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Robust Moving-Horizon Estimation for Quasi-LPV Discrete-Time Systems ⋆

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Abstract: This paper deals with moving horizon estimation for a class of quasi-LPV systems. Under both observability condition and incremental exponential input output-to-state stability (i-EIOSS) assumption, novel stability conditions of the moving horizon estimator (MHE) are proposed. Such conditions guarantee exponential robust stability of the MHE based on a particular prediction step that is independent of the dynamic of the system. An application to vehicle motion estimation, using the kinematic model, is provided to show the validity and effectiveness of the proposed method, and to support the theoretical results.

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Keywords: Quasi-LPV systems; moving horizon estimator; stability; incremental input output-to-state stability.

1. INTRODUCTION

State estimation is important for real-world applications, such as biomedical systems, robotics, and population dynamic models. Estimation is necessary because the system state variables are used in control design schemes, diagnosis, and monitoring. Estimation is also essential due to the unavailability of the system states in real-time. Indeed, full-state variables cannot be measured or are too expensive to measure in many applications due to the unavailability of sensors at any cost. To this end, several methods have been developed in the literature, among others, estimators based on moving horizon estimation (MHE) Alessandri et al. (2008, 2011). MHE reduces the effect of uncertainties by taking advantage of a smoothing estimate update. Given the same estimator tuning, model, and measurements, MHE provides improved state estimation and greater robustness to poor guesses of the initial state.

The notion of input-to-state stability (ISS) was introduced by Sontag et al. (1989): no matter what is the initial state, if the inputs are small, then the state must eventually be small. The importance of ISS is due to the fact that the concept has bridged the gap between input-output and state-space methods. Incremental input/output to-state stability (i-IOSS) was introduced in Sontag and Wang (1995) since it is not possible to deal with inputs and outputs separately in general. Sontag and Wang (1997) have shown that systems admitting robust estimators must be i-IOSS. However, these results do not apply to moving-horizon estimators.

The use of the notion incremental exponential input/output to-state stability (i-EIOSS) as defined in (Allan et al., 2021; Schiller et al., 2021) in the concept of the MHE, allows getting synthesis conditions of the parameters of the cost function that are numerically or analytically easy to verify. This objective is the main motivation for this work. This will lead to criteria, expressed in terms of inequalities, for instance, to synthesize the parameters of the cost function guaranteeing the robust convergence of the MHE.

In this paper, we develop a robust MHE scheme for a class of quasi-LPV systems. First, we will provide stability conditions of the MHE under both the i-EIOSS condition as well as with the observability assumption. These conditions exploit the observability of the system through a design parameter $\delta > 0$, which allows having degrees of freedom to tune the cost function-related parameters while ensuring robust stability of the MHE for any value of the size of the moving window $N \geq 1$.

An application to vehicle tracking problems based on the kinematic model (Rajamani et al., 2020) is provided to show the effectiveness of the proposed estimation scheme. It also shows the superiority of the estimation scheme compared to other estimation methods since this kinematic model cannot be handled by some observer design approaches, like the LMI-based techniques, high-gain observer, and sliding mode observer, without transforming it or switching between several regions, as shown in (Rajamani et al., 2020). Indeed, the MHE scheme developed in this paper provides an accurate estimation of the original states without performing any transformation of the kinematic model, and in the presence of noises.

The paper is structured as follows: In Section 2, we give some preliminaries on moving horizon estimation. In section 3, we develop a robust MHE which satisfies an exponential-like ISS condition. Such a robust estimator requires i-EIOSS property together with observability con-

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ditions as assumptions to guarantee that the ISS criterion holds. Simulation results are reported in Section 4. We finish by giving conclusions and prospects for future work in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

The moving horizon estimation technique (MHE) is a state estimation method that is particularly useful for nonlinear or constrained dynamic systems. Two sources of error are present during estimation: the state transition is affected by a disturbance, w, and the measurement process is affected by another disturbance, v. Consider the following linear discrete-time system:

$$\begin{cases} x_{t+1} = A(\rho_t)x_t + w_t \\ y_t = C(\rho_t)x_t + v_t \end{cases} \tag{1}$$

where $x_t \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state of the system; $\rho_t \subset \mathcal{X}_\rho \subset \mathbb{R}^n$; with $\mathcal{X}_\rho$ is compact; $y_t \in \mathbb{R}^n$ is the output vector and $w_t \in W \subseteq \mathbb{R}^n$ and $v_t \in V \subset \mathbb{R}^n$ are unknown external disturbances.

