in dimension three [44].

The central developments most relevant to this chapter are contained in a recent paper of Raicu and VandeBogert [116]. They used an established connection between the cohomology of line bundles on flag varieties and the groups $H^j(\mathbb{P}(V), \mathbb{S}_{\lambda\backslash\mu}\Omega)$, where $\mathbb{S}_{\lambda\backslash\mu}$ is a skew-Schur functor and Ω is the cotangent sheaf of the projective space $\mathbb{P}(V)$. Remarkably, they show that these groups carry a trivial action of the general linear group that depend only on the combinatorial data of $\lambda\backslash\mu$,and not on the ambient dimension $n = \dim_{\mathbf{k}} V$, once n is sufficiently large. They denote these dimensionally stable groups by $H^j_{st}(\mathbb{S}_{\lambda\backslash\mu}\Omega)$, referring to them as the *stable sheaf cohomology groups.* Furthermore, for certain skew-partitions they construct explicit *arithmetic complexes* $C^{\mathbf{k}}_{\bullet}(\underline{w})$ which computes the stable cohomology and reveal nontrivial symmetric relations.

The purpose of this chapter is to extend the framework of Raicu and VandeBogert to projective space over the integers. In the first section, we review the background on line bundles over flag varieties, present in detail the results of Raicu and VandeBogert, and introduce a generalization of their arithmetic complexes. The second section is devoted to Theorem 3.19, which constitutes the technical heart of the chapter: Here we establish a uniform identification between two arithmetic complexes arising from distinct data. The proof is intricate and requires a sequence of preparatory lemmata to make the argument transparent. This identification leads to the third and final section, where, employing the methods of Raicu and VandeBogert, we extend their results to the case of projective space defined over the integers.

The results presented in this chapter were published in [55] in collaboration with Ethan Reed, Shahriyar Roshan Zamir, and Hongmiao Yu. The project started at the research school *PRAGMATIC 2023 - Cohomology and Frobenius.*

3.1 Preliminaries

In this and following sections, $\mathbf{k}$ will be an algebraically closed field of characteristic $\operatorname{char}(\mathbf{k}) = p$, where p is either zero or a prime number.

Partitions and Schur functors

A partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is an element of $\mathbb{N}^n$ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_n$. We define the size of λ as $|\lambda| = \sum_i \lambda_i$. Then λ is an integer partition of the number $|\lambda|$, i.e. a possible way to write $|\lambda|$ as sum of positive numbers.

A nice way to visualise a partition is through its Young diagram. A Young diagram is a finite collection of boxes, disposed on a grid, align to the left, and with row-lengths in a non-increasing order. Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, its Young diagram consists of $|\lambda|$ boxes arranged in n rows: The i-th row consists of λ_i boxes.

Some special types of partitions, that will recur frequently in this chapter, are

- Hook partitions. They are of type $\lambda = (m, 1^d)$ where 1^d means that the number 1 is repeated d times. They take their name from the shape of their Young diagrams which resemble a hook.
- Two-column partitions. These partitions have a Young diagram consisting of two columns. They have the form $\lambda = (2^m, 1^{d-m})$ for $d \geqslant m \geqslant 1$. Its conjugate partition is $\lambda^\vee = (d, m)$. We recall from Definition 1.20 that the conjugate partition is defined as

$$\lambda^\vee = (\lambda_0^\vee, \dots, \lambda_r^\vee), \text{ where } \lambda_i^\vee = \#\{k : \lambda_k \geqslant i\}.$$

One of the main tools in studying cohomology groups of line bundles on flag varieties are Schur functors. These functors, each depending on a partition, interpolate between the exterior algebra functor and the symmetric algebra functor in an attempt to generalise them. In the theory of vector spaces over fields of characteristic zero, it is standard to define Schur functors using the Young symmetrisers. But, in our setting where the field has any characteristic this approach is not suitable and a different equivalent definition is needed. This uses the notions of exterior algebra and symmetric algebra. Let be V be a $\mathbf{k}$-vector space. For every $k \geqslant 0$, the antisystematization map $\Delta_k : \bigwedge^k V \hookrightarrow \otimes_k V$ is defined as

$$\Delta_k(f_1 \wedge \cdots \wedge f_k) = \sum_\sigma \operatorname{sgn}(\sigma) f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(k)}.$$

Let $\lambda \in \mathbb{N}^r$ be a partition with conjugate partition $\lambda^\vee = (\lambda_1^\vee, \dots, \lambda_t^\vee)$. Two natural maps are defined:

$$\alpha_\lambda \colon \bigwedge^{\lambda_1} V \otimes \cdots \otimes \bigwedge^{\lambda_r} V \longrightarrow \bigotimes_{|\lambda|} V$$

which is the tensor product of the antisystematization maps $\Delta_{\lambda_i} : \bigwedge^{\lambda_i} V \hookrightarrow \otimes_{\lambda_i} V$, and

$$\beta_\lambda \colon \bigotimes_{|\lambda|} V \longrightarrow \operatorname{Sym}^{\lambda_1^\vee} V \otimes \cdots \otimes \operatorname{Sym}^{\lambda_t^\vee} V$$

defined as

$$\beta_\lambda(f_1 \otimes \cdots \otimes f_{|\lambda|}) = f_1 f_{\lambda_1+1} f_{\lambda_1+\lambda_2+1} \cdots f_{\lambda_1+\cdots+\lambda_{r-1}+1} \otimes f_2 f_{\lambda_1+2} f_{\lambda_1+\lambda_2+2} \cdots \otimes \cdots .$$

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$ be a partition. The Schur functor associated with the partition λ is an endo-functor $\mathbb{S}_\lambda \colon \mathfrak{Vec}_{\mathbf{k}} \longrightarrow \mathfrak{Vec}_{\mathbf{k}}$, where $\mathfrak{Vec}_{\mathbf{k}}$ is the category of vector spaces over the field $\mathbf{k}$. For a $\mathbf{k}$-vector space V, it is defined as $\mathbb{S}_\lambda V = \operatorname{Im} \beta_\lambda \circ \alpha_\lambda$. Given a linear map $\varphi : V \to W$, the naturality of the tensor product, and the exterior and symmetric powers yields the commutative diagram

$$\begin{tikzcd}
\bigwedge^{\lambda_1} V \otimes \cdots \otimes \bigwedge^{\lambda_r} V \arrow[r, "\alpha_\lambda"] \arrow[d] & \bigotimes_{|\lambda|} V \arrow[r, "\beta_\lambda"] \arrow[d] & \operatorname{Sym}^{\lambda_1^\vee} V \otimes \cdots \otimes \operatorname{Sym}^{\lambda_t^\vee} V \arrow[d] \\
\bigwedge^{\lambda_1} W \otimes \cdots \otimes \bigwedge^{\lambda_r} W \arrow[r, "\alpha_\lambda"] & \bigotimes_{|\lambda|} W \arrow[r, "\beta_\lambda"] & \operatorname{Sym}^{\lambda_1^\vee} W \otimes \cdots \otimes \operatorname{Sym}^{\lambda_t^\vee} W
\end{tikzcd} .$$

The right-most vertical map restricts to a linear map $\mathbb{S}_\lambda\varphi \colon \mathbb{S}_\lambda V \to \mathbb{S}_\lambda W$.

Example 3.1. We see some easy examples of Schur functors.

- For $\lambda = (m, 0, \dots, 0)$, it follows $\mathbb{S}_\lambda = \operatorname{Sym}^m$.
- For $\lambda = (1^m, 0, \dots, 0)$, it follows $\mathbb{S}_\lambda = \bigwedge^m$. Here 1^m means that the integer 1 is repeated m times.
- Set $\lambda = (2, 1)$. Then $\mathbb{S}_\lambda(V)$ is the subspace of $\operatorname{Sym}^2 V \otimes V$ generated by

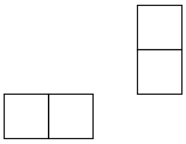
$$v_1 v_2 \otimes v_3 - v_2 v_3 \otimes v_1, \text{ with } v_1, v_2, v_3 \in V. \qquad \diamondsuit$$

Theorem 3.2. *If* $\operatorname{char}(\mathbf{k}) = 0$*, then for a given vector space* V *of dimension* n*, the set* $\{\mathbb{S}_\lambda V : \lambda_1 \leqslant n\}$ *gives a complete set of distinct irreducible polynomial representations of the general linear group* $\mathrm{GL}(V)$*.*

Proof. See [58, Section 8.2], in particular to Theorem 2. □

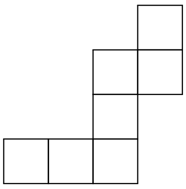
Remark 3.3. If $\operatorname{char}(\mathbf{k}) > 0$, then $\mathrm{GL}(V)$ is not longer linearly reductive. Moreover, the Schur functors $\mathbb{S}_\lambda V$ are indecomposable ([38, Thm 3.8]), but not irreducible in general. For example in characteristic p the Schur module $\mathbb{S}_{p,0}(\langle e_1, e_2 \rangle) = \operatorname{Sym}^p(\langle e_1, e_2 \rangle)$, the submodule generated by $\{e_1^p, e_2^p\}$ is proper and invariant under the action of the general linear group. A class of Schur functors that are irreducible in every characteristic is exterior algebras ([134, Ex. 2.7 (a)]). △

The concept of Schur functors can be generalised also to skew-partitions. A skew-partition $\lambda \backslash \mu$ is given by two partitions $\lambda = (\lambda_1, \dots, \lambda_n), \mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$ such that $\lambda_i \geqslant \mu_i$ for all $i = 1, \dots, n$. Every partition is also a skew-partition with $\mu = 0$. The best way to illustrate skew-partitions is by using Young diagrams: $\lambda \backslash \mu$ is obtained by removing the Young diagram of μ in the Young diagram of λ. For example, let $\lambda = (4, 4, 2)$ and $\mu = (3, 3, 0)$ then the skew-partition $\lambda \backslash \mu$ is



A skew-partition $\lambda \backslash \mu$ is said to be a ribbon if the resulting Young diagram is connected and it does not contain any 2×2 box. The connected condition is imposed by saying that $\mu_i < \lambda_{i+1}$ whenever $\lambda_{i+1} > 0$. The only ribbons that are also partitions are hook partitions.

The size of a ribbon is defined as $|\lambda \backslash \mu| = \sum(\lambda_i - \mu_i)$. It will often be convenient to encode the data of a ribbon of size m as a partition $w_1 + \cdots + w_d = m$ where w_i is the number of boxes appearing in the ith column. This is very useful since every ribbon of size m corresponds to a unique partition of m. The following figure is an example of a ribbon shape where $\mu = (3,2,2)$, $\lambda = (4,4,3,3)$ and $\underline{w} = (1,1,3,2)$.



Let $\lambda\backslash\mu$ be a skew-partition, and let $\lambda^\vee$, respectively $\mu^\vee$, be the conjugate partition of λ, respectively μ. As in the standard case we need to define two maps whose composition will define the skew-Schur functor. We consider the *Ferres matrix* associated with $\lambda\backslash\mu$ that is a square $\lambda_1 \times \lambda_1$-matrix $(a_{i,j})$ defined as follows. We set $a_{i,j} = 1$ for $\mu_i + 1 \leqslant j \leqslant \lambda_i$, and $a_{i,j} = 0$ for $1 \leqslant j \leqslant \mu_i$ or $\lambda_{i+1} \leqslant j \leqslant \lambda_1$. In other words, $a_{i,j}$ is 1 if in the Young diagram of $\lambda\backslash\mu$ there is a box in position (i,j), and 0 otherwise.

The map $\alpha_{\lambda\backslash\mu}\colon \bigwedge^{\lambda_1-\mu_1} V \otimes \cdots \otimes \bigwedge^{\lambda_n-\mu_n} V \longrightarrow \bigotimes_{a_{i,j}} V$ is defined as the tensor product of the antisystematization maps

$$\Delta_i\colon \bigwedge^{\lambda_i-\mu_i} V \longrightarrow \bigotimes_{a_{i,j}=1} V \longrightarrow \bigotimes_{a_{i,j}} V, \text{ with } i = 1,\dots,\lambda_1$$

The map $\beta_{\lambda\backslash\mu}\colon \bigotimes_{a_{i,j}} V \longrightarrow \mathrm{Sym}^{\lambda_1^\vee-\mu_1^\vee} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_n^\vee-\mu_n^\vee} V$ is defined as the tensor product of the multiplication maps

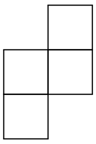
$$m_j\colon \bigotimes_{a_{i,j}} V \longrightarrow \bigotimes_{a_{i,j}=1} V \longrightarrow \mathrm{Sym}^{\lambda_j^\vee-\mu_j^\vee} V, \text{ with } j = 1,\dots,\lambda_1.$$

Eventually, the skew-Schur functor $\mathbb{S}_{\lambda\backslash\mu}\colon \mathfrak{Vec}_{\mathbf{k}} \longrightarrow \mathfrak{Vec}_{\mathbf{k}}$ associated with the skew-partition $\lambda\backslash\mu$ is defined similarly as in the standard case. For a vector space V we define $\mathbb{S}_{\lambda\backslash\mu}V := \operatorname{Im}\beta_{\lambda\backslash\mu}\circ\alpha_{\lambda\backslash\mu}$, while for a linear map $\varphi\colon V \to W$ we obtain a commutative diagram which defines $\mathbb{S}_{\lambda\backslash\mu}\varphi\colon \mathbb{S}_{\lambda\backslash\mu}V \to \mathbb{S}_{\lambda\backslash\mu}W$. Whenever $\mu = 0$, it follows the identity $\mathbb{S}_{\lambda\backslash\mu} = \mathbb{S}_\lambda$.

Example 3.4. Let $\lambda = (2,2,1), \mu = (1,0,0) \in \mathbb{N}^3$ be two partitions. Then for every vector space V, the skew-Schur functor $\mathbb{S}_{\lambda\backslash\mu}V$ is the subspace of $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 V$ generated by

$$v_2v_4 \otimes v_1v_3 - v_3v_4 \otimes v_1v_2, \text{ with } v_1,\dots,v_4 \in V.$$

This computation can be visualised using the Young diagram of $\lambda\backslash\mu$



and the following diagram

$$\begin{tikzcd} \begin{matrix} V \\ \otimes \\ \bigwedge^2 V \\ \otimes \\ V \end{matrix} \arrow[r, "\alpha_{\lambda\backslash\mu}"] & \begin{matrix} \phantom{V\otimes{}} V \\ \phantom{V\otimes{}} \otimes \\ V \otimes V \\ \otimes \phantom{{}\otimes V} \\ V \phantom{{}\otimes V} \end{matrix} \arrow[d, "\beta_{\lambda\backslash\mu}"] \\ & \mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 V \end{tikzcd}$$

The tensor product $V^{\otimes 4} \subset V^{\otimes 6}$ that corresponds to indices where $a_{i,j} = 1$, is aligned as the boxes in the skew-shape $\lambda\backslash\mu$. The map $\alpha_{\lambda\backslash\mu}$ pointing from left to right shows how the antisystematization maps are arranged. While, the map $\alpha_{\lambda\backslash\mu}$ pointing from top to bottom shows which elements need to be multiplied. ♢

As a final note, since sheaves over projective varieties are locally vector spaces over the field **k**, one can extend the definition of Schur functors to the category of quasi-coherent sheaves over some projective scheme/variety. More information on Schur functors can be found in [6], [25] and [134].

3.1.1 Line bundles over flag varieties and their cohomology

Let V be a **k**-vector space of dimension n. The (complete) flag variety Fl_n is the set of complete flags in V, that is

$$\mathrm{Fl}_n := \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V : \dim V_i = i\}.$$

It is trivial that $\mathrm{Fl}_n \subset \mathrm{Gr}(1, V) \times \cdots \times \mathrm{Gr}(n-1, V)$ where $\mathrm{Gr}(j, V)$ is the Grassmannian variety of j-dimensional vector subspaces of V. Using the Plücker embedding $\mathrm{Gr}(k, V) \hookrightarrow \mathbb{P}(\bigwedge^k V)$, Fl_n injects into a multi-projective space making it a multi-projective varieties. The subject of study are line bundles on a fixed flag variety Fl_n. The Picard group $\mathrm{Pic}(\mathrm{Fl}_n)$ is the set of line bundles on Fl_n up to isomorphisms of vector bundles together with the tensor product operation. To compute this group, we need to use some tools from Representation Theory.

The general linear group $\mathrm{GL}(V)$ acts on the flag variety by permuting its elements. In particular, every complete flag can be obtained as the image of the action of some element in $\mathrm{GL}(V)$ to the standard flag

$$0 \subsetneq \langle e_1\rangle_{\mathbf{k}} \subsetneq \langle e_1, e_2\rangle_{\mathbf{k}} \subsetneq \cdots \subsetneq \langle e_1, \ldots, e_{n-1}\rangle_{\mathbf{k}} \subsetneq V,$$

where $\mathcal{B} := \{e_1, \ldots, e_n\}$ is a fixed basis of V. Therefore, one obtains an isomorphism $\mathrm{Fl}_n \cong \mathrm{GL}(V)/B$. Here B is the Borel sub-group of the upper triangular matrices by fixing the basis $\mathcal{B}$ on V. It represents the elements that act trivially on the standard flag. Using Representation Theory arguments we obtain a complete description of the Picard group of Fl_n as

$$\mathrm{Pic}(\mathrm{Fl}_n) = \frac{\mathbb{Z}^n}{(1, \ldots, 1)\mathbb{Z}}.$$

This implies that there exist $\mathcal{L}_1, \ldots, \mathcal{L}_n$ line bundles on Fl_n such that every line bundle on Fl_n, up to isomoprhism, is parametrised by $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ and it can be written as

$$\mathcal{O}(\lambda) := \mathcal{L}_1^{\lambda_1} \otimes \cdots \otimes \mathcal{L}_n^{\lambda_n}.$$

Moreover such special line bundles satisfy the relation $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n = \mathcal{O}_{\mathrm{Fl}_n}$, where $\mathcal{O}_{\mathrm{Fl}_n}$ is the structure sheaf of the variety Fl_n. The $\mathcal{L}_i$'s are also called the tautological line bundles of Fl_n and they can be explicitly constructed. Consider the tautological quotient bundle $\mathcal{Q}_i$ with fiber V/V_{n-i} at the point $0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V \in \mathrm{Fl}_n$. Then we get $\mathcal{L}_i = \ker(\mathcal{Q}_i \twoheadrightarrow \mathcal{Q}_{i-1})$. The interested reader can fill in the gaps by reading [58, Sec. 9.1].

A central problem connecting Representation Theory and Algebraic Geometry is the calculation of sheaf cohomology of these line bundles. We recall that the sheaf cohomology is defined as follows: Let $(X, \mathcal{O}_X)$ be a ringed space, then one considers the global sections functor from $\mathfrak{Mod}_X$ the category of sheaves of $\mathcal{O}_X$-modules to $\mathfrak{Ab}$ the category of Abelian groups

$$\Gamma(X, -) \colon \mathfrak{Mod}_X \to \mathfrak{Ab}.$$

It is well-known that this functor is left-exact, but not right exact in general. So, it natural to consider its right derived functors $H^i(X, -)$. Then we define the sheaf cohomology modules of a sheaf $\mathcal{F}$ being the Abelian groups $H^i(X, \mathcal{F})$. From its very definition we have that $H^i(X, -) = 0$ for $i < 0$, and $H^0(X, -) = \Gamma(X, -)$. A famous result of Grothendieck states that if X is a Noetherian topological space of dimension n, then $H^i(X,-) = 0$ for $i > n$. For more and complete information on this topic we refer to [74, Chapter III].

In the characteristic zero case, the Borel-Weil-Bott theorem computes the sheaf cohomology groups and shows that line bundles have at most one non-zero cohomology group which is an irreducible representation of the general linear group. We recall the theorem below.

Theorem 3.5 (Borel-Weil-Bott). *If* $\mathrm{char}(\mathbf{k}) = 0$*, then the following holds true.*

- *There exists at most one index i such that* $H^i(\mathrm{Fl}_n, \mathcal{O}(\lambda)) \neq 0$.
- *When* $H^i(\mathrm{Fl}_n, \mathcal{O}(\lambda)) \neq 0$*, then it is an irreducible representation of* SL_n.
- *If* $\lambda_t - t = \lambda_s - s$ *for some* $t \neq s$*, then* $H^i(\mathrm{Fl}_n, \mathcal{O}(\lambda)) = 0$ *for every* i.

Proof. See [19] and [20]. □

In positive characteristic the sheaf cohomology groups can be much more complicated. For instance, Raicu and VandeBogert [116] showed that the number of irreducible factors in a filtration of the cohomology groups cannot be bounded by a polynomial in terms of the cohomological degree, dimension of the flag variety, and the combinatorial data defining the line bundle. Nonetheless, some general results have been proved. One of the first works in a positive characteristic setting is Griffith's Ph.D. thesis (cf. [69]) where he studied Fl_3 constructing families of line bundles having more than one non-vanishing cohomological group. Andersen in [9] gave a characterisation of non-vanishing of the first cohomology group, while Donkin in [44] presented a full description of the cohomology groups when the underlying vector space is 3-dimensional completing Griffith's work. The most notable result in a characteristic free settings is the Kempf vanishing theorem which states as follows.

Theorem 3.6 (Kempf vanishing theorem). *Given* $\lambda \in \mathbb{Z}^n$ *such that* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$*, we have* $H^i(\mathrm{Fl}_n, \mathcal{O}(\lambda)) = 0$ *for* $i > 0$*, and* $H^0(\mathrm{Fl}_n, \mathcal{O}(\lambda)) = \mathbb{S}_\lambda \mathbf{k}^n$*, where* $\mathbb{S}_\lambda$ *is a Schur functor.*

Proof. The first instance was proved by Kempf in [91], [92], and [93]. The final version of this theorem is due to Haboush in [70], and by Andersen in [10] where they improved and simplified Kempf results and arguments. □

The principal paper that we are now discussing, and also the one that inspired the work of this chapter, is due to Raicu and VandeBogert [116]. The idea behind their paper is one can study certain line bundles by instead studying Schur functors applied to the cotangent sheaf Ω of $\mathbb{P}(V)$. The coherent sheaf Ω is classically defined as the sheafification of $\bigoplus_{i=1}^n S(-1)dx_i / \sum x_i dx_i$, where $S = \mathrm{Sym}\, V = \mathbf{k}[x_1, \ldots, x_n]$.

Theorem 3.7. *Let* Ω *be the cotangent sheaf of the projective space* $\mathbb{P}^{n-1}$*. Let* $\lambda \in \mathbb{N}^n$ *be a partition with* $\lambda_n = 0$*. Then for every* $j \geqslant 0$ *we have*

$$H^j(\mathrm{Fl}_n, \mathcal{O}(\mu)) = H^j(\mathbb{P}^{n-1}, \mathbb{S}_\lambda \Omega(e)),$$

where $\mu = (e - |\lambda|, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \in \mathbb{Z}^n$*, and* $e \in \mathbb{Z}$.

Proof. This is a direct consequence of [25, Thm 9.8.5]. □

To give an idea of why this result is very important, we now give an example giving explicit computations. This example is claimed by many authors to be due to Mumford or his student Griffith. Despite, no proper references are given, it is undeniable that the two authors were very interested in the topic and Griffith's Ph.D. thesis [68] (cf. [69]) considered the case of flags in 3-dimensional vector spaces over fields of positive characteristic. Moreover, it shows that the zero and nonzero characteristic cases are much different and the Borel-Weil-Bott Theorem is false in this context.

Example 3.8. Let $n = 3$, $p = 2$, together with $\lambda = (2, 0, 0)$ and $\mu = (-2, 2, 0)$. Then one has

$$H^j(Fl_3, \mathcal{O}(\mu)) = H^j(\mathbb{P}^2, \mathbb{S}_\lambda \Omega) = H^j(\mathbb{P}^2, \operatorname{Sym}^2 \Omega).$$

Since, we are in characteristic 2, it holds the short exact sequence

$$0 \to F^2\Omega \to \operatorname{Sym}^2 \Omega \to \wedge^2 \Omega \to 0, \tag{3.1}$$

where F^2 is the 2-nd Frobenius power functor. The cohomology groups of $F^2\Omega$ and $\wedge^2 \Omega$ are known in this case and their are equal to

$$H^j(\mathbb{P}^2, F^2\Omega) = \begin{cases} \mathbf{k} & \text{if } j = 1 \\ 0 & \text{if } j \neq 1 \end{cases}, \qquad H^j(\mathbb{P}^2, \wedge^2 \Omega) = \begin{cases} \mathbf{k} & \text{if } j = 2 \\ 0 & \text{if } j \neq 2 \end{cases}.$$

Equation (3.1) induces a long exact sequence on the cohomology groups obtaining the final result

$$H^j(Fl_3, \mathcal{O}(\mu)) = \begin{cases} \mathbf{k} & \text{if } j = 1, 2 \\ 0 & \text{otherwise} \end{cases}. \qquad \diamondsuit$$

In their paper, Raicu and VandeBogert consider the idea of "stabilizing" the cohomology groups by letting the dimension of the vector space V tend to infinity. To make this coherent, every time a partition $\lambda \in \mathbb{N}^r$ is considered, the same partition can be viewed inside $\mathbb{N}^n$ for $n \geqslant r$ by adding as many 0s as needed at the partition's end. The same concept can be also applied to skew-partitions. With this setting they proved the following.

Theorem 3.9. *Let* $\mathbf{k}$ *be an algebraically closed field. Consider a skew-partition* $\lambda \backslash \mu$. *Then, for every* $j \geqslant 0$, *the cohomology group* $H^j(\mathbb{P}(\mathbf{k}^n), \mathbb{S}_{\lambda \backslash \mu}\Omega)$ *have a trivial* $GL_{n-1}(\mathbf{k})$*-action, and is independent on* n *as* $n \gg 0$. *We denote this group as* $H^j_{st}(\mathbb{S}_{\lambda \backslash \mu}\Omega)$.

Proof. See [116, Theorem 4.1 and Theorem 4.2]. In these theorems, the authors prove the result for general polynomial functors, but for our purposes we are only interested to their applications to the Schur functors. □

In the particular case that the skew-partition is a ribbon or a two-column partition, Raicu and VandeBogert constructed explicit "arithmetic complexes" $C^{\mathbf{k}}_\bullet(\underline{w})$, depending on a tuple $\underline{w}$, to compute these cohomology groups. Moreover, they are able to find a relation formula between two-column partitions and hook partitions. We postpone the definition of arithmetic complexes in Section 3.1.2: There we generalise Raicu and Keller definition of arithmetic complexes to prove results over the projective on $\mathbb{Z}$.

Theorem 3.10. *Let* Ω *be the cotangent sheaf over* $\mathbb{P}^n(\mathbf{k})$ *and* $\mathbb{S}_{\lambda/\mu}$ *be a skew Schur functor. The following is true for all* i.

i) If $\mathbb{S}_{\lambda/\mu}$ *is a ribbon Schur functor corresponding to* $\underline{w} = (w_0, ..., w_d)$, *then*

$$H^i_{st}(\mathbb{S}_{\lambda/\mu}\Omega) = H_{|\underline{w}|-i}(C^{\mathbf{k}}_\bullet(\underline{w})).$$

ii) If λ *is a two column partition with conjugate partition* $\lambda^\vee = (m, d)$ *for integers* $m, d \geqslant 0$, *then*

$$H^i_{st}(\mathbb{S}_\lambda\Omega) = H_{d+m-i}(\overset{\smile}{C}{}^{\mathbf{k}}_\bullet(-m-d-1, 1^d)[-d]),$$

where $\overset{\smile}{C}{}^{\mathbf{k}}_\bullet(-m-d-1, 1^d)$ *denotes the dual of the complex* $C^{\mathbf{k}}_\bullet(-m-d-1, 1^d)$.

iii) Consider λ *as in ii) and let* $d \geqslant 1$, *then*

$$H^i_{st}(\mathbb{S}_\lambda\Omega) = H^{2m+1-i}_{st}(\mathbb{S}_{(d+1, 1^{m-d})}\Omega).$$

3.1.2 Arithmetic Complexes

Let R be the ring of integer-valued polynomials, that is

$$R = \{f(x) \in \mathbb{Q}[x] : f(n) \in \mathbb{Z} \text{ for every } n \in \mathbb{Z}\} = \{f(x) \in \mathbb{Q}[x] : f(n) \in \mathbb{Z} \text{ for every } n \gg 0\}.$$

For every polynomial $p \in R$ and $n \in \mathbb{Z}_{>0}$, we define the (generalised) binomial coefficient

$$\begin{bmatrix} p \\ n \end{bmatrix} := \frac{p(p-1)\cdots(p-n+1)}{n!} \in R.$$

We recall the following equalities which hold for binomial coefficients:

1. If $k \in \mathbb{N}$ then $\left[\begin{smallmatrix} k \\ n \end{smallmatrix}\right] = \binom{k}{n}$. In particular, if $k < n$, then $\binom{k}{n} = 0$.
2. If $k \in \mathbb{Z}_{<0}$, then $\left[\begin{smallmatrix} k \\ n \end{smallmatrix}\right] = (-1)^n \binom{n-k-1}{n}$. More in general, $\left[\begin{smallmatrix} p \\ n \end{smallmatrix}\right] = (-1)^n \left[\begin{smallmatrix} n-p-1 \\ n \end{smallmatrix}\right]$ for any $p \in R$.

We also adopt the conventions $\left[\begin{smallmatrix} p \\ 0 \end{smallmatrix}\right] = 1$ and $\left[\begin{smallmatrix} p \\ n \end{smallmatrix}\right] = 0$ for $p \in R$ and $n \in \mathbb{Z}_{<0}$.

Proposition 3.11. *The ring R is a $\mathbb{Z}$-free sub-algebra of $\mathbb{Q}[x]$. Moreover, The binomial coefficients $\left[\begin{smallmatrix} x \\ n \end{smallmatrix}\right]$ for $n \geqslant 0$ form a $\mathbb{Z}$-module basis for R.*

The ring R is used for example in the study of Hilbert polynomials. We should remark that R is not Noetherian since the ascending chain of ideals

$$\begin{bmatrix} x \\ 1 \end{bmatrix} R \subseteq \begin{bmatrix} x \\ 1 \end{bmatrix} R + \begin{bmatrix} x \\ 2 \end{bmatrix} R \subseteq \begin{bmatrix} x \\ 1 \end{bmatrix} R + \begin{bmatrix} x \\ 2 \end{bmatrix} R + \begin{bmatrix} x \\ 3 \end{bmatrix} R \subseteq \cdots$$

does not stabilises. Information about this topic and more general constructions can be found in [29].

In this chapter, our main objects of study are $C_\bullet(\underline{w})$ with $\underline{w} = (w_0, \ldots, w_d) \in R \times \mathbb{N}^{d-1}$. For a combinatorial description, consider the undirected, weighted path graph of $d+1$ vertices and d edges:

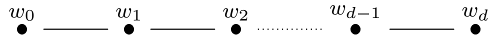


Figure 3.1: Undirected Weighted Path Graph with Vertex Weights $\underline{w}$

where w_i's denote weights on the vertices. Given $d, t \in \mathbb{N}$ such that $1 \leqslant t \leqslant d+1$ for any weak composition of $d-t+1$, that is non-negative integers $(\lambda_t, \ldots, \lambda_1)$ such that $\sum_{i=1}^{t} \lambda_i = d-t+1$, there is a unique decomposition of the above path graph into t disjoint intervals I_{λ_i} of length λ_i. If $\lambda_i = 0$ then I_{λ_i} consists of a single vertex.

The *weight of an interval* I_λ, denoted $\omega(I_\lambda)$, is the sum of the weights of its vertices. For instance, the weights of the intervals in Figure 3.2 are $\omega(I_{\lambda_4}) = w_0$, $\omega(I_{\lambda_3}) = w_1 + w_2 + w_3$, $\omega(I_{\lambda_2}) = w_4$, and $\omega(I_{\lambda_1}) = w_5 + w_6$.

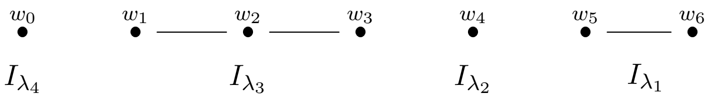


Figure 3.2: A decomposition for $d = 6$, $t = 4$, and $(\lambda_4, \lambda_3, \lambda_2, \lambda_1) = (0, 2, 0, 1)$.

Definition 3.12. Define the arithmetic complex, $C_\bullet(\underline{w}) = C_\bullet(w_0, \ldots, w_d)$, as follows: for each $k \in \mathbb{Z}$,

$$C_k(\underline{w}) := \bigoplus_{\sum_{i=1}^{t} \lambda_i = k} R \cdot f_{(\lambda_t, \ldots, \lambda_1)} \cong R^{\oplus \binom{d}{k}},$$

where $t = d-k+1 \geqslant 1$, and $f_{(\lambda_t, \ldots, \lambda_1)}$ corresponds to a basis element of $C_k(\underline{w})$. Note the basis elements of $C_k(\underline{w})$ are in bijective correspondence with decompositions of the path graph in Figure 3.1. The differential ∂_k is computed by removing a single edge from an interval of a basis element of $C_k(\underline{w})$. More precisely,

$$\begin{array}{rcl} \partial_k : C_k(\underline{w}) & \longrightarrow & C_{k-1}(\underline{w}) \\ f_{(\lambda_t,\ldots,\lambda_1)} & \mapsto & \displaystyle\sum_{j=1}^{t}\sum_{i=1}^{\lambda_j}(-1)^{t-j}\begin{bmatrix}\omega(I_{\lambda_j}) \\ \omega(I_{\lambda_j}\backslash\{i\})\end{bmatrix} f_{(\lambda_t,\ldots,\lambda_{j+1},i-1,\lambda_j-i,\lambda_{j-1},\ldots,\lambda_1)}, \end{array}$$

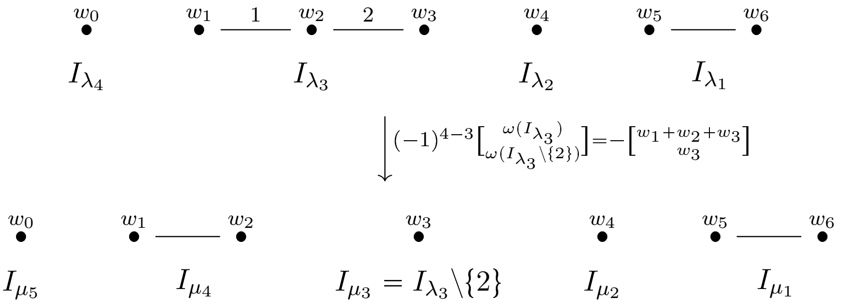
where $I_\lambda\backslash\{i\}$ denotes the subinterval of I_λ to the right of the i-th edge (counting starting at the left).

The definition of arithmetic complex $C_\bullet^{\mathbf{k}}$ defined by Raicu and VandeBogert in [116, Section 5] can be recovered as follows. Consider an element $w_0 \in \mathbf{k}$. Then the evalutation morphism $ev\colon R \to \mathbf{k}$ that maps x into w_0 gives to $\mathbf{k}$ a structure of R-module. Consider now the arithmetic complex $C_\bullet(x, w_1, ..., w_d)$ with $w_1, \ldots, w_d \in \mathbb{N}$. Then we can evaluate the complex at $x = w_0$ and tensor with $\mathbf{k}$ to obtain

$$C_\bullet^{\mathbf{k}}(w_0, w_1, ..., w_d) \cong C_\bullet(x, w_1, ..., w_d)_{|x=w_0} \otimes_R \mathbf{k}.$$

Similarly, one can define $C_\bullet^{\mathbb{Z}}(\underline{w})$ with $w_0 \in \mathbb{Z}$ by evaluating $C_\bullet(x, w_1, ..., w_d)$ on $x = w_0$ and then tensoring with $\mathbb{Z}$.

Example 3.13. Consider the graph in Figure 3.2. Let $j = 3$ and $i = 2$. Let $I_{\mu_3} = I_{\lambda_3}\backslash\{2\}$, the interval consisting of a single vertex with weight w_3. The corresponding coefficient is $-\left[\begin{smallmatrix} w_1+w_2+w_3 \\ w_3 \end{smallmatrix}\right]$.



◊

Our focus is when $w_0 = x$ or $w_0 = -x - 2d$, and $w_1 = \cdots = w_d = 1$ with the aim of establishing an isomorphism between $C_\bullet(x, 1^d)$ and $C_\bullet(-x - 2d, 1^d)$.

Remark 3.14. When $w_0 = -x-2d$ the following binomial identity is used in computing the differentials:

$$\begin{bmatrix} -x - m \\ n \end{bmatrix} = (-1)^n \begin{bmatrix} x + m + n - 1 \\ n \end{bmatrix}$$

for $m, n \in \mathbb{Z}$. △

Example 3.15. Computations with Macaulay2 [M2] exemplify an isomorphism from $C_\bullet(x, 1^d)$ to $C_\bullet(-x-2d, 1^d)$ for $d = 2, 3, 4$.