The estimation scheme can be obtained by defining $\hat{x}_{t-N+1|t}, ..., \hat{x}_{t|t}$ as estimates generated by $\hat{x}_{t-N|t}$ through the dynamics

$$\hat{x}_{i+1,t} = A(\hat{\rho}_{i|t})\hat{x}_{i|t} + \hat{w}_{i|t}, \quad i = t-N, ..., t-1$$

The MHE technique is based on the idea of minimizing a quadratic estimation cost function defined on a sliding window composed of a finite number of time stages. The cost function is written as the sum of an output error computed on the basis of the most recent measurements; the sum of estimated measurements disturbance, and the other terms are the sum of distances of the current estimated state $\hat{x}$ and the system parameter ρ from their predictions. All the terms are penalized with a power of an exponential decrease factor η. Consider the quadratic objective function:

$$\begin{aligned} J_t^N(\hat{x}_{t-N}, \hat{\rho}_{t-N}^{t-1}, \hat{w}_{t-N}^{t-1}) &= \mu_t|\hat{x}_{t-N} - \bar{x}_{t-N}|^2\eta^N \\ &+ \nu_t \sum_{i=t-N}^{t-1} \eta^{t-1-i}|y_i - C(\hat{\rho}_i)\hat{x}_i|^2 + \omega_t \sum_{i=t-N}^{t-1} \eta^{t-1-i}|\hat{w}_i|^2 \\ &+ \kappa_t \sum_{i=t-N}^{t-1} \eta^{t-1-i}|\hat{\rho}_i - \bar{\rho}_i|^2 \end{aligned} \tag{2}$$

under the constraints

$$\hat{x}_{t+1} = A(\hat{\rho}_t)\hat{x}_t + \hat{w}_t, \tag{3}$$

where $\eta \in (0,1)$ and $\mu_t, \nu_t, \omega_t > 0$, $\forall t \in \mathbb{N}$. We rely on the following prediction equations:

$$\bar{x}_{t-N} = \begin{cases} \bar{x}_0, & t = 1, ..., N \\ \hat{x}_{t-N|t-N}, & t \geq N+1 \end{cases} \tag{4}$$

$$\bar{\rho}_{t-N}^{t-1} = \begin{cases} \underbrace{(\bar{\rho}_0 \ \dots \ \bar{\rho}_0)}_{N \text{ times}}, & t = 1, ..., N \\ \hat{\rho}_{t-N|t-1}^{t-1}, & t \geq N+1. \end{cases} \tag{5}$$

For the purpose of this paper, we make the following assumptions:

***Assumption** 1.* The system state x_t and the unknown LPV parameter ρ_t satisfy the following conditions:

(1) There exists a compact set $\mathcal{X} \subset \mathbb{R}^n$ such that $x_t \in \mathcal{X}$ for all $t = 0, 1, \ldots$, and let $x_{\max} \triangleq \sup_{x \in \mathcal{X}} |x|$;

(2) The mappings $\rho \mapsto A(\rho) \in \mathbb{R}^{n \times n}$ and $\rho \mapsto C(\rho) \in \mathbb{R}^{m \times n}$ are Lipschitz continuous with respect to ρ_t.

***Assumption** 2.* System (1) is observable in $N-$steps according to the following condition (Alessandri et al., 2008):

$$\delta \triangleq \inf_{\rho_{t-N}^t \in \mathcal{X}_\rho{}^{N+1}} \lambda_{\min}\left(F\left(\rho_{t-N}^t\right)^\top F\left(\rho_{t-N}^t\right)\right) > 0 \tag{6}$$

where

$$F\left(\rho_{t-N}^t\right) \triangleq \begin{bmatrix} C(\rho_{t-N}) \\ C(\rho_{t-N+1})A(\rho_{t-N}) \\ \vdots \\ C(\rho_t)\prod_{i=1}^{N} A(\rho_{t-i}) \end{bmatrix}. \tag{7}$$

Before formulating the MHE problem, notice that the output measurement y_t obeys to the following equation:

$$\begin{aligned} y_{t-N+1}^t = F\left(\rho_{t-N}^t\right) x_{t-N} + v_{t-N}^t \\ + H\left(\rho_{t-N}^t\right) w_{t-N}^{t-1}, \ \forall t \geq N, \end{aligned} \tag{8}$$

where

$$H\left(\rho_{t-N}^t\right) \triangleq \begin{bmatrix} 0 & \dots & 0 \\ C(\rho_{t-N+1}) & \dots & 0 \\ C(\rho_{t-N+2})A(\rho_{t-N+1}) & \dots & 0 \\ \vdots & \ddots & \vdots \\ C(\rho_t)\prod_{i=1}^{N-1} A(\rho_{t-i}) & \dots & C(\rho_t) \end{bmatrix}.$$