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\left(\begin{smallmatrix} \left[\begin{smallmatrix} x+2 \\ 2 \end{smallmatrix}\right] \\ x+2 \end{smallmatrix}\right)"] \arrow[d, "(1)"'] & R^2 \arrow[r, "(-2,\ x+1)"] \arrow[d, "\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)"] & R \arrow[r] \arrow[d, "(1)"] & 0 \\
0 \arrow[r] & R \arrow[r, "\left(\begin{smallmatrix} \left[\begin{smallmatrix} x+3 \\ 2 \end{smallmatrix}\right] \\ -x-2 \end{smallmatrix}\right)"'] & R^3 \arrow[r, "(-2,\ -x-3)"'] & R \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large, row sep=huge]
0 \arrow[r] & R \arrow[r, "\begin{pmatrix} \genfrac{[}{]}{0pt}{}{x+3}{3} \\ \genfrac{[}{]}{0pt}{}{x+3}{2} \\ x+3 \end{pmatrix}"] \arrow[d, "(1)"'] & R^3 \arrow[r, "\begin{pmatrix} -3 & x+1 & 0 \\ -3 & 0 & \genfrac{[}{]}{0pt}{}{x+2}{2} \\ 0 & -2 & x+2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} -1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}" description] & R^3 \arrow[r, "(2,\ -2,\ x+1)"] \arrow[d, "\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{pmatrix}" description] & R \arrow[r] \arrow[d, "(-1)" description] & 0 \\
0 \arrow[r] & R \arrow[r, "\begin{pmatrix} -\genfrac{[}{]}{0pt}{}{x+5}{3} \\ \genfrac{[}{]}{0pt}{}{x+4}{2} \\ -x-3 \end{pmatrix}"'] & R^3 \arrow[r, "\begin{pmatrix} -3 & -x-5 & 0 \\ -3 & 0 & \genfrac{[}{]}{0pt}{}{x+5}{2} \\ 0 & -2 & -x-4 \end{pmatrix}"'] & R^3 \arrow[r, "(2,\ -2,\ -x-5)"'] & R \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large, row sep=huge]
0 \arrow[r] & R \arrow[r, "\begin{pmatrix} \genfrac{[}{]}{0pt}{}{x+4}{4} \\ \genfrac{[}{]}{0pt}{}{x+4}{3} \\ \genfrac{[}{]}{0pt}{}{x+4}{2} \\ x+4 \end{pmatrix}"] \arrow[d, "(1)" description] & R^4 \arrow[r, "\begin{pmatrix} -4 & x+1 & 0 & 0 \\ -6 & 0 & \genfrac{[}{]}{0pt}{}{x+2}{2} & 0 \\ -4 & 0 & 0 & \genfrac{[}{]}{0pt}{}{x+3}{3} \\ 0 & -3 & x+2 & 0 \\ 0 & -3 & 0 & \genfrac{[}{]}{0pt}{}{x+3}{2} \\ 0 & 0 & -2 & x+3 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}" description] & R^6 \arrow[r, "\begin{pmatrix} 3 & -2 & 0 & x+1 & 0 & 0 \\ 3 & 0 & -3 & 0 & x+1 & 0 \\ 0 & 2 & -3 & 0 & 0 & \genfrac{[}{]}{0pt}{}{x+2}{2} \\ 0 & 0 & 0 & 2 & -2 & x+2 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 5 & 1 & 4 \\ 0 & 0 & 1 & 0 & 4 & 6 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}" description] & R^4 \arrow[r, "\begin{pmatrix} -2 \\ 2 \\ -2 \\ x+1 \end{pmatrix}^T"] \arrow[d, "\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -1 \end{pmatrix}" description] & R \arrow[r] \arrow[d, "(1)" description] & 0 \\
0 \arrow[r] & R \arrow[r, "\begin{pmatrix} \genfrac{[}{]}{0pt}{}{x+7}{4} \\ -\genfrac{[}{]}{0pt}{}{x+6}{3} \\ \genfrac{[}{]}{0pt}{}{x+5}{2} \\ -x-4 \end{pmatrix}"'] & R^4 \arrow[r, "\begin{pmatrix} -4 & -x-7 & 0 & 0 \\ -6 & 0 & \genfrac{[}{]}{0pt}{}{x+7}{2} & 0 \\ -4 & 0 & 0 & -\genfrac{[}{]}{0pt}{}{x+7}{3} \\ 0 & -3 & -x-6 & 0 \\ 0 & -3 & 0 & \genfrac{[}{]}{0pt}{}{x+6}{2} \\ 0 & 0 & -2 & -x-5 \end{pmatrix}"'] & R^6 \arrow[r, "\begin{pmatrix} 3 & -2 & 0 & -x-7 & 0 & 0 \\ 3 & 0 & -3 & 0 & -x-7 & 0 \\ 0 & 2 & -3 & 0 & 0 & \genfrac{[}{]}{0pt}{}{x+7}{2} \\ 0 & 0 & 0 & 2 & -2 & -x-6 \end{pmatrix}"'] & R^4 \arrow[r, "\begin{pmatrix} -2 \\ 2 \\ -2 \\ -x-7 \end{pmatrix}^T"'] & R \arrow[r] & 0
\end{tikzcd}$$

◇

In Example 3.15, the isomorphism maps in homological degree $d-t+1$ can be obtained using the following formulas:

$$\begin{aligned}
t = 1 : f_{\lambda_1} &\mapsto f_{\mu_1}, \\
t = 2 : f_{(\lambda_2,\lambda_1)} &\mapsto \sum_{\mu_1+\mu_2=d-1} (-1)^{d-\mu_2} \binom{\mu_1}{\lambda_1} \cdot f_{(\mu_2,\mu_1)}, \\
t = 3 : f_{(\lambda_3,\lambda_2,\lambda_1)} &\mapsto \sum_{\mu_1+\mu_2+\mu_3=d-2} (-1)^{d-\mu_3} \binom{\mu_1}{\lambda_1} \binom{\lambda_1+\mu_1+\mu_2+2}{\lambda_1+\mu_1+\lambda_2+2} \cdot f_{(\mu_3,\mu_2,\mu_1)}.
\end{aligned}$$

A generalization of this pattern is given by $\alpha_\bullet : C_\bullet(x, 1^d) \longrightarrow C_\bullet(-x-2d, 1^d)$ where for an arbitrary d the map $\alpha_{d-t+1} : C_{d-t+1}(x, 1^d) \longrightarrow C_{d-t+1}(-x-2d, 1^d)$ is defined as

$$\alpha_{d-t+1}(f_{(\lambda_t,\lambda_{t-1},\ldots,\lambda_1)}) = \sum_{\mu_1+\cdots+\mu_t=d-t+1} (-1)^{d-\mu_t} \prod_{s=0}^{t-2} \binom{\sum_{j=1}^{s}(\lambda_j+\mu_j)+\mu_{s+1}+2s}{\sum_{j=1}^{s}(\lambda_j+\mu_j)+\lambda_{s+1}+2s} f_{(\mu_t,\mu_{t-1},\ldots,\mu_1)}.$$

The main result of the next section is that $\alpha_\bullet$ is indeed an isomorphism of complexes.

3.2 Uniform identification of aritmethic complexes

This section is devoted to the proof of Theorem 3.19 which is broken into two parts. First, it is shown that $\alpha_\bullet$ defines a homomorphism of complexes (using lemmata 3.16 and 3.17). Next, Lemma 3.21 shows that the matrices in $\alpha_\bullet$ are invertible by using an ordering of the basis, as explained in Remark 3.20. Finally, Corollary 3.22 provides a more specific description of those matrices.

Lemma 3.16. *Let $d, t \in \mathbb{N}$, $t \geqslant 1$. For weak compositions $\lambda = (\lambda_t, \dots, \lambda_1)$ of $d-t+1$, and $\mu = (\mu_{t+1}, \mu_t, \dots, \mu_1)$ of $d-t$, define the following quantities:*

$$A_k(\lambda,\mu) := \prod_{s=0}^{k-2} \binom{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\mu_{s+1}+2s}{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\lambda_{s+1}+2s},$$

$$B_k(\lambda,\mu) := \prod_{s=k+1}^{t-1} \binom{\sum_{m=1}^{s-1}(\mu_m+\lambda_m)+\mu_s+\mu_{s+1}+2s-1}{\sum_{m=1}^{s}(\mu_m+\lambda_m)+2s-1},$$

$$g_k(\lambda,\mu;z) := \binom{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\mu_k+2(k-1)}{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\lambda_k-z+2(k-1)},$$

$$h_k(\lambda,\mu;z) := \binom{\sum_{m=1}^{k}(\mu_m+\lambda_m)+\mu_{k+1}-z+2(k-1)+2}{\sum_{m=1}^{k}(\mu_m+\lambda_m)+2(k-1)+1},$$

where $k = 1, \dots, t$ and $z \in \mathbb{Z}$. Then the following hold

i) $g_1(\lambda,\mu;\lambda_1+1) = 0$;

ii) $A_k(\lambda,\mu) \cdot g_k(\lambda,\mu;0) = A_{k+1}(\lambda,\mu)$ *and* $h_k(\lambda,\mu;0) = g_{k+1}(\lambda,\mu;\lambda_{k+1}+1)$, *for* $1 \leqslant k \leqslant t-1$;

iii) $B_k(\lambda,\mu) = B_{k+1}(\lambda,\mu) \cdot h_{k+1}(\lambda,\mu;\lambda_{k+1}+1)$, *for* $1 \leqslant k \leqslant t-2$;

iv) $A_t(\lambda,\mu) \cdot g_t(\lambda,\mu;0) = 0$.

Proof. The proof proceeds by explicit computations. Notice that $g_1(\lambda,\mu;\lambda_1+1) = \binom{\mu_1}{-1} = 0$. Given $k = 1, \dots, t-1$, part *ii)* follows from the observations

$$\begin{aligned}A_k(\lambda,\mu)\cdot g_k(\lambda,\mu;0) &= \prod_{s=0}^{k-2}\binom{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\mu_{s+1}+2s}{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\lambda_{s+1}+2s}\cdot\binom{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\mu_k+2(k-1)}{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\lambda_k+2(k-1)}\\ &= \prod_{s=0}^{k-1}\binom{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\mu_{s+1}+2s}{\sum_{m=1}^{t}(\mu_m+\lambda_m)+\lambda_{s+1}+2s} = A_{k+1}(\lambda,\mu)\end{aligned}$$

and

$$\begin{aligned}g_{k+1}(\lambda,\mu;\lambda_{k+1}+1) &= \binom{\sum_{m=1}^{k}(\mu_m+\lambda_m)+\mu_{k+1}+2k}{\sum_{m=1}^{k}(\mu_m+\lambda_m)+2k-1}\\ &= \binom{\sum_{m=1}^{k}(\mu_m+\lambda_m)+\mu_{k+1}+2(k-1)+2}{\sum_{m=1}^{k}(\mu_m+\lambda_m)+2(k-1)+1} = h_k(\lambda,\mu;0).\end{aligned}$$

Furthermore, if $k = 1, \dots, t-2$ then

$$\begin{aligned}h_{k+1}(\lambda,\mu;\lambda_{k+1}+1) &= \binom{\sum_{m=1}^{k+1}(\mu_m+\lambda_m)+\mu_{k+2}-\lambda_{k+1}-1+2k+2}{\sum_{m=1}^{k+1}(\mu_m+\lambda_m)+2k+1}\\ &= \binom{\sum_{m=1}^{k}(\mu_m+\lambda_m)+\mu_{k+1}+\mu_{k+2}+2k+1}{\sum_{m=1}^{k+1}(\mu_m+\lambda_m)+2k+1},\end{aligned}$$

and the equality $B_{k+1}(\lambda,\mu)\cdot h_{k+1}(\lambda,\mu;\lambda_{k+1}+1) = B_k(\lambda,\mu)$ is obtained.

Finally, if $A_t(\lambda,\mu) = \prod_{s=0}^{t-2}\binom{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\mu_{s+1}+2s}{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\lambda_{s+1}+2s} \neq 0$ then

$$\binom{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\mu_{s+1}+2s}{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\lambda_{s+1}+2s} = \binom{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\mu_{s+1}+2s}{\mu_{s+1}-\lambda_{s+1}} \neq 0$$

for $s = 0,\dots,t-2$, or equivalently $\mu_j \geqslant \lambda_j$ for $j = 1,\dots,t-1$. Since $\sum_{j=1}^{t}\lambda_j = d-t+1$ and $\sum_{j=1}^{t+1}\mu_j = d-t$, it follows that $\mu_t - \lambda_t < 0$ and thus

$$g_t(\lambda,\mu;0)=\binom{\sum_{m=1}^{t-1}(\mu_m+\lambda_m)+\mu_t+2(t-1)}{\sum_{m=1}^{t-1}(\mu_m+\lambda_m)+\lambda_t+2(t-1)}=\binom{\sum_{m=1}^{t-1}(\mu_m+\lambda_m)+\mu_t+2(t-1)}{\mu_t-\lambda_t}=0.$$

These observations yield the identity below which completes the proof of part *iv*).

$$A_t(\lambda,\mu)\cdot g_t(\lambda,\mu;0) = \begin{cases} A_t(\lambda,\mu)\cdot 0 & \text{if } A_t(\lambda,\mu)\neq 0 \\ 0\cdot g_t(\lambda,\mu;0) & \text{if } A_t(\lambda,\mu) = 0 \end{cases} = 0. \qquad \square$$

Lemma 3.17. *Under the assumptions of Lemma 3.16*

$$\sum_{k=1}^{t-1}(-1)^{t-k}A_k(\lambda,\mu)\cdot B_k(\lambda,\mu)\cdot\Big[h_k(\lambda,\mu;\lambda_k+1)\cdot g_k(\lambda,\mu;\lambda_k+1)$$
$$+\,g_k(\lambda,\mu;0)\cdot h_k(\lambda,\mu;0)\Big] = -A_t(\lambda,\mu)\cdot g_t(\lambda,\mu;\lambda_t+1).$$

Proof. For $k = 1,\dots,t-2$, Lemma 3.16 *ii*) and *iii*) imply

$$A_k(\lambda,\mu){\cdot}B_k(\lambda,\mu){\cdot}g_k(\lambda,\mu;0){\cdot}h_k(\lambda,\mu;0)A_{k+1}(\lambda,\mu){\cdot}B_{k+1}(\lambda,\mu){\cdot}g_{k+1}(\lambda,\mu;\lambda_{k+1}+1){\cdot}h_{k+1}(\lambda,\mu;\lambda_{k+1}+1).$$

In the case $k = t-1$, $B_{t-1}(\lambda,\mu) = 1$ and so by Lemma 3.16 *ii*) the following equation holds:

$$A_{t-1}(\lambda,\mu)\cdot B_{t-1}(\lambda,\mu)\cdot g_{t-1}(\lambda,\mu;0)\cdot h_{t-1}(\lambda,\mu;0) = A_t(\lambda,\mu)\cdot g_t(\lambda,\mu;\lambda_t+1).$$

The following chain of equalities finish the proof, where $(*)$ uses Lemma 3.16 *i*):

$$\begin{aligned}
&\sum_{k=1}^{t-1}(-1)^{t-k}A_k(\lambda,\mu)\cdot B_k(\lambda,\mu)\cdot g_k(\lambda,\mu;0)\cdot h_k(\lambda,\mu;0)\\
&=\sum_{k=1}^{t-2}(-1)^{t-k}A_k(\lambda,\mu)\cdot B_k(\lambda,\mu)\cdot g_k(\lambda,\mu;0)\cdot h_k(\lambda,\mu;0) - A_{t-1}(\lambda,\mu)\cdot B_{t-1}(\lambda,\mu)\cdot g_{t-1}(\lambda,\mu;0)\cdot h_{t-1}(\lambda,\mu;0)\\
&=\sum_{k=1}^{t-2}(-1)^{t-k}A_{k+1}(\lambda,\mu)\cdot B_{k+1}(\lambda,\mu)\cdot h_{k+1}(\lambda,\mu;\lambda_{k+1}+1)\cdot g_{k+1}(\lambda,\mu;\lambda_{k+1}+1) - A_t(\lambda,\mu)\cdot g_t(\lambda,\mu;\lambda_t+1)\\
&\overset{*}{=}-\sum_{k=1}^{t-1}(-1)^{t-k}A_k(\lambda,\mu)\cdot B_k(\lambda,\mu)\cdot h_k(\lambda,\mu;\lambda_k+1)\cdot g_k(\lambda,\mu;\lambda_k+1) - A_t(\lambda,\mu)\cdot g_t(\lambda,\mu;\lambda_t+1). \qquad \square
\end{aligned}$$

The following combinatorial formula can be found in [124, Corollary 4]. We give here a different and easier proof.

Lemma 3.18. *For any non-negative integers $a,b,c \in \mathbb{N}$, the following identity of integer-valued polynomials in $\mathbb{Q}[y]$ holds true.*

$$\sum_{j\geqslant 0}\binom{b}{j}\begin{bmatrix} y+a \\ a-b+1+j\end{bmatrix}\begin{bmatrix} y+a+b+c-j \\ c-j\end{bmatrix} = \begin{bmatrix} y+a+c \\ a-b+c+1\end{bmatrix}\begin{bmatrix} a+c+1 \\ c\end{bmatrix}$$

Proof. Clearly Lemma 3.18 holds for $b = 0$. So we may assume $b \geqslant 1$. We will show both sides of the equation have the same generating function. Let $A = a + c$ and $B = y + a + c$. Let t and u be variables. First multiplying the right hand side by t^c, summing over $c \geqslant 0$, we get

$$\begin{bmatrix} B \\ A-b+1 \end{bmatrix} \sum_{c\geqslant 0} \begin{bmatrix} A+1 \\ c \end{bmatrix} t^c = \begin{bmatrix} B \\ A-b+1 \end{bmatrix} (1+t)^{A+1},$$

and then multiplying by u^A and summing over $A \geqslant 0$,

$$\sum_{A\geqslant 0} \begin{bmatrix} B \\ A-b+1 \end{bmatrix} (1+t)^{A+1} u^A = (1+t)^b u^{b-1} \sum_{A\geqslant 0} \begin{bmatrix} B \\ A-b+1 \end{bmatrix} [(1+t)\cdot u]^{A-b+1} = u^{b-1}(1+t)^b(1+u+tu)^B.$$

Note the index of summation for A starts at 0 because $b \geqslant 1$.

Replacing A and B in the left hand side, multiplying by u^A and summing over $A \geqslant 0$ we get

$$\sum_{j\geqslant 0} \binom{b}{j} \begin{bmatrix} B+b-j \\ c-j \end{bmatrix} \Big(\sum_{A\geqslant 0} \begin{bmatrix} B-c \\ A-c-b+1+j \end{bmatrix} u^A \Big) = \Big[\sum_{j\geqslant 0} \binom{b}{j} \begin{bmatrix} B+b-j \\ c-j \end{bmatrix} \Big] \Big(u^{b+c-1-j} (1+u)^{B-c} \Big).$$

We can start the summation from $A = 0$ because if $j > c$ the summand is zero, hence $j \leqslant c$ and we have $j + 1 - c - b \leqslant 0$. Then, multiplying by t^c and summing over $c \geqslant 0$ we get

$$\begin{aligned}
&\sum_{j\geqslant 0} \binom{b}{j} u^{b-1-j} (1+u)^B \cdot \sum_{c\geqslant 0} \begin{bmatrix} B+b-j \\ c-j \end{bmatrix} t^c (1+u)^{-c} u^c \\
=&\sum_{j\geqslant 0} \binom{b}{j} u^{b-1} (1+u)^{B-j} t^j \cdot \sum_{c\geqslant 0} \begin{bmatrix} B+b-j \\ c-j \end{bmatrix} \Big(\frac{tu}{1+u} \Big)^{c-j} \\
=&\sum_{j\geqslant 0} \binom{b}{j} u^{b-1} (1+u)^{B-j} t^j \cdot \left(1 + \frac{tu}{1+u} \right)^{B+b-j} = \sum_{j\geqslant 0} \frac{u^{b-1} t^j}{(1+u)^b} \cdot (1+u+tu)^{B+b-j} \\
=&\frac{u^{b-1}(1+u+tu)^B}{(1+u)^b} \cdot \sum_{j\geqslant 0} \binom{b}{j} \cdot t^j \cdot (1+u+tu)^{b-j} = \frac{u^{b-1}(1+u+tu)^B}{(1+u)^b} \cdot (1+t+u+tu)^b \\
=&u^{b-1}(1+t)^b(1+u+tu)^B.
\end{aligned}$$

We have obtained both the right and the left hand side of Lemma 3.18 are the coefficient of the term $u^A t^c$ of the same generating function, and are thus equal. □

We are ready to prove the main theorem of this section.

Theorem 3.19. *For every integer $d \geqslant 1$, there exists an isomorphism of complexes*

$$C_\bullet(x, 1^d) \cong C_\bullet(-x-2d, 1^d).$$

Proof. Consider the map $\alpha^d_{d-t+1} : C_{d-t+1}(x, 1^d) \longrightarrow C_{d-t+1}(-x-2d, 1^d)$ defined as

$$\begin{aligned}
&\alpha^d_{d-t+1}(f_{(\lambda_t,\lambda_{t-1},\ldots,\lambda_1)}) \\
=&\sum_{\mu_1+\cdots+\mu_t = d-t+1} (-1)^{d-\mu_t} \prod_{s=0}^{t-2} \binom{\sum_{j=1}^s (\lambda_j+\mu_j) + \mu_{s+1} + 2s}{\sum_{j=1}^s (\lambda_j+\mu_j) + \lambda_{s+1} + 2s} f_{(\mu_t,\mu_{t-1},\ldots,\mu_1)}
\end{aligned}$$

which is a homomorphism for $t \geqslant 1$, presented by a $\binom{d}{t-1} \times \binom{d}{t-1}$ matrix. When there is no ambiguity the superscript d is removed and the map is denoted by α_{d-t+1}. To prove $\alpha_\bullet$ is an isomorphism of complexes entails demonstrating $\alpha_\bullet$ is a morphism of chain complexes and each α_{d-t+1} is represented by an invertible matrix. The former is established first, that is

$$\alpha_{d-t}(\partial^{x}_{d-t+1}(f_{(\lambda_t,\dots,\lambda_1)})) = \partial^{-x-2d}_{d-t+1}(\alpha_{d-t+1}(f_{(\lambda_t,\dots,\lambda_1)})) \tag{3.2}$$

for each t and basis element $f_{(\lambda_t,\dots,\lambda_1)}$ of $C_{d-t+1}(x,1^d)$. Lemma 3.21 addresses the invertibility of α_{d-t+1}. Taking $\{f_{(\mu_{t+1},\dots,\mu_1)}\}$ as a basis of $C_{d-t}(-x-2d,1^d)$ results in equalities

$$\begin{aligned}
\partial^{-x-2d}_{d-t+1}(\alpha_{d-t+1}(f_{(\lambda_t,\dots,\lambda_1)})) &= \sum_{\mu_1+\dots+\mu_{t+1}=d-t} c_{(\mu_{t+1},\dots,\mu_1)} f_{(\mu_{t+1},\dots,\mu_1)},\\
\alpha_{d-t}(\partial^{x}_{d-t+1}(f_{(\lambda_t,\dots,\lambda_1)})) &= \sum_{\mu_1+\dots+\mu_{t+1}=d-t} \tilde{c}_{(\mu_{t+1},\dots,\mu_1)} f_{(\mu_{t+1},\dots,\mu_1)},
\end{aligned}$$

for some coefficients $c_{(\mu_{t+1},\dots,\mu_1)}$ and $\tilde{c}_{(\mu_{t+1},\dots,\mu_1)}$. Equation (3.2) is proved by fixing $\mu_{t+1},\dots,\mu_1$ and the following sequence of steps, the details of which constitute the remainder of this proof.

Step 1: Compute $\partial^{-x-2d}_{d-t+1}(\alpha_{d-t+1}(f_{(\lambda_t,\dots,\lambda_1)}))$ and $c_{(\mu_{t+1},\dots,\mu_1)}$.

Step 2: Compute $\alpha_{d-t}(\partial^{x}_{d-t+1}(f_{(\lambda_t,\dots,\lambda_1)}))$ and $\tilde{c}_{(\mu_{t+1},\dots,\mu_1)}$.

Step 3: Compare the coefficients $c_{(\mu_{t+1},\dots,\mu_1)}$ and $\tilde{c}_{(\mu_{t+1},\dots,\mu_1)}$.

Step 1. Let us first determine the basis elements $f_{(\epsilon_t,\dots,\epsilon_1)}$ of $C_{d-t+1}(-x-2d,1^d)$, such that in the sum for $\partial^{-x-2d}_{d-t+1}(f_{(\epsilon_t,\dots\epsilon_1)})$, the basis element $f_{(\mu_{t+1},\dots,\mu_1)}$ appears with non-zero coefficient. By definition of ∂^{-x-2d}_{d-t+1} such an $f_{(\epsilon_t,\dots,\epsilon_1)}$ is of the form

$$f_{(\mu_{t+1},\dots,\mu_{k+2},\mu_{k+1}+\mu_k+1,\mu_{k-1},\dots,\mu_1)}$$

for some $1 \leqslant k \leqslant t$. More concretely, for $k=t$, in the sum $\partial^{-x-2d}_{d-t+1}(f_{(\mu_{t+1}+\mu_t+1,\mu_{t-1},\dots,\mu_1)})$ the coefficient of $f_{(\mu_{t+1},\dots,\mu_1)}$ is

$$\begin{bmatrix} -x-2d+\mu_{t+1}+\mu_t+1 \\ \mu_t+1 \end{bmatrix} = (-1)^{\mu_t+1} \begin{bmatrix} x+2d-\mu_{t+1}-1 \\ \mu_t+1 \end{bmatrix}.$$

For $k<t$, in the sum $\partial^{-x-2d}_{d-t+1}(f_{(\mu_{t+1},\dots,\mu_{k+2},\mu_{k+1}+\mu_k+1,\mu_{k-1},\dots,\mu_1)})$ the coefficient of $f_{(\mu_{t+1},\dots,\mu_1)}$ is given by

$$(-1)^{t-k}\binom{\mu_{k+1}+\mu_k+2}{\mu_k+1}.$$

The coefficient of $f_{(\mu_{t+1},\dots,\mu_{k+2},\mu_{k+1}+\mu_k+1,\mu_{k-1},\dots,\mu_1)}$ in the sum for $\alpha_{d-t+1}(f_{(\lambda_t,\dots,\lambda_1)})$ is computed next. For $k<t$, the coefficient is given by

$$\begin{aligned}
&(-1)^{d-\mu_{t+1}} \cdot \prod_{s=k+1}^{t-1} \binom{\sum_{m=1}^{s-1}(\mu_m+\lambda_m)+\mu_s+\mu_{s+1}+2s-1}{\sum_{m=1}^{s}(\mu_m+\lambda_m)+2s-1}\\
&\prod_{s=0}^{k-2} \binom{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\mu_{s+1}+2s}{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\lambda_{s+1}+2s} \cdot \binom{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\mu_k+\mu_{k+1}+2k-1}{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\lambda_k+2k-2}\\
&= (-1)^{d-\mu_{t+1}} B_k(\lambda,\mu)\cdot A_k(\lambda,\mu)\cdot \binom{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\mu_k+\mu_{k+1}+2k-1}{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\lambda_k+2k-2}
\end{aligned}$$

where A_k, B_k are as defined in Lemma 3.16.

The coefficient for $k=t$ is

$$(-1)^{d-(\mu_{t+1}+\mu_t+1)} \prod_{s=0}^{t-2} \binom{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\mu_{s+1}+2s}{\sum_{m=1}^{s}(\mu_m+\lambda_m)+\lambda_{s+1}+2s} = (-1)^{d-(\mu_{t+1}+\mu_t+1)} A_t(\lambda,\mu).$$

Thus Equation (3.3) displays the coefficient of $f_{(\mu_{t+1},\mu_t,\ldots,\mu_1)}$ in the composition $\partial_{d-t+1}^{-x-2d}(\alpha_{d-t+1}(f_{\lambda_t,\ldots,\lambda_1}))$.

$$\begin{aligned} c_{(\mu_{t+1},\ldots,\mu_1)} = (-1)^{d-\mu_{t+1}} \Bigg[\sum_{k=1}^{t-1} \Bigg((-1)^{t-k} \binom{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\mu_k+\mu_{k+1}+2k-1}{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\lambda_k+2k-2} \cdot \\ \cdot \binom{\mu_{k+1}+\mu_k+2}{\mu_k+1} \cdot A_k(\lambda,\mu)\cdot B_k(\lambda,\mu) \Bigg) + \binom{x+2d-\mu_{t+1}-1}{\mu_t+1} A_t(\lambda,\mu) \Bigg]. \end{aligned} \tag{3.3}$$

Step 2. By Definition 3.12

$$\begin{aligned} \partial_{d-t+1}^{x}(f_{(\lambda_t,\ldots,\lambda_1)}) &= \sum_{i=1}^{\lambda_t} \binom{x+\lambda_t}{\lambda_t-i+1} f_{(i-1,\lambda_t-i,\lambda_{t-1},\ldots,\lambda_1)} \\ &+ \sum_{k=1}^{t-1}\sum_{i=1}^{\lambda_k} (-1)^{t-k}\binom{\lambda_k+1}{i} f_{(\lambda_t,\ldots,\lambda_{k+1},i-1,\lambda_k-i,\lambda_{k-1},\ldots,\lambda_1)}. \end{aligned}$$

Applying α_{d-t} to the basis elements appearing in the above summation, it follows

$$\begin{aligned} \alpha_{d-t}(f_{(i-1,\lambda_t-i,\lambda_{t-1},\ldots,\lambda_1)}) &= \sum_{\mu_1+\cdots+\mu_{t+1}=d-t} (-1)^{d-\mu_{t+1}} \prod_{s=0}^{t-2} \binom{(\sum_{t=1}^{s}\lambda_t+\mu_t)+\mu_{s+1}+2s}{(\sum_{t=1}^{s}\lambda_t+\mu_t)+\lambda_{s+1}+2s} \cdot \\ &\cdot \binom{\sum_{m=1}^{t-1}(\mu_m+\lambda_m)+\mu_t+2(t-1)}{\sum_{m=1}^{t-1}(\mu_m+\lambda_m)+\lambda_t-i+2(t-1)} f_{(\mu_{t+1},\mu_t,\ldots,\mu_1)} \\ &= \sum_{\mu_1+\cdots+\mu_{t+1}=d-t} (-1)^{d-\mu_{t+1}} A_t(\lambda,\mu)\cdot g_t(\lambda,\mu;i) f_{(\mu_{t+1},\mu_t,\ldots,\mu_1)}, \end{aligned}$$

and

$$\begin{aligned} &\alpha_{d-t}(f_{(\lambda_t,\ldots,\lambda_{k+1},i-1,\lambda_k-i,\lambda_{k-1},\ldots,\lambda_1)}) \\ &= \sum_{\mu_1+\cdots+\mu_{t+1}=d-t} (-1)^{d-\mu_{t+1}} \prod_{s=0}^{k-2} \binom{(\sum_{t=1}^{s}\lambda_t+\mu_t)+\mu_{s+1}+2s}{(\sum_{t=1}^{s}\lambda_t+\mu_t)+\lambda_{s+1}+2s} \cdot \\ &\cdot \binom{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\mu_k+2(k-1)}{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\lambda_k-i+2(k-1)} \cdot \binom{\sum_{m=1}^{k}(\mu_m+\lambda_m)+\mu_{k+1}-i+2k}{\sum_{m=1}^{k}(\mu_m+\lambda_m)+2k-1} \cdot \\ &\cdot \prod_{s=k+1}^{t-1} \binom{\sum_{m=1}^{s-1}(\mu_m+\lambda_m)+\mu_s+\mu_{s+1}+2s-1}{\sum_{m=1}^{s}(\mu_m+\lambda_m)+2s-1} f_{(\mu_{t+1},\mu_t,\ldots,\mu_1)} \\ &= \sum_{\mu_1+\cdots+\mu_{t+1}=d-t} (-1)^{d-\mu_{t+1}} A_k(\lambda,\mu)\cdot g_k(\lambda,\mu;i)\cdot h_k(\lambda,\mu;i)\cdot B_k(\lambda,\mu) f_{(\mu_{t+1},\mu_t,\ldots,\mu_1)}, \end{aligned}$$

where $g_k(\lambda,\mu;-)$, and $h_k(\lambda,\mu;-)$ are defined in Lemma 3.16. The above computations result in

$$\begin{aligned} (\alpha_{d-t}\circ\partial_{d-t+1}^{x})(f_{(\lambda_t,\ldots,\lambda_1)}) &= \sum_{i=1}^{\lambda_t} \begin{bmatrix} x+\lambda_t \\ \lambda_t-i+1 \end{bmatrix} \alpha_{d-t}(f_{(i-1,\lambda_t-i,\lambda_{t-1},\ldots,\lambda_1)}) \\ &+ \sum_{k=1}^{t-1}\sum_{i=1}^{\lambda_k} (-1)^{t-k}\binom{\lambda_k+1}{i} \alpha_{d-t}(f_{(\lambda_t,\ldots,\lambda_{k+1},i-1,\lambda_k-i,\lambda_{k-1},\ldots,\lambda_1)}) \\ &= \sum_{\mu_1+\cdots+\mu_{t+1}=d-t} \tilde{c}_{(\mu_{t+1},\ldots,\mu_1)} f_{(\mu_{t+1},\mu_t,\ldots,\mu_1)}, \end{aligned}$$

where

$$\begin{aligned} \tilde{c}_{(\mu_{t+1},\ldots,\mu_1)} = (-1)^{d-\mu_{t+1}} \Bigg[\sum_{i=1}^{\lambda_t} \begin{bmatrix} x+\lambda_t \\ \lambda_t-i+1 \end{bmatrix} A_t(\lambda,\mu)\cdot g_t(\lambda,\mu;i) \\ + \sum_{k=1}^{t-1} (-1)^{t-k} \Bigg(\sum_{i=1}^{\lambda_k} \binom{\lambda_k+1}{i} A_k(\lambda,\mu)\cdot g_k(\lambda,\mu;i)\cdot B_k(\lambda,\mu)\cdot h_k(\lambda,\mu;i) \Bigg) \Bigg]. \end{aligned} \tag{3.4}$$

Step 3. The final step is to exhibit the coefficient $\tilde{c}_{(\mu_{t+1},\dots,\mu_1)}$ of $f_{(\mu_{t+1},\dots,\mu_1)}$ in Equation (3.4) is equal to $c_{(\mu_{t+1},\dots,\mu_1)}$, i.e. the expression in Equation (3.3). This is proven with the aid of Lemma 3.18, and the the well-known Vandermonde's identity

$$\binom{m+n}{r} = \sum_{k\geqslant 0}\binom{m}{k}\cdot\binom{n}{r-k} \quad \text{with } m,n,r\in\mathbb{N}. \tag{3.5}$$

These formulas along with Lemma 3.16 are used to rewrite the inner sums in $\tilde{c}_{(\mu_{t+1},\dots,\mu_1)}$ as having index $i\geqslant 0$, which are later recognized as a product of binomial coefficients. By the above discussion and the fact that $\begin{bmatrix} x+\lambda_t \\ \lambda_t-i+1\end{bmatrix}=0$ for $i>\lambda_t+1$,

$$\begin{aligned}
&\sum_{i=1}^{\lambda_t}\begin{bmatrix} x+\lambda_t \\ \lambda_t-i+1\end{bmatrix}A_t(\lambda,\mu)\cdot g_t(\lambda,\mu;i)\\
=&A_t(\lambda,\mu)\left(\sum_{i\geqslant 0}\begin{bmatrix} x+\lambda_t \\ \lambda_t-i+1\end{bmatrix}g_t(\lambda,\mu;i)-g_t(\lambda,\mu;\lambda_t+1)\right)-\begin{bmatrix} x+\lambda_t \\ 1+\lambda_t\end{bmatrix}\underbrace{A_t(\lambda,\mu)g_t(\lambda,\mu;0)}_{=0 \text{ by Lemma 3.16 } iv)}\\
=&A_t(\lambda,\mu)\left(\underbrace{\sum_{i\geqslant 0}\begin{bmatrix} x+\lambda_t \\ \lambda_t-i+1\end{bmatrix}\binom{\sum_{m=1}^{t-1}(\mu_m+\lambda_m)+\mu_t+2(t-1)}{\mu_t-\lambda_t+i}}_{=\binom{\sum_{m=1}^{t-1}(\mu_m+\lambda_m)+\mu_t+\lambda_t+x+2(t-1)}{\mu_t+1}\text{ by Equation (3.5)}}-g_t(\lambda,\mu;\lambda_t+1)\right)\\
=&A_t(\lambda,\mu)\left(\begin{bmatrix} x+2d-\mu_{t+1}-1 \\ \mu_t+1\end{bmatrix}-g_t(\lambda,\mu;\lambda_t+1)\right),
\end{aligned}$$

as $\lambda_1+\cdots+\lambda_t=d-t+1=\mu_1+\cdots+\mu_{t+1}+1$. For the other summation appearing,

$$\begin{aligned}
&\sum_{k=1}^{t-1}(-1)^{t-k}\left[\sum_{i=1}^{\lambda_k}\binom{\lambda_k+1}{i}A_k(\lambda,\mu)g_k(\lambda,\mu;i)B_k(\lambda,\mu)h_k(\lambda,\mu;i)\right]\\
=&\sum_{k=1}^{t-1}(-1)^{t-k}A_k(\lambda,\mu)B_k(\lambda,\mu)\left[\sum_{i\geqslant 0}\binom{\lambda_k+1}{i}g_k(\lambda,\mu;i)h_k(\lambda,\mu;i)\right.\\
&\left.-\,g_k(\lambda,\mu;0)h_k(\lambda,\mu;0)-g_k(\lambda,\mu;\lambda_k+1)h_k(\lambda,\mu;\lambda_k+1)\right]\\
=&\sum_{k=1}^{t-1}(-1)^{t-k}A_k(\lambda,\mu)B_k(\lambda,\mu)\left[\sum_{i\geqslant 0}\binom{\lambda_k+1}{i}g_k(\lambda,\mu;i)h_k(\lambda,\mu;i)\right]\\
&-\sum_{k=1}^{t-1}(-1)^{t-k}A_k(\lambda,\mu)B_k(\lambda,\mu)\left[g_k(\lambda,\mu;0)h_k(\lambda,\mu;0)+g_k(\lambda,\mu;\lambda_k+1)h_k(\lambda,\mu;\lambda_k+1)\right]\\
=&\sum_{k=1}^{t-1}(-1)^{t-k}A_k(\lambda,\mu)B_k(\lambda,\mu)\left[\sum_{i\geqslant 0}\binom{\lambda_k+1}{i}g_k(\lambda,\mu;i)h_k(\lambda,\mu;i)\right]+A_t(\lambda,\mu)g_t(\lambda,\mu;\lambda_t+1)
\end{aligned}$$

where the last equality follows by Lemma 3.17. Moreover, by applying Lemma 3.18 with $y=\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+2(k-1)$, $a=\mu_k$, $b=\lambda_k+1$ and $c=\mu_{k+1}+1$, we obtain

$$\begin{aligned}
\sum_{i\geqslant 0}\binom{\lambda_k+1}{i}g_k(\lambda,\mu;i)\cdot h_k(\lambda,\mu;i)&=\sum_{i\geqslant 0}\binom{\lambda_k+1}{i}\binom{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+2(k-1)+\mu_k}{\mu_k-\lambda_k+i}\\
&\quad\cdot\binom{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+2(k-1)+\mu_k+(\lambda_k+1)+(\mu_{k+1}+1)-i}{\mu_{k+1}+1-i}\\
&=\binom{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\mu_k+\mu_{k+1}+2k-1}{\mu_k-\lambda_k+\mu_{k+1}+1}\cdot\binom{\mu_k+\mu_{k+1}+2}{\mu_{k+1}+1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sum_{k=1}^{t-1}(-1)^{t-k}\Big(\sum_{i=1}^{\lambda_k}\binom{\lambda_k+1}{i}A_k(\lambda,\mu)\cdot g_k(\lambda,\mu;i)\cdot B_k(\lambda,\mu)\cdot h_k(\lambda,\mu;i)\Big)\\
=\ &\sum_{k=1}^{t-1}(-1)^{t-k}A_k(\lambda,\mu)\cdot B_k(\lambda,\mu)\left(\binom{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\mu_k+\mu_{k+1}+2k-1}{\mu_k-\lambda_k+\mu_{k+1}+1}\cdot\binom{\mu_k+\mu_{k+1}+2}{\mu_{k+1}+1}\right)\\
&+A_t(\lambda,\mu)\cdot g_t(\lambda,\mu;\lambda_t+1)
\end{aligned}$$

and the claimed equality below concludes that $\alpha_\bullet$ is a morphism of chain complexes.

$$\begin{aligned}
\tilde{c}_{(\mu_{t+1},\dots,\mu_1)}=(-1)^{d-\mu_{t+1}}\Bigg[&A_t(\lambda,\mu)\binom{x+2d-\mu_{t+1}-1}{\mu_t+1}+\sum_{k=1}^{t-1}(-1)^{t-k}A_k(\lambda,\mu)B_k(\lambda,\mu)\\
&\cdot\binom{\sum_{m=1}^{k-1}(\mu_m+\lambda_m)+\mu_k+\mu_{k+1}+2k-1}{\mu_k-\lambda_k+\mu_{k+1}+1}\binom{\mu_k+\mu_{k+1}+2}{\mu_{k+1}+1}\Bigg]=c_{(\mu_{t+1},\dots,\mu_1)}.
\end{aligned}$$

□

The following remark introduces an order on the basis of $C_k(\underline{w})$ to facilitate understanding of the matrices corresponding to the maps constructed in Theorem 3.19.

Remark 3.20. Let $t,k\in\mathbb{N}$, $t>0$. Define an order $\succeq$ on the set of weak compositions of $\mathbf{k}$ with t parts

$$S_{t,k}=\left\{(a_t,a_{t-1}\dots,a_1)\in\mathbb{N}^t \mid \sum_{i=1}^{t}a_i=k\right\}$$

as follows: for $a=(a_t,\dots,a_1)$, $b=(b_t,\dots,b_1)\in S_{t,k}$, consider the associated monomials $x_1^{a_1}\cdots x_t^{a_t}$ and $x_1^{b_1}\cdots x_t^{b_t}$. Define $a\succeq b$ if $x_1^{a_1}\cdots x_t^{a_t}\geqslant_{\text{RLex}}x_1^{b_1}\cdots x_t^{b_t}$ with respect to the reverse lexicographic order on the monomials. In other words, $a\succeq b$ if $a=b$ or the first non-zero entry of the vector of integers $a-b=(a_t-b_t,\dots,a_1-b_1)$ is negative.