Now we focus on a moving horizon estimation MHE_N resulting from the minimization of the cost function (2) given by the following:

$$\begin{cases} \left(\hat{x}_{0|t}, \hat{\rho}_{0|t}^{t-1}, \hat{w}_{0|t}^{t-1}\right) \in \\ \quad \left\{ \underset{\hat{x}_0 \in \mathcal{X}, \hat{\rho}_0^{t-1} \in \mathcal{X}_\rho^t, \hat{w}_0^{t-1} \in W^t}{\operatorname{argmin}} J_t^t\left(\hat{x}_0, \hat{\rho}_0^{t-1}, \hat{w}_0^{t-1}\right) \right. \\ \qquad \left. \text{s.t. (3) holds for} \quad t = 1, ..., N \right\} \\ \left(\hat{x}_{t-N|t}, \hat{\rho}_{t-N|t}^{t-1}, \hat{w}_{t-N|t}^{t-1}\right) \in \\ \quad \left\{ \underset{\hat{x}_{t-N} \in \mathcal{X}, \rho_{t-N}^{t-1} \in \mathcal{X}_\rho^t, \hat{w}_{t-N}^{t-1} \in W^t}{\operatorname{argmin}} J_t^N\left(\hat{x}_{t-N}, \hat{\rho}_{t-N}^{t-1}, \hat{w}_{t-N}^{t-1}\right) \right. \\ \qquad \left. \text{s.t. (3) holds for} \quad t = N+1, N+2, ... \right\}. \end{cases}$$

We introduce the following definition of the exponential robust stability of an estimator. For that, we focus only on MHE_N. Since the estimation of the system state, x_t depends on the quality of estimation of the LPV parameter ρ_t, we introduce a specific definition.

***Definition** 1.* An MHE_N is *exponentially robustly quasi-stable* (ERqS), with respect to ρ, if the following inequality holds:

$$\begin{aligned} |x_t - \hat{x}_{t|t}| \leq \alpha_1|x_0 - \bar{x}_0|\eta^t + \alpha_2 \sum_{i=0}^{t-1} \eta^{t-1-i}|v_i|^2 \\ + \alpha_3 \sum_{i=0}^{t-1} \eta^{t-1-i}|w_i|^2 + \alpha_4 \sum_{i=0}^{t-1} \eta^{t-1-i}|\rho_i - \bar{\rho}_i|^2 \end{aligned} \tag{9}$$

for some $\eta \in (0,1)$ and $\alpha_i > 0, i = 1, \ldots, 4$.

In order to establish robust stability of the proposed MHE$_N$ scheme, an appropriate detectability characterization is required. We suppose that system (1) is incrementally exponentially input/output-to-state stable. As in the Definition 1, we specify the definition of i-IOSS for the quasi-LPV systems.

***Definition** 2.* System (1) is *incrementally Exponentially Input Output-to-State quasi-Stable* (i-EIOSqS), with respect to ρ, if there exist constants $c_x, c_v, c_w, c_\rho > 0$ and $\eta \in (0,1)$ such that for each pair of initial conditions $\bar{x}_0, \tilde{x}_0 \in \mathcal{X}$ and each two disturbance sequences $\bar{w}_t, \tilde{w}_t \in \Omega$, and parameters $\bar{\rho}_t, \tilde{\rho}_t \in \mathcal{X}_\rho$, the following holds:

$$\begin{aligned}|x_t(\bar{x}, w_0^{t-1}) - \tilde{x}_t(\tilde{x}, \tilde{w}_0^{t-1})|^2 &\leq c_x|\bar{x}_0 - \tilde{x}_0|^2\eta^t \\ &+ c_v\sum_{i=0}^{t-1}\eta^{t-1-i}|y_i(\bar{x}, \bar{w}_0^{i-1}, \bar{v}_0^{i-1}) - y_i(\tilde{x}, \tilde{w}_0^{i-1}, \tilde{v}_0^{i-1})|^2 \\ &+ c_w\sum_{i=0}^{t-1}\eta^{t-1-i}|\bar{w}_i - \tilde{w}_i|^2 + c_\rho\sum_{i=0}^{t-1}\eta^{t-1-i}|\bar{\rho}_i - \tilde{\rho}_i|^2. \end{aligned} \tag{10}$$

The above definitions will be exploited in the next section for robust stability analysis of the MHE$_N$. At first, we provide sufficient conditions ensuring the ERqS of the MHE$_N$, according to the Definition 1, by assuming that system (1) satisfies both the observability condition (6) and the i-EIOSqS assumption as stated in Definition 2, of course, as well as Assumption 1.