For each $d\in\mathbb{N}$ and for each $\underline{w}=(w_0,\dots,w_d)\in R\times\mathbb{N}^{d-1}$, denote the basis of $C_{d-t+1}(\underline{w})$ by

$$\mathcal{B}(C_{d-t+1}(w_0,w_1,\dots,w_d))=\{f_{a^{[1]}},f_{a^{[2]}},\cdots\mid a^{[i]}\in S_{t,d-t+1}\},$$

where superscript $[i]$ denotes the i-th basis element under the ordering $a^{[i]}\succeq a^{[i+1]}$ for all $i\geqslant 1$. For example,Robinson-Schensted correspondence. The article contains a section treating the basic results about the passage to initial ideals and algebras. $a^{[1]}=(0,\dots,0,d-t+1)$ is the biggest element in $S_{t,d-t+1}$ and $a^{[\binom{d}{t-1}]}=(d-t+1,0,\dots,0)$ is the smallest one. Fix w_0 and let $d=4$ and $t=3$. The basis of $C_2(w_0,\dots,w_4)$ is ordered as $\mathcal{B}(C_2(w_0,\dots,w_4))=\{f_{(0,0,2)},f_{(0,1,1)},f_{(0,2,0)},f_{(1,0,1)},f_{(1,1,0)},f_{(2,0,0)}\}$. In particular, denote the bases of $C_{d-t+1}(x,1^d)$ and $C_{d-t+1}(-x-2d,1^d)$ by

$$\begin{aligned}
\mathcal{B}(C_{d-t+1}(x,1^d)) &= \{f_{\lambda^{[1]}},f_{\lambda^{[2]}},\cdots\mid\lambda^{[i]}\in S_{t,d-t+1}\},\\
\mathcal{B}(C_{d-t+1}(-x-2d,1^d)) &= \{f_{\mu^{[1]}},f_{\mu^{[2]}},\cdots\mid\mu^{[j]}\in S_{t,d-t+1}\}.
\end{aligned}$$

Since $\lambda^{[i]}$ and $\mu^{[j]}$ are in $S_{t,d-t+1}$ for each $i,j\geqslant 1$, they are comparable and $\lambda^{[i]}=\mu^{[j]}$ if and only if $i=j$. Whence $\lambda^{[i]}\succeq\mu^{[j]}$ if and only if $i\leqslant j$. Moreover for the map α_{d-t+1} constructed in Theorem 3.19, the (i,j)-th entry of the matrix of α_{d-t+1} with respect to the above order is $(\lambda^{[i]},\mu^{[j]})=[\alpha_{d-t+1}]_{(f_{\lambda^{[i]}},f_{\mu^{[j]}})}$ for each $1\leqslant i,j\leqslant\binom{d}{t-1}$, that is,

$$\alpha_{d-t+1}=\begin{bmatrix}
(\lambda^{[1]},\mu^{[1]}) & (\lambda^{[2]},\mu^{[1]}) & \dots & (\lambda^{[i]},\mu^{[1]}) & \dots\\
(\lambda^{[1]},\mu^{[2]}) & (\lambda^{[2]},\mu^{[2]}) & \dots & (\lambda^{[i]},\mu^{[2]}) & \dots\\
\vdots & \vdots & & \vdots & \\
(\lambda^{[1]},\mu^{[j]}) & (\lambda^{[2]},\mu^{[j]}) & \dots & (\lambda^{[i]},\mu^{[j]}) & \dots\\
\vdots & \vdots & & \vdots &
\end{bmatrix}.$$

△

Lemma 3.21. *With respect to the basis order introduced in Remark 3.20, the matrices corresponding to the maps* α_{d-t+1}*,* $1 \leqslant t \leqslant d+1$*, constructed in Theorem 3.19 are upper triangular, with diagonals made up of* 1*'s and* -1*'s, and hence are invertible.*

Proof. Let $\lambda = (\lambda_t, \lambda_{t-1}, \ldots, \lambda_1)$ and $\mu = (\mu_t, \mu_{t-1}, \ldots, \mu_1)$ be compositions such that $f_\lambda \in \mathcal{B}(C_{d-t+1}(x, 1^d))$ and $f_\mu \in \mathcal{B}(C_{d-t+1}(-x-2d, 1^d))$. Observe that

$$[\alpha_{d-t+1}]_{(f_\lambda, f_\mu)} = (-1)^{d-\mu_t} \prod_{s=0}^{t-2} \binom{\sum_{j=1}^{s}(\lambda_j + \mu_j) + \mu_{s+1} + 2s}{\sum_{j=1}^{s}(\lambda_j + \mu_j) + \lambda_{s+1} + 2s} \neq 0$$

if and only if $\binom{\sum_{j=1}^{s}(\lambda_j+\mu_j)+\mu_{s+1}+2s}{\mu_{s+1}-\lambda_{s+1}} \neq 0$ for $s = 0, \ldots, t-2$. This is equivalent to $\mu_j \geqslant \lambda_j$ for $j = 1, \ldots, t-1$, which means

$$\lambda = \mu \quad \text{or} \quad \begin{cases} \mu_j \geqslant \lambda_j & \text{for each } 1 \leqslant j \leqslant t-1, \\ \mu_j > \lambda_j & \text{for some } 1 \leqslant j \leqslant t-1, \text{ and} \\ \mu_t < \lambda_t \end{cases} \tag{3.6}$$

as $\sum_{i=1}^{t} \lambda_i = \sum_{i=1}^{t} \mu_i$. Hence by the definition of $\succeq$, $[\alpha_{d-t+1}]_{(f_\lambda, f_\mu)} \neq 0$ implies $\mu \succeq \lambda$. Consequently

$$[\alpha_{d-t+1}]_{(f_\lambda, f_\mu)} = \begin{cases} 0 & \text{if } \lambda \succeq \mu \text{ and } \mu \neq \lambda, \\ (-1)^{d-\mu_t} & \text{if } \mu = \lambda. \end{cases}$$

Thus for $f_{\lambda[i]}$ and $f_{\mu[j]}$, the identity

$$[\alpha_{d-t+1}]_{(f_{\lambda[i]} f_{\mu[j]})} = \begin{cases} 0 & \text{if } i < j, \\ 1 \text{ or } -1 & \text{if } i = j \end{cases}$$

proves α_{d-t+1} is upper triangular, with diagonals made up of 1's and -1's. □

Corollary 3.22. *Using the basis order introduced in Remark 3.20, and for each integer* $1 \leqslant t \leqslant d+1$*, the matrices corresponding to the maps constructed in Theorem 3.19 are of the form*

$$\alpha^d_{d-t+1} = \left(\begin{array}{c|c} (-1)^d \mathbb{1}_{\binom{d-1}{t-2}} & \gamma^d_{d-t+1} \\ \hline \mathbf{0} & \alpha^{d-1}_{d-t} \end{array} \right)$$

with γ^d_{d-t+1} *a* $\binom{d-1}{t-2} \times \binom{d-1}{t-1}$ *matrix, and* $\mathbb{1}_{\binom{d-1}{t-2}}$ *is the identity matrix of size* $\binom{d-1}{t-2}$*.*

Remark 3.23. The blocks on the left and right correspond to the basis elements f_λ in the domain with $\lambda_t = 0$ and $\lambda_t \neq 0$ respectively. Likewise, the blocks on top and bottom correspond to the basis elements f_μ in the codomain satisfying $\mu_t = 0$ and $\mu_t \neq 0$ respectively. This decomposition reflects the fact that $C_\bullet(x, 1^d)$ is the mapping cone of a morphism $\phi_x : C_\bullet(x+1, 1^{d-1}) \longrightarrow C_\bullet(1^d)$ as described in [116]. There the proof is for w_0 an integer, but it extends immediately to the case w_0 a polynomial (checking this requires showing the commutativity of some diagrams, which if true for infinitely many specializations to w_0 an integer must also be true for w_0 a polynomial). As a basic property of mapping cones, the fact that $\alpha^d_\bullet$ is a morphism of complexes means that the following diagram commutes up to homotopy given by $\gamma_\bullet$:

$$\begin{tikzcd} C_\bullet(x+1, 1^{d-1}) \arrow[r, "\alpha^{d-1}_\bullet"] \arrow[d, "\phi_x"] & C_\bullet(-x-2d+1, 1^{d-1}) \arrow[d, "\phi_{-x-2d}"] \\ C_\bullet(1^d) \arrow[r, "(-1)^d"] & C_\bullet(1^d) \end{tikzcd}$$

Note that $\alpha^{d-1}_\bullet : C_\bullet(x+1, 1^{d-1}) \to C_\bullet(-x-2d+1, 1^{d-1})$ makes sense as the formula for $\alpha^{d-1}_\bullet$ is independent of x.

Eventually, using induction on d, Corollary 3.22 implies that α^d_k has determinant ± 1 giving a second proof of its invertibility. △

Proof of Corollary 3.22. Let $\alpha_{d-t}^{d-1} : C_{d-t}(x, 1^{d-1}) \longrightarrow C_{d-t}(-x-2d+2, 1^{d-1})$ be as defined in Theorem 3.19. For f_λ, f_μ satisfying $\lambda_t, \mu_t \geqslant 1$, by the definition of $\alpha_\bullet^d$

$$\begin{aligned}[][\alpha_{d-t+1}^d]_{(f_\lambda, f_\mu)} &= \underbrace{(-1)^{d-\mu_t}}_{(-1)^{(d-1)-(\mu_t-1)}} \prod_{s=0}^{t-2} \binom{\sum_{j=1}^s(\lambda_j + \mu_j) + \mu_{s+1} + 2s}{\sum_{j=1}^s(\lambda_j + \mu_j) + \lambda_{s+1} + 2s} \\ &= [\alpha_{d-t}^{d-1}]_{(f_{(\lambda_t-1, \lambda_{t-1}, \ldots, \lambda_1)}, f_{(\mu_t-1, \mu_{t-1}, \ldots, \mu_1)})}\end{aligned}$$

Thus, the lower right block of α_{d-t+1}^d is indeed α_{d-t}^{d-1} as desired.

If $\mu \succeq \lambda$ and $\lambda_t = 0$, then $[\alpha_{d-t+1}]_{(f_\lambda, f_\mu)} \neq 0$ if and only if $\lambda = \mu$ by Equation (3.6). In this case

$$[\alpha_{d-t+1}^d]_{(f_\lambda, f_\mu)} = \begin{cases} (-1)^{d-0} \prod_{s=0}^{t-2} \binom{\sum_{j=1}^s(\mu_j+\mu_j)+\mu_{s+1}+2s}{\sum_{j=1}^s(\mu_j+\mu_j)+\mu_{s+1}+2s} = (-1)^d & \text{if } \lambda = \mu, \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

Moreover, notice that there are $\binom{d-1}{t-2}$-many $(\mu_t, \ldots, \mu_1)$'s in $S_{t,d-t+1}$ such that $\mu_t = 0$. It follows that the upper left corner is $(-1)^d \mathbb{1}_{\binom{d-1}{t-2}}$ and the lower left corner is $\mathbf{0}$ as desired. □

3.3 Flag varieties over the integers

This section explains the relationship between stable sheaf cohomology for $\mathbb{P}^n(\mathbf{k})$ and $\mathbb{P}^n(\mathbb{Z})$ and the homology groups of $C_\bullet(\underline{w})$ as well as a streamlined proof of [116, Theorem 6.12] via Theorem 3.19. Recall, as given in [116], the connection between $C_\bullet(\underline{w})$ and stable sheaf cohomology of ribbon and two column partition Schur functors applied to Ω, the cotangent sheaf of projective space. The case $w_0 \geqslant 1$ corresponds to a ribbon, λ/μ. In [116, Theorem 5.4] it is proven that the stable cohomology of $\mathbb{S}_{\lambda/\mu}\Omega$ is the same as the homology of $C_\bullet^{\mathbf{k}}(\underline{w})$ with an appropriate shift.

Additionally, in [116, Theorem 5.7(2)] it is shown for λ a two column partition, i.e. the conjugate partition $\lambda' = (m, d)$ with integers $m, d \geqslant 0$, the stable cohomology groups of $\mathbb{S}_\lambda\Omega$ are given by the homology groups of $C_\bullet^{\mathbf{k}}(-m-d-1, 1^d)$ up to a shift. Notice the first weight is negative, whereas in the previous case all of the weights are positive. Further, they demonstrate the ranks of the homology groups of $C_\bullet^{\mathbf{k}}(-m-d-1, 1^d)$ and $C_\bullet^{\mathbf{k}}(m-d+1, 1^d)$ are the same via recursive formulas for the ranks and induction, motivating the conjecture that forms the basis of this paper. Theorem 3.19 gives a direct proof of this identification, and shows moreover that for fixed d this identification is uniform with respect to these complexes.

Theorem 3.24 (Raicu-VandeBogert)**.** *Let λ be a partition with two columns, i.e. the conjugate partition is of the form $\lambda' = (m, d)$ for some integers $m, d \geqslant 0$. Then for all i,*

$$H_{st}^i(\mathbb{S}_\lambda\Omega) = H_{st}^{2m+1-i}(\mathbb{S}_{(d+1, 1^{m-d})}\Omega).$$

Proof. The composition associated with the hook $\mu = (d+1, 1^{m-d})$ is $w = (m-d+1, 1^d)$. If x is evaluated to be $m-d+1$ in Theorem 3.19, then

$$\begin{aligned} H_{st}^i(\mathbb{S}_\lambda\Omega) &\overset{[116,\text{ Theorem 5.7(2)}]}{=} H_{i-m}(C_\bullet(-m-d-1, 1^d) \otimes_R \mathbf{k}) \\ &\overset{\text{Theorem 3.19}}{=} H_{i-m}(C_\bullet(m-d+1, 1^d) \otimes_R \mathbf{k}) \overset{[116,\text{ Theorem 5.4}]}{=} H_{st}^{2m+1-i}(\mathbb{S}_{(d+1, 1^{m-d})}\Omega). \end{aligned}$$

□

Theorem 3.19 also gives a uniform identification for projective space over the integers.

Theorem 3.25. *Let Ω be the cotangent sheaf over $\mathbb{P}^n(\mathbb{Z})$ and $\mathbb{S}_{\lambda/\mu}$ be a skew Schur functor. The following is true for all i.*

i) $H^i(\mathbb{P}^n(\mathbb{Z}), \mathbb{S}_{\lambda/\mu}\Omega)$ is independent of n for $n \gg 0$, and is denoted by $H_{st}^i(\mathbb{S}_{\lambda/\mu}\Omega)$.

ii) If $\mathbb{S}_{\lambda/\mu}$ *is a ribbon Schur functor corresponding to* $(w_0, ..., w_d)$*, then*

$$H^i_{st}(\mathbb{S}_{\lambda/\mu}\Omega) = H_{|w|-i}(C_\bullet(\underline{w}) \otimes_R \mathbb{Z}).$$

iii) If λ *is a two column partition, i.e. the conjugate partition for* λ *is of the form* $\lambda' = (m, d)$ *for integers* $m, d \geqslant 0$*, then*

$$H^i_{st}(\mathbb{S}_\lambda\Omega) = H_{d+m-i}(\breve{C}_\bullet(-m-d-1, 1^d)[-d] \otimes_R \mathbb{Z}),$$

where $\breve{C}_\bullet(-m-d-1, 1^d)$ *denotes the dual of the complex* $C_\bullet(-m-d-1, 1^d)$*.*

iv) Consider λ *as in iii) and let* $d \geqslant 1$*, then*

$$H^i_{st}(\mathbb{S}_\lambda\Omega) = H^{2m+2-i}_{st}(\mathbb{S}_{(d+1,1^{m-d})}\Omega).$$

Proof. By [5], skew Schur functors have universal resolutions by tensor products of exterior powers over arbitrary commutative rings. Therefore, $\mathbb{S}_{\lambda/\mu}\Omega$ admits a resolution which has terms given by direct sums of tensor products of exterior powers of Ω. If $d = |\lambda| - |\mu| = (\sum \lambda_i) - (\sum \mu_i)$, then these tensor powers of exterior powers of Ω will be of the form

$$\left(\bigwedge^{d_1}\Omega\right) \otimes_{\mathscr{O}_{\mathbb{P}^n(\mathbb{Z})}} \cdots \otimes_{\mathscr{O}_{\mathbb{P}^n(\mathbb{Z})}} \left(\bigwedge^{d_h}\Omega\right), \text{ for some } h \text{ and } \sum d_i = d.$$

The cohomology of such sheaves is given by

$$H^i\left(\mathbb{P}^n(\mathbb{Z}), \left(\bigwedge^{d_1}\Omega\right) \otimes_{\mathscr{O}_{\mathbb{P}^n(\mathbb{Z})}} \cdots \otimes_{\mathscr{O}_{\mathbb{P}^n(\mathbb{Z})}} \left(\bigwedge^{d_h}\Omega\right)\right) = \begin{cases} \mathbb{Z}, & i = d, \\ 0, & \text{otherwise,} \end{cases} \tag{3.7}$$

assuming that $n \geqslant d$. This calculation is followed by a double induction argument on the number of factors h and the size of the last factor d_h using the long exact sequence on cohomology and the short exact sequence

$$0 \to \bigwedge^d\Omega \to \bigwedge^d(\mathbb{Z}^{n+1}) \otimes_{\mathscr{O}_{\mathbb{P}^n(\mathbb{Z})}} \mathscr{O}_{\mathbb{P}^n(\mathbb{Z})}(-d) \to \bigwedge^{d-1}\Omega \to 0 \tag{3.8}$$

for $d \geqslant 1$. Also, necessary for this calculation is the fact that

$$H^j\left(\mathbb{P}^n(\mathbb{Z}), \left(\bigwedge^{d_1}\Omega\right) \otimes_{\mathscr{O}_{\mathbb{P}^n(\mathbb{Z})}} \cdots \otimes_{\mathscr{O}_{\mathbb{P}^n(\mathbb{Z})}} \left(\bigwedge^{d_h}\Omega\right)(-i)\right) = 0$$

for all i in the range $1 \leqslant i \leqslant n-d$ and all j (again under the assumption that $n \geqslant d$). This fact can similarly be proven using induction and Equation (3.8).

By Equation (3.7), applying the hypercohomology spectral sequence [132, Proposition 5.7.9] to this resolution results in a single nonzero row on the first page. This nonzero row is then a chain complex of free abelian groups, which has homology groups matching the sheaf cohomology groups for $\mathbb{S}_{\lambda/\mu}$ up to a shift. Further, these chain complexes will be the same for $n \geqslant d$, so $H^i(\mathbb{P}^n(\mathbb{Z}), \mathbb{S}_{\lambda/\mu}\Omega)$ is independent of n for $n \geqslant d$ proving *i)*.

In the special case when $\mathbb{S}_{\lambda/\mu}$ is a ribbon or two column Schur functor, the complexes of free abelian groups described above can be derived from explicit resolutions for the skew Schur functor. Such a resolution for ribbons is given by [116, Theorem 5.1], and such a resolution for two column partitions is given by [4, Section 4] or [116, Theorem 5.6]. Finally, the differential maps can be derived by noting that [116, Theorem 4.9 ii)] generalises to projective space over the integers, which states that the map induced on cohomology

$$H^j(\bigwedge^{a+b}\Omega) \to H^j(\bigwedge^a\Omega) \otimes_{\mathbb{Z}} H^j(\bigwedge^b\Omega)$$

is multiplication by $\binom{a+b}{a}$. The statement of *ii)* and *iii)* is then obtained by taking homology, in analogy with the statement of [116, Theorem 5.4] and [116, Theorem 5.7(2)].

As $\mathcal{C} := C_\bullet(-m-d-1, 1^d) \otimes_R \mathbb{Z}$ is a complex of finitely generated free abelian groups, the universal coefficient theorem [75, Thm 3.2]

$$0 \to \operatorname{Ext}^1_R(H_{n-1}(\mathcal{C}), \mathbb{Z}) \longrightarrow H^n(\check{\mathcal{C}}) \longrightarrow \operatorname{Hom}_R(H_n(\mathcal{C}), \mathbb{Z}) \to 0 \tag{3.9}$$

gives a relationship between its homology groups with the homology groups of its dual. Since $d > 1$, the complex $C_\bullet(-m-d-1, 1^d) \otimes_R \mathbb{Q}$ is exact, as the sheaf $\mathbb{S}_\lambda \Omega$ now considered over $\mathbb{Q}$ has zero sheaf cohomology groups by the Borel-Weil-Bott theorem. Thus, the homology of $\mathcal{C}$ is only torsion.

Since the homology $\mathcal{C}$ is only torsion we have

$$\operatorname{Hom}_R(H_n(\mathcal{C}), \mathbb{Z}) = 0 \text{ for every } n \geqslant 0.$$

Moreover, for every $n \geqslant 1$, using the exact sequence

$$0 \to \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \to 0,$$

we obtain a long exact sequence by applying the covariant functor $\operatorname{Hom}(H_{n-1}(\mathcal{C}), -)$ that implies

$$\operatorname{Ext}^1_R(H_{n-1}(\mathcal{C}), \mathbb{Z}) = \operatorname{Hom}_R(H_{n-1}(\mathcal{C}), \mathbb{Q}/\mathbb{Z}) = H_{n-1}(\mathcal{C}). \tag{3.10}$$

Equation (3.9), and Equation (3.10) yields

$$H^i_{st}(\mathbb{S}_\lambda \Omega) = H_{d+m-i}(\widecheck{C_\bullet}(-m-d-1, 1^d)[-d] \otimes_R \mathbb{Z}) = H_{i-m-1}(C_\bullet(-m-d-1, 1^d) \otimes_R \mathbb{Z}).$$

The rest of the proof of *iv)* proceeds in analogy with our proof of Theorem 3.24 above now using *ii)* and *iii)* instead of [116, Theorem 5.4] and [116, Theorem 5.7(2)]. □

Remark 3.26. Of particular interest is the difference between Theorem 3.24 and Theorem 3.25 (iv). We can see they are almost equal, but for a homological degree shift of one on the right-hand side. △

Chapter 4

Vasconcelos invariant for graded modules

> *"Algebra is the offer made by the devil to the mathematician...All you need to do, is give me your soul: give up geometry"*
>
> – Michael Atiyah

Coding theory lies in the intersection theoretical computer science, combinatorics, and algebra, studying the design and properties of *codes*. Codes are fundamental tools for error detection and correction in data transmission and storage, and they also find applications in cryptography and data compression. In this context, a code is not necessarily an encryption algorithm, but rather a structured way of encoding messages to ensure reliability, efficiency, and/or security.

The main idea of coding theory is quite simple. It wants to introduce redundancy into a message so that errors occurring during transmission or storage can be detected and, in an ideal world, corrected. This principle is crucial in many aspects of modern life. For instance, compact discs (CDs), DVDs, and Blu-ray discs rely on error-correcting codes to ensure the proper functioning of the device even when the surface is damaged. In a similar way, while using solid-state drives, and cloud storage services, it guarantees data integrity against hardware failures or corruption. On the communications side, it ensures the reliability of data transmission in mobile phones, Wi-Fi networks, and satellite communications, where signals must pass through noisy channels. In cryptography, coding-theoretic ideas contribute to the design of secure communication protocols and post-quantum cryptographic schemes.

As an example, suppose a communication channel has been established between Genoa and Eraclea. This channel may be formed in many different ways, such as a wire connection or a satellite link. Information is transmitted through the channel as a sequence of bits, where each bit takes one of two values: 0 or 1. Inevitably, the channel is subject to *noise*, which introduces the possibility that a bit is altered during transmission—flipping from 0 to 1, or vice-versa with some small probability. Examples of noise are: interference with other devices, distortion, and natural electric disturbance coming from the atmosphere.

To ensure that the receiver reads our message correctly, error-correcting codes are employed. We present three of the most classical examples: parity-check code, repetition code, and Hamming codes.

- *Parity bit.* The parity bit is a single bit that counts whenever the number of ones appear in a given string is even 0 or odd 1. Starting from a string of data, a single parity bit is added (usually in the last position) to the string forming the messages that needs to be delivered. This methods ensures that if a single error occurs, the receiver can detect the error, but it cannot be corrected. For instance, both the messages $(1,0,0,1;0)$ and $(1,1,0,1;1)$ can be influenced by a single error to become $(1,1,0,1;0)$.

- *Repetition code.* In this code every bit of the string is repeated n times while transmitting our message. The length of the message increases by n times; but, as this length increases, more errors can be detected and corrected. For example, for $n = 3$ the string $(1,0,0)$ is sent as $(1,1,1,0,0,0,0,0,0)$. This code can detect and fix at most $\frac{n-1}{2}$ errors using a majority-decision based system.

- *Hamming (7,4).* A Hamming code (m,n) is used with messages of length n, encoding them into a message of length m by adding as many parity bits as needed. In this specific case, we require encoding a 4-bit string into a 7-bit message. Each of the 3 parity bits is not associated with all the data, but only with part of it. The following Venn diagram illustrates how to explicitly employ the code.

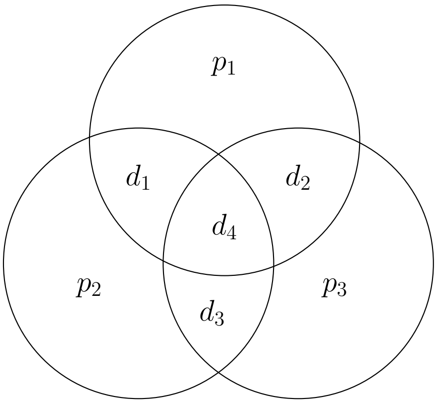


Figure 4.1: The message (d_1, d_2, d_3, d_4) is encrypted as $(p_1, p_2, d_1, p_3, d_2, d_3, d_4)$

We see that every circle represents a parity bit of the data contained in the circle itself, while in the possible intersections of the three circles lie the message. This code is able to detect at most two errors, and it can fix the message if only one error has occurred.

In coding theory, as we can notice in these examples, two opposing needs must be balanced. On one side, we seek codes powerful enough to detect and correct as many errors as possible with high reliability. On the other side, we aim to keep the encoded message as short as possible, ideally close in length to the original data. The three examples above lie in the family of *linear codes*, which we now give a proper definition.

Consider a field $\mathbf{k}$ that contains the possible values available for a single bit of information. Consider any injective linear map $G^T : \mathbf{k}^s \to \mathbf{k}^n$ given by the multiplication of the transpose of a matrix G. The linear code C associated with G is the image of the map G^T. In other words, a linear code is the set of vectors lying in a certain vector sub-space. To emphasize the dimensions of the domain and co-domain, we say that C is a $[n,s]$ code. In real-world situations, the base field is finite; therefore, one should imagine a finite number of codewords.

The principal tool used to evaluate the efficiency of the code is given by the *minimum distance* or *Hamming distance*. First, one can define a metric on $\mathbf{k}^n$ using the weight function

$$wt(v) := \#\{j : v_j \neq 0\}, \text{ for } v \in \mathbf{k}^n.$$

In particular, the distance between the disturbed message and the original message is equal to the number of errors. The minimum distance function is given by

$$\delta(C) := \min\{wt(v) : v \in C \backslash \{0\}\}.$$

By linearity, this number describes the minimum distance between two different codewords. Therefore, $\delta(C)$ describes how many errors the code is able to handle. This is the case since the ball of radius $\delta(C) - 1$ centred in a codeword defines an area where all disturbed messages should collide to the centre in the processing of correcting the errors.

A generalization of this function is given by the r-generalised Hamming weight of a linear code C which is defined as

$$\delta_r(C) := \min_{D\subset C,\,\dim D = r} \#\{j : \exists v \in D,\ v_j \neq 0\}.$$

This gives a finer criterion in comparing different linear codes. In fact, by taking $r = 1$, one recovers the minimum distance function $\delta(C)$. A first property that can be proved is their monotonicity.

Proposition 4.1. *Given a $[n, s]$ code C, then we have the following series of inequalities*

$$1 \leqslant \delta_1(C) < \delta_2(C) < \cdots < \delta_s(C) \leqslant n.$$

Proof. See [112, Prop. 3.3.2]. □

A different way to appreciate this invariant is by defining the concept of dual code. Given a $[n, s]$ code C, then the dual code $C^{\perp}$ is the orthogonal space to C with respect to the usual scalar product: this is a $[n, n - s]$ code. Its defining matrix H is called the *parity-check matrix* of the code C and it represents the linear equations of the vector space C.

Proposition 4.2. *Given a $[n, s]$ code C, then we have*

$$\{n + 1 - \delta_r(C) : 1 \leqslant r \leqslant s\} \sqcup \{\delta_t(C^{\perp}) : 1 \leqslant t \leqslant n - s\} = [n].$$

Proof. See [112, Prop. 3.3.5]. □

Projective Reed-Muller type codes

Let $\mathbb{X} = \{P_1, \dots, P_n\} \subset \mathbb{P}^{s-1} = \mathbb{P}(\mathbf{k}^s)$ be a finite set of points in some projective space. Consider the polynomial ring $S = \mathbf{k}[t_1, \dots, t_s] = \bigoplus_{d\geqslant 0} S_d$, and let $I = I(\mathbb{X})$ the coordinate ideal of $\mathbb{X}$. For every positive integer $d \geqslant 1$, it is defined the evaluation map

$$\begin{aligned} ev_d : S_d &\longrightarrow \mathbf{k}^n \\ f &\mapsto (f(P_1), \dots, f(P_n)) \end{aligned}$$

by evaluating a homogeneous polynomial in every point of $\mathbb{X}$. The function is not well-defined since a point in a projective space has coordinates defined up to a non-zero scalar. To avoid this problem, since the homogeneous coordinates of a point in $\mathbb{P}^{s-1}$ are defined up to a scalar multiple, we assume that the first non-zero coordinate of each point in $\mathbb{X}$ is 1, obtained by multiplying by an appropriate non-zero scalar. The *projective Reed-Muller* type code associated with the set $\mathbb{X}$ and the integer d is given by $C_d(\mathbb{X}) := \operatorname{Im} ev_d$. The r-generalised Hamming weight is indicated as $\delta_{\mathbb{X}}(d, r) := \delta_r(C_{\mathbb{X}}(d))$. The general focus in studying these codes is establishing connections between the cryptographic invariants of $C_{\mathbb{X}}$ and the algebraic invariants of I(see for example [45], [65]). We focus our attention to two particular papers [64], [39] in which the authors gave particular attention to the generalised Hamming distance. In particular, a general formula for computing the Hamming distance is provided.

Theorem 4.3. *[64, Thm 4.5] Let $\mathcal{F}_{d,r}$ be the (possibly empty) set of r-tuples of d-forms in S that are linear independent in S/I. Then*

$$\delta_{\mathbb{X}}(d, r) = \deg(S/I) - \max_{F\in\mathcal{F}_{d,r}} \left\{ \deg \frac{S}{I + (F)} \right\}.$$

This result inspired the authors of [39] in introducing the *Vasconcelos invariant* of a homogeneous ideal in a polynomial ring over a field. If $I \subset S$ is a homogeneous ideal, then for every associated prime $\mathfrak{p} \in \operatorname{Ass}_R(I)$ the local Vasconcelos number is

$$v_{\mathfrak{p}}(I) := \inf\{d \geqslant 0 : \text{there exists } f \in R_d \text{ such that } \mathfrak{p} = I :_R f\}.$$

Meanwhile, the Vasconcelos number is defined as $v(I) := \inf\{v_{\mathfrak{p}}(I) : \mathfrak{p} \in \operatorname{Ass}(I)\}$. This invariant has been named after the mathematician Wolmer V. Vasconcelos (1937-2021). Recent treatises about his most known results, and his legacy are [21] and [114].

Using this invariant they are able to prove the following result.

Theorem 4.4. *[39, Cor 4.7] Let $I = I(\mathbb{X}) \subsetneq S$ be the defining ideal of a set of points $\mathbb{X} \subset \mathbb{P}^{s-1}$. Then*

$$\delta_{\mathbb{X}}(1,1) > \delta_{\mathbb{X}}(2,1) > \cdots > \delta_{\mathbb{X}}(v,1) = \delta_{\mathbb{X}}(k,1), \quad \text{for all } k \geqslant v,$$

where $v = v(I)$ is the Vasconcelos number of I.

The seminal results in [39] inspired mathematicians to investigate this invariant in different areas spanning between Commutative Algebra and Combinatorics, discovering connections with other different invariants. For instance, a combinatorial interpretation is shown in [89, Thm. A] for the Vasconcelos invariant of a binomial edge ideal using the connected domination number. A similar result is also obtained in [88, Thm. 3.5] for square-free monomial ideals. In [39, p. 16], the authors connect the Vasconcelos invariant to the degree of finite projective varieties. Various (in)equalities between Castelnuovo-Mumford regularity and v-number are established in [39, Thm. 4.10], [88, Thm. 3.13], [121, p. 905], [119, Thm. 3.8], [16, Thm. 4.19], [53, Prop. 2.2 and Rmk. 2.3] and [120, Thm. A].

The aim of this chapter is to generalise the definition of (local) Vasconcelos number for graded module M over a finitely generated Noetherian ring R. In particular, we want to extend the outcomes of [50], and [36] in this setting by proving results on the asymptotic behaviour of the local Vasconcelos numbers of $M/I_1^{n_1} \cdots I_r^{n_r} N$ where $N \subset M$ is a graded sub-module, and $I_1, \dots, I_r$ are homogeneous ideal of R. A key tool is Lemma 4.15 which relates the problem of computing the Vasconcelos invariants with the computation of the initial degree of some graded R-module. To this end we introduce proper preliminaries and general considerations on the theory of initial degree, its iteration with the theory of Gröbner degenerations, and Vasconcelos number. Then, the focus is shifted, in Section 4.2, to the asymptotic properties of both the Vasconcelos number and the initial degree. We distinguish the case when only the power of a single ideal is considered (i.e. $r = 1$), and the general case. One should notice that Theorem 4.31 is a generalization of Theorem 4.22, as well as Theorem 4.35 is a generalization of Theorem 4.28. However, in the single ideal case, more specific results and descriptions are obtained. We end this section with a series of (non)examples, and explicit computations. The original results of Section 4.2 are based on the work present in [53], and [52] in collaboration with Dipankar Ghosh. The chapter ends with considerations about Vasconcelos number in local rings. This is based on discussions with Luís Duarte.

4.1 Preliminaries

We start by giving some preliminary definition and results about *initial degree* and *Vasconcelos number*. Results about Gröbner degeneration and initial degree have been inspired by [25, Sec. 8.3].

Setup 4.5. *Let $R = R_0[x_1, \dots, x_d]$ be a commutative Noetherian $\mathbb{N}$-graded ring, not necessarily standard graded. Let M be a finitely generated $\mathbb{Z}$-graded R-module. Let N be a graded submodule of M (e.g., $N = \mathfrak{a}M$ for some homogeneous ideal $\mathfrak{a}$ of R).*

Let I be a homogeneous ideal of R. Let $J \subset I$ be a homogeneous reduction ideal of I (possibly, $J = I$), i.e. J is a homogeneous ideal such that $JI^m = I^{m+1}$ for some $m \in \mathbb{N}$. Further, suppose J is generated by homogeneous elements $y_1, \dots, y_c$ of degree $d_1 \leqslant \cdots \leqslant d_c$ respectively.

For each $1 \leqslant i \leqslant r$, suppose I_i is a homogeneous ideal of R generated in degrees $d_{i,j}$ for $1 \leqslant j \leqslant a_i$. Set $\mathbf{I} := I_1 \cdots I_r$. For $\underline{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$, denote $\mathbf{I}^{\underline{n}} := I_1^{n_1} \cdots I_r^{n_r}$.

We use the following notations frequently. With Setup 4.5,

$$N :_M I := \{x \in M : Ix \subseteq N\}, \quad \Gamma_I(M) := \bigcup_{n \geqslant 1} \left(0 :_M I^n\right)$$

and $\operatorname{Ann}_M(I) := 0 :_M I$.

The first instances of the notion *ideal reduction* can be mapped to the work of Northcott, and Rees [109]. Their object of study were blow-up algebras which are strictly connected to the one of Rees algebras. Given an ideal I in some ring R, the Rees algebra associated with an ideal I is the standard $\mathbb{N}$-graded ring defined as

$$\mathscr{R}(I) := \bigoplus_{i \geqslant 0} I^i t^i = R \oplus It \oplus I^2t^2 \oplus \dots .$$

The variables t is a silent variable which is usually added to recall easily the degree of a certain element in $\mathscr{R}(I)$, and to make this object easier to handle. The concept of blow-up algebra is closely related to the one of Rees algebra, since the coordinate ring of a blow-up of a variety $V(I) \subset \operatorname{Proj} R$ is $\mathscr{R}(I)$. In the same way, given an R-module M, the Rees module as

$$\mathscr{R}(I, M) = \bigoplus_{i \geqslant 0} I^i M t^i$$

which is a $\mathbb{N}$-graded $\mathscr{R}(I)$-module. At the same time, when dealing with a finite number of ideals $I_1, \dots, I_r$, the Rees algebra of $I_1, \dots, I_r$ is defined as

$$\mathscr{R}(I_1, \dots, I_r) := \bigoplus_{\underline{n} \in \mathbb{N}^r} I_1^{n_1} \cdots I_r^{n_r} t_1^{n_1} \cdots t_r^{n_r},$$

and the Rees module of M with respect to the ideals $I_1, \dots, I_r$ is

$$\mathscr{R}(I_1, \dots, I_r; M) := \bigoplus_{\underline{n} \in \mathbb{N}^r} I_1^{n_1} \cdots I_r^{n_r} M t_1^{n_1} \cdots t_r^{n_r}.$$

As already mentioned, a reduction of I is an ideal $J \subset I$ such that $JI^k = I^{k+1}$ for some $k \in \mathbb{N}$. It follows immediately that $I^n = J^{n-k} I^k$ for every $n \geqslant k$. Whenever R is a graded ring, and I is a homogeneous ideal, it comes in handy to consider homogeneous ideal reductions. Strictly related to reductions, and in particular to minimal reductions are superficial elements and superficial sequences. Another way of introducing ideal reductions is through Rees algebras. In fact, J is a reduction of I if and only if $\mathscr{R}(J)$ is a finitely generated algebra over $\mathscr{R}(I)$. For more details on this topic, we refer to the classic reference [82].

4.1.1 Initial degree

Following Setup 4.5, the *initial degree* of a $\mathbb{Z}$-graded module M is defined as

$$\operatorname{indeg}(M) := \inf\{n \in \mathbb{Z} : M_n \neq 0\}.$$

For consistency, we impose $\operatorname{indeg}(0) = +\infty$. If M is non-zero and finitely generated, then its initial degree is finite.

Lemma 4.6. *Let R be a Noetherian $\mathbb{N}$-graded ring, and L, M, N be $\mathbb{Z}$-graded R-module. Then the following holds true.*

(1) $\operatorname{indeg}(M(a)) = \operatorname{indeg}(M) + a$*, where $M(a)$ is the graded $\mathbb{Z}$-graded R-module whose nth graded part is M_{n+a} for every $n \in \mathbb{Z}$.*

(2) If $L \hookrightarrow M$ is a graded injection of degree 0, then $\operatorname{indeg}(M) \leqslant \operatorname{indeg}(L)$.

(3) If $0 \to L \to M \to N \to 0$ is a graded short exact sequence, then $\operatorname{indeg}(M) = \min\{\operatorname{indeg}(L), \operatorname{indeg}(N)\}$.

(4) If $0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_q = M$ is a chain of graded submodules, then $\operatorname{indeg}(M) = \min\{\operatorname{indeg}(M_{i+1}/M_i) : 0 \leqslant i \leqslant q-1\}$.

Proof. Statements (1)-(2) follow from the very definition of initial degree. For Statement (3), we can restrict the exact sequence to the nth graded part, and obtain that $M_n = 0$ if and only if $N_n = L_n = 0$. Finally, from the chain of submodules, one derives the following graded exact sequences

$$0 \to M_i \to M_{i+1} \to M_{i+1}/M_i \to 0 \quad \text{for } 1 \leqslant i \leqslant q-1.$$

The last result is then a direct consequence of Statement (3). □

We are interested in the behaviour of initial degree under Gröbner perturbation. The theory of Gröbner basis can be defined also in polynomial rings over general rings with some precautions.

Consider a polynomial ring $S = A[X_1, \dots, X_n]$ over a Noetherian ring A. Similarly to polynomial rings over a field, the ring S is a free A-algebra whose canonical basis is given by the set of monomials of S namely $Mon(S)$.

Definition 4.7. A *monomial order* is a total order $<$ on $Mon(S)$ satisfying the properties:

(1) $1 < m$ for every $m \in Mon(S)$;

(2) if $m_1, m_2, m_3 \in Mon(S)$ with $m_1 < m_2$, then $m_1 m_3 < m_2 m_3$.

Proposition 4.8. *Every monomial order $<$ on $Mon(S)$ is a well-ordering, that is every non empty subset of $Mon(S)$ has a minimal element.*

Proof. Consider a non empty subset M of $Mon(S)$, and consider the ideal I generated by M in R. Since the ring A is Noetherian, by the Hilbert Basis Theorem, the ideal I is finitely generated. In particular, by [107, Thm 1.3.6], it has a unique set of minimal generators all lying in M. Now we notice that, by the definition of monomial order, if $m_1 | m_2$, then $m_1 < m_2$ for every pair of monomials. Therefore, every element in the generating set has to be minimal for S. □

Given an element $f \in S$, we can write this element as $f = a_1 m_1 + \cdots + a_t m_t$ where $a_i \in A \backslash \{0\}$ and $m_1, \dots, m_t$ are monomials listed in descending order. Then m_1 is called the *initial monomial* of f, while the *initial term* is $in_<(f) = a_1 m_1$. For a given ideal $I \subset S$, the *initial ideal* is a monomial ideal defined as $in_<(I) = (in_<(g) : g \in I)$. By convention, $in_<(0) = 0$.

Remark 4.9. In the case that A is a field, when considering the initial ideal, it is alike to define the initial ideal with the initial monomial or the initial term. However, in the general case when A is not a field, the two notions arise two different ideals.

For example, consider the ideal $I = (4x + y, 5y + z, zx - y^2) \subset \mathbb{Z}[x, y, z]$. We define the following term oder: Consider $m_a = x^{a_1} y^{a_2} z^{a_3}, m_b = x^{b_1} y^{b_2} z^{b_3} \in Mon(\mathbb{Z}[x, y, z])$ with $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3) \in \mathbb{N}^3$. Then

$$\begin{aligned} m_a < m_b \text{ if and only if} & \deg m_a < \deg m_b, \text{ or} \\ & \deg m_a = \deg m_b \text{ and the last nonzero entry of } b - a \in \mathbb{Z}^3 \text{ is negative.} \end{aligned}$$

This term order is known in the literature as *Graded Reverse Lexicographic* order or GRevLex in short. Then, using [M2], the initial ideal is $in_<(I) = (5y, 4x, z^2, yz, xz, y^2, xy)$, while by considering the initial monomials, the obtained ideal would have been (x, y, z^2). △

Theorem 4.10. *Fix a term order $<$ on S. Let I be an ideal of S, then there exists a finite system of generators $\mathcal{G} = \{f_1, \dots, f_q\}$ of I such that the ideal $in_<(I)$ is generated by $\{in_<(f_1), \dots, in_<(f_q)\}$. The set $\mathcal{G}$ is called a Gröbner basis for I.*

Proof. See [3, Corollary 4.1.17]. □

Remark 4.11. Theorem 4.10 and Definition 4.7 Property(2) ensure every monomial in $in_<(I)$ can be written as the initial term of some element in I. In fact, let $\mathcal{G} = \{f_1, \dots, f_q\}$ be a Gröbner basis for I. Then given a monomial $cm \in in_<(I)$ with $c \in A$, and $m \in Mon(S)$, it follows $cm = \sum_{j=1}^{u} c_j m_j in_<(f_{i_j})$ for $c_j \in A$, $m_j \in Mon(S)$, and for some indices $j_1, \dots, j_r$ which can repeat. Using Property (2), one has the equality $c_k m_j in_<(f_{i_j}) = in_<(c_k m_j m' f_{i_j})$. Therefore we get $cm = \sum_{j=1}^{u} in_<(c_k m_j f_{i_j})$.

When summing two, or in general more, elements, it is not true that $in_<(f + g) = in_<(f) + in_<(g)$ because some cancellations can occur. For example, in $\mathbf{k}[x]$, for any term order $<$, we have $in_<(x^2 + x^3) = x^2$, and $in_<(x^4 - x^2) = -x^2$, but $x^3 = in_<((x^2 + x^3) + (x^4 - x^2)) \gneq 0 = in_<(x^2 + x^3) + in_<(x^4 - x^2)$.

However, in this case we have that the sum of the initial terms is cm, and the only monomial in $Mon(M)$ appearing in this sum is exactly m. Therefore, no cancellations occur and one obtain the equality $cm = in_<(\sum_{j=1}^{u} c_j m_j f_{i_j})$ where $\sum_{j=1}^{u} c_j m_j f_{i_j} \in I$. △

4.12. For our purposes, we consider the generalized theory of Gröbner bases, extending it from ideals to modules. Let F be a finitely generated free S-module, and let $\mathcal{B}_F := \{e_1, \dots, e_s\}$ be the standard basis of F. Then, as an A-module, F is generated by the set $\{Xe_i : X \in \operatorname{Mon}(S), 1 \leqslant i \leqslant s\}$ of *terms.* A natural way to construct a term order on F is to fix a term order $<$ on S, and define

$$X_i e_i < X_j e_j \text{ if and only if } i < j, \text{ or } i = j \text{ and } X_i < X_j.$$

A more general approach is to consider the symmetric algebra $R[e_1, \dots, e_s]$ and view F as the R-submodule generated by $e_1, \dots, e_s$. Then any term order on the symmetric algebra restricts to a term order on the free module. Finally, as in the classical case, one can now consider initial term of every element of F. Given a subset U of F, the initial submodule of U is defined to be

$$\operatorname{in}_<(U) := \text{the } A\text{-submodule of } F \text{ generated by } \{\operatorname{in}_<(u) : u \in U\}.$$

It turns out that if U is an S-submodule of F, then $\operatorname{in}_<(U)$ is also an S-submodule of F. Indeed, $\operatorname{in}_<(U)$ is always an A-submodule of F. Moreover, if U is an S-submodule of F, then $\operatorname{in}_<(U)$ is closed under multiplication by elements of S, since $\operatorname{in}_<(U)$ is closed under multiplication by monomials and any polynomial in S is an A-linear combination of monomials. Analogous results to Proposition 4.8 and Theorem 4.10 continue to hold in this setting. More details on this topic can be found in [3, Chapter 3] and [96, Sec. 1.4].

We are now ready to prove that the initial degree behaves nicely under Gröbner deformation. The main idea of the following result is that by taking the initial module, no graded components can appear and/or disappear.

Proposition 4.13. *Let $\mathscr{F}$ be a $\mathbb{Z}^{r+1}$-graded finitely generated free S-module. Set a term order $<$ on $\mathscr{F}$. Let $\mathscr{U}$ be a nonzero multigraded submodule of $\mathscr{F}$. For every $\underline{n} \in \mathbb{Z}^r$, denote $\mathscr{U}_{(\underline{n},*)} := \bigoplus_{l \in \mathbb{Z}} \mathscr{U}_{(\underline{n},l)}$ and $\mathscr{F}_{(\underline{n},*)} := \bigoplus_{l \in \mathbb{Z}} \mathscr{F}_{(\underline{n},l)}$. Then*

$$\operatorname{indeg}\left(\mathscr{F}_{(\underline{n},*)}/\mathscr{U}_{(\underline{n},*)}\right) = \operatorname{indeg}\left(\mathscr{F}_{(\underline{n},*)}/\operatorname{in}_<(\mathscr{U})_{(\underline{n},*)}\right) \text{ for every } \underline{n} \in \mathbb{Z}^r.$$

Proof. We use the setup as discussed in 4.12, i.e., $\operatorname{rank}(\mathscr{F}) = s$, and $\mathscr{F}$ is generated as an A-module by the terms $\{Xe_i : X \in \operatorname{Mon}(S), 1 \leqslant i \leqslant s\}$. Fix $\underline{n} \in \mathbb{Z}^r$. In order to prove the desired equality, it is enough to show that, for every $k \in \mathbb{Z}$, $\mathscr{F}_{(\underline{n},k)} = \mathscr{U}_{(\underline{n},k)}$ if and only if $\mathscr{F}_{(\underline{n},k)} = \operatorname{in}_<(\mathscr{U})_{(\underline{n},k)}$. We fix $k \in \mathbb{Z}$.

Suppose $\mathscr{F}_{(\underline{n},k)} = \mathscr{U}_{(\underline{n},k)}$. Since $\mathscr{F}_{(\underline{n},k)}$ is generated, as an A-module, by the terms of $\mathscr{F}$ of degree $(\underline{n}, k)$, it remains unaffected by taking the initial submodule. Therefore, one has the equality $\mathscr{F}_{(\underline{n},k)} = \operatorname{in}_<(\mathscr{U}_{(\underline{n},k)})$. Now we notice that $\operatorname{in}_<(\mathscr{U}_{(\underline{n},k)}) \subset \operatorname{in}_<(\mathscr{U})_{(\underline{n},k)}$. Indeed, given an element $u \in \mathscr{U}_{(\underline{n},k)} \subset \mathscr{U}$, its initial term $\operatorname{in}_<(u)$ has degree $(\underline{n}, k)$, hence $\operatorname{in}_<(u) \in \operatorname{in}_<(\mathscr{U})_{(\underline{n},k)}$. Thus $\mathscr{F}_{(\underline{n},k)} = \operatorname{in}_<(\mathscr{U}_{(\underline{n},k)}) \subseteq \operatorname{in}_<(\mathscr{U})_{(\underline{n},k)} \subseteq \mathscr{F}_{(\underline{n},k)}$, which implies that $\mathscr{F}_{(\underline{n},k)} = \operatorname{in}_<(\mathscr{U})_{(\underline{n},k)}$.

For the reverse implication, we proceed by contradiction: Suppose that $\mathscr{F}_{(\underline{n},k)} = \operatorname{in}_<(\mathscr{U})_{(\underline{n},k)}$, but $\mathscr{F}_{(\underline{n},k)} \neq \mathscr{U}_{(\underline{n},k)}$. Since $\mathscr{F}_{(\underline{n},k)}$ is generated as an A-module by the terms of $\mathscr{F}$ of degree $(\underline{n}, k)$, there exists an element $Xe_i \in \mathscr{F}_{(\underline{n},k)} \smallsetminus \mathscr{U}_{(\underline{n},k)}$ for some $X \in \operatorname{Mon}(S)$ and $1 \leqslant i \leqslant s$. By the well-ordering of monomial orders, we can suppose that this is the smallest term in $\mathscr{F}_{(\underline{n},k)} \smallsetminus \mathscr{U}_{(\underline{n},k)}$.

Since $\mathscr{F}_{(\underline{n},k)} = \operatorname{in}_<(\mathscr{U})_{(\underline{n},k)}$, there exists a homogeneous element $u \in \mathscr{U}_{(\underline{n},k)}$ such that $\operatorname{in}_<(u) = Xe_i$ by Remark 4.11. The element u can be written as $u = Xe_i + u_1$, where the monomials in u_1 are smaller than Xe_i. By the minimality of Xe_i, it follows that u_1 must lie in $\mathscr{U}_{(\underline{n},k)}$. Hence $Xe_i = u - u_1 \in \mathscr{U}_{(\underline{n},k)}$, which is a contradiction. ☐

4.1.2 Vasconcelos number

The definition of (local) Vasconcelos number in [39] can be very easly generalised to our setup as follows.

Definition 4.14. For each $\mathfrak{p} \in \operatorname{Ass}_R(M)$, the local Vasconcelos number of M at $\mathfrak{p}$ is the number $v_{\mathfrak{p}}(M) := \inf\{u \in \mathbb{Z} : \text{there exists } x \in M_u \text{ such that } \mathfrak{p} = 0 :_R x\}$. Then, the Vasconcelos number of M is $v(M) := \inf\{v_{\mathfrak{p}}(M) : \mathfrak{p} \in \operatorname{Ass}_R(M)\}$. By convention, $v(0) = \infty$.

Inspired by [36, Lem. 1.2], we interpret $v_{\mathfrak{p}}(M)$ as the initial degree of certain graded module as follows. It highly generalises [39, Prop. 4.2].

Lemma 4.15. *With* Setup 4.5*, let* $\mathfrak{p} \in \operatorname{Ass}_R(M)$*. Set* $X_{\mathfrak{p}} := \{\mathfrak{q} \in \operatorname{Ass}_R(M) : \mathfrak{p} \subsetneq \mathfrak{q}\}$*. Let* $V = R$ *if* $X_{\mathfrak{p}} = \varnothing$*, otherwise* $V = \prod_{\mathfrak{q} \in X_{\mathfrak{p}}} \mathfrak{q}$*. Then*

$$v_{\mathfrak{p}}(M) = \operatorname{indeg}\big(\operatorname{Ann}_M(\mathfrak{p}) / \operatorname{Ann}_M(\mathfrak{p}) \cap \Gamma_V(M)\big).$$

Proof. Let $v = v_{\mathfrak{p}}(M)$ and $w = \operatorname{indeg}\big(\operatorname{Ann}_M(\mathfrak{p}) / \operatorname{Ann}_M(\mathfrak{p}) \cap \Gamma_V(M)\big)$. Then, by definition of $v_{\mathfrak{p}}(M)$, there exists a non-zero element $x \in M_v$ such that $\mathfrak{p} = (0 :_R x)$. Hence $\mathfrak{p}x = 0$, i.e., $x \in \operatorname{Ann}_M(\mathfrak{p})$. We prove that $x \notin \operatorname{Ann}_M(\mathfrak{p}) \cap \Gamma_V(M)$, equivalently, $x \notin \Gamma_V(M)$. If $V = R$, then $\Gamma_V(M) = 0$, and there is nothing to prove. We may assume that $V \neq R$, i.e., $X_{\mathfrak{p}} \neq \varnothing$. If possible, assume that $x \in \Gamma_V(M)$. Then $V^m x = 0$ for some integer $m \geqslant 1$. Thus $V^m \subseteq (0 :_R x) = \mathfrak{p}$, which implies that $V \subseteq \mathfrak{p}$. Hence, since $V = \prod_{\mathfrak{q} \in X_{\mathfrak{p}}} \mathfrak{q}$, it follows that $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{q} \in X_{\mathfrak{p}}$. This is a contradiction because $\mathfrak{p} \subsetneq \mathfrak{q}$ (by the definition of $X_{\mathfrak{p}}$). So $x \notin \operatorname{Ann}_M(\mathfrak{p}) \cap \Gamma_V(M)$. Thus the image of x in $\operatorname{Ann}_M(\mathfrak{p}) / \operatorname{Ann}_M(\mathfrak{p}) \cap \Gamma_V(M)$ is a non-zero homogeneous element of degree v. Therefore $v \geqslant w$. It remains to prove the other inequality, i.e., $v \leqslant w$.

Suppose $y \in M_w$ induces a non-zero element of $\operatorname{Ann}_M(\mathfrak{p}) / \operatorname{Ann}_M(\mathfrak{p}) \cap \Gamma_V(M)$. Then $y \in \operatorname{Ann}_M(\mathfrak{p}) \smallsetminus \Gamma_V(M)$. In particular, $\mathfrak{p}y = 0$, i.e., $\mathfrak{p} \subseteq (0 :_R y)$. If the equality holds, i.e., $\mathfrak{p} = (0 :_R y)$, then $v \leqslant w$. So it is enough to prove that $\mathfrak{p} = (0 :_R y)$. If possible, assume that $\mathfrak{p} \subsetneq (0 :_R y)$. Note that $(0 :_R y)$ is a proper ideal, i.e., $\operatorname{Ass}_R(R/(0 :_R y)) \neq \varnothing$. Considering the map $R \to M$ given by $r \mapsto ry$, there is an injective R-module homomorphism $R/(0 :_R y) \hookrightarrow M$. So $\operatorname{Ass}_R(R/(0 :_R y)) \subseteq \operatorname{Ass}_R(M)$. Thus, for each $\mathfrak{q} \in \operatorname{Ass}_R(R/(0 :_R y))$, one has that $\mathfrak{p} \subsetneq (0 :_R y) \subseteq \mathfrak{q} \in \operatorname{Ass}_R(R/(0 :_R y)) \subseteq \operatorname{Ass}_R(M)$, which yields that $\mathfrak{q} \in X_{\mathfrak{p}}$. Therefore $\operatorname{Ass}_R(R/(0 :_R y)) \subseteq X_{\mathfrak{p}}$. In particular, $X_{\mathfrak{p}} \neq \varnothing$. Set $V_1 := \prod_{\mathfrak{q} \in \operatorname{Ass}_R(R/(0 :_R y))} \mathfrak{q}$. Hence, from a primary decomposition of $(0 :_R y)$, one deduces that there is an integer $m \geqslant 1$ such that $V_1^m \subseteq (0 :_R y)$, i.e., $V_1^m y = 0$. Since $\operatorname{Ass}_R(R/(0 :_R y)) \subseteq X_{\mathfrak{p}}$, it follows that $V \subseteq V_1$. So $V^m y \subseteq V_1^m y = 0$. Thus $y \in \Gamma_V(M)$, which is a contradiction. Therefore $\mathfrak{p} = (0 :_R y)$. □

4.16. With Setup 4.5, further assume that R is standard graded. Set $R_+ := \bigoplus_{n \geqslant 1} R_n$. Consider R-modules $H^i_{R_+}(M)$, the ith local cohomology module of M with respect to the ideal R_+. Set $a_i(M) := \operatorname{end}\big(H^i_{R_+}(M)\big) = \max\{j \in \mathbb{Z} : H^i_{R_+}(M)_j \neq 0\}$ for every $j \geqslant 0$. Then, the Castelnuovo-Mumford regularity of M is given by

$$\operatorname{reg}(M) := \max\{a_i(M) + i : 0 \leqslant i \leqslant \dim(R)\},$$

Note that $H^0_{R_+}(M) = \Gamma_{R_+}(M)$. It follows that $a_0(M) = \operatorname{end}\big(\Gamma_{R_+}(M)\big) \leqslant \operatorname{reg}(M)$. For a recent account and overview on this topic, in the settings of this chapter, see [25, Chapter 8] and [26].

The v-number and regularity of a graded module can be compared as follows.

Proposition 4.17. *With* Setup 4.5*, assume that* R *is standard graded. Denote* $R_+ := \bigoplus_{n \geqslant 1} R_n$*. Assume there exists* $\mathfrak{p} \in \operatorname{Ass}_R(M)$ *such that* $R_+ \subseteq \mathfrak{p}$*, then* $v_{\mathfrak{p}}(M) \leqslant \operatorname{reg}(M)$*. In particular, if* $\operatorname{depth}(R_+, M) = 0$*, then* $v(M) \leqslant \operatorname{reg}(M)$*.*

Proof. Consider $\mathfrak{p} \in \operatorname{Ass}_R(M)$ such that $R_+ \subseteq \mathfrak{p}$, and set $v = v_{\mathfrak{p}}(M)$. Then there exists a non-zero element $x \in M_v$ such that $\mathfrak{p} = (0 :_R x)$. Since $R_+ \subseteq \mathfrak{p}$, it follows that $R_+ x = 0$. Thus $x \in \Gamma_{R_+}(M) \cap M_v = \big(\Gamma_{R_+}(M)\big)_v$, which implies that $v \leqslant \operatorname{end}\big(\Gamma_{R_+}(M)\big) \leqslant \operatorname{reg}(M)$. This proves the first part. If $\operatorname{depth}(R_+, M) = 0$, then $R_+ \subseteq \bigcup_{\mathfrak{q} \in \operatorname{Ass}_R(M)} \mathfrak{q}$, which yields that $R_+ \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R(M)$ (by prime avoidance), and hence $v(M) \leqslant v_{\mathfrak{p}}(M) \leqslant \operatorname{reg}(M)$. □

From now on, we go back to the general case (Setup 4.5). The v-number of a module is less than or equal to that of any of its submodule.

Proposition 4.18. *Let L be a graded submodule of M. Then*

(1) $v_{\mathfrak{p}}(M) \leqslant v_{\mathfrak{p}}(L)$ *for each* $\mathfrak{p} \in \operatorname{Ass}_R(L)$.

(2) $v(M) \leqslant v(L)$.

Proof. Let $\mathfrak{p} \in \operatorname{Ass}_R(L)$, and $v = v_{\mathfrak{p}}(L)$. Then there exists $x \in L_v$ such that $\mathfrak{p} = (0 :_R x)$. Since $x \in L_v \subseteq M_v$, one obtains that $\mathfrak{p} \in \operatorname{Ass}_R(M)$ and $v_{\mathfrak{p}}(M) \leqslant v = v_{\mathfrak{p}}(L)$. It remains to prove the second part. Since $v(0) = \infty$, we may assume that $L \neq 0$, equivalently, $\operatorname{Ass}_R(L)$ is non-empty. Let $w = v(L)$. Then $w = v_{\mathfrak{q}}(L)$ for some $\mathfrak{q} \in \operatorname{Ass}_R(L)$. Since $\operatorname{Ass}_R(L) \subseteq \operatorname{Ass}_R(M)$, it follows that $\mathfrak{q} \in \operatorname{Ass}_R(M)$. By the first part, $v_{\mathfrak{q}}(M) \leqslant v_{\mathfrak{q}}(L) = w$. Hence $v(M) \leqslant v_{\mathfrak{q}}(M) \leqslant w = v(L)$. □

4.2 Asymptotic properties of Vasconcelos number and initial degree

A classical result of Brodmann [23] states that both the sets $\operatorname{Ass}_R(I^nM/I^{n+1}M)$ and $\operatorname{Ass}_R(M/I^nM)$ are eventually constants. Set $\mathcal{A}(I) := \operatorname{Ass}_R(R/I^n)$ for all $n \gg 0$. Recently, in [36], Conca proved that when R is domain, for each $\mathfrak{p} \in \mathcal{A}(I)$, the function $v_{\mathfrak{p}}(R/I^n)$ is eventually linear in n, i.e., $v_{\mathfrak{p}}(R/I^n) = an+b$ for all $n \gg 0$, where a and b are some constants. When R is a polynomial ring over a field, this result has been shown independently by Ficarra-Sgroi in [50, Thm. 3.1]. With Setup 4.5, the sets $\operatorname{Ass}_R(I^nM/I^nN)$ and $\operatorname{Ass}_R(M/I^nN)$ are also eventually constants due to McAdam-Eakin [102, Prop. 2] (cf. 4.20) and Katz-West [90, Prop. 5.2] respectively. So a natural question arises whether the functions $v(I^nM/I^nN)$ and $v(M/I^nN)$ are eventually linear in n? Another motivation of this question came from a result of Trung-Wang [126, Thm. 3.2] that $\operatorname{reg}(I^nM)$ is eventually a linear function of n. This was proved earlier for polynomial rings over a field by Cutkosky-Herzog-Trung [41, Thm. 1.1] and Kodiyalam [95] independently.

At the same time similar results are also proved when multiple ideals are involved. By [76, Cor. 1.2], the sets $\operatorname{Ass}_R(M/\mathbf{I}^{\underline{n}}N)$ and $\operatorname{Ass}_R(\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}}N)$ stabilize (possibly to two different sets) for all $\underline{n} \gg \underline{0}$. Similar results regarding Castelnuovo-Munford regularity are proved in this general setting: when R is a standard graded algebra over a field, in [61, Cor. 4.4], it is shown that $\operatorname{reg}(\mathbf{I}^{\underline{n}}M)$ is bounded above by a linear function in $\underline{n}$. Later, Bruns-Conca in [24, Thm. 2.2] proved that asymptotically $\operatorname{reg}(\mathbf{I}^{\underline{n}}M)$ is, in fact, the maximum of finitely many linear functions in $\underline{n}$.

Notation 4.19. With Setup 4.5, we denote

$$\mathcal{A}_N^M(I) := \operatorname{Ass}_R(M/I^nN) \ \text{ and } \ \mathcal{B}_N^M(I) := \operatorname{Ass}_R(I^nM/I^nN) \ \text{ for all } n \gg 0,$$

and

$$\mathcal{A}_N^M(\mathbf{I}) := \operatorname{Ass}_R(M/\mathbf{I}^{\underline{n}}N) \ \text{ and } \ \mathcal{B}_N^M(\mathbf{I}) := \operatorname{Ass}_R(\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}}N) \ \text{ for all } \underline{n} \gg \underline{0}.$$

4.2.1 Single ideal case

4.20. In [102, Prop. 2], McAdam-Eakin proved that if $\mathcal{R}$ is a Noetherian standard graded algebra over R, then $\operatorname{Ass}_R(\mathcal{R}_n)$ is eventually constant. This proof can be easily modified to give the result for a finitely generated graded $\mathcal{R}$-module $\mathcal{M}$, see [133, Thm. 3.4]. Note that the Rees module $\mathscr{R}(I,M)$ is finitely generated graded over the Rees algebra $\mathscr{R}(I)$. Since N is a submodule of M, the Rees module $\mathscr{R}(I,N)$ is a graded submodule of $\mathscr{R}(I,M)$. So the quotient $\mathscr{H} := \bigoplus_{n\geqslant 0} I^nM/I^nN$ is a finitely generated graded module over $\mathscr{R}(I)$. Thus, by [133, Thm. 3.4], it follows that the set $\operatorname{Ass}_R(I^nM/I^nN)$ is constant for all $n \gg 0$.

With Setup 4.5, we actually have a bigrading structure on $\mathscr{H} = \bigoplus_{n\geqslant 0} I^nM/I^nN$.

4.21. With Setup 4.5, let $\deg(x_i) = f_i$ for $1 \leqslant i \leqslant d$. Let $\mathscr{R}(J) = \bigoplus_{n\in\mathbb{N}} J^n$ be the Rees algebra of J. We consider it as an $\mathbb{N}^2$-graded ring by setting

$$\mathscr{R}(J)_{(n,l)} = (J^n)_l$$

for each $(n,l) \in \mathbb{N}^2$. Then $\mathscr{R}(J)$ can be written as $\mathscr{R}(J) = R_0[x_1,\ldots,x_d,y_1,\ldots,y_c]$, where $\deg(x_i) = (0,f_i)$ for $1 \leqslant i \leqslant d$ and $\deg(y_j) = (1,d_j)$ for $1 \leqslant j \leqslant c$. We recall that in Setup 4.5 we have supposed $J = (y_1,\ldots,y_c)$ and $\deg_R y_j = d_j$. Consider the Rees module $\mathscr{R}(I,M) = \bigoplus_{n\in\mathbb{Z}} I^nM$. By convention, $I^nM = 0$ whenever $n < 0$. Setting $\mathscr{R}(I,M)_{(n,l)} := (I^nM)_l$ for $(n,l) \in \mathbb{Z}^2$, we make $\mathscr{R}(I,M)$ a $\mathbb{Z}^2$-graded module over $\mathscr{R}(J)$. Since J is a reduction ideal of I, the module $\mathscr{R}(I,M)$ is finitely generated over $\mathscr{R}(J)$. With similar gradation, $\mathscr{R}(I,N)$ is a $\mathbb{Z}^2$-graded $\mathscr{R}(J)$-submodule of $\mathscr{R}(I,M)$. So the quotient $\mathscr{H} := \mathscr{R}(I,M)/\mathscr{R}(I,N)$ is a finitely generated $\mathbb{Z}^2$-graded module over $\mathscr{R}(J)$.

We deduce Theorem 4.25 from the following more general result.

Theorem 4.22. *Let $T = R_0[x_1,\ldots,x_d,y_1,\ldots,y_c]$ be a bigraded ring over a commutative Noetherian ring R_0, where $\deg(x_i) = (0,f_i)$ for $1 \leqslant i \leqslant d$ and $\deg(y_j) = (1,d_j)$ for $1 \leqslant j \leqslant c$, where $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_c$. Let $\mathscr{L}$ be a finitely generated $\mathbb{Z}^2$-graded T-module. Set $\delta := \inf\left\{j : y_j \notin \sqrt{\operatorname{Ann}_T(\mathscr{L})}\right\}$. Set $R := R_0[x_1,\ldots,x_d]$, where $\deg(x_i) = f_i$ for $1 \leqslant i \leqslant d$. Denote $\mathscr{L}_{(n,*)} := \bigoplus_{l\in\mathbb{Z}} \mathscr{L}_{(n,l)}$ for each $n \in \mathbb{Z}$.*

Then, $\mathscr{L}_{(n,*)}$ is a $\mathbb{Z}$-graded R-module for each $n \in \mathbb{Z}$. Moreover, either $\mathscr{L}_{(n,*)} = 0$ for all $n \gg 0$, or $\mathscr{L}_{(n,*)} \neq 0$ for all $n \gg 0$. In the second case, δ is finite, and the following holds true:

(1) *$\operatorname{indeg}\left(\mathscr{L}_{(n,*)}\right) = d_\delta \cdot n + b_1$ for all $n \gg 0$, and for some $b_1 \in \mathbb{Z}$;*

(2) *$v_{\mathfrak{p}}\left(\mathscr{L}_{(n,*)}\right) = a_{\mathfrak{p}} \cdot n + b_{\mathfrak{p}}$ for all $n \gg 0$ whenever $\mathfrak{p} \in \operatorname{Ass}_R\left(\mathscr{L}_{(n,*)}\right)$ for all $n \gg 0$, and for some $b_{\mathfrak{p}} \in \mathbb{Z}$ and $a_{\mathfrak{p}} \in \{d_j : \delta \leqslant j \leqslant c\}$ (both integers depending only on the prime $\mathfrak{p}$);*

(3) *$v\left(\mathscr{L}_{(n,*)}\right) = d_\delta \cdot n + b_2$ for all $n \gg 0$, and for some $b_2 \in \mathbb{Z}$.*

Proof. Note that $x_i\mathscr{L}_{(n,*)} \subseteq \mathscr{L}_{(n,*)}$ for each $n \in \mathbb{Z}$ and $1 \leqslant i \leqslant d$. Thus, restricting the scalars from T to R, the set $\mathscr{L}_{(n,*)}$ forms a $\mathbb{Z}$-graded R-module. From the construction of $\mathscr{L}_{(n,*)}$, the bigraded module $\mathscr{L}$ can be written as $\mathscr{L} = \bigoplus_{n\in\mathbb{Z}} \mathscr{L}_{(n,*)}$. Since $y_j\mathscr{L}_{(n,*)} \subseteq \mathscr{L}_{(n+1,*)}$, we may consider $\mathscr{L} = \bigoplus_{n\in\mathbb{Z}} \mathscr{L}_{(n,*)}$ as a $\mathbb{Z}$-graded module over an $\mathbb{N}$-graded ring $T = R[y_1,\ldots,y_c]$, where $\deg(y_j) = 1$ for $1 \leqslant j \leqslant c$, and R is the 0th graded component of T in this gradation. Since we are only changing the grading (from bigraded to graded), $\mathscr{L} = \bigoplus_{n\in\mathbb{Z}} \mathscr{L}_{(n,*)}$ is also finitely generated as a graded module over $T = R[y_1,\ldots,y_c]$. Consequently, by [133, Thm. 3.4], there exists n_0 such that $\operatorname{Ass}_R\left(\mathscr{L}_{(n,*)}\right) = \operatorname{Ass}_R\left(\mathscr{L}_{(n_0,*)}\right)$ for all $n \geqslant n_0$. Denote $\mathcal{A}_{\mathscr{L}} := \operatorname{Ass}_R\left(\mathscr{L}_{(n_0,*)}\right)$. Clearly, if $\mathcal{A}_{\mathscr{L}}$ is an empty-set, then $\mathscr{L}_{(n,*)} = 0$ for all $n \geqslant n_0$. In the other case, $\mathcal{A}_{\mathscr{L}} \neq \varnothing$, and we have that $\mathscr{L}_{(n,*)} \neq 0$ for all $n \geqslant n_0$. Since $\mathscr{L}_{(n,*)} \neq 0$ for $n \gg 0$, we must have that $(y_1,\ldots,y_c) \nsubseteq \sqrt{\operatorname{Ann}_T(\mathscr{L})}$, and hence δ is a finite number.

(1) Consider an $\mathbb{N}^2$-graded polynomial ring $S = R_0[X_1,\ldots,X_d,Y_1,\ldots,Y_c]$ over R_0, where $\deg(X_i) = (0,f_i)$ for $1 \leqslant i \leqslant d$ and $\deg(Y_j) = (1,d_j)$ for $1 \leqslant j \leqslant c$. There is a natural graded ring homomorphism $S \to T$. Via this homomorphism, $\mathscr{L}$ is a finitely generated $\mathbb{Z}^2$-graded S-module as well. Hence, by [32, Prop. 3.1][1], there exist $a_1 \in \{d_j : 1 \leqslant j \leqslant c\}$ and $b_1 \in \mathbb{Z}$ such that

$$\operatorname{indeg}\left(\mathscr{L}_{(n,*)}\right) = \operatorname{indeg}\left(\operatorname{Tor}_0^R(\mathscr{L}_{(n,*)}, R_0)\right) = a_1 n + b_1 \text{ for all } n \gg 0. \tag{4.1}$$

We show that $a_1 = d_\delta$, where $\delta = \inf\left\{j : y_j \notin \sqrt{\operatorname{Ann}_T(\mathscr{L})}\right\}$. As a graded module over $T = R[y_1,\ldots,y_c]$, let

$$\mathscr{L} = \textstyle\bigoplus_{n\in\mathbb{Z}} \mathscr{L}_{(n,*)} \text{ be generated by homogeneous elements of degree } \leqslant n_1. \tag{4.2}$$

For $1 \leqslant j < \delta$, since $y_j \in \sqrt{\operatorname{Ann}_T(\mathscr{L})}$, there exist k_j such that $y_j^{k_j} \in \operatorname{Ann}_T(\mathscr{L})$. Set $n_2 := (\delta-1) \cdot \max\{k_j : 1 \leqslant j < \delta\}$. So $(y_1,\ldots,y_{\delta-1})^k\mathscr{L} = 0$ for all $k \geqslant n_2$. Thus, for every $n \geqslant n_1 + n_2$, one has

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[1] Note that [32, Prop. 3.1] is proved using a result [14, Thm. 4.6] of Bagheri-Chardin-Hà. As [14, Thm. 4.6] holds with this grading, we have the output of [32, Prop. 3.1] in this setup as well.

that

$$\begin{aligned}\mathscr{L}_{(n,*)} &= \bigoplus_{i\leqslant n_1} (y_1,\dots,y_c)^{n-i}\mathscr{L}_{(i,*)}\\ &= \bigoplus_{i\leqslant n_1}\ \bigoplus_{k\leqslant n-i} (y_1,\dots,y_{\delta-1})^k(y_\delta,\dots,y_c)^{n-i-k}\mathscr{L}_{(i,*)}\\ &= \bigoplus_{i\leqslant n_1}\ \bigoplus_{k\leqslant n_2} (y_1,\dots,y_{\delta-1})^k(y_\delta,\dots,y_c)^{n-i-k}\mathscr{L}_{(i,*)}.\end{aligned}$$

Moreover, for every $n \geqslant n_1+n_2$, note that $y_\delta\mathscr{L}_{(n,*)} \neq 0$. Indeed, if $y_\delta\mathscr{L}_{(n,*)} = 0$ for some $n \geqslant n_1+n_2$, then $y_\delta^{n-i+1}\mathscr{L}_{(i,*)} \subseteq y_\delta\mathscr{L}_{(n,*)} = 0$ for all $i \leqslant n_1$, and hence $y_\delta \in \sqrt{\operatorname{Ann}_T(\mathscr{L})}$, a contradiction. Thus one concludes that

$$\operatorname{indeg}\left(\mathscr{L}_{(n+1,*)}\right) - \operatorname{indeg}\left(\mathscr{L}_{(n,*)}\right) = \deg(y_\delta) = d_\delta \text{ for all } n \gg 0. \tag{4.3}$$

Consequently, the statement (1) follows from (4.1) and (4.3).

(2) Let $\mathfrak{p} \in \operatorname{Ass}_R\left(\mathscr{L}_{(n,*)}\right)$ for all $n \gg 0$, equivalently, $\mathfrak{p} \in \mathcal{A}_{\mathscr{L}}$. Set $X_\mathfrak{p} := \{\mathfrak{q} \in \mathcal{A}_{\mathscr{L}} : \mathfrak{p} \subsetneq \mathfrak{q}\}$. Let $V = R$ if $X_\mathfrak{p} = \varnothing$, otherwise $V = \prod_{\mathfrak{q}\in X_\mathfrak{p}} \mathfrak{q}$. Let $\mathscr{M} = \operatorname{Ann}_{\mathscr{L}}(\mathfrak{p})/\operatorname{Ann}_{\mathscr{L}}(\mathfrak{p}) \cap \Gamma_V(\mathscr{L})$. Note that $y_j \in \sqrt{\operatorname{Ann}_T(\mathscr{L})} \subseteq \sqrt{\operatorname{Ann}_T(\mathscr{M})}$ for all $1 \leqslant j < \delta$. Since T is Noetherian, and $\mathscr{L}$ is finitely generated, the (sub)quotient $\mathscr{M}$ is also a finitely generated $\mathbb{Z}^2$-graded module over T, where the grading of $\mathscr{M}$ is induced by that of $\mathscr{L}$. In particular, $\mathscr{M}_{(n,*)} = \bigoplus_{l\in\mathbb{Z}} \mathscr{M}_{(n,l)}$ is same as $\operatorname{Ann}_{\mathscr{L}_{(n,*)}}(\mathfrak{p})/\operatorname{Ann}_{\mathscr{L}_{(n,*)}}(\mathfrak{p}) \cap \Gamma_V(\mathscr{L}_{(n,*)})$. Therefore, in view of Lemma 4.15 and (1), one has that $v_\mathfrak{p}\left(\mathscr{L}_{(n,*)}\right) = \operatorname{indeg}(\mathscr{M}_{(n,*)}) = a_\mathfrak{p} n + b_\mathfrak{p}$ for all $n \gg 0$, and for some $a_\mathfrak{p} \in \{d_j : \delta \leqslant j \leqslant c\}$ and $b_\mathfrak{p} \in \mathbb{Z}$.

(3) Note that $\mathcal{A}_{\mathscr{L}}$ is a (non-empty) finite set. By the definition of v-numbers, $v\left(\mathscr{L}_{(n,*)}\right) = \inf\{v_\mathfrak{p}\left(\mathscr{L}_{(n,*)}\right) : \mathfrak{p} \in \mathcal{A}_{\mathscr{L}}\}$ for all $n \geqslant n_0$. Hence from (2) and the observation made in 4.23, the function $v\left(\mathscr{L}_{(n,*)}\right)$ is eventually linear, i.e., $v\left(\mathscr{L}_{(n,*)}\right) = a\cdot n + b_2$ for all $n \gg 0$, and for some $a \in \{d_j : \delta \leqslant j \leqslant c\}$ and $b_2 \in \mathbb{Z}$. Clearly, $d_\delta \leqslant a$. It is enough to show that $a \leqslant d_\delta$.