3. ROBUST STABILITY ANALYSIS OF THE MHE

We consider the MHE$_N$ (9) with time-varying cost parameters

$$(\mu_t, \nu_t, \omega_t, \kappa_t) = \begin{cases} (\bar{\mu}, \bar{\nu}, \bar{\omega}, \bar{\kappa}), \ t = 1, ..., N \\ (\mu, \nu, \omega, \kappa), \ t = N+1, N+2, ... \end{cases}$$

The procedure is divided into two parts: for $t = 1, \ldots, N-1$, we need only the system (1) to be i-EIOSqS to prove that MHE$_N$ satisfies ERqS according to to (9). Whether for the second part, for $t \geq N$, we use both observability and i-EIOSqS condition.

***Lemma** 1.* Assume that system (1) is i-EIOSqS in the sense of (10). Then the MHE$_N$ with cost weights $\bar{\mu}$, $\bar{\nu}$, $\bar{\omega}$, and $\bar{\kappa}$ provides an estimate $\hat{x}_{t|t}$ satisfying the following inequality $\forall t \leq N-1$:

$$\begin{aligned}|x_t - \hat{x}_{t|t}|^2 &\leq 4\bar{\mu}|x_0 - \bar{x}_0|^2\eta^t + 2\bar{\nu}\sum_{i=0}^{t-1}|v_i|^2\eta^{t-1-i} \\ &+ 4\bar{\omega}\sum_{i=0}^{t-1}|w_i|^2\eta^{t-1-i} + 4\bar{\kappa}\sum_{i=0}^{t-1}|\rho_i - \bar{\rho}_i|^2\eta^{t-1-i} \end{aligned} \tag{11}$$

$\forall x_0 \in \mathcal{X}, v_0^{t-1} \in V, w_0^{t-1} \in W$ if the parameters of the cost function as tuned such that $\bar{\mu} \geq c_x, \bar{\nu} \geq \frac{c_v}{2}$, $\bar{\omega} \geq c_w$, and $\bar{\kappa} \geq c_\rho$.

Proof. for $t = 1$ the minimizer $(\hat{x}_{0|1}, \hat{\rho}_{0|1}, \hat{w}_{0|1})$ provides the estimate $\hat{x}_{1|1} = A(\hat{\rho}_{0|1})\hat{x}_{0|1} + \hat{w}_{0|1}$ and hence, by definition, we have $J_1^1\left(\hat{x}_{0|1}, \hat{\rho}_{0|1}^{t-1}, \hat{w}_{0|1}\right) \leq J_1^1(x_0, \rho_0, w_0)$

$$\begin{aligned}&\bar{\mu}|\hat{x}_{0|1} - \bar{x}_0|^2 + \bar{\nu}|y_0 - C(\hat{\rho}_i)x_i|^2 + \bar{\omega}|\hat{w}_{0|1}|^2 \\ &+ \kappa_t|\hat{\rho}_{0|1} - \bar{\rho}_0|^2 \leq \bar{\mu}|x_0 - \bar{x}_0|^2\eta + \bar{\nu}|y_0 - C(\rho_0)x_0|^2 \\ &+ \bar{\omega}|w_0|^2 + \kappa_t|\rho_0 - \bar{\rho}_0|^2. \end{aligned} \tag{12}$$

Using the square sum bound

$$\begin{aligned}|\bar{x}_0 - \hat{x}_{0|1}|^2 &\geqslant \frac{1}{2}|x_0 - \hat{x}_{0|1}|^2 - |x_0 - \bar{x}_0|^2 \\ |\hat{w}_{0|1}|^2 &\geqslant \frac{1}{2}|w_0 - \hat{w}_{0|1}|^2 - |w_0|^2 \\ |\bar{\rho}_0 - \hat{\rho}_0|^2 &\geqslant \frac{1}{2}|\rho_0 - \hat{\rho}_{0|1}|^2 - |\rho_0 - \bar{\rho}_{0|1}|^2\end{aligned}$$

we get

$$\begin{aligned}&\mu|x_0 - \hat{x}_{0|1}|^2\eta + 2\bar{\nu}|y_0 - C(\hat{\rho}_{0|1})\hat{x}_{0|1}|^2 + \bar{\omega}|w_0 - \hat{w}_{0|1}|^2 \\ &+ \bar{\kappa}|\rho_0 - \bar{\rho}_{0|1}|^2 \leqslant 4\bar{\mu}|x_0 - \bar{x}_0|^2\eta + 2\bar{\nu}|v_0|^2 + 4\bar{\omega}|w_0|^2 \\ &+ 4\bar{\kappa}|\rho_0 - \bar{\rho}_0|^2. \end{aligned} \tag{13}$$

It follows that

$$\begin{aligned}&c_x|x_0 - \hat{x}_{0|1}|^2 + c_v|y_0 - C(\hat{\rho}_{0|1})\hat{x}_{0|1}|^2 + c_w|w_0 - \hat{w}_{0|1}|^2 \\ &+ 4\bar{c}_\rho|\rho_0 - \bar{\rho}_0|^2 \leq 4\bar{\mu}|x_0 - \bar{x}_0|^2\eta + 2\bar{\nu}|v_0|^2 + 4\bar{\omega}|w_0|^2 \\ &+ 4\bar{\kappa}|\rho_0 - \bar{\rho}_0|^2 \end{aligned} \tag{14}$$

due to the assumptions on $\bar{\mu}, \bar{\nu}$ and $\bar{\omega}$. Therefore, by using the assumption i-EIOSqS (10), the inequality (11) is inferred for $t = 1$. We can proceed similarly for $t = 2, ..., N-1$. This ends the proof of Lemma 1.

In the following, we will exploit the observability condition and i-EIOSqS (10) of system (1) to derive general sufficient stability conditions ensuring the MHE$_N$ to be ERqS. Indeed, the observability parameter δ may improve the performance of the MHE$_N$.

***Theorem** 1.* Assume that system (1) is i-EIOSqS according to (10) and satisfies Assumption 1 and Assumption 2. Then, the MHE$_N$ is ERqS according to the following inequality:

$$\begin{aligned}|x_t - \hat{x}_{t|t}| &\leq \max\left(4\bar{\mu},\ \frac{12\eta c_x\mu}{3\eta\mu + 2\delta\nu_1}\right)|x_0 - \bar{x}_0|^2\eta^t \\ &+ \max\left(2\bar{\nu},\ \frac{6\eta c_x(3\nu_1 + v_2)}{3\eta\mu + 2\delta\nu_1}\right)\sum_{i=0}^{t-1}\eta^{t-1-i}|v_i|^2 \\ &+ \max\left(4\bar{\omega},\ \frac{12\eta c_x\omega}{3\eta\mu + 2\delta\nu_1}\right)\sum_{i=0}^{t-1}\eta^{t-1-i}|w_i|^2 \\ &+ \max\left(4\bar{\kappa},\ \frac{12\kappa\eta c_x}{3\eta\mu + 2\delta\nu_1}\right)\sum_{i=0}^{t-1}\eta^{t-1-i}|\rho_i - \bar{\rho}_i|^2 \end{aligned} \tag{15}$$

if the parameters $\mu, \nu_1, \nu_2, \omega, \kappa$ satisfy the condition:

$$c_\nu < \frac{6\eta c_x\nu_2}{3\eta\mu + 2\delta\nu_1} \tag{16a}$$

$$c_\omega < \frac{3\eta c_x(\omega - 8\nu_1\sigma_H)}{3\eta\mu + 2\delta\nu_1} \tag{16b}$$

$$c_\rho < \frac{3\kappa\eta c_x - 12\eta c_x\nu_1\left(\sigma_{x,C}^2 + 2\sigma_{x,A}^2\sigma_H\right)}{3\eta\mu + 2\delta\nu_1} \tag{16c}$$

$$\frac{12\eta\mu c_x}{3\eta\mu + 2\delta\nu_1} < 1. \tag{16d}$$

with $\nu = \nu_1 + \nu_2$.

Proof. We focus on the case $t \geq N$ with $(\mu_t, \nu_t, \omega_t) = (\mu, \nu, \omega)$, and then we combine with the bounds (11) of Lemma 1. From the definition of minimizer in (2), we have

$$J_t^N(\hat{x}_{t-N|t}, \hat{\rho}_{t-N|t}^{t-1}, \hat{w}_{t-N|t}^{t-1}) \leq J_t^N(x_{t-N}, \rho_{t-N}^{t-1}, w_{t-N}^{t-1})$$

which means that

$$\mu|\hat{x}_{t-N|t}-\bar{x}_{t-N}|^2\eta^N+\nu\sum_{i=t-N}^{t-1}\eta^{t-1-i}|y_i-C(\hat{\rho}_{i|t})\hat{x}_{i|t}|^2$$
$$+\omega\sum_{i=t-N}^{t-1}\eta^{t-1-i}|\hat{w}_{i|t}|^2+\kappa\sum_{i=0}^{t-1}\eta^{t-1-i}|\hat{\rho}_{i|t}-\bar{\rho}_i|^2$$
$$\leq\mu|x_{t-N}-\bar{x}_{t-N}|^2\eta^N+\nu\sum_{i=t-N}^{t-1}\eta^{t-1-i}|v_i|^2$$
$$+\omega\sum_{i=t-N}^{t-1}\eta^{t-1-i}|\hat{w}_i|^2+\kappa\sum_{i=t-N}^{t-1}\eta^{t-1-i}|\rho_i-\bar{\rho}_i|^2. \quad (17)$$