Since the module $\mathscr{L}$ is Noetherian, the chain of submodules

$$(0 :_{\mathscr{L}} y_\delta) \subseteq (0 :_{\mathscr{L}} y_\delta^2) \subseteq (0 :_{\mathscr{L}} y_\delta^3) \subseteq \cdots$$

stabilizes. So there exists $m_0 \geqslant 1$ such that $(0 :_{\mathscr{L}} y_\delta^m) = (0 :_{\mathscr{L}} y_\delta^{m_0})$ for all $m \geqslant m_0$. In particular, $(0 :_{\mathscr{L}_{(n_1,*)}} y_\delta^m) = (0 :_{\mathscr{L}_{(n_1,*)}} y_\delta^{m_0})$ for all $m \geqslant m_0$, where n_1 is as in (4.2). We prove that $(0 :_{\mathscr{L}_{(n_1,*)}} y_\delta^{m_0})$ is a proper R-submodule of $\mathscr{L}_{(n_1,*)}$. If possible, let $(0 :_{\mathscr{L}_{(n_1,*)}} y_\delta^{m_0}) = \mathscr{L}_{(n_1,*)}$. Then $y_\delta^{m_0}\mathscr{L}_{(n_1,*)} = 0$. Denote $\operatorname{indeg}(\mathscr{L}) := \inf\{n : \mathscr{L}_{(n,*)} \neq 0\}$. Setting $l := \max\{0, -\operatorname{indeg}(\mathscr{L})\}$, for all $\operatorname{indeg}(\mathscr{L}) \leqslant n \leqslant n_1$, one has that $y_\delta^{m_0+n_1+l}\mathscr{L}_{(n,*)} \subseteq y_\delta^{m_0+n+l}\mathscr{L}_{(n_1,*)} = 0$ as $n+l \geqslant 0$. Therefore, since $\mathscr{L}$ is generated by homogeneous elements of degree $\leqslant n_1$, one concludes that $y_\delta^{m_0+n_1+l}\mathscr{L}_{(n,*)} = 0$ for all $n \in \mathbb{Z}$, and hence $y_\delta \in \sqrt{\operatorname{Ann}_T \mathscr{L}}$, a contradiction. Thus $(0 :_{\mathscr{L}_{(n_1,*)}} y_\delta^{m_0}) \subsetneq \mathscr{L}_{(n_1,*)}$.

We show that $v\left(\mathscr{L}_{(n,*)}\right) \leqslant v\Big(\mathscr{L}_{(n_1,*)}/\big(0 :_{\mathscr{L}_{(n_1,*)}} y_\delta^{m_0}\big)\Big) + (n-n_1)\cdot d_\delta$ for all $n \geqslant n_1+m_0$. For every $n \geqslant m \geqslant 0$, the natural map $\mathscr{L}_{(n-m-1,*)}(-d_\delta) \xrightarrow{y_\delta} \mathscr{L}_{(n-m,*)}/\big(0 :_{\mathscr{L}_{(n-m,*)}} y_\delta^m\big)$ induces a graded injective R-module homomorphism

$$\frac{\mathscr{L}_{(n-m-1,*)}}{\big(0 :_{\mathscr{L}_{(n-m-1,*)}} y_\delta^{m+1}\big)}(-d_\delta) \xrightarrow{y_\delta} \frac{\mathscr{L}_{(n-m,*)}}{\big(0 :_{\mathscr{L}_{(n-m,*)}} y_\delta^{m}\big)}. \tag{4.4}$$

Here $M(-h)$ denotes a graded R-module with M_{n-h} as its n-graded component. So, by definition of v-numbers, $v(M(-h)) = v(M) + h$. Thus, by Proposition 4.18.(2), for every $n \geqslant m \geqslant 0$, the map (4.4) yields that

$$v\left(\frac{\mathscr{L}_{(n-m,*)}}{\big(0 :_{\mathscr{L}_{(n-m,*)}} y_\delta^{m}\big)}\right) \leqslant v\left(\frac{\mathscr{L}_{(n-m-1,*)}}{\big(0 :_{\mathscr{L}_{(n-m-1,*)}} y_\delta^{m+1}\big)}\right) + d_\delta. \tag{4.5}$$

Using this inequality repeatedly starting from $m = 0$, one obtains that

$$\begin{aligned} v\left(\mathscr{L}_{(n,*)}\right) &\leqslant v\left(\frac{\mathscr{L}_{(n-1,*)}}{\left(0 :_{\mathscr{L}_{(n-1,*)}} y_\delta^1\right)}\right) + d_\delta \leqslant v\left(\frac{\mathscr{L}_{(n-2,*)}}{\left(0 :_{\mathscr{L}_{(n-2,*)}} y_\delta^2\right)}\right) + 2 \cdot d_\delta \leqslant \cdots \\ &\leqslant v\left(\frac{\mathscr{L}_{(n_1,*)}}{\left(0 :_{\mathscr{L}_{(n_1,*)}} y_\delta^{n-n_1}\right)}\right) + (n - n_1) \cdot d_\delta \\ &= n \cdot d_\delta + v\Big(\mathscr{L}_{(n_1,*)}/\big(0 :_{\mathscr{L}_{(n_1,*)}} y_\delta^{m_0}\big)\Big) - n_1 \cdot d_\delta \ \text{ for all } n \geqslant n_1 + m_0. \end{aligned}$$

It follows that $a \leqslant d_\delta$, and hence $a = d_\delta$. □

4.23. Let r be a positive integer. Let $a_i, b_i \in \mathbb{Z}$ for $i = 1, \ldots, r$. Define $f(n) = \inf\{a_i n + b_i : 1 \leqslant i \leqslant r\}$ for all $n \in \mathbb{N}$. Then $f(n) = an + b$ for all $n \gg 0$, where $a := \inf\{a_1, \ldots, a_r\}$ and $b := \inf\{b_i : a_i = a, 1 \leqslant i \leqslant r\}$.

Now we are in a position to prove Theorem 4.25. Here, in order to describe the leading coefficient of $v(I^n M/I^n N)$, we need the following.

4.24. With Setup 4.5, considering the (graded) module $\mathscr{H} := \mathscr{R}(I, M)/\mathscr{R}(I, N)$ over the Rees algebra $\mathscr{R}(J)$, set $\delta := \inf\left\{j : y_j \notin \sqrt{\operatorname{Ann}_{\mathscr{R}(J)}(\mathscr{H})}, 1 \leqslant j \leqslant c\right\}$.

Theorem 4.25. *With* Setup 4.5 *and* Notation 4.19, *let* $\mathcal{B}_N^M(I)$ *be a non-empty set. Then, for every* $\mathfrak{p} \in \mathcal{B}_N^M(I)$, *there exist* $a \in \{d_\delta, \ldots, d_c\}$ *and* $b \in \mathbb{Z}$ *such that* $v_{\mathfrak{p}}(I^n M/I^n N) = an + b$ *for all* $n \gg 0$, *where* δ *is as in* 4.24. *Furthermore, both the functions* $\operatorname{indeg}(I^n M/I^n N)$ *and* $v(I^n M/I^n N)$ *are eventually linear in* n *with the same leading coefficient* $d_\delta \in \{d_1, \ldots, d_c\}$.

Proof. With the discussion made in 4.21, $\mathscr{H} := \mathscr{R}(I, M)/\mathscr{R}(I, N)$ is a finitely generated $\mathbb{Z}^2$-graded module over $\mathscr{R}(J) = R_0[x_1, \ldots, x_d, y_1, \ldots, y_c]$, where $\deg(x_i) = (0, f_i)$ for $1 \leqslant i \leqslant d$ and $\deg(y_j) = (1, d_j)$ for $1 \leqslant j \leqslant c$. Note that $\mathscr{H}_{(n,*)} := \bigoplus_{l \in \mathbb{Z}} \mathscr{H}_{(n,l)}$ is given by $\mathscr{R}(I, M)_{(n,*)}/\mathscr{R}(I, N)_{(n,*)}$, which is same as $I^n M/I^n N$. Therefore, in view of Theorem 4.22, one deduces that:

(1) $\operatorname{indeg}(I^n M/I^n N)$ is eventually linear in n with the leading coefficient d_δ;

(2) for every $\mathfrak{p} \in \mathcal{B}_N^M(I)$, the function $v_{\mathfrak{p}}(I^n M/I^n N)$ is eventually linear in n with the leading coefficient inside the set $\{d_\delta, \ldots, d_c\}$;

(3) $v(I^n M/I^n N)$ is eventually linear in n with the leading coefficient d_δ.

This completes the proof of the theorem. □

As a consequence of Theorem 4.25, we obtain a linear bound of $v(M/I^n N)$.

Corollary 4.26. *With* Setup 4.5 *and* Notation 4.19, *let* $\mathfrak{p} \in \mathcal{B}_N^M(I)$. *Then* $\mathfrak{p} \in \mathcal{A}_N^M(I)$, *and there exist* $a \in \{d_\delta, \ldots, d_c\}$ *and* $b \in \mathbb{Z}$ *such that* $v_{\mathfrak{p}}(M/I^n N) \leqslant an + b$ *for all* $n \gg 0$, *where* δ *is as in* 4.24. *Furthermore,* $v(M/I^n N) \leqslant d_\delta n + e$ *for all* $n \geqslant 1$, *and for some* $e \in \mathbb{Z}$.

Proof. Let $\mathfrak{p} \in \mathcal{B}_N^M(I)$. Since $I^n M/I^n N$ is a (graded) R-submodule of $M/I^n N$ for all $n \geqslant 0$, it follows that $\mathcal{B}_N^M(I) \subseteq \mathcal{A}_N^M(I)$. So $\mathfrak{p} \in \mathcal{A}_N^M(I)$. In view of Theorem 4.25, there exist $a \in \{d_\delta, \ldots, d_c\}$ and $b \in \mathbb{Z}$ such that $v_{\mathfrak{p}}(I^n M/I^n N) = an + b$ for all $n \gg 0$. On the other hand, by Proposition 4.18.(1), $v_{\mathfrak{p}}(M/I^n N) \leqslant v_{\mathfrak{p}}(I^n M/I^n N)$ for all $n \gg 0$. Combining these two results, $v_{\mathfrak{p}}(M/I^n N) \leqslant an + b$ for all $n \gg 0$.

For the second part, in view of Proposition 4.18.(2) and Theorem 4.25, there exists n_0 such that $v(M/I^n N) \leqslant v(I^n M/I^n N) = d_\delta n + b'$ for all $n \geqslant n_0$, and for some $b' \in \mathbb{Z}$. Note that $M \neq IN$. Otherwise, if $M = IN$, then $M = I^n N$ for all $n \geqslant 1$, and hence $\mathcal{A}_N^M(I)$ is an empty set, a contradiction. So $M \neq IN$. Consequently, $M \neq I^n N$, and $v(M/I^n N)$ is finite for every $n \geqslant 1$. Set e as the maximum value among b' and $\big(v(M/I^n N) - d_\delta n\big)$, $1 \leqslant n < n_0$. It follows that $v(M/I^n N) \leqslant d_\delta n + e$ for all $n \geqslant 1$. □

In the proof of Theorem 4.28, we use the following lemma.

Lemma 4.27. *Consider the setting defined in* Setup 4.5. *We suppose also that* $(0 :_M I) = 0$. *Let* $\mathfrak{u}$ *and* $\mathfrak{a}$ *be homogeneous ideals of* R *such that* $I \subseteq \mathfrak{u}$. *Then, for all* $n \gg 0$,

$$\frac{\operatorname{Ann}_{M/I^{n+1}N}(\mathfrak{u})}{\operatorname{Ann}_{M/I^{n+1}N}(\mathfrak{u}) \cap \Gamma_{\mathfrak{a}}(M/I^{n+1}N)} = \frac{\operatorname{Ann}_{I^nM/I^{n+1}N}(\mathfrak{u})}{\operatorname{Ann}_{I^nM/I^{n+1}N}(\mathfrak{u}) \cap \Gamma_{\mathfrak{a}}(I^nM/I^{n+1}N)}.$$

Proof. Note that $(I^{n+1}N :_M \mathfrak{u}) \subseteq (I^{n+1}M :_M \mathfrak{u}) \subseteq (I^{n+1}M :_M I) = I^nM$ for all $n \gg 0$, where the last equality follows from [23, Lem. (4)]. So

$$(I^{n+1}N :_M \mathfrak{u}) = (I^{n+1}N :_M \mathfrak{u}) \cap I^nM = (I^{n+1}N :_{I^nM} \mathfrak{u}) \text{ for all } n \gg 0.$$

Going modulo $I^{n+1}N$ both sides, as graded submodules of $I^nM/I^{n+1}N$,

$$\operatorname{Ann}_{M/I^{n+1}N}(\mathfrak{u}) = \operatorname{Ann}_{I^nM/I^{n+1}N}(\mathfrak{u}) \text{ for all } n \gg 0. \tag{4.6}$$

As $(I^nM/I^{n+1}N) \cap \Gamma_{\mathfrak{a}}(M/I^{n+1}N) = \Gamma_{\mathfrak{a}}(I^nM/I^{n+1}N)$, (4.6) further induces that

$$\begin{aligned} \operatorname{Ann}_{M/I^{n+1}N}(\mathfrak{u}) \cap \Gamma_{\mathfrak{a}}(M/I^{n+1}N) &= \operatorname{Ann}_{I^nM/I^{n+1}N}(\mathfrak{u}) \cap \Gamma_{\mathfrak{a}}(M/I^{n+1}N) \qquad (4.7)\\ &= \operatorname{Ann}_{I^nM/I^{n+1}N}(\mathfrak{u}) \cap \Gamma_{\mathfrak{a}}(I^nM/I^{n+1}N) \end{aligned}$$

for all $n \gg 0$. Combining (4.6) and (4.7), one obtains the desired equalities. □

Now we give the following.

Theorem 4.28. *With* Setup 4.5 *and* Notation 4.19, *let* $(0 :_M I) = 0$.

(1) *Let* $\mathfrak{p} \in \mathcal{A}_N^M(I)$ *be such that* $I \subseteq \mathfrak{p}$. *Then, there exist* $a \in \{d_1, \ldots, d_c\}$ *and* $b \in \mathbb{Z}$ *such that* $v_{\mathfrak{p}}(M/I^nN) = an + b$ *for all* $n \gg 0$. *Moreover, if* $\mathcal{B}_{IN}^M(I) = \mathcal{A}_N^M(I)$, *then* $v_{\mathfrak{p}}(I^nM/I^{n+1}N) = v_{\mathfrak{p}}(M/I^{n+1}N)$ *for all* $n \gg 0$.

(2) *Let* $\mathcal{A}_N^M(I) \neq \varnothing$, *and* $I^{n_0}M \subseteq N$ *for some* n_0 *(e.g.,* $N = M$, *or* $N = \mathfrak{a}M$ *for some homogeneous ideal* $\mathfrak{a}$ *satisfying* $I \subseteq \sqrt{\mathfrak{a}}$*). Then, the functions*

$$\operatorname{indeg}\big(I^nM/I^{n+1}N\big), \quad v\big(I^nM/I^{n+1}N\big) \quad \textit{and} \quad v\big(M/I^{n+1}N\big)$$

all are eventually linear in n *with the same leading coefficient* $d_\gamma \in \{d_1, \ldots, d_c\}$, *where*

$$\gamma = \inf\Big\{j : y_j \notin \sqrt{\operatorname{Ann}_{\mathscr{R}(J)}(\mathscr{G})}, 1 \leqslant j \leqslant c\Big\}$$

and $\mathscr{G} := \mathscr{R}(I,M)/\mathscr{R}(I,IN)$. *In addition, if* $\mathcal{B}_{IN}^M(I) = \mathcal{A}_N^M(I)$, *then*

$$v\big(I^nM/I^{n+1}N\big) = v\big(M/I^{n+1}N\big) \textit{ for all } n \gg 0.$$

(3) *When* $(0 :_M y_1) = 0$ *and* $d_1 \geqslant 1$, *the leading coefficient in* (2) *is* $d_\gamma = d_1$.

Proof. (1) Considering IN in place of N in 4.21, one has that $\mathscr{G} = \mathscr{R}(I,M)/\mathscr{R}(I,IN)$ is a finitely generated $\mathbb{Z}^2$-graded module over $\mathscr{R}(J)$, where $\mathscr{R}(J)$ is an $\mathbb{N}^2$-graded ring with the same gradation as in 4.21. Using the prime ideal $\mathfrak{p} \in \mathcal{A}_N^M(I)$, set $X_{\mathfrak{p}} := \{\mathfrak{q} \in \mathcal{A}_N^M(I) : \mathfrak{p} \subsetneq \mathfrak{q}\}$. Let $V = R$ if $X_{\mathfrak{p}} = \varnothing$, otherwise $V = \prod_{\mathfrak{q} \in X_{\mathfrak{p}}} \mathfrak{q}$. Let $\mathscr{L} = \operatorname{Ann}_{\mathscr{G}}(\mathfrak{p}) / \operatorname{Ann}_{\mathscr{G}}(\mathfrak{p}) \cap \Gamma_V(\mathscr{G})$. Then $\mathscr{L}$ is also a finitely generated $\mathbb{Z}^2$-graded module over $\mathscr{R}(J)$, where the bigrading in $\mathscr{L}$ is induced by that of $\mathscr{G}$. So $\mathscr{L}_{(n,*)} = \bigoplus_{l \in \mathbb{Z}} \mathscr{L}_{(n,l)}$ is same as $\operatorname{Ann}_{\mathscr{G}_{(n,*)}}(\mathfrak{p}) / \operatorname{Ann}_{\mathscr{G}_{(n,*)}}(\mathfrak{p}) \cap \Gamma_V(\mathscr{G}_{(n,*)})$, where $\mathscr{G}_{(n,*)} = I^nM/I^{n+1}N$. Hence, since $I \subseteq \mathfrak{p}$, in view of Lemma 4.27,

$$\mathscr{L}_{(n,*)} = \frac{\operatorname{Ann}_{M/I^{n+1}N}(\mathfrak{p})}{\operatorname{Ann}_{M/I^{n+1}N}(\mathfrak{p}) \cap \Gamma_V(M/I^{n+1}N)} \text{ for all } n \gg 0.$$

Finally, by Lemma 4.15 and Theorem 4.22.(1), there exists an integer b_1 such that $v_{\mathfrak{p}}(M/I^{n+1}N) = \operatorname{indeg}(\mathscr{L}_{(n,*)}) = d_{\delta_{\mathfrak{p}}} n + b_1$ for all $n \gg 0$, where

$$\delta_{\mathfrak{p}} = \inf\left\{ j : y_j \notin \sqrt{\operatorname{Ann}_{\mathscr{R}(J)}(\mathscr{L})}, 1 \leqslant j \leqslant c \right\}. \tag{4.8}$$

Setting $b_{\mathfrak{p}} := b_1 - d_{\delta_{\mathfrak{p}}}$, one obtains that $v_{\mathfrak{p}}(M/I^nN) = d_{\delta_{\mathfrak{p}}} n + b_{\mathfrak{p}}$ for all $n \gg 0$. This proves the first part of (1). For the second part of (1), further assume that $\mathcal{B}_{IN}^M(I) = \mathcal{A}_N^M(I)$. Analyzing the proof above, one obtains that $v_{\mathfrak{p}}(I^nM/I^{n+1}N) = v_{\mathfrak{p}}(\mathscr{G}_{(n,*)}) = \operatorname{indeg}(\mathscr{L}_{(n,*)}) = v_{\mathfrak{p}}(M/I^{n+1}N)$ for all $n \gg 0$, where the last two equalities follow from Lemma 4.15 and the two different expressions of $\mathscr{L}_{(n,*)}$.

(2) Note that $\mathcal{A}_N^M(I)$ is a finite non-empty set. Since $I^{n_0}M \subseteq N$ for some n_0, it can be verified that each $\mathfrak{p} \in \mathcal{A}_N^M(I)$ satisfies $I \subseteq \mathfrak{p}$. Hence, from the proof of (1), one observes that $\mathscr{G}_{(n,*)} = I^nM/I^{n+1}N \neq 0$ for all $n \gg 0$ (otherwise, if $\mathscr{G}_{(n,*)} = 0$, then $\mathscr{L}_{(n,*)} = 0$, and hence $v_{\mathfrak{p}}(M/I^{n+1}N) = \infty$ for all $n \gg 0$, a contradiction). Thus $\mathcal{B}_{IN}^M(I)$ is also a non-empty set. This will be used later while applying Theorem 4.25. Note that $v(M/I^{n+1}N) = \inf\{v_{\mathfrak{p}}(M/I^{n+1}N) : \mathfrak{p} \in \mathcal{A}_N^M(I)\}$ for all $n \gg 0$. Therefore, since each $\mathfrak{p} \in \mathcal{A}_N^M(I)$ contains I, by (1) and 4.23, one concludes that $v(M/I^{n+1}N)$ is eventually linear in n with the leading coefficient d_τ, where $\tau := \inf\{\delta_{\mathfrak{p}} : \mathfrak{p} \in \mathcal{A}_N^M(I)\}$ and $\delta_{\mathfrak{p}}$ is described in (4.8). We prove that $\tau = \gamma$. First, notice that in the proof of (1), if $y_j \in \sqrt{\operatorname{Ann}_{\mathscr{R}(J)}(\mathscr{G})}$, then $y_j \in \sqrt{\operatorname{Ann}_{\mathscr{R}(J)}(\mathscr{L})}$. This yields that $\delta_{\mathfrak{p}} \geqslant \gamma$ for every $\mathfrak{p} \in \mathcal{A}_N^M(I)$. Hence $\tau \geqslant \gamma$. Secondly, in view of Proposition 4.18.(2), $v(M/I^{n+1}N) \leqslant v(I^nM/I^{n+1}N)$ for all $n \geqslant 0$. Here the leading coefficient of the asymptotic linear function $v(M/I^{n+1}N)$ is same as d_τ, while the leading coefficients of $\operatorname{indeg}(I^nM/I^{n+1}N)$ and $v(I^nM/I^{n+1}N)$ are equal to d_γ by Theorem 4.25. Thus, comparing the leading coefficients, it follows that $d_\tau \leqslant d_\gamma$, which implies that $\tau \leqslant \gamma$. So $\tau = \gamma$. It proves the first part of (2). Since each $\mathfrak{p} \in \mathcal{A}_N^M(I)$ contains I, the second part of (2) follows from (1).

(3) Assume that $(0 :_M y_1) = 0$ and $d_1 \geqslant 1$. In view of (2), it is enough to show that $\gamma = 1$, i.e., $y_1 \notin \sqrt{\operatorname{Ann}_{\mathscr{R}(J)}(\mathscr{G})}$. If possible, let $y_1 \in \sqrt{\operatorname{Ann}_{\mathscr{R}(J)}(\mathscr{G})}$. Then $y_1^s \mathscr{G} = 0$ for some $s \geqslant 1$. Therefore, since $\mathscr{G}_{(n,*)} = I^nM/I^{n+1}N$, one obtains that $y_1^s I^n M \subseteq I^{n+s+1}N$ for all $n \geqslant 0$. Since J is a reduction ideal of I, there exists n_1 such that $JI^{n_1} = I^{n_1+1}$, and hence $J^nI^{n_1} = I^{n_1+n}$ for all $n \geqslant 1$. Therefore, $y_1^s I^{n_1} M \subseteq I^{n_1+s+1}N = J^{s+1}I^{n_1}N$. Since $(0 :_M I) = 0$, $I^{n_0}M \subseteq N$ and $M \neq 0$, it follows that $I^{n_1}N \neq 0$ and $I^{n_1}M \neq 0$. Moreover, since $J = (y_1, \ldots, y_c)$ and $(0 :_M y_1) = 0$, one derives that $\operatorname{indeg}(y_1^s I^{n_1} M) = sd_1 + \operatorname{indeg}(I^{n_1}M)$ and $\operatorname{indeg}(J^{s+1}I^{n_1}N) = (s+1)d_1 + \operatorname{indeg}(I^{n_1}N)$. Thus

$$\begin{aligned} sd_1 + \operatorname{indeg}(I^{n_1}M) &= \operatorname{indeg}(y_1^s I^{n_1} M) \\ &\geqslant \operatorname{indeg}(J^{s+1}I^{n_1}N) \quad [\text{as } y_1^s I^{n_1} M \subseteq J^{s+1}I^{n_1}N] \\ &= (s+1)d_1 + \operatorname{indeg}(I^{n_1}N) \\ &\geqslant (s+1)d_1 + \operatorname{indeg}(I^{n_1}M) \quad [\text{as } I^{n_1}N \subseteq I^{n_1}M], \end{aligned}$$

which is a contradiction as $d_1 \geqslant 1$. So $y_1 \notin \sqrt{\operatorname{Ann}_{\mathscr{R}(J)}(\mathscr{G})}$, and hence $d_\gamma = d_1$. □

Analyzing the proof of Theorem 4.28, we make the following remarks.

Remark 4.29. Let $n \geqslant 1$ be such that $\operatorname{Ass}_R(I^nM/I^{n+1}N) = \operatorname{Ass}_R(M/I^{n+1}N)$.

(1) In Theorem 4.28.(1), $v_{\mathfrak{p}}(I^nM/I^{n+1}N) = v_{\mathfrak{p}}(M/I^{n+1}N)$ whenever $I \subseteq \mathfrak{p} \in \operatorname{Ass}_R(I^nM/I^{n+1}N)$ and $(I^{n+1}N :_M I) = I^nM$, because the equality of the quotients in Lemma 4.27 holds whenever $(I^{n+1}N :_M I) = I^nM$.

(2) Thus, in Theorem 4.28.(2), one has that $v(I^nM/I^{n+1}N) = v(M/I^{n+1}N)$ whenever $(I^{n+1}N :_M I) = I^nM$ and $I^{n_0}M \subseteq N$ for some n_0. △

4.2.2 Asymptotic behaviour - general case

4.30. The additive group $\mathbb{Z}^r$, of r-tuples $\underline{n} = (n_1, \ldots, n_r)$ of integers with componentwise addition, is endowed with the componentwise order, that is $\underline{n} \geqslant \underline{m}$ if $n_i \geqslant m_i$ for all $i = 1, \ldots, r$. By $\underline{0}$ and $\underline{1}$, we denote the r-tuples $(0, \ldots, 0)$ and $(1, \ldots, 1)$ respectively. Let $\underline{e_j}$ for $1 \leqslant j \leqslant r$ denote the standard basis of $\mathbb{Z}^r$ as a free $\mathbb{Z}$-module. For $\underline{m}, \underline{n} \in \mathbb{Z}^r$, set $\underline{m} \cdot \underline{n} := m_1 n_1 + \cdots + m_r n_r$, which is the usual dot product of $\underline{m}$ and $\underline{n}$. For a property (P), by writing "(P) holds true for all $\underline{n} \gg \underline{0}$", we mean "there exists $\underline{m} \in \mathbb{N}^r$ such that (P) holds true for all $\underline{n} \geqslant \underline{m}$".

The following result is a generalization of Theorem 4.22. We use the theory developed above for the initial degree.

Theorem 4.31. *Let $T = R_0[x_1, \ldots, x_d, y_{1,1}, \ldots, y_{1,a_1}, \ldots, y_{r,1}, \ldots, y_{r,a_r}]$ be a $\mathbb{Z}^{r+1}$-graded ring over a commutative Noetherian ring R_0, where $\deg(x_i) = (\underline{0}, f_i)$ for $1 \leqslant i \leqslant d$ and $\deg(y_{i,j}) = (\underline{e_i}, d_{i,j})$ for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant a_i$. Assume that $f_i \geqslant 0$ for $1 \leqslant i \leqslant d$. Let $\mathscr{L}$ be a finitely generated $\mathbb{Z}^{r+1}$-graded T-module. Set $R := R_0[x_1, \ldots, x_d]$, where $\deg(x_i) = f_i$ for $1 \leqslant i \leqslant d$. Denote $\mathscr{L}_{(\underline{n},*)} := \bigoplus_{l \in \mathbb{Z}} \mathscr{L}_{(\underline{n},l)}$ for each $\underline{n} \in \mathbb{Z}^r$.*

Note that R is an $\mathbb{N}$-graded ring, and $\mathscr{L}_{(\underline{n},*)}$ is a $\mathbb{Z}$-graded R-module for each $\underline{n} \in \mathbb{Z}^r$. Moreover, the set $\operatorname{Ass}_R(\mathscr{L}_{(\underline{n},*)})$ stabilizes to a set, say $\mathcal{A}_{\mathscr{L}}$, for all $\underline{n} \gg \underline{0}$. It follows that $\mathscr{L}_{(\underline{n},*)} = 0$ for all $\underline{n} \gg \underline{0}$, or $\mathscr{L}_{(\underline{n},*)} \neq 0$ for all $\underline{n} \gg \underline{0}$. Assume the second case. Suppose $F(\underline{n}) = \operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right)$, or $F(\underline{n}) = v_{\mathfrak{p}}\left(\mathscr{L}_{(\underline{n},*)}\right)$ for $\mathfrak{p} \in \mathcal{A}_{\mathscr{L}}$, or $F(\underline{n}) = v\left(\mathscr{L}_{(\underline{n},*)}\right)$ for all $\underline{n} \in \mathbb{Z}^r$.

Then, there exist $\underline{\omega}_1, \ldots, \underline{\omega}_s \in \mathbb{Z}^r$ and $c_1, \ldots, c_s \in \mathbb{Z}$, depending on F, such that

$$F(\underline{n}) = \min\{\underline{\omega}_j \cdot \underline{n} + c_j : 1 \leqslant j \leqslant s\} \quad \text{for all } \underline{n} \gg \underline{0},$$

where the ith component ω_{ji} of the coefficient vector $\underline{\omega}_j$ lies in $\{d_{i,1}, \ldots, d_{i,a_i}\}$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$. Recall that $\underline{\omega}_j \cdot \underline{n} = \omega_{j1} n_1 + \cdots + \omega_{jr} n_r$ for $\underline{n} \in \mathbb{Z}^r$.

Proof. By writing $T = R[y_{1,1}, \ldots, y_{1,a_1}, \ldots, y_{r,1}, \ldots, y_{r,a_r}]$ with $\deg(y_{i,j}) = \underline{e_i}$ for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant a_i$, we can realize T as a Noetherian standard $\mathbb{N}^r$-graded ring over $T_{\underline{0}} = R$. Thus $\mathscr{L} = \bigoplus_{\underline{n} \in \mathbb{Z}^r} \mathscr{L}_{(\underline{n},*)}$ becomes a finitely generated $\mathbb{Z}^r$-graded T-module. So, by [133, Thm. 3.4.(i)], the set $\operatorname{Ass}_R(\mathscr{L}_{(\underline{n},*)})$ stabilizes to a set, say $\mathcal{A}_{\mathscr{L}}$, for $\underline{n} \gg \underline{0}$. If $\mathcal{A}_{\mathscr{L}}$ is an empty set, then $\mathscr{L}_{(\underline{n},*)} = 0$ for all $\underline{n} \gg \underline{0}$. In the second case, assume that $\mathcal{A}_{\mathscr{L}} \neq \varnothing$. In this case, $\mathscr{L}_{(\underline{n},*)} \neq 0$ for all $\underline{n} \gg \underline{0}$.

(1) We first prove that $\operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right)$ is asymptotically the minimum of finitely many linear functions. Consider the polynomial ring

$$\mathcal{T} := R[Y_{1,1}, \ldots, Y_{1,a_1}, \ldots, Y_{r,1}, \ldots, Y_{r,a_r}],$$

where $\deg(f) = (\underline{0}, \deg_R(f))$ for $f \in R$ and $\deg(Y_{i,j}) = (\underline{e_i}, d_{i,j})$. Then $\mathscr{L}$ can be regarded as a $\mathcal{T}$-module via the natural ring homomorphism $\mathcal{T} \to T$. We start by presenting $\mathscr{L}$ as a quotient $\mathscr{F}/\mathscr{U}$, where $\mathscr{F}$ is a $\mathbb{Z}^{r+1}$-graded free $\mathcal{T}$-module, and $\mathscr{U}$ is a multigraded submodule of $\mathscr{F}$. Then, by taking any term order $<$ on $\mathscr{F}$, as explained in 4.12, consider the initial submodule $\operatorname{in}_<(\mathscr{U})$. From Proposition 4.13, it follows that $\operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right) = \operatorname{indeg}\left(\mathscr{F}_{(\underline{n},*)}/(\operatorname{in}_<(\mathscr{U}))_{(\underline{n},*)}\right)$. Next consider a chain of multigraded submodules

$$0 = \mathscr{M}^0 \subsetneq \mathscr{M}^1 \subsetneq \cdots \subsetneq \mathscr{M}^s = \mathscr{F}/(\operatorname{in}_<(\mathscr{U}))$$

in such a way that any consecutive quotient $\mathscr{M}^j/\mathscr{M}^{j-1}$ is isomorphic to a quotient of $\mathcal{T}$ by a monomial prime ideal (up to a degree shift). In particular, by restricting the chain to the $(\underline{n}, *)$ graded component and by applying Lemma 4.6.(4), we obtain

$$\operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right) = \min\left\{\operatorname{indeg}\left(\mathscr{M}^j_{(\underline{n},*)}/\mathscr{M}^{j-1}_{(\underline{n},*)}\right) : 1 \leqslant j \leqslant s\right\}.$$

Hence it is enough to show that $\operatorname{indeg}\left(\mathscr{M}^j_{(\underline{n},*)}/\mathscr{M}^{j-1}_{(\underline{n},*)}\right)$ is asymptotically a linear function in $\underline{n}$ whose leading coefficients are taken from the degrees $d_{i,j}$. Thus, we reduce to the case that $\mathscr{L} = (\mathcal{T}/\mathcal{J})(-\underline{u}, -b)$

for some $\underline{u} \in \mathbb{Z}^r$ and $b \in \mathbb{Z}$, where $\mathcal{J} = J_0\mathcal{T} + (Y_{i,j} : Y_{i,j} \notin V)$ for some prime ideal J_0 of R and for some subset V of the set of the variables $\{Y_{1,1},\dots,Y_{1,a_1},\dots,Y_{r,1},\dots,Y_{r,a_r}\}$. Since $\mathscr{L}_{(\underline{n},*)} \neq 0$ for all $\underline{n} \gg \underline{0}$, the intersection $V \cap \{Y_{i,1},\dots Y_{i,a_i}\}$ is not an empty set for every $1 \leqslant i \leqslant r$. Set $w_i := \min\{d_{i,j} : 1 \leqslant j \leqslant a_i, Y_{i,j} \in V\}$ for $i = 1,\dots,r$, and $\underline{w} := (w_1,\dots,w_r)$. Hence, since $\mathscr{L} = (\mathcal{T}/\mathcal{J})(-\underline{u},-b)$, it follows that

$$\operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right) = \underline{w}\cdot(\underline{n}-\underline{u}) + b = \underline{w}\cdot\underline{n} + \tilde{b},$$

where $\tilde{b} := b - \underline{w}\cdot\underline{u}$. Note that here we need the condition that $f_i \geqslant 0$ for $1 \leqslant i \leqslant d$.

(2) We now prove that $v_{\mathfrak{p}}\left(\mathscr{L}_{(\underline{n},*)}\right)$ for $\mathfrak{p} \in \mathcal{A}_{\mathscr{L}}$ is asymptotically the minimum of finitely many linear functions. Given $\mathfrak{p} \in \mathcal{A}_{\mathscr{L}}$, denote $X_{\mathfrak{p}} := \{\mathfrak{q} \in \mathcal{A}_{\mathscr{L}} : \mathfrak{p} \subsetneq \mathfrak{q}\}$. Set $V := \prod_{\mathfrak{q}\in X_{\mathfrak{p}}} \mathfrak{q}$ (in the critical case $X_{\mathfrak{p}} = \varnothing$, set $V := R$). Let $\mathscr{M} = \operatorname{Ann}_{\mathscr{L}}(\mathfrak{p})/\operatorname{Ann}_{\mathscr{L}}(\mathfrak{p}) \cap \Gamma_V(\mathscr{L})$. Since T is Noetherian, and $\mathscr{L}$ is finitely generated, the quotient $\mathscr{M}$ is also a finitely generated $\mathbb{Z}^{r+1}$-graded T-module, whose grading is induced by that of $\mathscr{L}$. In particular, $\mathscr{M}_{(\underline{n},*)} = \bigoplus_{l\in\mathbb{Z}} \mathscr{M}_{(\underline{n},l)}$ is same as $\operatorname{Ann}_{\mathscr{L}_{(\underline{n},*)}}(\mathfrak{p})/\operatorname{Ann}_{\mathscr{L}_{(\underline{n},*)}}(\mathfrak{p}) \cap \Gamma_V(\mathscr{L}_{(\underline{n},*)})$ for every $\underline{n} \in \mathbb{Z}^r$. Therefore, Lemma 4.15 together with (1) yields the desired result for $v_{\mathfrak{p}}\left(\mathscr{L}_{(\underline{n},*)}\right)$.

(3) Note that $\mathcal{A}_{\mathscr{L}}$ is a non-empty finite set. By the definition of Vasconcelos invariant, $v\left(\mathscr{L}_{(\underline{n},*)}\right) = \inf\{v_{\mathfrak{p}}\left(\mathscr{L}_{(\underline{n},*)}\right) : \mathfrak{p} \in \mathcal{A}_{\mathscr{L}}\}$ for all $\underline{n} \gg \underline{0}$. Hence, by (2), $v\left(\mathscr{L}_{(n,*)}\right)$ is eventually the minimum of finitely many linear functions whose leading coefficients are taken from the degrees $d_{i,j}$. □

Remark 4.32. The condition $f_i \geqslant 0$ is a strong condition that makes sure the previous theorem holds true. Indeed, suppose that $f_i < 0$ for some $i = 1,\dots,d$, and x_i is $\mathscr{L}$-regular. Then, by taking $0 \neq \ell \in \mathscr{L}_{(\underline{n},*)}$, the element $x_i^k\cdot\ell$ is non-zero in $\mathscr{L}_{(\underline{n},*)}$ for every $k \in \mathbb{N}$, which implies that $\operatorname{indeg}(\mathscr{L}_{(\underline{n},*)}) = -\infty$. △

Under some additional conditions, the functions $\operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right)$ and $v\left(\mathscr{L}_{(\underline{n},*)}\right)$ in Theorem 4.31 are eventually linear in $\underline{n}$, as shown below.

Theorem 4.33. *With the hypotheses as in* Theorem 4.31*, without loss of generality, assume that* $d_{i,1} \leqslant d_{i,2} \leqslant \cdots \leqslant d_{i,a_i}$ *for* $1 \leqslant i \leqslant r$. *Suppose* $y_{1,1}\cdots y_{r,1} \notin \sqrt{\operatorname{Ann}\mathscr{L}}$. *Then, the functions* $\operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right)$ *and* $v\left(\mathscr{L}_{(\underline{n},*)}\right)$ *become linear for all* $\underline{n} \gg \underline{0}$ *with the same leading coefficients given by* $\underline{\delta} := (d_{1,1}, d_{2,1},\dots,d_{r,1})$.

Proof. Set $\mathbf{y}^{\underline{n}} := y_{1,1}^{n_1}\cdots y_{r,1}^{n_r}$ for $\underline{n} \in \mathbb{N}^r$. Then $\deg(\mathbf{y}^{\underline{n}}) = (\underline{n}, \underline{\delta}\cdot\underline{n})$ for all $\underline{n} \in \mathbb{N}^r$. Suppose $\mathscr{L} = \bigoplus_{\underline{n}\in\mathbb{Z}^r} \mathscr{L}_{(\underline{n},*)}$ is generated by homogeneous elements of degree $\leqslant \underline{m}$. We first prove the following claims:

Claim 1. There exists $\ell \in \mathbb{N}$ such that $(0 :_{\mathscr{L}} \mathbf{y}^{\underline{n}}) = (0 :_{\mathscr{L}} \mathbf{y}^{\ell\cdot\underline{1}})$ for every $\underline{n} \geqslant \ell\cdot\underline{1}$.

Claim 2. For every $n \in \mathbb{N}$, $\left(0 :_{\mathscr{L}_{(\underline{m},*)}} \mathbf{y}^{n\cdot\underline{1}}\right)$ is a proper submodule of $\mathscr{L}_{(\underline{m},*)}$.