The main idea to perform stability analysis for MHE consists in handling the term containing the output error, namely the term

$$\nu\sum_{i=t-N}^{t-1}\eta^{t-1-i}|y_i-C(\hat{\rho}_i)\hat{x}_{i|t}|^2.$$

Since we will exploit both observability condition and i-EIOSqS of system (1), and both cases depend on such output error, then we need to put $\nu=\nu_1+\nu_2$. The first term corresponding to ν_1 will be used to obtain a lower bound, while the second one, namely that of ν_2, will be exploited in the application of the i-EIOSqS of system (1) to get an appropriate upper bound of the estimation error. To this end, as the proof is long and needs several intermediate results, it will be divided into four steps to increase its clarity and readability.

Step 1 (Simplifications and preliminary bounds):
First, let us introduce the following notations:

$$\bar{w}_t\triangleq(A(\rho_t)-A(\hat{\rho}_t))x_t+w_t$$
$$\bar{v}_t\triangleq(C(\rho_t)-C(\hat{\rho}_t))x_t+v_t. \quad (18)$$

Then, by substituting $\bar{w}_t$ and $\bar{v}_t$ from (18), system (1) is rewritten as

$$\begin{cases}x_{t+1}=A(\hat{\rho}_t)x_t+\bar{w}_t\\ y_t=C(\hat{\rho}_t)x_t+\bar{v}_t.\end{cases} \quad (19)$$

Hence, we can write

$$y_k-C(\hat{\rho}_k)\hat{x}_k=C(\hat{\rho}_k)(x_k-\hat{x}_k)+\bar{v}_k,\ \forall k\geq 1. \quad (20)$$

By exploiting (20) and using the quadratic inequality, we obtain the following bound:

$$\frac{1}{2}\left|C(\hat{\rho}_{t-N+i|t})(x_{t-N+i}-\hat{x}_{t-N+i})\right|^2-|\bar{v}_{t-N+i}|^2$$
$$\leq|y_{t-N+i}-C(\hat{\rho}_{t-N+i|t})\hat{x}_{t-N+i}|^2. \quad (21)$$

To avoid cumbersome notations, we will use F_t and H_t instead of $F(\rho_t)$ and $H(\rho_t)$, respectively. Again, we simplify the term $\sum_{i=t-N}^{t-1}\eta^{t-1-i}|y_i-C(\hat{\rho}_{i|t})\hat{x}_{i|t}|^2$ by rewriting it as

$$\sum_{i=t-N}^{t-1}\eta^{t-1-i}|y_i-C(\hat{\rho}_{i|t})\hat{x}_{i|t}|^2$$
$$=\overbrace{\left(y_{t-N}^{k-1}-F_t\hat{x}_{t-N}-H_t\hat{w}_{t-N}^{t-1}\right)^\top}^{X_y^\top}$$
$$\times\overbrace{\text{diag}(\eta^{N-j}\mathbb{I},j=1,...,N)}^{\Delta_\eta}X_y.$$

where $\mathbb{I}$ denotes the identity matrix of an appropriate dimension. Also, we have

$$F_te_{t-N}=F_tx_{t-N}-F_t\hat{x}_{t-N}$$
$$=\overbrace{\left(F_tx_{t-N}+H_t\bar{w}_{t-N}^{t-1}+\bar{v}_{t-N}^{t-1}\right)}^{y_{t-N}^{t-1}}$$
$$-\left(H_t\bar{w}_{t-N}^{t-1}+\bar{v}_{t-N}^{t-1}\right)$$
$$+H_t(\hat{w}_{t-N}^{t-1}-\hat{w}_{t-N}^{t-1})-F_t\hat{x}_{t-N}$$
$$=\overbrace{y_{t-N}^{t-1}-F_t\hat{x}_{t-N}-H_t\hat{w}_{t-N}^{t-1}}^{X_y}$$
$$+H_t(\hat{w}_{t-N}^{t-1}-\hat{w}_{t-N}^{t-1})-\bar{v}_{t-N}^{t-1}. \quad (22)$$