Claim 3. For every $\underline{n} \in \mathbb{Z}^r$ and $\underline{\nu} \in \mathbb{N}^r$, one has

$$\operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right) \leqslant \operatorname{indeg}\left(\frac{\mathscr{L}_{(\underline{n}-\underline{\nu},*)}}{\left(0 :_{\mathscr{L}_{(\underline{n}-\underline{\nu},*)}} \mathbf{y}^{\underline{\nu}}\right)}\right) + \underline{\delta}\cdot\underline{\nu}.$$

The same inequality holds for the v-numbers and the local v-numbers at every associate prime of the quotient R-module in the right hand side.

Proof of Claim 1. Since the module $\mathscr{L}$ is Noetherian, the chain of submodules

$$(0 :_{\mathscr{L}} \mathbf{y}^{\underline{1}}) \subseteq (0 :_{\mathscr{L}} \mathbf{y}^{2\cdot\underline{1}}) \subseteq (0 :_{\mathscr{L}} \mathbf{y}^{3\cdot\underline{1}}) \subseteq \cdots$$

stabilizes. So there exists $\ell \geqslant 1$ such that $(0 :_{\mathscr{L}} \mathbf{y}^{n\cdot\underline{1}}) = (0 :_{\mathscr{L}} \mathbf{y}^{\ell\cdot\underline{1}})$ for every $n \geqslant \ell$. Fix $\underline{n} \in \mathbb{N}^r$ such that $\underline{n} \geqslant \ell\cdot\underline{1}$. Set $\alpha := \max\{n_i : 1 \leqslant i \leqslant r\}$. Then one has $\ell\cdot\underline{1} \leqslant \underline{n} \leqslant \alpha\cdot\underline{1}$, which implies that

$$(0 :_{\mathscr{L}} \mathbf{y}^{\ell\cdot\underline{1}}) \subseteq (0 :_{\mathscr{L}} \mathbf{y}^{\underline{n}}) \subseteq (0 :_{\mathscr{L}} \mathbf{y}^{\alpha\cdot\underline{1}}).$$

Since the submodules on both sides coincide by the construction of ℓ, they must all coincide to $(0 :_{\mathscr{L}} \mathbf{y}^{\ell\cdot\underline{1}})$. This proves Claim 1.

Proof of Claim 2. If possible, let $(0 :_{\mathscr{L}_{(\underline{m},*)}} \mathbf{y}^{n\cdot\underline{1}}) = \mathscr{L}_{(\underline{m},*)}$. Then $\mathbf{y}^{n\cdot\underline{1}}\mathscr{L}_{(\underline{m},*)} = 0$. Since $\mathscr{L}$ is finitely generated in degrees $\leqslant \underline{m}$, it follows that $\mathbf{y}^{\underline{1}} = y_{1,1}\cdots y_{r,1} \in \sqrt{\operatorname{Ann}\mathscr{L}}$, which is a contradiction. So $(0 :_{\mathscr{L}_{(\underline{m},*)}} \mathbf{y}^{n\cdot\underline{1}}) \subsetneqq \mathscr{L}_{(\underline{m},*)}$.

Proof of Claim 3. Fix $\underline{n} \in \mathbb{Z}^r$ and $\underline{\nu} \in \mathbb{N}^r$. Consider the T-module homomorphism $\mathscr{L} \to \mathscr{L}$ given by multiplication with $\mathbf{y}^{\underline{\nu}}$. Since $\deg(\mathbf{y}^{\underline{\nu}}) = (\underline{\nu}, \underline{\delta}\cdot\underline{\nu})$, it induces an injective graded R-module homomorphism

$$\frac{\mathscr{L}_{(\underline{n}-\underline{\nu},*)}}{\left(0 :_{\mathscr{L}_{(\underline{n}-\underline{\nu},*)}} \mathbf{y}^{\underline{\nu}}\right)}(-(\underline{\delta}\cdot\underline{\nu})) \xrightarrow{\mathbf{y}^{\underline{\nu}}} \mathscr{L}_{(\underline{n},*)}. \tag{4.9}$$

Here $M(-m)$ denotes the graded R-module with M_{n-m} as its nth graded component. By the definition of v-numbers, $v_{\mathfrak{p}}(M(-m)) = v_{\mathfrak{p}}(M) + m$ for all $\mathfrak{p} \in \operatorname{Ass}_R(M)$. Claim 3 now follows from (4.9) using the basic properties of initial degrees and [53, Prop. 2.5].

Set $\underline{n}_0 := \underline{m} + \ell\cdot\underline{1}$. Combining the three claims above, for every $\underline{n} \geqslant \underline{n}_0$, considering $\underline{\nu} = \underline{n} - \underline{m}$ in Claim 3, one obtains that

$$\operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right) \leqslant \operatorname{indeg}\left(\frac{\mathscr{L}_{(\underline{m},*)}}{\left(0 :_{\mathscr{L}_{(\underline{m},*)}} \mathbf{y}^{\underline{n}-\underline{m}}\right)}\right) + (\underline{\delta}\cdot(\underline{n}-\underline{m})) \tag{4.10}$$

$$= \underline{\delta}\cdot\underline{n} + \operatorname{indeg}\left(\frac{\mathscr{L}_{(\underline{m},*)}}{\left(0 :_{\mathscr{L}_{(\underline{m},*)}} \mathbf{y}^{\ell\cdot\underline{1}}\right)}\right) - (\underline{\delta}\cdot\underline{m}) < \infty. \tag{4.11}$$

Thus, there exists $c \in \mathbb{Z}$ such that

$$\operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right) \leqslant \underline{\delta}\cdot\underline{n} + c \text{ for all } \underline{n} \geqslant \underline{n}_0. \tag{4.12}$$

On the other hand, in Theorem 4.31, it is shown that there exist $\underline{\omega}_1, \ldots, \underline{\omega}_s \in \mathbb{Z}^r$ and $c_1, \ldots, c_s \in \mathbb{Z}$ such that

$$\operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right) = \min\{\underline{\omega}_j\cdot\underline{n} + c_j : 1 \leqslant j \leqslant s\} \quad \text{for all } \underline{n} \gg \underline{0}, \tag{4.13}$$

where the ith component ω_{ji} of the coefficient vector $\underline{\omega}_j$ lies in $\{d_{i,1}, \ldots, d_{i,a_i}\}$ for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$. In particular, by the given hypothesis, $\underline{\omega}_j \geqslant \underline{\delta}$ for $1 \leqslant j \leqslant s$, which yields that $\underline{\omega}_j\cdot\underline{n} \geqslant \underline{\delta}\cdot\underline{n}$ for all $\underline{n} \in \mathbb{N}^r$. Thus, combining (4.12) and (4.13), there exists $b \in \mathbb{Z}$ such that

$$\underline{\delta}\cdot\underline{n} + b \leqslant \operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right) \leqslant \underline{\delta}\cdot\underline{n} + c \quad \text{for all } \underline{n} \gg \underline{0}. \tag{4.14}$$

Hence, for every fixed $\underline{\nu} \gg \underline{0}$, one has that

$$m(\underline{\delta}\cdot\underline{\nu}) + b \leqslant \operatorname{indeg}\left(\mathscr{L}_{(m\underline{\nu},*)}\right) \leqslant m(\underline{\delta}\cdot\underline{\nu}) + c \quad \text{for all } m \gg 0. \tag{4.15}$$

On the other hand, for every fixed $\underline{\nu} \gg \underline{0}$, by (4.13), the function $\operatorname{indeg}\left(\mathscr{L}_{(m\underline{\nu},*)}\right)$ is linear in m for all $m \gg 0$, in fact, there exists some $j \in \{1, \ldots, s\}$ such that $\operatorname{indeg}\left(\mathscr{L}_{(m\underline{\nu},*)}\right) = m(\underline{w}_j\cdot\underline{\nu}) + c_j$ for all $m \gg 0$. In view of (4.15), the leading coefficient must be the same as $\underline{\delta}\cdot\underline{\nu}$. So $\underline{w}_j\cdot\underline{\nu} = \underline{\delta}\cdot\underline{\nu}$ for all $\underline{\nu} \gg \underline{0}$. Since $\underline{w}_j \geqslant \underline{\delta}$, it follows that $\underline{w}_j = \underline{\delta}$. Thus there exists $j \in \{1, \ldots, s\}$ such that $\underline{w}_j = \underline{\delta}$. Set $a := \min\{c_l : 1 \leqslant l \leqslant s, \underline{w}_l = \underline{\delta}\}$. Then, by (4.13),

$$\operatorname{indeg}\left(\mathscr{L}_{(\underline{n},*)}\right) = \underline{\delta}\cdot\underline{n} + a \text{ for all } \underline{n} \gg \underline{0}.$$

Similar inequalities as in (4.10) and (4.11) for v-numbers yield that

$$v(\mathscr{L}_{(\underline{n},*)}) \leqslant \underline{\delta}\cdot\underline{n} + e \text{ for all } \underline{n} \gg \underline{0},$$

where $e \in \mathbb{Z}$. These are the inequalities like (4.12). Now, arguing in the same manner as for the function $\operatorname{indeg}(\mathscr{L}_{(\underline{n},*)})$, one obtains that $v(\mathscr{L}_{(\underline{n},*)})$ is eventually linear in $\underline{n}$ with the leading coefficients given by $\underline{\delta}$. □

Remark 4.34. In the proof of Theorem 4.33, denote the quotient R-module considered in (4.11) by V, i.e., $V := \mathscr{L}_{(\underline{m},*)}/\big(0 :_{\mathscr{L}_{(\underline{m},*)}} \mathbf{y}^{\ell\cdot\underline{1}}\big)$. Then, the injective homomorphisms in (4.9) yield that $\operatorname{Ass}_R(V) \subseteq \operatorname{Ass}_R(\mathscr{L}_{(\underline{n},*)})$ for all $\underline{n} \geqslant \underline{n}_0$. Hence, for every fixed $\mathfrak{p} \in \operatorname{Ass}_R(V)$, following the same steps as (4.10) and (4.11),

$$v_{\mathfrak{p}}(\mathscr{L}_{(\underline{n},*)}) \leqslant \underline{\delta} \cdot \underline{n} + h \text{ for all } \underline{n} \gg \underline{0},$$

where $h \in \mathbb{Z}$. These inequalities are obtained under the same considerations as (4.12). Now, arguing in the same manner, for $\mathfrak{p} \in \operatorname{Ass}_R(V)$, one sees that $v_{\mathfrak{p}}(\mathscr{L}_{(\underline{n},*)})$ is eventually linear in $\underline{n}$ with the leading coefficients given by $\underline{\delta}$. △

We are now in a position to prove the main theorems.

Theorem 4.35. *With* Setup 4.5 *and* Notation 4.19*, the following statements hold.*

(1) *For each* $\mathfrak{p} \in \mathcal{B}_N^M(\mathbf{I})$*, there exist* $\underline{w}_1, \ldots, \underline{w}_s \in \mathbb{N}^r$ *and* $c_1, \ldots, c_s \in \mathbb{Z}$ *such that*

$$v_{\mathfrak{p}}(\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}}N) = \min\{\underline{w}_k \cdot \underline{n} + c_k : 1 \leqslant k \leqslant s\}$$

for $\underline{n} \gg \underline{0}$*. Moreover, if* $\underline{w}_k = (w_{k1}, w_{k2}, \ldots, w_{kr})$*, then* $w_{ki} \in \{d_{i,1}, \ldots, d_{i,a_i}\}$*.*

(2) *If* $(0 :_M I_k) = 0$ *for all* $k = 1, \ldots, r$*, and* $\mathbf{I}^{\underline{s}}M \subseteq N$ *for some* $\underline{s} \in \mathbb{N}^r$*, then for each* $\mathfrak{p} \in \mathcal{A}_N^M(\mathbf{I})$*, the same result holds true for* $v_{\mathfrak{p}}(M/\mathbf{I}^{\underline{n}}N)$*, i.e.,* $v_{\mathfrak{p}}(M/\mathbf{I}^{\underline{n}}N)$ *is asymptotically the minimum of finitely many linear functions in* $\underline{n}$*.*

(3) *With the same hypotheses of* (2)*, given* $\mathfrak{p} \in \mathcal{B}_{\mathbf{I}N}^M(\mathbf{I})$ *(hence* $\mathfrak{p} \in \mathcal{A}_N^M(\mathbf{I})$*), the functions* $v_{\mathfrak{p}}(\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}+\underline{1}}N)$ *and* $v_{\mathfrak{p}}(M/\mathbf{I}^{\underline{n}+\underline{1}}N)$ *coincide for all* $\underline{n} \gg \underline{0}$*.*

Proof. Suppose $R = R_0[x_1, \ldots, x_d]$, where $\deg(x_i) = f_i$ for $1 \leqslant i \leqslant d$. Let I_i be generated by homogeneous elements $y_{i,1}, \ldots, y_{i,a_i}$, where $\deg(y_{i,j}) = d_{i,j}$ for $1 \leqslant j \leqslant a_i$. We consider the Rees ring $\mathscr{R} = \mathscr{R}(I_1, \ldots, I_r)$ with $\mathbb{N}^{r+1}$-graded structure given by $\mathscr{R}_{(\underline{n},m)} = (\mathbf{I}^{\underline{n}})_m$ for all $(\underline{n}, m) \in \mathbb{N}^{r+1}$. Thus, $\mathscr{R}$ can be identified with the multigraded ring T as described in Theorem 4.31.

(1) Let $\mathscr{R}(I_1, \ldots, I_r; M)$ denote the Rees module of M with respect to the ideals $I_1, \ldots, I_r$. Set $\mathscr{L} := \mathscr{R}(I_1, \ldots, I_r; M)/\mathscr{R}(I_1, \ldots, I_r; N)$, where the grading is given by $\mathscr{L}_{(\underline{n},l)} := (\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}}N)_l$. Clearly, $\mathscr{L}$ is a finitely generated $\mathbb{Z}^{r+1}$-graded $\mathscr{R}$-module. Hence Theorem 4.35.(1) is a direct consequence of Theorem 4.31.

(2) Let $\mathfrak{p} \in \mathcal{A}_N^M(\mathbf{I})$. Set $X_{\mathfrak{p}} := \{\mathfrak{q} \in \mathcal{A}_N^M(\mathbf{I}) : \mathfrak{p} \subsetneq \mathfrak{q}\}$. Let $V = R$ if $X_{\mathfrak{p}} = \varnothing$, otherwise $V = \prod_{\mathfrak{q} \in X_{\mathfrak{p}}} \mathfrak{q}$. Consider $\mathscr{G} := \mathscr{R}(I_1, \ldots, I_r; M)/\mathscr{R}(I_1, \ldots, I_r; \mathbf{I}N)$, which is a finitely generated $\mathbb{Z}^{r+1}$-graded $\mathscr{R}$-module. We now consider $\mathscr{L} := \operatorname{Ann}_{\mathscr{G}}(\mathfrak{p})/\operatorname{Ann}_{\mathscr{G}}(\mathfrak{p}) \cap \Gamma_V(\mathscr{G})$. This is also a finitely generated $\mathbb{Z}^{r+1}$-graded $\mathscr{R}$-module, where the grading is induced by the one in $\mathscr{G}$. Using the notations as in Theorem 4.31, observe that

$$\mathscr{L}_{(\underline{n},*)} = \frac{\operatorname{Ann}_{\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}+\underline{1}}N}(\mathfrak{p})}{\operatorname{Ann}_{\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}+\underline{1}}N}(\mathfrak{p}) \cap \Gamma_V(\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}+\underline{1}}N)} \text{ for all } \underline{n} \in \mathbb{N}^r.$$

Since $\mathbf{I}^{\underline{s}}M \subseteq N$, it follows that $\mathbf{I} \subseteq \mathfrak{p}$. Therefore

$$(\mathbf{I}^{\underline{n}+\underline{1}}N :_M \mathfrak{p}) \subseteq (\mathbf{I}^{\underline{n}+\underline{1}}M :_M \mathfrak{p}) \subseteq (\mathbf{I}^{\underline{n}+\underline{1}}M :_M \mathbf{I}) = \mathbf{I}^{\underline{n}}M \text{ for all } \underline{n} \gg \underline{0},$$

where the last equality is obtained by [94, Lem. 1.3.(ii)]. Hence, a similar proof as that of Lemma 4.27 yields

$$\mathscr{L}_{(\underline{n},*)} = \frac{\operatorname{Ann}_{M/\mathbf{I}^{\underline{n}+\underline{1}}N}(\mathfrak{p})}{\operatorname{Ann}_{M/\mathbf{I}^{\underline{n}+\underline{1}}N}(\mathfrak{p}) \cap \Gamma_V(M/\mathbf{I}^{\underline{n}+\underline{1}}N)} \text{ for all } \underline{n} \gg \underline{0}.$$

By Lemma 4.15, one has the equality $v_{\mathfrak{p}}(M/\mathbf{I}^{\underline{n}+\underline{1}}N) = \operatorname{indeg}(\mathscr{L}_{(\underline{n},*)})$ for all $\underline{n} \gg \underline{0}$. Theorem 4.35.(2) is now a consequence of Theorem 4.31.

(3) Given $\mathfrak{p} \in \mathcal{B}_{\mathbf{I}N}^M(\mathbf{I})$. Then $\mathfrak{p} \in \mathcal{A}_N^M(\mathbf{I})$. Following the notations as in the proof of (2), the functions $v_{\mathfrak{p}}(\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}+\underline{1}}N)$ and $v_{\mathfrak{p}}(M/\mathbf{I}^{\underline{n}+\underline{1}}N)$ coincide for all $\underline{n} \gg \underline{0}$ since they both are asymptotically equal to $\operatorname{indeg}(\mathscr{L}_{(\underline{n},*)})$ by Lemma 4.15. □

Remark 4.36. Using the assumptions and notations of Theorem 4.35, it is clear that given $\mathfrak{p} \in \mathcal{B}^M_{\mathbf{I}N}(\mathbf{I}) \subseteq \mathcal{A}^M_N(\mathbf{I})$, the functions $v_{\mathfrak{p}}(\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}+\underline{1}}N)$ and $v_{\mathfrak{p}}(M/\mathbf{I}^{\underline{n}+\underline{1}}N)$ coincide as long as $(\mathbf{I}^{\underline{n}+\underline{1}}N :_M \mathfrak{p}) \subseteq \mathbf{I}^{\underline{n}}M$. △

The following result is a direct consequence of Theorem 4.35.

Corollary 4.37. *With* Setup 4.5*, the v-number $v(\mathbf{I}^{\underline{n}}M/\mathbf{I}^{\underline{n}}N)$ eventually becomes either ∞, or the minimum of finitely many linear functions in $\underline{n}$. The same holds for the function $v(M/\mathbf{I}^{\underline{n}}N)$ under the additional conditions that $(0 :_M I_k) = 0$ for all $k = 1, \ldots, r$, and $\mathbf{I}^{\underline{s}}M \subseteq N$ for some $\underline{s} \in \mathbb{N}^r$.*

When $R = R_0[X_1, \ldots, X_d]$ is a (graded) polynomial ring over a Noetherian integral domain R_0, Corollary 4.37 yields that $v(R/\mathbf{I}^{\underline{n}})$ eventually is the minimum of finitely many linear functions in $\underline{n}$. Our next theorem shows that $v(R/\mathbf{I}^{\underline{n}})$ is, in fact, eventually a linear function in $\underline{n}$, where the leading coefficients are given by the initial degrees of $I_1, \ldots, I_r$. This result is surprising because $\operatorname{reg}(R/\mathbf{I}^{\underline{n}})$ is not always eventually linear even when R is a polynomial ring over a field, as shown in [24, Ex. 3.1] by Bruns-Conca.

Theorem 4.38. *Let $R = R_0[X_1, \ldots, X_d]$ be an $\mathbb{N}$-graded polynomial ring (not necessarily standard graded) over a Noetherian integral domain R_0, and let $I_1, \ldots, I_r$ be non-zero homogeneous ideals such that* $\operatorname{indeg}(I_i) \geqslant 1$ *for at least one i. Then, the functions $v(R/\mathbf{I}^{\underline{n}})$, $v(\mathbf{I}^{\underline{n}}/\mathbf{I}^{\underline{n}+\underline{1}})$ and* $\operatorname{indeg}(\mathbf{I}^{\underline{n}}/\mathbf{I}^{\underline{n}+\underline{1}})$ *eventually become linear in $\underline{n}$ with the same leading coefficients given by $(d_1, \ldots, d_r)$, where $d_i :=$* $\operatorname{indeg}(I_i)$ *for $1 \leqslant i \leqslant r$.*

Proof. Suppose $R = R_0[X_1, \ldots, X_d]$, where $\deg(X_i) = f_i$ for $1 \leqslant i \leqslant d$. Here $f_i \geqslant 0$ for $1 \leqslant i \leqslant d$. Let I_i be generated by homogeneous elements $y_{i,1}, \ldots, y_{i,a_i}$, where $\deg(y_{i,j}) = d_{i,j}$ for $1 \leqslant j \leqslant a_i$. Without loss of generality, we may assume that $d_{i,1} \leqslant d_{i,2} \leqslant \cdots \leqslant d_{i,a_i}$ for $1 \leqslant i \leqslant r$. Then $\operatorname{indeg}(I_i) = d_{i,1}$ for $1 \leqslant i \leqslant r$. Consider the Rees ring $\mathscr{R} = \mathscr{R}(I_1, \ldots, I_r)$, which can be identified with the multigraded ring T as described in Theorem 4.31. Set $\mathscr{L} := \mathscr{R}(I_1, \ldots, I_r)/\mathbf{I}\mathscr{R}(I_1, \ldots, I_r)$. Then $\mathscr{L}$ is a finitely generated $\mathbb{N}^{r+1}$-graded $\mathscr{R}$-module. Now, we follow the notations as in Theorems 4.31 and 4.33.

We prove that the initial degree and the global v-number of $\mathscr{L}_{(\underline{n},*)} = \mathbf{I}^{\underline{n}}/\mathbf{I}^{\underline{n}+\underline{1}}$ are eventually linear in $\underline{n}$ with the same leading coefficients given by $\underline{\delta}$. For this, in view of Theorem 4.33, it is enough to show that $\mathbf{y} := y_{1,1} \cdots y_{r,1} \notin \sqrt{\operatorname{Ann}_{\mathscr{R}}(\mathscr{L})}$. If possible, let $\mathbf{y} \in \sqrt{\operatorname{Ann}_{\mathscr{R}}(\mathscr{L})}$. Then $\mathbf{y}^s\mathscr{L} = 0$ for some $s \geqslant 1$. Since $\mathscr{L}_{(\underline{n},*)} = \mathbf{I}^{\underline{n}}/\mathbf{I}^{\underline{n}+\underline{1}}$ and $\deg(\mathbf{y}) = (\underline{1}, \underline{\delta} \cdot \underline{1})$, it follows that $\mathbf{y}^s\mathbf{I}^{\underline{n}} \subseteq \mathbf{I}^{\underline{n}+(s+1)\cdot\underline{1}}$ for all $\underline{n} \in \mathbb{N}^r$. Denote $|\underline{\delta}| := \underline{\delta} \cdot \underline{1}$. As R is an integral domain, $\mathbf{I}^{\underline{n}} \neq 0$, in addition $\operatorname{indeg}(\mathbf{y}^s\mathbf{I}^{\underline{n}}) = s|\underline{\delta}| + \operatorname{indeg}(\mathbf{I}^{\underline{n}})$ and $\operatorname{indeg}(\mathbf{I}^{\underline{n}+(s+1)\cdot\underline{1}}) = (s+1)|\underline{\delta}| + \operatorname{indeg}(\mathbf{I}^{\underline{n}})$. Thus

$$\begin{aligned} s|\underline{\delta}| + \operatorname{indeg}(\mathbf{I}^{\underline{n}}) &= \operatorname{indeg}(\mathbf{y}^s\mathbf{I}^{\underline{n}}) \\ &\geqslant \operatorname{indeg}(\mathbf{I}^{\underline{n}+(s+1)\cdot\underline{1}}) \quad [\text{as } \mathbf{y}^s\mathbf{I}^{\underline{n}} \subseteq \mathbf{I}^{\underline{n}+(s+1)\cdot\underline{1}}] \\ &= (s+1)|\underline{\delta}| + \operatorname{indeg}(\mathbf{I}^{\underline{n}}), \end{aligned}$$

which is a contradiction as $|\underline{\delta}| \geqslant 1$. So $\mathbf{y} \notin \sqrt{\operatorname{Ann}\mathscr{L}}$. This proves the result for the functions $v(\mathbf{I}^{\underline{n}}/\mathbf{I}^{\underline{n}+\underline{1}})$ and $\operatorname{indeg}(\mathbf{I}^{\underline{n}}/\mathbf{I}^{\underline{n}+\underline{1}})$.

Note that $(0 :_R I_i) = 0$ for $1 \leqslant i \leqslant r$. So, by Theorem 4.35.(2), there exist $\underline{u}_1, \ldots, \underline{u}_s \in \mathbb{Z}^r$ and $g_1, \ldots, g_s \in \mathbb{Z}$ such that

$$v(R/\mathbf{I}^{\underline{n}}) = \min\{\underline{u}_j \cdot \underline{n} + g_j : 1 \leqslant j \leqslant s\} \quad \text{for all } \underline{n} \gg \underline{0}, \tag{4.16}$$

where the ith component u_{ji} of the coefficient vector $\underline{u}_j$ lies in $\{d_{i,1}, \ldots, d_{i,a_i}\}$ for $1 \leqslant i \leqslant r$. In particular, $\underline{u}_j \geqslant \underline{\delta}$ for $1 \leqslant j \leqslant s$. Hence, since $v(R/\mathbf{I}^{\underline{n}+\underline{1}}) \leqslant v(\mathbf{I}^{\underline{n}}/\mathbf{I}^{\underline{n}+\underline{1}})$ for all $\underline{n} \in \mathbb{N}^r$ (cf. [53, Prop. 2.5.(2)]), there exist $g, h \in \mathbb{Z}$ such that

$$\underline{\delta} \cdot \underline{n} + g \leqslant v(R/\mathbf{I}^{\underline{n}}) \leqslant \underline{\delta} \cdot \underline{n} + h \quad \text{for all } \underline{n} \gg \underline{0}. \tag{4.17}$$

Following the arguments as shown in the proof of Theorem 4.33, one obtains that $v(R/\mathbf{I}^{\underline{n}})$ is eventually linear with the leading coefficients given by $\underline{\delta}$. □

4.2.3 Examples

Here we present several examples that complement our main results. Computations using Macaulay2 [M2] were helpful in constructing some of these examples. For the reader's convenience, we describe the Macaulay2 commands used.

4.39. Using Lemma 4.15, we only need to compute generators of (usually complicated) modules. This can be done using the Macaulay2 command `mingens`, which also orders the generators in increasing degree. Other commands we use include `ass`, which computes the associated prime ideals of modules over a polynomial ring or over quotients of a polynomial ring by a homogeneous ideal, and `saturate`, which computes modules of the form $I^mJ^n :_R \mathfrak{m}^\infty$. For modules over other rings, for example in 4.52, the command `primaryDecomposition` is required, since the command `ass` does not return any result. By applying `radical` to the output of `primaryDecomposition`, one can compute the associated primes.

Example 4.40. Let $R = k[X,Y]$ be a standard graded polynomial ring in two variables over a field $\mathbf{k}$. Set $M := R/(XY^b)$, $I := (X^a)$, $\mathfrak{p} := (X)$, $\mathfrak{q} := (Y)$ and $\mathfrak{m} := (X,Y)$, where a and b are some positive integers. Then $(0 :_M I) = \mathfrak{q}^bM \neq 0$. Moreover, the following hold true.

(1) $\operatorname{Ass}_R(I^nM) = \{\mathfrak{q}\}$, $\operatorname{indeg}(I^nM/I^{n+1}M) = \operatorname{indeg}(I^nM) = an$ and

$$v(I^nM) = v_{\mathfrak{q}}(I^nM) = an + (b-1) \text{ for all } n \geqslant 1.$$

(2) $\operatorname{Ass}_R(M/IM) = \{\mathfrak{p}\}$ if $a = 1$, and $\operatorname{Ass}_R(M/I^nM) = \{\mathfrak{p}, \mathfrak{m}\}$ whenever $an \geqslant 2$.

(3) $\operatorname{Ass}_R(I^nM/I^{n+1}M) = \{\mathfrak{m}\}$ and $v(I^nM/I^{n+1}M) = an + (a+b-2)$ for all $n \geqslant 1$.

(4) $v_{\mathfrak{p}}(M/IM) = 0$ if $a = 1$, and $v_{\mathfrak{p}}(M/I^nM) = b$ whenever $an \geqslant 2$.

(5) $v_{\mathfrak{m}}(M/I^nM) = an + (b-2)$ whenever $an \geqslant 2$.

(6) $v(M/IM) = 0$ if $a = 1$, and $v(M/I^nM) = b$ whenever $an \geqslant 2$. ♢

Proof. Let $n \geqslant 1$. Then, $I^nM = (X^{an}, XY^b)/(XY^b)$. It follows that

$$\operatorname{indeg}(I^nM/I^{n+1}M) = \operatorname{indeg}(I^nM) = an.$$

As $\operatorname{Ass}_R(I^nM) \subseteq \operatorname{Ass}_R(M) = \{\mathfrak{p}, \mathfrak{q}\}$, and $(I^nM)_{\mathfrak{p}} = 0$, one gets that $\operatorname{Ass}_R(I^nM) = \{\mathfrak{q}\}$. Write the images of X and Y in M as x and y respectively. The main relation of x and y that we have in M is $xy^b = 0$. So $(0 :_M I) = \mathfrak{q}^bM \neq 0$. Note that each element of I^nM can be written as $x^{an}f(x,y)$ for some polynomial $f(x,y)$ over $\mathbf{k}$. Therefore $x^{an}y^{b-1} \in (I^nM)_{an+b-1}$ and $\mathfrak{q} = (0 :_R x^{an}y^{b-1})$. Clearly, $an + b - 1$ is the least possible degree of a homogeneous element of I^nM whose annihilator is $\mathfrak{q}$. It follows that $v(I^nM) = v_{\mathfrak{q}}(I^nM) = an + (b-1)$ for all $n \geqslant 1$. This proves (1).

The quotient $M/I^nM \cong R/(X^{an}, XY^b)$ for all $n \geqslant 1$. So $\operatorname{Ass}_R(M/I^nM) = \{\mathfrak{p}, \mathfrak{m}\}$ if $an \geqslant 2$. If $a = 1$, then $M/IM \cong R/(X)$, hence $\operatorname{Ass}_R(M/IM) = \{\mathfrak{p}\}$ and $v_{\mathfrak{p}}(M/IM) = 0$. In the case, when $an \geqslant 2$, one has that $\mathfrak{p} = \big((X^{an}, XY^b) :_R Y^b\big)$ and $\mathfrak{m} = \big((X^{an}, XY^b) :_R X^{an-1}Y^{b-1}\big)$. These two equalities do not hold if Y and $X^{an-1}Y^{b-1}$ are replaced respectively by any other homogeneous element of lower degree. Thus, one obtains (2), (4) and (5). Consequently, (6) follows.

For (3), let $n \geqslant 1$. Note that $\operatorname{Ass}_R(I^nM/I^{n+1}M) \subseteq \operatorname{Ass}_R(M/I^{n+1}M) = \{\mathfrak{p}, \mathfrak{m}\}$ by (2). Therefore, since $(I^nM)_{\mathfrak{p}} = 0$, one concludes that $\operatorname{Ass}_R(I^nM/I^{n+1}M) = \{\mathfrak{m}\}$. Since $I^nM = (X^{an}, XY^b)/(XY^b)$, the element $\overline{x^{an+a-1}y^{b-1}} \in I^nM/I^{n+1}M$ has the smallest possible degree such that $\mathfrak{m} = \operatorname{Ann}_R(\overline{x^{an+a-1}y^{b-1}})$. Therefore $v(I^nM/I^{n+1}M) = an + (a+b-2)$. □

Remark 4.41. In Example 4.40, we notice the following.

(1) The functions $v_{\mathfrak{p}}(M/I^nM)$ and $v(M/I^nM)$ of n are eventually constants. Note that $(0 :_M I) \neq 0$. It particularly ensures that the hypothesis $(0 :_M I) = 0$ in Theorem 4.28 cannot be removed.

(2) The functions $\operatorname{indeg}(I^nM/I^{n+1}M)$ and $v(I^nM/I^{n+1}M)$ (for all $n \geqslant 1$) both are linear with the same leading coefficient (as in Theorem 4.25), however their constant terms are different, namely 0 and $(a+b-2)$ respectively. Moreover, the difference $(a+b-2)$ can be arbitrarily large depending on a and b. △

Example 4.42. Let $R = k[X,Y]/(XY)$ over a field $\mathbf{k}$ with $\deg(X) = \deg(Y) = 1$. Write the images of X and Y in R as x and y respectively. Then $R = k[x,y]$. Set $\mathfrak{m} := (x,y)$ and $I := (x^{d_1}, y^{d_2})$, where $d_1 \leqslant d_2$ are some positive integers. Then, $v(R/I^n) = v_{\mathfrak{m}}(R/I^n) = d_1n - 1$ and $\operatorname{reg}(R/I^n) = d_2n - 1$ for every $n \geqslant 1$. ♢

Proof. Let $n \geqslant 1$. Since $xy = 0$, it follows that $I^n = (x^{d_1n}, y^{d_2n})$. Therefore R/I^n along with the gradation can be written as

$$k \oplus (kx \oplus ky) \oplus (kx^2 \oplus ky^2) \oplus \cdots \oplus (kx^{d_1n-1} \oplus ky^{d_1n-1}) \oplus ky^{d_1n} \oplus \cdots \oplus ky^{d_2n-1}.$$

Clearly, $\mathfrak{m} = (0 :_R x^{d_1n-1})$, and $d_1n - 1$ is the least possible degree of a homogeneous element of R/I^n whose annihilator is $\mathfrak{m}$. So $v(R/I^n) = v_{\mathfrak{m}}(R/I^n) = d_1n - 1$. Since R/I^n has finite length, $\operatorname{reg}(R/I^n) = \operatorname{end}(R/I^n) = d_2n - 1$. □

Remark 4.43. (1) Unlike [36, Thm. 1.1], Theorem 4.28 can be applied for a Noetherian graded ring which is not a domain. The ring R in Example 4.42 is not a domain, however $(0 :_R I) = 0$, and $v(R/I^n)$ is linear with the leading coefficient d_1.

(2) In Example 4.42, if $d_1 < d_2$, then the difference $\operatorname{reg}(R/I^n) - v(R/I^n) = (d_2 - d_1)n$ can be arbitrarily large depending on n. △

Example 4.44. Let $R = k[X,Y,Z]$ be a standard graded polynomial ring in three variables over a field $\mathbf{k}$. Set $\mathfrak{m} := (X,Y,Z)$. Consider $M := R/(X^3, XY^4)$ and $I := (X, Y^2, Z^3)$. Then $(0 :_M I) = 0$. Moreover, the following hold true.

(1) $\operatorname{Ass}_R(I^nM/I^{n+1}M) = \operatorname{Ass}_R(M/I^nM) = \{\mathfrak{m}\}$ for all $n \geqslant 1$.

(2) $$\operatorname{indeg}(I^nM/I^{n+1}M) = \begin{cases} n & \text{if } n = 0,1,2 \\ n+1 & \text{if } n = 3 \\ n+3 & \text{if } n = 4 \\ 2n & \text{if } n \geqslant 5 \end{cases}$$

(3) $$v(M/I^{n+1}M) = v(I^nM/I^{n+1}M) = \begin{cases} n+3 & \text{if } n = 0,1,2 \\ n+4 & \text{if } n = 3 \\ n+6 & \text{if } n = 4 \\ 2n+3 & \text{if } n \geqslant 5 \end{cases}$$ ♢

Proof. Since $M = R/(X^3, XY^4)$, the element $Z^3 \in I$ is M-regular, and hence $(0 :_M I) = 0$. Note that $I^nM = \big(I^n + (X^3, XY^4)\big)/(X^3, XY^4)$, $M/I^nM \cong R/\big(I^n + (X^3, XY^4)\big)$ and $I^nM/I^{n+1}M \cong \big(I^n + (X^3, XY^4)\big)/\big(I^{n+1} + (X^3, XY^4)\big)$ for all $n \geqslant 0$. We use x, y, z for the classes of X, Y, Z in M respectively.

(1) Let $n \geqslant 1$. Note that I and I^n annihilate $I^nM/I^{n+1}M$ and M/I^nM respectively. Therefore every associated prime ideal of each of these modules contains $I = (X, Y^2, Z^3)$, and hence this prime ideal must be same as $\mathfrak{m}$.

(2) A non-zero element of the least possible degree in the module $I^nM/I^{n+1}M$ for $0 \leqslant n \leqslant 4$ is given by 1, x, x^2, x^2y^2 and $x^2y^2z^3$ of degree 0, 1, 2, 4 and 7 respectively. For $n \geqslant 5$, the module I^nM is generated by

$$x^2y^2(z^3)^{n-3},\ x^2(z^3)^{n-2},\ xy^2(z^3)^{n-2},\ x(z^3)^{n-1} \text{ and } (y^2)^j(z^3)^{n-j} \text{ for } 0 \leqslant j \leqslant n,$$

and their total degrees in M are respectively

$$3n-5, 3n-4, 3n-3, 3n-2 \text{ and } 3n-j \text{ for } 0 \leqslant j \leqslant n.$$

Among these degrees, $2n$ is the least possible value. Thus $\overline{y^{2n}}$ is a non-zero element of the least possible degree in $I^nM/I^{n+1}M$ proving (2).

(3) Consider $n \geqslant 0$. By (1), $v(M/I^{n+1}M) = v_{\mathfrak{m}}(M/I^{n+1}M)$. Moreover, in view of Lemma 4.15, one has that

$$v(M/I^{n+1}M) = v_{\mathfrak{m}}(M/I^{n+1}M) = \operatorname{indeg}\big((I^{n+1}M :_M \mathfrak{m})/I^{n+1}M\big). \tag{4.18}$$

Claim: We claim that $(I^{n+1}M :_M I) = I^nM$ for all $n \geqslant 0$. The claim is equivalent to that $\big(\big((X^3, XY^4) + I^{n+1}\big) :_R I\big) = (X^3, XY^4) + I^n$ for all $n \geqslant 0$.

For $n \geqslant 5$, the monomial ideal $((X^3, XY^4) + I^{n+1})$ is minimally generated by

$$\begin{gathered} X^3, XY^4, X^2Y^2(Z^3)^{n-2}, X^2(Z^3)^{n-1}, XY^2(Z^3)^{n-1}, X(Z^3)^n \\ \text{and } (Y^2)^j(Z^3)^{n+1-j} \text{ for } 0 \leqslant j \leqslant n+1. \end{gathered}$$

Therefore, using [49, Sec. 3.2.2], one has that

$$\begin{aligned} \Big(\big((X^3, XY^4) + I^{n+1}\big) :_R X\Big) &= \Big(X^2, Y^4, XY^2(Z^3)^{n-2}, X(Z^3)^{n-1}, \\ &\qquad Y^2(Z^3)^{n-1}, (Z^3)^n\Big), \\ \Big(\big((X^3, XY^4) + I^{n+1}\big) :_R Y^2\Big) &= \Big(X^3, XY^2, X^2(Z^3)^{n-2}, X(Z^3)^{n-1}, \\ &\qquad (Y^2)^j(Z^3)^{n-j}, 0 \leqslant j \leqslant n\Big) \text{ and} \\ \Big(\big((X^3, XY^4) + I^{n+1}\big) :_R Z^3\Big) &= \Big(X^3, XY^4, X^2Y^2(Z^3)^{n-3}, X^2(Z^3)^{n-2}, \\ &\qquad XY^2(Z^3)^{n-2}, X(Z^3)^{n-1}, (Y^2)^j(Z^3)^{n-j}, 0 \leqslant j \leqslant n\Big). \end{aligned}$$

Now $\big(\big((X^3, XY^4) + I^{n+1}\big) :_R I\big)$ is the intersection of the three ideals shown above. Moreover, the intersection of two monomial ideals is constructed by taking the lcm of pairs of generators one from each ideal. So the resulting ideal is exactly $(X^3, XY^4) + I^n$, which is minimally generated by

$$X^3, XY^4, X^2Y^2(Z^3)^{n-3}, X^2(Z^3)^{n-2}, XY^2(Z^3)^{n-2}, X(Z^3)^{n-1} \text{ and } (Y^2)^j(Z^3)^{n-j}$$

where j is varying in $0 \leqslant j \leqslant n$.