By using Young's inequality, it follows that

$$e_{t-N}^\top F_t^\top\Delta_\eta F_te_{t-N}\leq(1+\varepsilon_1+\varepsilon_2)X_y^\top\Delta_\eta X_y$$
$$+\left(1+\frac{1}{\varepsilon_1}+\varepsilon_3\right)(\hat{w}_{t-N}^{t-1}-\bar{w}_{t-N}^{t-1})^\top$$
$$H_t^\top\Delta_\eta H_t(\hat{w}_{t-N}^{t-1}-\bar{w}_{t-N}^{t-1})$$
$$+\left(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}\right)(\bar{v}_{t-N}^{t-1})^\top\Delta_\eta\bar{v}_{t-N}^{t-1}.$$

In particular, for $\varepsilon_i=1,i=1,2,3$, we have

$$\frac{1}{3}e_{t-N}^\top F_t^\top\Delta_\eta F_te_{t-N}\leq X_y^\top\Delta_\eta X_y+(\bar{v}_{t-N}^{t-1})^\top\Delta_\eta\bar{v}_{t-N}^{t-1}$$
$$+(\hat{w}_{t-N}^{t-1}-\bar{w}_{t-N}^{t-1})^\top H_t^\top\Delta_\eta H_t(\hat{w}_{t-N}^{t-1}-\bar{w}_{t-N}^{t-1}).$$

Now, we are ready to derive a lower bound of the term

$$\nu_1\sum_{i=t-N}^{t-1}\eta^{t-1-i}|y_i-C(\hat{\rho}_{i|t})\hat{x}_{i|t}|^2$$

which will be given in the next step.

Step 2 (Lower bound– Use of observability condition):
Now we exploit the observability assumption. Since from (6) we have $F_t^\top F_t\geq\delta\mathbb{I}$, then from the definition of Δ_η, we have $\lambda_{\min}(F_t^\top\Delta_\eta F_t)\geq\delta\eta^{N-1}$, which leads to the following inequality:

$$\frac{\delta\eta^{N-1}}{3}|e_{t-N}|^2\leq\sum_{i=t-N}^{t-1}\eta^{t-1-i}|y_i-C(\hat{\rho}_{i|t})\hat{x}_{i|t}|^2$$
$$+\sum_{i=t-N}^{t-1}\eta^{t-1-i}|\bar{v}_i|^2$$
$$+2\lambda_{\max}(H_t^\top H_t)\sum_{i=t-N}^{t-1}\eta^{t-1-i}|\hat{w}_{i|t}-\bar{w}_i|^2. \quad (23)$$

Being the functions $A(\cdot)$ and $C(\cdot)$ Lipschitz and the state x_t is bounded (from Assumption 1), there exist constants $\sigma_{x,C},\sigma_{x,A}>0$ such that

$$|(C(\rho_i)-C(\hat{\rho}_i))x_i|\leq\sigma_{x,C}|\rho_i-\hat{\rho}_i|$$
$$|(A(\rho_i)-A(\hat{\rho}_i))x_i|\leq\sigma_{x,A}|\rho_i-\hat{\rho}_i|.$$

Then, by using the expressions (18) of $\bar{v}$ and $\bar{w}$, we get

$$|\bar{w}_i-\hat{w}_i|^2\leq 4|w_i|^2+4|\hat{w}_i|^2+2\sigma_{x,A}^2|\rho_i-\hat{\rho}_i|^2$$
$$|\bar{v}_i|^2\leq 2|v_i|^2+2\sigma_{x,C}^2|\rho_i-\hat{\rho}_i|^2. \quad (24)$$

Consequently, (23) and (24) lead to

$$\frac{\delta\eta^{N-1}}{3}\nu_1|e_{t-N}|^2 \le \nu_1\sum_{i=t-N}^{t-1}\eta^{t-1-i}|y_i - C(\hat{\rho}_i)\hat{x}_{i|t}|^2 + 2\nu_1\sum_{i=t-N}^{t-1}\eta^{t-1-i}|v_i|^2 + 2\nu_1\left(\sigma_{x,C}^2 + 2\sigma_{x,A}^2\sigma_H\right)\sum_{i=t-N}^{t-1}\eta^{t-1-i}\left|\rho_i - \hat{\rho}_{i|t}\right|^2 + 8\nu_1\sigma_H\sum_{i=t-N}^{t-1}\eta^{t-1-i}\left(|w_i|^2 + |\hat{w}_{i|t}|^2\right) \quad (25)$$

where $\sigma_H = \sup_t \lambda_{\max}(H_t^\top H_t)$. This lower bound will be exploited in the next step to obtain an adequate upper bound on the remaining output error term

$$\nu_2\sum_{i=t-N}^{t-1}\eta^{t-1-i}|y_i - C(\hat{\rho}_{i|t})\hat{x}_{i|t}|^2. \quad (26)$$