The cases $n = 0, \ldots, 4$ would require more attention, but instead they can be verified using any mathematical software (e.g., Macaulay2 [M2]). Thus the claim is verified.

Using the above claim, since $I \subseteq \mathfrak{m}$, it follows that

$$(I^{n+1}M :_M \mathfrak{m}) \subseteq (I^{n+1}M :_M I) = I^nM \text{ for every } n \geqslant 0. \tag{4.19}$$

For $n = 0, 1, \ldots, 4$, using Macaulay2 [M2], one obtains that a non-zero homogeneous element in $(I^{n+1}M :_M \mathfrak{m})/I^{n+1}M$ of minimum possible degree is given by

$$yz^2, xyz^2, x^2yz^2, x^2y^3z^2 \text{ and } x^2y^3z^5 \text{ respectively.}$$

Hence, in view of (4.18), for $0 \leqslant n \leqslant 4$, $v(M/I^{n+1}M) = 3, 4, 5, 7$ and 10 respectively. Let $n \geqslant 5$. Then, since $\mathfrak{m} \subseteq (I^{n+1}M :_R y^{2n+1}z^2)$ and $y^{2n+1}z^2 \notin I^{n+1}M$, one has that $\mathfrak{m} = (I^{n+1}M :_R y^{2n+1}z^2)$. Moreover, one can check that if $g \in I^nM$ with $\deg(g) < 2n+3$, then $\mathfrak{m} \neq (I^{n+1}M :_R g)$. Therefore, using (4.18) and (4.19), it follows $v_{\mathfrak{m}}(M/I^{n+1}M) = 2n+3$.

In view of Remark 4.29 and the above claim, one obtains that $v(I^nM/I^{n+1}M) = v(M/I^{n+1}M)$ for all $n \geqslant 0$. □

Remark 4.45. In Theorem 4.28, the leading coefficient of the function $v(M/I^nN)$ is not necessarily same as $\operatorname{indeg}(I)$. In Example 4.48, the leading coefficient of $v(M/I^nM)$ is 2, however $\operatorname{indeg}(I) = 1$. In this example, $X \in \sqrt{\operatorname{Ann}_R(M)} \subseteq \sqrt{\operatorname{Ann}_{\mathscr{R}(I)}(\mathscr{G})}$, but $Y^2 \notin \sqrt{\operatorname{Ann}_{\mathscr{R}(I)}(\mathscr{G})}$, where $\mathscr{G} = \mathscr{R}(I, M)/\mathscr{R}(I, IM)$. It also ensures that the condition $(0 :_M y_1) = 0$ in Theorem 4.28.(3) cannot be removed. △

Example 4.46. Let $R = \mathbf{k}[x, y]$ be a standard graded polynomial ring in two variables x and y over a field $\mathbf{k}$. Set $I := (x, y^2)$, $J := (x^2, y)$, and $\mathfrak{m} := (x, y)$. Then, for all $m, n \in \mathbb{N}$ with $m + n \geqslant 1$, the following hold.

(1) $\operatorname{Ass}_R(R/I^mJ^n) = \{\mathfrak{m}\}$ and $v(R/I^mJ^n) = v_{\mathfrak{m}}(R/I^mJ^n) = m + n$.

(2) [24, Ex. 3.1] $\operatorname{reg}_R(R/I^mJ^n) = \max\{m + 2n - 1, 2m + n - 1\}$. ♢

Proof. Fix $m, n \in \mathbb{N}$ not both zero. Since $x, y \in \sqrt{I^mJ^n}$, $\operatorname{Ass}_R(R/I^mJ^n) = \{\mathfrak{m}\}$. Note that the ideal $I^mJ^n = (x, y^2)^m(x^2, y)^n$ is given by

$$\begin{aligned}
&(x^m, x^{m-1}y^2, x^{m-2}y^4, \ldots, xy^{2m-2}, y^{2m})(x^{2n}, x^{2n-2}y, x^{2n-4}y^2, \ldots, x^2y^{n-1}, y^n)\\
&= (x^{m+2n}, x^{m+2n-2}y, x^{m+2n-4}y^2, \ldots, x^{m+2}y^{n-1}, x^my^n,\\
&\quad x^{m+2n-1}y^2, x^{m+2n-3}y^3, x^{m+2n-5}y^4, \ldots, x^{m+1}y^{n+1}, x^{m-1}y^{n+2}, \ldots,\\
&\quad x^{2n}y^{2m}, x^{2n-2}y^{2m}, x^{2n-4}y^{2m+2}, \ldots, x^2y^{2m+n-1}, y^{2m+n}).
\end{aligned}$$

Clearly, $\mathfrak{m} = (I^mJ^n :_R x^{m-1}y^{n+1}) = (I^mJ^n :_R x^{m+1}y^{n-1})$, and $m + n$ is the least possible degree of a homogeneous element of R/I^mJ^n whose annihilator is $\mathfrak{m}$. So the assertion in (1) follows. For the equality in (2), note that $\operatorname{reg}_R(R/I^mJ^n) = \operatorname{reg}_R(I^mJ^n) - 1 = \max\{m + 2n - 1, 2m + n - 1\}$ by Corollary A.17. □

In the following example, none of $v(R/I^mJ^n)$ and $v(I^mJ^n/I^{m+1}J^{n+1})$ are eventually linear in (m, n). Here, we use the notation $\operatorname{end}(M) := \sup\{n : M_n \neq 0\}$, where M is a non-zero graded R-module.

Example 4.47. Let $S = \mathbf{k}[X, Y]$ be a standard graded polynomial ring in two variables X and Y over a field $\mathbf{k}$. Set $R := \mathbf{k}[X, Y]/(XY)$, and denote the images of X and Y in R as x and y respectively. Then $R = \mathbf{k}[x, y]$. Set $I := (x, y^2)$, $J := (x^2, y)$, and $\mathfrak{m} := (x, y)$. Then, $(0 :_R I) = 0$ and $(0 :_R J) = 0$. Moreover,

(1) $\operatorname{Ass}_R(R/I^mJ^n) = \{\mathfrak{m}\} = \operatorname{Ass}_R(I^{m-1}J^{n-1}/I^mJ^n)$ whenever $m, n \geqslant 1$.

(2) $v(R/I^mJ^n) = \min\{m + 2n - 1, 2m + n - 1\}$ for all $m, n \in \mathbb{N}$ with $m + n \geqslant 2$.

(3) $\operatorname{reg}_S(R/I^mJ^n) = \max\{m + 2n - 1, 2m + n - 1\}$ for all $m, n \in \mathbb{N}$ with $m + n \geqslant 1$.

(4) $v(I^{m-1}J^{n-1}/I^mJ^n) = v(R/I^mJ^n)$ for all $m, n \geqslant 1$. ♢

Proof. Fix $m, n \in \mathbb{N}$ not both zero. It follows

$$I^mJ^n = (x, y^2)^m(x^2, y)^n = (x^m, y^{2m})(x^{2n}, y^n) = (x^{m+2n}, y^{2m+n}).$$

Then, $\operatorname{Ass}_R(R/I^mJ^n) = \{\mathfrak{m}\}$. Since IJ annihilates $I^{m-1}J^{n-1}/I^mJ^n$, every associated prime of this module will contain IJ, and hence must be the same as $\mathfrak{m}$. So (1) follows. Considering the gradation of R/I^mJ^n, since R/I^mJ^n has finite length, by Corollary A.17 it follows $\operatorname{reg}_S(R/I^mJ^n) = \operatorname{end}(R/I^mJ^n) = \max\{m + 2n - 1, 2m + n - 1\}$ whenever $m + n \geqslant 1$. It shows (3). When $m + n \geqslant 2$, one has that $\mathfrak{m} = (I^mJ^n :_R x^{m+2n-1}) = (I^mJ^n :_R y^{2m+n-1})$. Moreover, there is no other homogeneous element $f \in R$ of degree different from $m + 2n - 1$ and $2m + n - 1$ such that $\mathfrak{m} = (I^mJ^n :_R f)$. Thus, (2) follows. For (4), observe that the images of x^{m+2n-1} and y^{2m+n-1} in $I^{m-1}J^{n-1}/I^mJ^n$ are non-zero elements, where $m, n \geqslant 1$. □

In the next example, both $v_{\mathfrak{m}}(M/I^mJ^nM)$ and $v(M/I^mJ^nM)$ are asymptotically not linear in (m, n). Moreover, in this example, all four functions induced by the local and global v-numbers are asymptotically distinct functions.

Example 4.48. Let $R = \mathbf{k}[X,Y,Z]$ be a standard graded polynomial ring in three variables over a field $\mathbf{k}$. Consider the module $M := R/(XY)$, and the ideals $I := (X,Z^2)$, $J := (Y,Z^3)$, $\mathfrak{p} := (X,Z)$, $\mathfrak{q} := (Y,Z)$ and $\mathfrak{m} := (X,Y,Z)$. Then, $(0 :_M I) = 0$ and $(0 :_M J) = 0$. Moreover, for all $m,n \geqslant 1$, the following hold.

(1) $\operatorname{Ass}_R(M/I^mJ^nM) = \{\mathfrak{p},\mathfrak{q},\mathfrak{m}\}$.

(2) $v_{\mathfrak{m}}(M/I^mJ^nM) = \min\{2m+n+1, m+3n\}$.

(3) $v_{\mathfrak{p}}(M/I^mJ^nM) = 2m+n-1$ and $v_{\mathfrak{q}}(M/I^mJ^nM) = m+3n-1$.

(4) $v(M/I^mJ^nM) = \min\{2m+n-1, m+3n-1\}$. ♢

Proof. Fix $m,n \geqslant 1$. The ideal $I^mJ^n + (XY)$ is generated by

$$\begin{gathered} XY, X^mZ^{3n}, X^{m-1}Z^{3n+2}, X^{m-2}Z^{3n+4}, \ldots, XZ^{2m+3n-2}, \\ Y^nZ^{2m}, Y^{n-1}Z^{2m+3}, Y^{n-2}Z^{2m+6}, \ldots, YZ^{2m+3n-3}, Z^{2m+3n}. \end{gathered}$$

Since $M/I^mJ^nM \cong R/(I^mJ^n + (XY))$, considering the primary decomposition of the monomial ideal $I^mJ^n + (XY)$, one obtains (1).

Denote the images of X, Y and Z in M as x, y and z respectively. Then $xy = 0$. Moreover, the R-module I^mJ^nM is generated by

$$\begin{gathered} x^mz^{3n}, x^{m-1}z^{3n+2}, x^{m-2}z^{3n+4}, \ldots, xz^{2m+3n-2}, \\ y^nz^{2m}, y^{n-1}z^{2m+3}, y^{n-2}z^{2m+6}, \ldots, yz^{2m+3n-3}, z^{2m+3n}. \end{gathered}$$

Therefore, $\mathfrak{m} = (I^mJ^nM :_R y^{n-1}z^{2m+2}) = (I^mJ^nM :_R x^{m-1}z^{3n+1})$. On the other hand, there are no elements of smaller degree in M/I^mJ^nM whose annihilator is $\mathfrak{m}$. Therefore, $v_{\mathfrak{m}}(M/I^mJ^nM) = \min\{2m+n+1, m+3n\}$, which shows (2).

In the same manner, one sees that

- an element which realises $v_{\mathfrak{p}}(M/I^mJ^nM)$ is y^nz^{2m-1};
- an element which realises $v_{\mathfrak{q}}(M/I^mJ^nM)$ is x^mz^{3n-1}.

This implies (3) and (4), which completes the proof. □

Remark 4.49. In Example 4.48, both $v_{\mathfrak{p}}(M/I^mJ^nM)$ and $v_{\mathfrak{q}}(M/I^mJ^nM)$ eventually become linear, but $v(M/I^mJ^nM) = \min\{v_{\mathfrak{p}}(M/I^mJ^nM), v_{\mathfrak{q}}(M/I^mJ^nM)\}$ is not eventually linear. Moreover, asymptotically, $v(M/I^mJ^nM)$ and $v_{\mathfrak{m}}(M/I^mJ^nM)$ are two different functions. △

The following example ensures that, despite Theorem 4.38, one cannot expect that every local v-number for products and powers of several ideals eventually becomes linear even over a polynomial ring over a field.

Example 4.50. Let $R = \mathbf{k}[x,y,z]$ be a standard graded polynomial ring over a field $\mathbf{k}$. Consider the ideals $I = (x^2, yz^2)$ and $J = (y^2, xz^2)$. Set $\mathfrak{p} := (x,y)$, $\mathfrak{q} := (x,z)$, $\mathfrak{r} := (y,z)$, and $\mathfrak{m} := (x,y,z)$. Then, for every $m,n \geqslant 1$, the following hold.

(1) $\operatorname{Ass}_R(R/I^mJ^n) = \{\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{m}\}$.

(2) $v_{\mathfrak{m}}(R/I^mJ^n) = 2m+2n+2$.

(3) $v_{\mathfrak{p}}(R/I^mJ^n) = \min\{3m+2n+1, 2m+3n+1\}$.

(4) $v(R/I^mJ^n) = v_{\mathfrak{q}}(R/I^mJ^n) = v_{\mathfrak{r}}(R/I^mJ^n) = 2m+2n+1$.

(5) $\operatorname{Ass}_R(I^{m-1}J^{n-1}/I^mJ^n) = \{\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{m}\}$, and the (local) v-numbers of the modules $I^{m-1}J^{n-1}/I^mJ^n$ and R/I^mJ^n coincide. ♢

Proof. Fix $m, n \geqslant 1$. The ideals I^m and J^n are generated by

$$\{(x^2)^s(yz^2)^{m-s} = x^{2s}y^{m-s}z^{2m-2s} : 0 \leqslant s \leqslant m\} \text{ and}$$
$$\{(y^2)^t(xz^2)^{n-t} = x^{n-t}y^{2t}z^{2n-2t} : 0 \leqslant t \leqslant n\} \text{ respectively.}$$

Hence $I^mJ^n = \left(x^{n+2s-t}y^{m-s+2t}z^{2(m+n-s-t)} : 0 \leqslant s \leqslant m, 0 \leqslant t \leqslant n\right)$. The statement in (1) follows from the primary decomposition of this monomial ideal. For better understanding, the reader may consider the case $m, n = 1$.

In view of Lemma 4.15, the local v-numbers are given by

$$v_{\mathfrak{m}}(R/I^mJ^n) = \operatorname{indeg}\left((I^mJ^n :_R \mathfrak{m})/I^mJ^n\right) \text{ and}$$
$$v_{\mathfrak{a}}(R/I^mJ^n) = \operatorname{indeg}\left(\frac{I^mJ^n :_R \mathfrak{a}}{(I^mJ^n :_R \mathfrak{a}) \cap (I^mJ^n :_R \mathfrak{m}^\infty)}\right) \text{ for } \mathfrak{a} \in \{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}\},$$

where $I^mJ^n :_R \mathfrak{m}^\infty = \bigcup_{\ell \geqslant 1}(I^mJ^n :_R \mathfrak{m}^\ell)$. Note that

$$(I^mJ^n :_R \mathfrak{p}) \cap (I^mJ^n :_R \mathfrak{m}^\infty) = (I^mJ^n :_R \mathfrak{p}) \cap (I^mJ^n :_R z^\infty),$$
$$(I^mJ^n :_R \mathfrak{q}) \cap (I^mJ^n :_R \mathfrak{m}^\infty) = (I^mJ^n :_R \mathfrak{q}) \cap (I^mJ^n :_R y^\infty),$$
$$(I^mJ^n :_R \mathfrak{r}) \cap (I^mJ^n :_R \mathfrak{m}^\infty) = (I^mJ^n :_R \mathfrak{r}) \cap (I^mJ^n :_R x^\infty).$$

The assertions (2), (3) and (4) can be obtained from the following observations:

$$x^{2m-1}y^{2n}z^3, x^{2m}y^{2n-1}z^3 \in (I^mJ^n :_R \mathfrak{m}) \smallsetminus I^mJ^n, \tag{4.20}$$
$$y^{m+2n-1}z^{2m+2}, x^{2m+n-1}z^{2n+2} \in (I^mJ^n :_R \mathfrak{p}) \smallsetminus (I^mJ^n :_R z^\infty), \tag{4.21}$$
$$x^{2m-1}y^{2n+1}z \in (I^mJ^n :_R \mathfrak{q}) \smallsetminus (I^mJ^n :_R y^\infty), \tag{4.22}$$
$$\text{and} \quad x^{2m+1}y^{2n-1}z \in (I^mJ^n :_R \mathfrak{r}) \smallsetminus (I^mJ^n :_R x^\infty). \tag{4.23}$$

These are the monomials of the minimum possible degree contained in the right-hand sides above, and therefore they compute the respective local v-numbers.

The monomials listed in (4.20), (4.21), (4.22) and (4.23) all lie in $I^{m-1}J^{n-1}$. The annihilator ideals of these monomials in $I^{m-1}J^{n-1}/I^mJ^n$ provide the associated prime ideals $\{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{m}\}$ of the module $I^{m-1}J^{n-1}/I^mJ^n$. Due to the minimality of the degrees, these monomials also compute the respective local v-numbers of the same module, which coincide with that of R/I^mJ^n. Thus, assertion (5) follows. □

Remark 4.51. The assumption $\mathbf{I}^{\underline{s}}M \subseteq N$ for some $\underline{s} \in \mathbb{N}^r$, in Theorem 4.35.(2) and Corollary 4.37, seems to be necessary because in the critical case $\mathbf{I} \subseteq \sqrt{\operatorname{Ann}_R(N)}$, the function $v_{\mathfrak{p}}(M/\mathbf{I}^{\underline{n}}N)$ is eventually constant for each $\mathfrak{p} \in \mathcal{A}_N^M(\mathbf{I})$. However, we are unaware of any example in which $\mathbf{I}^{\underline{n}}M \nsubseteq N$ for every $\underline{n} \in \mathbb{N}^r$, $\mathbf{I} \nsubseteq \sqrt{\operatorname{Ann}_R(N)}$, and $v_{\mathfrak{p}}(M/\mathbf{I}^{\underline{n}}N)$ is not eventually the minimum of linear functions. Some attempts of explicit computations have been made in the case $r = 1$. As shown in [53], the same hypotheses lead to a linear behaviour. Due to these attempts, we feel that the condition $\mathbf{I}^{\underline{s}}M \subseteq N$, which is merely technical but still true in many cases, could be relaxed to a more natural one. △

The following example shows that the condition "R_0 is an integral domain" in Theorem 4.38 is crucial.

Example 4.52. Set $R_0 := K[a, b]/(ab)$, where a, b are indeterminates over a field K. Let $R = R_0[x, y]$ be a polynomial ring over R_0, and consider the homogeneous ideals $I = (x^2, bx)$ and $J = (y^2, ay)$. Then, for any $m, n \geqslant 1$, one has

$$\operatorname{indeg}(I^mJ^n/I^{m+1}J^{n+1}) = \min\{2m + n, m + 2n\}.$$

◇

Proof. Fix $m, n \geqslant 1$. The ideals I^m and J^n are generated by

$$\{(x^2)^s(bx)^{m-s} = x^{s+m}b^{m-s} : 0 \leqslant s \leqslant m\} \text{ and}$$
$$\{(y^2)^t(ay)^{n-t} = y^{t+n}a^{n-t} : 0 \leqslant t \leqslant n\} \text{ respectively.}$$

Since $ab = 0$ in R_0, it follows that the product I^mJ^n is generated by

$$\left(x^{2m}y^{t+n}a^{n-t}, x^{s+m}y^{2n}b^{m-s} : 0 \leqslant s \leqslant m, 0 \leqslant t \leqslant n\right).$$

We note that the elements in R_0 have zero degree. Moreover, we can see that $x^{2m}y^n$ and x^my^{2n} do not lie in $I^{m+1}J^{n+1}$. Therefore, the result follows. □

4.3 Vasconcelos number for local rings

In this section, we explore the case of local rings trying to understand the problems and questions that this setting could have. Let $(A, \mathfrak{m})$ be a local ring. The $\mathfrak{m}$-adic evaluation is a map $\nu : A \to \mathbb{N}$ defined as

$$\nu(a) = \inf\{n \geqslant 0 : a \in \mathfrak{m}^n \backslash \mathfrak{m}^{n+1}\}.$$

As in the graded case, given a non-zero ideal $I \subset \mathfrak{m}$ we can define the local Vasconcelos number as

$$v_{\mathfrak{p}}(I) := \inf\{n \geqslant 0 : \exists f \in A \text{ such that } \nu(f) = n, \mathfrak{p} = I :_A f\} \text{ for } \mathfrak{p} \in \operatorname{Ass}_A(I),$$

and the Vasconcelos number as $v(I) := \inf\{v_{\mathfrak{p}}(I) : \mathfrak{p} \in \operatorname{Ass}_A(I)\}$.

Example 4.53. Consider the Noetherian local domain $A = \mathbf{k}[\![x, y]\!]/(x^2 - y^3)$ with maximal ideal $\mathfrak{m} = (x, y) \subset A$. Then the $\mathfrak{m}$-adic evaluation of the powers of $x \in A$ can be computed as follows

$$\nu(x^n) = \begin{cases} 3k & \text{if } n = 2k, \\ 3k + 1 & \text{if } n = 2k + 1. \end{cases}$$

This non-linear behaviour can be used to construct a interesting example of Vasconcelos number in the local case. Consider the ideal $I = (x)$. First, we see that the radical ideal $\sqrt{I^n} = (x, y)$ given $n \geqslant 2$. Therefore the set of associated primes are $\operatorname{Ass}(I^n) = \{(x, y)\}$. Using a similar argument as given in the proof of Lemma 4.15, we get

$$\{f \in A : \mathfrak{p} = I^n :_A f\} = (I^n :_A \mathfrak{p}) \backslash I^n.$$

Therefore, for $n \geqslant 2$, the Vasconcelos number is computed as

$$v(I^n) = v_{\mathfrak{m}}(I^n) = \nu(x^{n-1}y^2) = \begin{cases} 3k & \text{if } n = 2k, \\ 3k + 2 & \text{if } n = 2k + 1. \end{cases}$$

We can see that the Vasconcelos number fails to be linear. Instead, there exists two linear functions

$$\alpha(n) = \frac{3}{2}n, \text{ and } \beta(n) = \frac{3}{2}n + \frac{1}{2}$$

such that $v(I^n) = \alpha(n)$ on even integers, and $v(I^n) = \beta(n)$ on odd integers. This kind of function is called *periodically linear*. Moreover, we have

$$\lim_{n \to \infty} \frac{v(I^n)}{n} = \frac{3}{2}$$

which is not a integer, but rather a rational number. This was not the case in the graded setting where the above limit was always an integer corresponding to the degree of a generator of the ideal. ♢

The above examples shows that specific hypothesis for the ring are needed to obtain valuable results. In contrast, to the graded case, being a quotient of a regular local ring is not enough to ensure the right setting. The crucial point is that the $\mathfrak{m}$-adic evaluation does not handle good enough the product operation. For a general local ring the inequality $\nu(ab) \geqslant \nu(a) + \nu(b)$, for $a, b \in A$ holds true, but equality could not hold as we have seen in Example 4.53. One way to ensure equality is by introducing the associated graded ring of a local ring. Let $(A, \mathfrak{m})$ be a local ring, then the *associated graded ring* is the $\mathbb{N}$-graded ring

$$\mathrm{gr}_{\mathfrak{m}}(A) := \bigoplus_{n \geqslant 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

It is immediate to see that the degree of a element in this graded ring corresponds to its $\mathfrak{m}$-adic evaluation. If $\mathrm{gr}_{\mathfrak{m}}(A)$ is a domain, then the product of two elements there is always non-zero implying that $\nu(ab) = \nu(a) + \nu(b)$, for any $a, b \in A$. For more information about this topic, see for example [108].

Theorem 4.54. *Let $(R, \mathfrak{m})$ be a Noetherian local ring such that $\mathrm{gr}_m(R)$ is a domain. Let I be an ideal of R such that $(0 : I) = 0$. By [23], the set $\mathrm{Ass}(I^n)$ stabilizes to a set $\mathrm{Ass}^{\infty}(I)$ for n sufficiently large. Consider $\mathfrak{p} \in \mathrm{Ass}^{\infty}(I)$. Then $v_{\mathfrak{p}}(I^n)$ is eventually a linear function of n with leading coefficient in $\{\nu(a_1), \ldots, \nu(a_r)\}$, where $\{a_1, \ldots, a_r\}$ is a minimal set of generators of I.*

Proof. The first part proceeds like the proof of Theorem 4.28, so we are going to just give a quick idea. Let $\mathcal{R}(I) = R[a_1t, \ldots, a_rt]$ be the Rees algebra of I and, following Lemma 4.15, let

$$H = \bigoplus_{n \geqslant 0} \frac{I^n : P}{(I^n : P) \cap \Gamma_V(R/I^n)}.$$

We know that $v_P(I^n) = \inf\{v(a) : a \text{ is nonzero in } H_n\}$, where $H_n = \frac{I^n:P}{I^n:(P+Q^{\infty})}$. In general $\mathcal{R}(I)$ and H are not bigraded, but H is a finitely generated $\mathcal{R}(I)$-module because it eventually coincides with the $\mathcal{R}(I)$-module $\mathrm{Ann}_{\mathcal{S}}(\mathfrak{p})/\mathrm{Ann}_{\mathcal{R}}(I)(\mathfrak{p}) \cap \Gamma_V(\mathcal{S})$ where $\mathcal{S} = \mathcal{R}(I)/I\mathcal{R}(I)$. Using the property $v(ab) = v(a) + v(b)$ for all $a, b \in R$, we can construct a bigraded structure on both $\mathcal{R}(I)$, and H. Then one proceeds as in the proof of Theorem 4.28. □

Appendix A

Minimal free resolutions and related invariants

> *"There's a syzygy in the alignment of our hearts that leaps beyond the mere meandering of stars. "*
>
> – Beryl Dov

This appendix is a overview on the theory of minimal free resolutions, regularity, and Koszul algebras containing no original results. We use as basis reference [25, 30, 46, 47, 87, 132].

Throughout this chapter let $\mathbf{k}$ be a field, $S = \mathbf{k}[x_1, \dots, x_n] = \oplus_{n \geqslant 0} S_n$ be the polynomial ring in n variables over $\mathbf{k}$, and $R = S/I = \oplus_{n \geqslant 0} R_n$ be a graded quotient. Set $\mathfrak{m}_S := \oplus_{n \geqslant 1} S_n$, and $\mathfrak{m}_R = \oplus_{n \geqslant 1} R_n$ be respectively the maximal irrelevant ideals of S and R. Eventually, let $\pi : S \to R$ be the canonical projection sending each polynomial into its equivalent class in R.

For a graded R-module M, a modern approach on studying and understanding the nature of M is to look at free resolutions of M, with particular attention to its minimal free resolution.

A *free resolution* of M (as a R-module) is a chain complex of free graded R-modules

$$\mathcal{F}_\bullet : \cdots \to F_i \to \cdots \to F_2 \to F_1 \to F_0,$$

together with a graded map $\epsilon : F_0 \to M$ such that

$$\mathcal{F}_\bullet \xrightarrow{\epsilon} M \to 0$$

is an exact complex. In literature, it is common to add "$\to 0$" at the end of the complex, and say that $\mathcal{F}_\bullet$ is a free resolution for M if and only if it is quasi-isomorphic to the chain complex $0 \to M \to 0$. A free resolution $\mathcal{F}_\bullet$ of M is said to be *minimal* if for every chain map $\varphi_i : F_i \to F_{i-1}$ we have $\varphi_i(F_i) \subset \mathfrak{m} F_{i-1}$ for every $i \geqslant 1$.

Theorem A.1. *Let M be a graded R-module. Let $\mathcal{F}_\bullet$ and $\mathcal{G}_\bullet$ be two minimal free resolutions of M. Then there exists a graded isomorphism of chain complexes $\mathcal{F}_\bullet \longrightarrow \mathcal{G}_\bullet$ inducing the identity map on M. Therefore, there exists a unique minimal free resolution up to isomorphism. Moreover, every free resolution of M contains the minimal free resolution as a direct summand.*

Proof. See [46, Theorem 20.2]. □

Let $\mathcal{F}_\bullet$ be the minimal free resolution of M. Since every term of the complex can be written as

$$F_i = \bigoplus_{j \geqslant 0} R(-j)^{\beta^R_{i,j}(M)},$$

we name $\beta^R_{i,j}(M)$ the *graded Betti numbers* of the module M, while $\beta^R_i(M) := \sum_j \beta_{i,j}(M) = \operatorname{rank} F_i$ denotes the *ith Betti number* of M. The superscript indicates the ring R used for resolving the module M. As the next example will show, the ambient ring is crucial in studying the minimal free resolution of M. We should say that free resolutions study the properties of M seen as module over some suitable ring R.

Example A.2. Let $S = \mathbf{k}[x]$, $R = \mathbf{k}[x]/(x^n)$ and consider the module structure on the residue field given by the quotient $\mathbf{k} = R/\mathfrak{m}_R$. Then its minimal free resolution as R-module is

$$\cdots \to R(-(k+1)n) \xrightarrow{x^{n-1}} R(-kn-1) \xrightarrow{x} R(-kn) \xrightarrow{x^{n-1}} \cdots \xrightarrow{x} R(-n) \xrightarrow{x^{n-1}} R(-1) \xrightarrow{x} R.$$

However, $\mathbf{k}$ has also a module structure over S, in fact $\mathbf{k} = S/\mathfrak{m}_S$. As a S-module, the minimal free resolution of $\mathbf{k}$ is

$$0 \to S(-1) \xrightarrow{x} S \to 0.$$

The two resolutions behave in a very different way. More in general, the field $\mathbf{k}$ as a module over polynomial ring S has a very distinguished minimal free resolution called *Koszul complex*, while by considering it as a module over a quotient ring $R = S/I$, its minimal resolution can be quite unsteady. ♢

Example A.2 gives a first example that shows the importance of the ambient ring. More in general given a graded R-module M, the canonical projection $S \to R$ induces a natural structure of graded S-module on M. The double behaviour of M as a S-module and R-module has been a central research topic in the last decades.

The crucial point on the uniqueness of the minimal free resolution is that the graded Betti numbers depends only on the module M and on its module structure and not on the resolution itself. This fact is more clearer by using the Tor functor, that we now briefly introduce. For a more extensive treat see [132, Chapter 2] or [27, Sec 6.2].

For any R-module M consider the covariant endo-functor $- \otimes_R M : R - \mathfrak{mod} \to R - \mathfrak{mod}$ that sends a module N into the tensor product $N \otimes_R M$. This functor restricts to an endo-functor of the sub-category of finitely generated graded R-modules if the the starting module M is graded and finitely generated. This functor is right-exact, i.e. for any exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$, the sequence $N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M \to 0$ is exact. Therefore, we may consider the derived functors L_i of $- \otimes_R M$ and define the Tor functor as $\operatorname{Tor}_i(N, M) := L_i(N)$. More concretely, $\operatorname{Tor}_i(N, M)$ is defined as follows: Consider $\mathcal{F}^N_\bullet$ a free resolution (non necessarily graded) of the module N, then one has $\operatorname{Tor}^R_i(N, M) = H_i(\mathcal{F}^N_\bullet \otimes_R M)$. The definition is independent on the choice of the free resolution.

Analogously, one may consider the functor $N \otimes_R -$ which takes the tensor product on the left. In this case, a free-resolution $\mathcal{F}^M_\bullet$ of the module M is needed and one obtains the equality $\operatorname{Tor}^R_i(N, M) = H_i(M \otimes_R \mathcal{F}^M_\bullet)$. More in general, one has the equality $\operatorname{Tor}^R_i(N, M) = H_i(\mathcal{F}^N_\bullet \otimes_R \mathcal{F}^M_\bullet)$. If both M and N are graded modules, $\operatorname{Tor}^R_i(N, M)$ inherits a graded structure for every $n \geqslant 0$.

Proposition A.3. *Let M be a graded R-module. Consider the R-module structure on $\mathbf{k}$ given by the quotient $R/\mathfrak{m}_R$. Then for every $i, j \geqslant 0$ one has the equality*

$$\beta^R_{i,j}(M) = \dim_{\mathbf{k}} \operatorname{Tor}^R_i(M, \mathbf{k})_j.$$

Proof. See [47, Proposition 1.7]. □

Two important invariants that measure the behaviour of the minimal free resolution are the projective dimension and the regularity. Consider a graded R-module M with minimal resolution $\mathcal{F}_\bullet$. We define the *projective dimension* of M to be

$$\operatorname{pd}_R M = \inf\{i : F_i \neq 0\}.$$

We say that M has a finite resolution if its projective dimension is finite. The (relative) regularity of M is defined as

$$\operatorname{reg}_R M = \sup\{j - i : \beta_{i,j}(M) \neq 0\}.$$

In the particular case when M is a module over S, the integer $\operatorname{reg}_S(M)$ is also called in literature the Castelnuovo-Mumford regularity of M. We will see that despite, the relative regularity which depends on the ring R, the Castelnuovo-Mumford regularity is a more an intrinsic property of the module M itself.

The condition $\varphi_i(F_i) \subset \mathfrak{m}F_{i-1}$ for every $i \geqslant 1$ of the minimal resolution implies that in the matrices representing each chain map are filled with elements of positive degree, and therefore $\beta^R_{i,j}(M) = 0$ for every $j < i$. To better visualise the graded Betti numbers in a compact way, it is useful to define the *Betti table* of a module. This is a table where the position (i, j) is filled with the value $\beta^R_{i,i+j}(M)$. Even if at first glance it seems counter-intuitive, displaying Betti numbers turns out to be quite useful and practical. For instance, the last non-zero column represents the projective dimension, while the last non-zero row represents the regularity of the module.

Example A.4. Let $S = \mathbf{k}[x, y, z]$, and consider the module $M = R/(x^2, xyz, z^2y^2, z^5)$. Using [M2], we obtain the minimal free resolution of M as S-module.

$$0 \to R(-8)^2 \xrightarrow{\begin{pmatrix} z^4 & 0 \\ 0 & z^3 \\ -y & 0 \\ x & -y \\ 0 & x \end{pmatrix}} \begin{array}{c} R(-4) \\ \oplus \\ R(-5) \\ \oplus \\ R(-7)^3 \end{array} \xrightarrow{\begin{pmatrix} -yz & 0 & -z^5 & 0 & 0 \\ x & -yz & 0 & -z^4 & 0 \\ 0 & x & 0 & 0 & -z^3 \\ 0 & 0 & x^2 & xy & y^2 \end{pmatrix}} \begin{array}{c} R(-2) \\ \oplus \\ R(-3) \\ \oplus \\ R(-4) \\ \oplus \\ R(-5) \end{array} \xrightarrow{\begin{pmatrix} x^2 \\ xyz \\ z^2y^2 \\ z^5 \end{pmatrix}^T} R$$

Therefore the Betti table of M is

```
+-----------------+
|       0  1  2  3|
|total: 1  4  5  2|
|    0: 1  -  -  -|
|    1: -  1  -  -|
|    2: -  1  1  -|
|    3: -  1  1  -|
|    4: -  1  -  -|
|    5: -  -  3  2|
+-----------------+
```

The zeros appearing in the Betti table are replaced with a dash symbol "-". ◇

Construction A.5. We want to compute the graded Betti number of $\mathbf{k} = S/\mathfrak{m}_S$ as an S-module. Following [27, Section 1.6], we consider the exterior algebra associated with the ring S. This is denoted by $\bigwedge S$ and it has a graded structure defined as

$$\bigwedge S = \bigoplus_{i=0}^{n} \bigwedge\nolimits^i S.$$

The exterior algebra construction defines a wedge product which is an alternating operation, that is

$$\begin{aligned} x \wedge y &= (-1)^{a+b} y \wedge x, && \text{where } x \in \textstyle\bigwedge^a S, y \in \bigwedge^b S, \text{ and} \\ x \wedge x &= 0, && \text{if } x \in \textstyle\bigwedge S \text{ has a odd degree.} \end{aligned}$$

Every homogeneous part $\bigwedge^s S$ is a S-module generated by $\{x_{j_1} \wedge \cdots \wedge x_{j_s} : 1 \leqslant j_1 < \cdots < j_s \leqslant n\}$. On $\bigwedge S$ is defined a differential operator $\partial : \bigwedge S \to \bigwedge S$ that is a graded R-linear homomorphism of degree -1. The map ∂ is constructed by setting

$$\partial(x_{j_1} \wedge \cdots \wedge x_{j_s}) = \sum_{k=1}^{s} (-1)^{k+1} x_{j_k} x_{j_1} \wedge \cdots \wedge \widehat{x_{j_k}} \wedge \cdots \wedge x_{j_s},$$

where $\widehat{x_{j_k}}$ means that the element x_{j_k} is omitted.

By explicit computations, it is possible to check that $\partial^2 = 0$. This fact is equivalent to say that

$$K_\bullet(x_1, \ldots, x_n) : 0 \to \bigwedge^n S \to \bigwedge^{n-1} S \to \cdots \to \bigwedge^1 S \to S \to 0$$

is a chain complex. Moreover, the homomorphism ∂ satisfies the *Leibniz rule*: For any pair of homogeneous elements $x, y \in \bigwedge S$ we have

$$\partial(x \wedge y) = \partial(x) \wedge y + (-1)^{\deg x} x \wedge \partial(y).$$

The pair $(\bigwedge S, \partial)$ is a finitely generated DG-algebra (The interested reader can look in [11] for more information on DG-algebras).

The complex $K_\bullet(x_1, \ldots, x_n)$ is called the *Koszul complex* associated with the sequence $x_1, \ldots, x_n$. In literature, it can be denoted also as $K_\bullet(\mathfrak{m}_S)$ by considering a coordinate-free construction. Since $x_1, \ldots, x_n$ is a regular sequence in S, the Koszul complex $K_\bullet(x_1, \ldots, x_n)$ is a minimal free resolution of $\mathbf{k}$. In particular, $\beta^S_{i,i}(\mathbf{k}) = \operatorname{rank} \bigwedge^i S = \binom{n}{i}$, and $\beta^S_{i,j}(\mathbf{k}) = 0$ for $i \neq j$, implying $\operatorname{pd}_S \mathbf{k} = n$, and $\operatorname{reg}_S \mathbf{k} = 0$. ♢

Definition A.6. Construction A.5 can be generalised to any graded quotient $R = S/I$. We define the *Koszul complex* $K_\bullet(\mathfrak{m}_R)$ obtained from the DG-algebra $\bigwedge R$. For any R-module M, it is also defined the chain complex $K_\bullet(\mathfrak{m}_R; M) := K_\bullet(\mathfrak{m}_R) \otimes_R M$ called the *Koszul homology* of M

The most important result regarding free resolution of modules over a polynomial ring is **Hilbert's Syzygy theorem** [81] which states as follows.

Theorem A.7. *Let M be a $\mathbb{Z}$-graded module over a polynomial ring $S = \mathbf{k}[x_1, \ldots, x_n]$. Then the minimal free resolution of M is finite, and $\operatorname{pd}_S M \leqslant \operatorname{pd}_S \mathbf{k} = \dim S = n$.*

Proof. From Proposition A.3, for every $i \geqslant 0$ we have the formula

$$\beta_i(M) = \dim_{\mathbf{k}} \operatorname{Tor}_i(M, \mathbf{k}).$$

To compute $\operatorname{Tor}_i(\mathbf{k}, M)$ we can proceed as follows. We consider the minimal free resolution of $\mathbf{k}$ given by the Koszul complex $K_\bullet(x_1, \ldots, x_n)$. Then

$$\operatorname{Tor}_i(M, \mathbf{k}) = H_i(K_\bullet(\mathfrak{m}_S; M)).$$

Since the the Koszul complex has length n, then we have $\beta_i(M) = 0$ for $i \geqslant n+1$ giving the desired result. □

Remark A.8. In Theorem A.7, the difference between the dimension of S and the projective dimension of M can be computed with the Auslander–Buchsbaum formula [46, Thm 19.9] and it turns out being the depth of M, i.e. the maximal length of a regular sequence in M. △

As shown in the following theorem, when considering a graded quotient R, $K_\bullet(\mathfrak{m}_R)$ is not a minimal free resolution of $\mathbf{k}$ as a R-module except for few exceptions. In particular, whenever the minimal resolution of $\mathbf{k}$ is finite then $K_\bullet(\mathfrak{m}_R)$ resolves $\mathbf{k}$. The following theorem can be seen as a generalization of Hilbert's Syzygy theorem.

Theorem A.9 (Auslander-Buchsbaum-Serre). *Let $R = S/I$ be a standard graded $\mathbf{k}$-algebra, with S a polynomial ring. Then the following are equivalent.*

1. *R is a polynomial ring, i.e. I is generated by linear forms;*
2. *$\operatorname{pd}_R \mathbf{k}$ is finite;*
3. *$\operatorname{pd}_R M$ is finite for every R-module M.*

In particular, $\operatorname{pd}_R M \leqslant \operatorname{pd}_R \mathbf{k}$ for any R-module M, and the Koszul complex $K_\bullet(\mathfrak{m}_R)$ is a minimal free resolution of $\mathbf{k} = R/\mathfrak{m}_R$.

Remark A.10. Tate in [125] studied a inductive construction called Tate complex to obtain the minimal free resolution of $\mathbf{k}$ starting from $K_\bullet(\mathfrak{m}_R)$. We refer to [11] for more generalities and proofs on this topic. △

The interest in Betti numbers is also due to their relation with the information given by the Hilbert function.

Proposition A.11. *Let M be a finitely generated graded S-module with Betti numbers $\beta^S_{i,j}(M)$. Set $\phi_j := \sum_{i\geqslant 0}(-1)^i\beta^S_{i,j}(M) \in \mathbb{Z}$ for every $j \in \mathbb{Z}$. By Theorem A.7 and since M is finitely generated, only finitely many ϕ_j are non-zero. Then the Hilbert function of M is determined by the formula*

$$\mathrm{HF}_M(d) = \sum_{j\in\mathbb{Z}} \phi_j \binom{n+d-j-1}{n-1}.$$

Equivalently, the Hilbert series of M can be written as

$$\mathrm{HS}_M = \frac{\sum_{j\in\mathbb{Z}} \phi_j t^j}{(1-t)^n}.$$

Proof. By Theorem A.7, the minimal free resolution of M looks like

$$0 \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta^S_{p,j}(M)} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta^S_{1,j}(M)} \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta^S_{0,j}(M)} \xrightarrow{\epsilon} M \to 0,$$

where p is the projective dimension of M. By restricting the above exact sequence to its d-th homogeneous part, we obtain a exact sequence of finitely dimensional $\mathbf{k}$-vector spaces where the right-most part is M_d. Therefore, the formulae are direct consequence of the Rank–nullity theorem for exact sequences. □

Remark A.12. The integers ϕ_j in Proposition A.11 can be obtained by the Hilbert function inductively by inverting the given formula. But, the exact value of the Betti numbers cannot be deducted in general by only the Hilbert function. In fact, there are plenty examples of modules having same Hilbert function, but different Betti table. △

Koszul homology

As we have seen in the proof of Theorem A.7, in the polynomial case, the Koszul homology $K_\bullet(\mathfrak{m}_S; -)$ measures the graded Betti numbers of a given graded S-module. Consider now a graded quotient $R = S/I$. Given a graded R-module M, what does $K_\bullet(\mathfrak{m}_R; M)$ computes? It turns out that it still computes the graded Betti numbers of M as S-module as shown below.

Proposition A.13. *Let $R = S/I$ be a graded quotient of a polynomial ring $S = \mathbf{k}[x_1, \dots, x_n]$ by a homogeneous ideal I. Consider a graded R-module M. The projection $\pi : S \to R$ induces a S-module structure on M compatible with its grading. Then, it follows*

$$\beta^S_{i,j}(M) = \dim_{\mathbf{k}} H_i(K_\bullet(\mathfrak{m}_R; M))_j, \text{ for every } i, j \in \mathbb{N}.$$

Therefore, one has the formula

$$\operatorname{reg}_S(M) = \sup\{j - i : H_i(K_\bullet(\mathfrak{m}_R; M))_j \neq 0\}.$$

Proof. By Theorem A.7, we have the formula $\beta^S_{i,j}(M) = \dim_{\mathbf{k}} H_i(K_\bullet(\mathfrak{m}_S; M))_j$ for every $i, j \in \mathbb{N}$. It remains to compare the two complexes $K_\bullet(\mathfrak{m}_S; M)$ and $K_\bullet(\mathfrak{m}_R; M)$. By definition, given an element $f \in S$ we have that the multiplication map $\cdot f : M \to M$ is defined as $m \cdot f := \pi(f)m$ for every $m \in M$. Consider now $\phi_s : \bigwedge^s M \to \bigwedge^{s-1} M$ the s-th chain map of $K_\bullet(\mathfrak{m}_S; M)$. From the definition of Koszul complex, ϕ_i is a $\mathbf{k}$-linear combination of multiplication maps. Since any element in S acts on M through the action of R on M, we obtain $H_i(K_\bullet(\mathfrak{m}_S; M)) \cong H_i(K_\bullet(\mathfrak{m}_R; M))$ for every $i = 1, \ldots, n$. □

Characterization using local cohomology

We now breafly introduce the local cohomology of a R-module M with support on an ideal I, and its relation with the Betti numbers of M.

Consider a homogeneous ideal $I \subset R$. Then the *Gamma functor* $\Gamma_I : R-\mathfrak{mod} \to R-\mathfrak{mod}$ is defined as $\Gamma_I(M) := \cup_{i \geqslant 0} 0 :_M I^n$. Since Γ_I is left-exact, we may consider its right derived functors that we denote as H^j_I. For every R-module M, we name $H^j_I(M)$ the j-th local cohomology module of M with support on I. Here we state some basic properties of local cohomology.

Proposition A.14. *Let M be an R-module.*

(1) For each $j \geqslant 0$, $H^j_I(M)$ is I-torsion, that is $I^t H^j_I(M) = 0$ for some $t \in N$.

(2) If $\sqrt{I} = \sqrt{J}$, then $H^j_I(M) \cong H^j_J(M)$ for every $j \geqslant 0$.

(3) The projection $\pi : S \to R$ induces a graded isomorphism $H^j_J(M) \cong H^j_{\pi(J)}(M)$ for every $j \geqslant 0$, where J is a homogeneous ideal in S.

Proof. See [87, Prop. 7.3] and [87, Prop. 7.15 (2)]. □

One of the first vanishing theorems for local cohomology states that the cohomology modules vanish after the dimension of the base ring.

Proposition A.15. *Let $I \subset R$ an ideal. Then for every M finitely generated module it follows:*

(1) For $i \geqslant \dim M + 1$, it follows $H^i_I(M) = 0$. In particular, $H^i_I(M) = 0$ for every $i \geqslant \dim R$.

(2) Let $\operatorname{depth}(I, M)$ be the maximal length of a M-regular sequence in I, then

$$\operatorname{depth}(I, M) = \inf\{i \in \mathbb{N} : H^i_I(M) \neq 0\}.$$

Proof. For (1) see [87, Prop 9.15], while for (2) see [87, Thm 9.1]. □

We now focus on the case $I = \mathfrak{m}_R$. For every R-module M, we define the integers

$$a_i(M) := \sup\{j : H^i_{\mathfrak{m}_R}(M)_j \neq 0\}, \text{ for } i \geqslant 0.$$

The invariant $a_i(M)$ is always a finite number since $H^i_{\mathfrak{m}_R}(M)$ is an Artinian module by Proposition A.14(1). These invariant can be used to compute the regularity of M as a S-module.

Theorem A.16. *Let M be finitely generated graded R-module. Then*

$$\operatorname{reg}_S(M) = \max\{a_i(M) + i : 0 \leqslant i \leqslant n\}.$$

Proof. The case $R = S$ is proved in [47, Thm 4.3]. For the general case, Proposition A.14(3) implies $H^j_{\mathfrak{m}_R}(M) \cong H^j_{\mathfrak{m}_S}(M)$ as graded S-modules which concludes the proof. □

Corollary A.17. *Let M be a graded R-module of finite length. Then* $\operatorname{reg}_S(M) = \max\{d : M_d \neq 0\}$.

Proof. Since M has finite length, a power of $\mathfrak{m}_R$ annihilates M. This implies that $H^0_{\mathfrak{m}_R}(M) = \Gamma_{\mathfrak{m}_R}(M) = M$, and $H^j_{\mathfrak{m}_R}(M) = 0$ for $j \geqslant 1$. Then the final result is a consequence of Theorem A.16. □

Castelnuovo-Mumford regularity for Noetherian rings

In recent time, the theory of regularity has been generalised to the case when the residue field $\mathbf{k}$ is replaced with a Noetherian ring. In this case, the classical condition of minimality does not subsist, and one needs to adjust it. For a recent account and overview on this topic, in the settings of this part, see [25, Chapter 8] and [26].

Consider a polynomial ring $S = S_0[x_1, \dots, x_n]$ where S_0 is a Noetherian ring. Let $R = S/I$ be a graded quotient. Set $S_+ = \oplus_{i \geqslant 1} S_i = (x_1, \dots, x_n)S$, and $R_+ = \oplus_{i \geqslant 1} R_i$. Finally, let M be a finitely generated $\mathbb{Z}$-graded R-module. Using the projection $\pi : S \to R$, M has also a structure of finitely generated graded S-module.

A free resolution $\mathcal{F}_\bullet$ of M as a S-module is minimal if at each step

$$\cdots \to F_{i+1} \xrightarrow{\phi_{i+1}} F_i \xrightarrow{\phi_i} F_{i-1} \to \dots,$$

the kernel of ϕ_i is minimally generated by the image of a basis of F_{i+1} through ϕ_{i+1}. In this section, we will only consider free resolutions over the ring S.

In this setting, unless S_0 is a local ring, the same module can be minimally generated by set of elements of different cardinality. Therefore, Theorem A.1 cannot be generalised. However, the degrees in which these generators lie are an invariant of the module. For every R-module M, one can define

$$t_0(M) := \min\{d \in \mathbb{Z} : M \text{ is minimally generated by } \oplus_{i \leqslant d} M_i\}.$$

Then, even if there is no notion of Betti numbers, one can define the Castelnuovo-Mumford regularity of M by mean of different, but equivalent definitions.

- **Local cohomology.** Let $H^i_{R_+}(M)$ be the ith cohomology module of M with support on R_+. Then we set $a_i(M) = \sup\{j \in \mathbb{Z} : H^i_{R_+}(M)_j \neq 0\}$. Let also $\operatorname{grade}(R_+, M) = \min\{i \in \mathbb{N} : H^i_{R_+}(M) \neq 0\}$. The regularity of M is defined as

  $$\operatorname{reg}(M) = \max\{a_i(M) + i : 0 \leqslant i \leqslant n\} = \max\{a_i(M) + i : \operatorname{depth}(R_+, M) \leqslant i \leqslant n\},$$

  where the second equality is a consequence of Proposition A.15(2).

- **Koszul homology.** Consider the Koszul homology $H_\bullet(R_+, M)$ that is the homology groups of the Koszul complex. As in Proposition A.13, we have $H_\bullet(R_+, M) = H_\bullet(S_+, M)$. We generalise the t_0 invariant as

  $$t_i(M) = \max\{j \in \mathbb{Z} : H_i(S_+, M)_j \neq 0\}, \text{ for } i = 0, \dots, n.$$

  Then the regularity given by Koszul homology is defined as

  $$\operatorname{reg}_1(M) = \max\{t_i(M) - i : i = 0, \dots, n\} = \max\{t_i(M) - i : i = 0, \dots, n - \operatorname{depth}(S_+, M)\},$$

  where the second equality is a consequence of [27, Thm 1.6.17].

- **Minimal free resolutions.** Let $\mathcal{F}_\bullet$ be a minimal free resolution of M. The regularities of $\mathcal{F}_\bullet$ are defined as

  $$\operatorname{reg}_2(\mathcal{F}_\bullet) = \sup\{t_0(F_i) - i : i = 0, \dots, n - \operatorname{depth}(S_+, M)\},$$

  and

  $$\operatorname{reg}_3(\mathcal{F}_\bullet) = \sup\{t_0(F_i) - i : i \in \mathbb{N}\}.$$

Theorem A.18. *Let M be a finitely generated graded R-module, and let $\mathcal{F}_\bullet$ be a minimal free resolution of M. Then*

$$\operatorname{reg}(M) = \operatorname{reg}_1(M) = \operatorname{reg}_2(\mathcal{F}_\bullet) = \operatorname{reg}_3(\mathcal{F}_\bullet).$$

Proof. See [25, Thm 8.1.3]. □

A hystorical note on Castelnuovo-Mumford regularity

The first step through the formal definition of regularity has been attributed to Castelnuovo in his classical paper [31]. There, he was studying complete linear systems, or divisors in modern terminology, over projective curves using the so called "basedpoint-free pencil trick". It is said that this trick was taught by Zariski, student of Castelnuovo, to all his students including Mumford. In the aim of generalizing Castelnuovo's results, Mumford introduced the concept of d-regularity for coherent sheaves on some projective space. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^N_{\mathbf{k}}$, where $\mathbf{k}$ is an algebraically closed field of characteristic zero. We say that $\mathcal{F}$ is d-regular if $H^i(\mathbb{P}^N, \mathcal{F}(d-i)) = 0$ for every $i \geqslant 1$. Then one defines the Castelnuovo-Mumford regularity $\operatorname{reg} \mathcal{F}$ as the least integer r such that $\mathcal{F}$ is r-regular. If $M = \oplus_{d \in \mathbb{Z}} H^0(\mathbb{P}^N_{\mathbf{k}}, \mathcal{F}(d))$, then one has $\operatorname{reg}_S M = \operatorname{reg} \mathcal{F}$, where $S = \mathbf{k}[x_0, \ldots, x_N]$. This is direct consequence of the characterisation of regularity with local cohomology, together with the theory of Čech complexes [87, Thm 7.13], and their application on the computation of sheaf cohomology [87, Thm 13.21]. The modern and standard definition of regularity as given in this appendix is due to Eisenbud and Goto in [48].

References

[1] J. Abbott, A. M. Bigatti, and L. Robbiano. *CoCoA: a system for doing Computations in Commutative Algebra.* Available at `http://cocoa.dima.unige.it`.

[2] N. Abdallah, N. Altafi, P. De Poi, L. Fiorindo, A. Iarrobino, P. Macias Marques, E. Mezzetti, R. M. Miró-Roig, and L. Nicklasson. "Hilbert functions and Jordan type of Perazzo Artinian algebras". In: *Lefschetz properties–current and new directions.* Vol. 59. Springer INdAM Ser. Springer, Singapore, 2024, pp. 59–80.

[3] W. W. Adams and P. Loustaunau. *An introduction to Gröbner bases.* Vol. 3. Graduate Studies in Mathematics. American Mathematical Society, 1994.

[4] K. Akin and D. A. Buchsbaum. "Characteristic-free representation theory of the general linear group". In: *Advances in Mathematics* 58.2 (1985), pp. 149–200.

[5] K. Akin and D. A. Buchsbaum. "Characteristic-free representation theory of the general linear group. II. Homological considerations". In: *Advances in Mathematics* 72.2 (1988), pp. 171–210.

[6] K. Akin, D. A. Buchsbaum, and J. Weyman. "Schur functors and Schur complexes". In: *Advances in Mathematics* 44.3 (1982), pp. 207–278.

[7] N. Altafi. "Hilbert functions of Artinian Gorenstein algebras with the strong Lefschetz property". In: *Proceedings of the American Mathematical Society* 150 (2022), pp. 499–513.

[8] A. Alzati and R. Re. "Complete intersections of quadrics and the weak Lefschetz property". In: *Collectanea Mathematica* 70.2 (2019), pp. 283–294.

[9] H. H. Andersen. "The first cohomology group of a line bundle on G/B". In: *Inventiones Mathematicae* 51.3 (1979), pp. 287–296.

[10] H. H. Andersen. "The Frobenius morphism on the cohomology of homogeneous vector bundles on G/B". In: *Annals of Mathematics (2)* 112.1 (1980), pp. 113–121.

[11] L. L. Avramov. "Infinite free resolutions". In: *Six lectures on commutative algebra (Bellaterra, 1996).* Vol. 166. Progress in Mathematics. Birkhäuser, Basel, 1998, pp. 1–118.

[12] L. L. Avramov and D. Eisenbud. "Regularity of modules over a Koszul algebra". In: *Journal of Algebra* 153.1 (1992), pp. 85–90.

[13] L. L. Avramov and I. Peeva. "Finite regularity and Koszul algebras". In: *American Journal of Mathematics* 123.2 (2001), pp. 275–281.

[14] A. Bagheri, M. Chardin, and H. T. Hà. "The eventual shape of Betti tables of powers of ideals". In: *Mathematical Research Letters* 20 (2013), pp. 1033–1046.

[15] A. Bernardi, A. Gimigliano, and M. Idà. "Computing symmetric rank for symmetric tensors". In: *Journal of Symbolic Computation* 46.1 (2011), pp. 34–53.

[16] P. Biswas and M. Mandal. "A study of v-number for some monomial ideals". In: *Collectanea Mathematica* 76 (2025), pp. 667–682.

[17] M. Boij, J. Migliore, R. M. Miró-Roig, and U. Nagel. "On the weak Lefschetz property for height four equigenerated complete intersections". In: *Transactions of the American Mathematical Society Series B* 10 (2023), pp. 1254–1286.

[18] M. de Bondt. "Homogeneous quasi-translations in dimension 5". In: *Beiträge zur Algebra und Geometrie. Contributions to Algebra and Geometry* 59.2 (2018), pp. 295–326.

[19] A. Borel and A. Weil. "Représentations linéaires et espaces homogènes kählériens des groupes de Lie compacts (exposé by J.P. Serre)". In: *Séminaire Bourbaki : années 1951/52 - 1952/53 - 1953/54, exposés 50-100, Séminaire Bourbaki* 2 (1954), pp. 447–454.

[20] R. Bott. "Homogeneous vector bundles". In: *Annals of mathematics* 66.2 (1957), pp. 203–248.

[21] J. Brennan and A. Simis. *The Mathematical Legacy of Wolmer V. Vasconcelos.* De Gruyter, 2025.

[22] H. Brenner and A. Kaid. "Syzygy bundles on $\mathbf{P}^2$ and the weak Lefschetz property". In: *Illinois Journal of Mathematics* 51.4 (2007), pp. 1299–1308.

[23] M. Brodmann. "Asymptotic stability of $\operatorname{Ass}(M/I^nM)$". In: *Proceedings of the American Mathematical Society* 74 (1979), pp. 16–18.

[24] W. Bruns and A. Conca. "A remark on regularity of powers and products of ideals". In: *Journal of Pure and Applied Algebra* 221 (2017), pp. 2861–2868.

[25] W. Bruns, A. Conca, C. Raicu, and M. Varbaro. *Determinants, Gröbner Bases and Cohomology.* Springer Monographs in Mathematics. Cham: Springer, 2022.

[26] W. Bruns, A. Conca, and M. Varbaro. "Castelnuovo-Mumford Regularity and Powers". In: *Commutative Algebra: Expository Papers Dedicated to David Eisenbud on the Occasion of his 75th Birthday.* Cham: Springer, 2022, pp. 147–158.

[27] W. Bruns and J. Herzog. *Cohen-Macaulay rings.* Vol. 39. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1993.

[28] W. Bruns, J. Herzog, and U. Vetter. "Syzygies and walks". In: *Commutative algebra (Trieste, 1992).* World Sci. Publ., River Edge, NJ, 1994, pp. 36–57.

[29] Paul-Jean Cahen and Jean-Luc Chabert. *Integer-valued polynomials.* Vol. 48. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997.

[30] H. Cartan and S. Eilenberg. *Homological Algebra.* Princeton mathematical series. Princeton University Press, 1956.

[31] G. Castelnuovo. "Sui multipi di una seire lineare di gruppi di punti appartenente ad una curva algebrica". In: *Rendiconti del Circolo Matematico di Palermo* 7 (1893), pp. 89–110.

[32] M. Chardin, D. Ghosh, and N. Nemati. "The (ir)regularity of Tor and Ext". In: *Transactions of the American Mathematical Society* 375 (2022), pp. 47–70.

[33] C. Ciliberto, F. Russo, and A. Simis. "Homaloidal hypersurfaces and hypersurfaces with vanishing Hessian". In: *Advances in Mathematics* 218.6 (2008), pp. 1759–1805.

[34] L. Colarte-Gómez, E. Mezzetti, and R. M. Miró-Roig. "On the arithmetic Cohen-Macaulayness of varieties parameterized by Togliatti systems". In: *Annali di Matematica Pura ed Applicata (4)* 200.4 (2021), pp. 1757–1780.

[35] A. Conca. "Koszul algebras and their syzygies". In: *Combinatorial algebraic geometry.* Vol. 2108. Lecture Notes in Math. Springer, Cham, 2014, pp. 1–31.

[36] A. Conca. "A note on the v-invariant". In: *Proceedings of the American Mathematical Society* 152.6 (2023), pp. 2349–2351.

[37] A. Conca, E. De Negri, and M. E. Rossi. "Koszul algebras and regularity". In: *Commutative algebra.* Springer, New York, 2013, pp. 285–315.

[38] C. de Concini, D. Eisenbud, and C. Procesi. "Young diagrams and determinantal varieties". In: 56.2 (1980), pp. 129–165.

[39] S. M. Cooper, A. Seceleanu, S. O. Tohăneanu, M. Pinto Vaz, and R. H. Villarreal. "Generalized minimum distance functions and algebraic invariants of Geramita ideals". In: *Advances in Applied Mathematics* 112 (2020), p. 101940.

[40] B. Costa and R. Gondim. "The Jordan type of graded Artinian Gorenstein algebras". In: *Advances in Applied Mathematics* 111 (2019), 101941, 27 pp.

[41] S. D. Cutkosky, J. Herzog, and N. V. Trung. "Asymptotic behaviour of the Castelnuovo-Mumford regularity". In: *Compositio Mathematica* 118 (1999), pp. 243–261.

[42] A. D'Alì. *On strongly Koszul algebras and tidy Gröbner bases.* 2025. arXiv: `2512.11778`.

[43] R. Di Gennaro and R. M. Miró-Roig. "Complete Intersection Algebras with Binomial Macaulay Dual Generator". In: *Mediterranean Journal of Mathematics* 22.7 (2025), Paper No. 178.

[44] S. Donkin. "The cohomology of line bundles on the three- dimensional flag variety". In: *Journal of Algebra* 307.2 (2007), pp. 570–613.

[45] I. M. Duursma, C. Rentería, and H. Tapia-Recillas. "Reed-Muller codes on complete intersections". In: *Applicable Algebra in Engineering, Communication and Computing* 11.6 (2001), pp. 455–462.

[46] D. Eisenbud. *Commutative algebra. With a view toward algebraic geometry.* Vol. 150. Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.

[47] D. Eisenbud. *The geometry of syzygies.* Vol. 229. Graduate Texts in Mathematics. A second course in commutative algebra and algebraic geometry. Springer-Verlag, New York, 2005.

[48] D. Eisenbud and S. Goto. "Linear free resolutions and minimal multiplicity". In: *Journal of Algebra* 88.1 (1984), pp. 89–133.

[49] V. Ene and J. Herzog. *Gröbner Bases in Commutative Algebra.* Vol. 130. Graduate Studies in Mathematics. Providence, RI: American Mathematical Society, 2012.

[50] A. Ficarra and E. Sgroi. *Asymptotic behaviour of the v-number of homogeneous ideals.* 2023. arXiv: `2306.14243`.

[51] L. Fiorindo. "Polynomials with vanishing Hessian and Lefschetz properties". arXiv:2212.11801. MA thesis. University of Trieste, 2022.

[52] L. Fiorindo and D. Ghosh. *Asymptotic behaviour of Vasconcelos invariants for products and powers of graded ideals.* 2025. arXiv: `2401.17815`.

[53] L. Fiorindo and D. Ghosh. "On the asymptotic behaviour of the Vasconcelos invariant for graded modules". In: *Nagoya Mathematical Journal* 258 (2025), pp. 296–310.

[54] L. Fiorindo, E. Mezzetti, and R. M. Miró-Roig. "Perazzo 3-folds and the weak Lefschetz property". In: *Journal of Algebra* 626 (2023), pp. 56–81.

[55] L. Fiorindo, E. Reed, S. Roshan Zamir, and H. Yu. "A uniform identification of stable sheaf cohomology". In: *Proceedings of the American Mathematical Society* 153.10 (2025), pp. 4197–4213.

[56] A. Franchetta. "Sulle forme algebriche di S_4 aventi l'hessiana indeterminata". In: *Rendiconti di Matematica e Applicazioni (5)* 14 (1954), pp. 252–257.

[57] R. Fröberg. "Determination of a class of Poincaré series". In: *Mathematica Scandinavica* 37.1 (1975), pp. 29–39.

[58] W. Fulton. *Young tableaux.* Vol. 35. London Mathematical Society Student Texts. With applications to representation theory and geometry. Cambridge University Press, Cambridge, 1997.

[59] Z. Gao, C. Raicu, and K. VandeBogert. "Some questions arising from the study of cohomology on flag varieties". In: *Open problems in algebraic combinatorics.* Vol. 110. Proceedings of Symposia in Pure Mathematics. American Mathematical Society, 2024, pp. 333–348.

[60] A. Garbagnati and F. Repetto. "A geometrical approach to Gordan-Noether's and Franchetta's contributions to a question posed by Hesse". In: *Collectanea Mathematica* 60 (2009), pp. 27–41.

[61] D. Ghosh. "Asymptotic linear bounds of Castelnuovo-Mumford regularity in multigraded modules". In: *Journal of Algebra* 445 (2016), pp. 103–114.

[62] R. Gondim. "On higher Hessians and the Lefschetz properties". In: *Journal of Algebra* 489 (2017), pp. 241–263.

[63] R. Gondim and G. Zappalà. "On mixed Hessians and the Lefschetz properties". In: *Journal of Pure and Applied Algebra* 223.10 (2019), pp. 4268–4282.

[64] M. González-Sarabia, J. Martínez-Bernal, R. H. Villarreal, and C. E. Vivares. "Generalized minimum distance functions". In: *Journal of Algebraic Combinatorics. An International Journal* 50.3 (2019), pp. 317–346.

[65] M. González-Sarabia, C. Rentería, and H. Tapia-Recillas. "Reed-Muller-type codes over the Segre variety". In: *Finite Fields and their Applications* 8.4 (2002), pp. 511–518.

[66] P. Gordan and M. Nöther. "Über die algebraischen Formen, deren Hesse'sche Determinante identisch verschwinde". In: *Mathematische Annalen* 10 (1876), pp. 547–568.

[M2] D. R. Grayson and M. E. Stillman. *Macaulay2, a software system for research in algebraic geometry.* `http://www2.macaulay2.com`.

[67] M. Green. "Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann". In: *Algebraic Curves and Projective Geometry (Trento, 1988).* Vol. 1389. Lecture Notes in Mathematics. 1989, pp. 76–86.

[68] W. L. Griffith Jr. *Cohomology of flag varieties in characteristic p.* Thesis (Ph.D.)–Harvard University. ProQuest LLC, Ann Arbor, MI, 1975.

[69] W. L. Griffith Jr. "Cohomology of flag varieties in characteristic p". In: *Illinois Journal of Mathematics* 24.3 (1980), pp. 452–461.

[70] W. J. Haboush. "A short proof of the Kempf vanishing theorem". In: *Inventiones Mathematicae* 56.2 (1980), pp. 109–112.

[71] T. Harima. "Characterization of Hilbert Functions of Gorenstein Artin Algebras with the Weak Stanley Property". In: *Proceedings of the American Mathematical Society* 12.123 (1995), pp. 3631–3638.

[72] T. Harima, T. Maeno, H. Morita, Y. Numata, A. Wachi, and J. Watanabe. *The Lefschetz Properties.* Lecture Notes in Mathematics 2080. Springer, 2013.

[73] T. Harima, J. Migliore, U. Nagel, and J. Watanabe. "The Weak and Strong Lefschetz properties for Artinian K-algebras". In: *Journal of Algebra* 262.1 (2003), pp. 99–126.

[74] R. Hartshorne. *Algebraic geometry.* Vol. No. 52. Graduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1977.

[75] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[76] F. Hayasaka. "Asymptotic stability of primes associated to homogeneous components of multigraded modules". In: *Journal of Algebra* 306 (2006), pp. 535–543.

[77] J. Herzog, T. Hibi, and G. Restuccia. "Strongly Koszul algebras". In: *Mathematica Scandinavica* 86.2 (2000), pp. 161–178.

[78] O. Hesse. "Über die Bedingung, unter welche eine homogene ganze Function on n unabhängigen Variabeln durch lineäre Substitutionen von n andern unabhängigen Variabeln auf eine homogene Function sich zurückführen läst, die eine Variable weniger enthält". In: *Journal für die reine und angewandte Mathematik.* 42nd ser. (1851), pp. 117–124.

[79] O. Hesse. "Zur Theorie der ganzen homogenen Functionen". In: *Journal für die reine und angewandte Mathematik.* 56th ser. (1859), pp. 263–269.

[80] T. Hibi, K. Matsuda, and H. Ohsugi. "Strongly Koszul edge rings". In: *Acta Mathematica Vietnamica* 41.1 (2016), pp. 69–76.

[81] D. Hilbert. "Über die Theorie der algebraischen Formen". In: *Mathematische Annalen* 36 (1980), pp. 473–534.

[82] C. Huneke and I. Swanson. *Integral closure of ideals, rings, and modules.* Vol. 336. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2006.

[83] A. Iarrobino. "Compressed algebras: Artin algebras having given socle degrees and maximal length". In: *Transactions of the American Mathematical Society* 285 (1984), pp. 337–378.

[84] A. Iarrobino and V. Kanev. *Power Sums, Gorenstein Algebras, and Determinantal Loci.* Vol. 1721. Lecture Notes in Mathematics. Springer-Verlag, 1999.

[85] A. Iarrobino, P. Macias Marques, and C. McDaniel. "Artinian algebras and Jordan type". In: *Journal of Commutative Algebra* 14.3 (2022), pp. 365–414.

[86] H. Ikeda. "Results on Dilworth and Rees numbers of Artinian local rings". In: *Japan Journal of Mathematics* 22 (1996), pp. 147–158.

[87] S. B. Iyengar, G. J. Leuschke, A. Leykin, C. Miller, E. Miller, A. K. Singh, and U. Walther. *Twenty-four hours of local cohomology.* Vol. 87. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2007.

[88] D. Jaramillo and R. Villarreal. "The v-number of edge ideals". In: *Journal of Combinatorial Theory, Series A* 177 (2021). Paper No. 105310.

[89] D. Jaramillo-Velez and L. Seccia. "Connected domination in graphs and v-numbers of binomial edge ideals". In: *Collectanea Mathematica* 75.3 (2024), pp. 771–793.

[90] D. Katz and E. West. "A linear function associated to asymptotic prime divisors". In: *Proceedings of the American Mathematical Society* 132 (2004), pp. 1589–1597.

[91] G. R. Kempf. "On the collapsing of homogeneous bundles". In: *Inventiones mathematicae* (1967), pp. 561–574.

[92] G. R. Kempf. "Linear systems on homogeneous spaces". In: *Annals of Mathematics (2)* 103.3 (1976), pp. 557–591.

[93] G. R. Kempf. "Vanishing theorems for flag manifolds". In: *American Journal of Mathematics* 98.2 (1976), pp. 325–331.

[94] A. Kingsbury and R. Sharp. "Asymptotic behaviour of certain sets of prime ideals". In: *Proceedings of the American Mathematical Society* 124 (1996), pp. 1703–1711.

[95] V. Kodiyalam. "Asymptotic behaviour of Castelnuovo-Mumford regularity". In: *Proceedings of the American Mathematical Society* 128 (2000), pp. 407–411.

[96] M. Kreuzer and L. Robbiano. *Computational commutative algebra. 1.* Springer-Verlag, Berlin, 2000.

[97] S. Lefschetz. *L'analysis situs et la géométrie algébrique.* Gauthier-Villars, Paris, 1950.

[98] C. Lossen. "When does the Hessian determinant vanish identically? (On Gordan and Noether's Proof of Hesse's Claim)". In: *Bulletin of the Brazilian Mathematical Society* 35 (2004), pp. 71–82.

[99] F. S. Macaulay. "Some Properties of Enumeration in the Theory of Modular Systems". In: *Proceedings of the London Mathematical Society* 26.2 (1927), pp. 531–555.

[100] T. Maeno and J. Watanabe. "Lefschetz elements of Artinian Gorenstein algebras and Hessians of homogeneous polynomials". In: *Illinois Journal of Mathematics* 53 (2009), pp. 593–603.

[101] K. Matsuda and H. Ohsugi. "Reverse lexicographic Gröbner bases and strongly Koszul toric rings". In: *Mathematica Scandinavica* 119.2 (2016), pp. 161–168.

[102] S. McAdam and P. Eakin. “The asymptotic ass”. In: *Jounral of Algebra* 61 (1979), pp. 71–81.

[103] E. Mezzetti, R. M. Miró-Roig, and G. Ottaviani. “Laplace Equations and the Weak Lefschetz Property”. In: *Canadian Journal of Mathematics* 65 (2013), pp. 634–654.

[104] M. Mezzetti and R. M. Miró-Roig. “Perazzo n-folds and the weak Lefschetz property”. In: *Rendiconti del Circolo Matematico di Palermo Series 2* 73 (2024), pp. 2277–2295.

[105] J. Migliore, R. M. Mirò-Roig, and U. Nagel. “Monomial ideals, almost complete intersections and the Weak Lefschetz Property”. In: *Transactions of the American Mathematical Society* 363 (2011), pp. 229–257.

[106] J. Migliore, U. Nagel, and F. Zanello. “Bounds and asymptotic minimal growth for Gorenstein Hilbert functions”. In: *Journal of Algebra* 321.5 (2009), pp. 1510–1521.

[107] W. F. Moore, M. Rogers, and S. Sather-Wagstaff. *Monomial ideals and their decompositions.* Universitext. Springer, Cham, 2018.

[108] M. Nagata. *Local rings.* Robert E. Krieger Publishing Co., Huntington, NY, 1975.

[109] D. G. Northcott and D. Rees. “Reductions of ideals in local rings”. In: *Proceedings of the Cambridge Philosophical Society* 50 (1954), pp. 145–158.

[110] G. Ottaviani. *Lectures on the Geometry of Tensors.* Notes for the Nordfjordeid (Norway) Summer School (2010). `http://web.math.unifi.it/users/ottaviani/nord/sylv.pdf`.

[111] I. Peeva. “Consecutive cancellations in Betti numbers”. In: *Proceedings of the American Mathematical Society* 132.12 (2004), pp. 3503–3507.

[112] R. Pellikaan, X.-W. Wu, S. Bulygin, and R. Jurrius. *Codes, cryptology and curves with computer algebra.* Cambridge University Press, Cambridge, 2018.

[113] U. Perazzo. “Sulle varietà cubiche la cui hessiana svanisce identicamente”. In: *Giornale di Matematiche di Battaglini* 38 (1900), pp. 337–354.

[114] M. V. Pinto and R. H. Villarreal. *Graph rings and ideals: Wolmer Vasconcelos’ contributions.* 2025. arXiv: `2305.06270`.

[115] S. B. Priddy. “Koszul resolutions”. In: *Transactions of the American Mathematical Society* 152 (1970), pp. 39–60.

[116] C. Raicu and K. VandeBogert. *Stable sheaf cohomology on flag varieties.* 2023. arXiv: `2306.14282`.

[117] L. Reid, L. G. Roberts, and M. Roitman. “On complete intersections and their Hilbert functions”. In: *Canad. Math. Bull.* 34.4 (1991), pp. 525–535.

[118] F. Russo. *On the Geometry of Some Special Projective Varieties.* Vol. 18. Lecture Notes of the Unione Matematica Italiana. Springer, Cham; Unione Matematica Italiana, Bologna, 2016.

[119] K. Saha. “The v-Number and Castelnuovo-Mumford Regularity of Cover Ideals of Graphs”. In: *International Mathematics Research Notices* (2023), pp. 9010–9019.

[120] K. Saha and N. Kotal. “On the v-number of Gorenstein ideals and Frobenius powers”. In: *Bulletin of the Malaysian Mathematical Sciences Society* 47.6 (2024). Paper No. 167.

[121] K. Saha and I. Sengupta. “The v-number of monomial ideals”. In: *Journal of Algebraic Combinatorics* 56 (2022), pp. 903–927.

[122] R. P. Stanley. “Hilbert functions of graded algebras”. In: *Advances in Mathematics* 28.1 (1978), pp. 57–83.

[123] R. P. Stanley. “Weyl groups, the hard Lefschetz theorem, and the Sperner property”. In: *SIAM Journal on Algebraic Discrete Methods* 1.2 (1980), pp. 168–184.

[124] L. A. Székely. “Common origin of cubic binomial identities; A generalization of Surányi’s proof on Le Jen Shoo’s formula”. In: *Journal of Combinatorial Theory, Series A* 40 (1985), pp. 171–174.

[125] J. Tate. “Homology of Noetherian rings and local rings”. In: 1 (1957), pp. 14–27.

[126] N. V. Trung and H.-J. Wang. “On the asymptotic linearity of Castelnuovo-Mumford regularity”. In: *Journal of Pure and Applied Algebra* 201 (2005), pp. 42–48.

[127] Keller VandeBogert. *Iterated Mapping Cones for Strongly Koszul Algebras.* 2022. arXiv: `2104.00037`.

[128] J. Watanabe. “The Dilworth number of Artinian rings and finite posets with rank function”. In: *Commutative Algebra and Combinatorics.* Vol. 11. Advanced Studies in Pure Mathematics. Kinokuniya Co. / North-Holland, 1987, pp. 303–312.

[129] J. Watanabe. “A remark on the Hessian of homogeneous polynomials”. In: The Curves Seminar at Queen’s, Volume XIII, Queen’s Papers in Pure and Applaied Mathematics 119. 2000, pp. 171–178.

[130] J. Watanabe. “On the Theory of Gordan-Noether on Homogeneous Forms with Zero Hessian”. In: *Proceedings of the School of Science of Tokai University* 49 (2014), pp. 1–21.

[131] J. Watanabe and M. de Bondt. “On the theory of Gordan-Noether on homogeneous forms with zero Hessian (Improved version)”. In: *”Polynomial rings and affine algebraic geometry”.* Springer Proc. Math. Stat. 319. Springer, Cham, 2020, pp. 73–107.

[132] C. A. Weibel. *An introduction to homological algebra.* Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994.

[133] E. West. “Primes associated to multigraded modules”. In: *Journal of Algebra* 271 (2004), pp. 427–453.

[134] J. Weyman. *Cohomology of vector bundles and syzygies.* Vol. 149. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2003.