Step 3 (Upper bound– Use of quadratic inequality): Now, since we have found a lower bound to

$$\nu_1\sum_{i=t-N}^{t-1}\eta^{t-1-i}|y_i - C(\hat{\rho}_{i|t})\hat{x}_{i|t}|^2.$$

We go back to (17), we substitute (25) in (17) and use the following quadratic inequality

$$\left|\hat{x}_{t-N|t} - \bar{x}_{t-N}\right|^2 \ge \frac{1}{2}\left|x_{t-N} - \hat{x}_{t-N|t}\right|^2 - \left|x_{t-N} - \bar{x}_{t-N}\right|^2$$

we obtain the following upper bound on the term in (26) and then by using the following upper bounds

$$\frac{1}{2}\left|w_i - \hat{w}_{i|t}\right|^2 - |w_i|^2 \le |w_i|^2$$
$$\frac{1}{2}\left|\rho_i - \hat{\rho}_{i|t}\right|^2 - \left|\rho_i - \bar{\rho}_i\right|^2 \le \left|\bar{\rho}_i - \hat{\rho}_{i|t}\right|^2$$

we get

$$\begin{aligned} & c_x\left|x_{t-N} - \hat{x}_{t-N|t}\right|^2\eta^N \\ & + \frac{6\eta c_x\nu_2}{3\eta\mu + 2\delta\nu_1}\sum_{i=t-N}^{t-1}\eta^{t-1-1}\left|y_i - C\left(\hat{\rho}_{i|t}\right)\hat{x}_{i|t}\right|^2 \\ & + \frac{3\kappa\eta c_x - 12\eta c_x\nu_1\left(\sigma_{x,C}^2 + 2\sigma_{x,A}^2\sigma_H\right)}{3\eta\mu + 2\delta\nu_1} \\ & \times \sum_{i=t-N}^{t-1}\eta^{t-1-i}\left|\rho_i - \hat{\rho}_{i|t}\right|^2 \\ & + \frac{3\eta c_x\left(\omega - 8\nu_1\sigma_H\right)}{3\eta\mu + 2\delta\nu_1}\sum_{i=t-N}^{t-1}\eta^{t-1-i}\left|w_i - \hat{w}_i\right|^2 \\ \le & \frac{12\eta c_x\mu}{3\eta\mu + 2\delta\nu_1}\left|x_{t-N} - \bar{x}_{t-N}\right|^2\eta^N \\ & + \frac{6\eta c_x\left(3\nu_1 + \nu_2\right)}{3\eta\mu + 2\delta\nu_1}\sum_{t-N}^{t-1}\eta^{t-1-i}\left|v_i\right|^2 \\ & + \frac{12\eta c_x\omega}{3\eta\mu + 2\delta\nu_1}\sum_{i=t-N}^{t-1}\eta^{t-1-i}\left|w_i\right|^2 \\ & + \frac{12\kappa\eta c_x}{3\eta\mu + 2\delta\nu_1}\sum_{i=t-N}^{t-1}\eta^{t-1-i}\left|\rho_i - \bar{\rho}_i\right|^2. \end{aligned} \quad (27)$$

Step 4 (Final bound– Use of i-EIOSqS assumption): Since the system (1) is i-EIOSqS according to (10), then by choosing $\omega, \nu_1, \nu_2, \mu$ and κ such that

$$\begin{aligned} c_\nu &< \frac{6\eta c_x\nu_2}{3\eta\mu + 2\delta\nu_1} \\ c_\omega &< \frac{3\eta c_x\left(\omega - 8\nu_1\sigma_H\right)}{3\eta\mu + 2\delta\nu_1} \\ c_\rho &< \frac{3\kappa\eta c_x - 12\eta c_x\nu_1\left(\sigma_{x,C}^2 + 2\sigma_{x,A}^2\sigma_H\right)}{3\eta\mu + 2\delta\nu_1} \end{aligned} \quad (28)$$

we obtain the following inequality:

$$\begin{aligned} \left|x_t - \hat{x}_{t|t}\right| \le & \frac{12\eta c_x\mu}{3\eta\mu + 2\delta\nu_1}\left|x_{t-N} - \bar{x}_{t-N}\right|^2\eta^N \\ & + \frac{6\eta c_x\left(3\nu_1 + \nu_2\right)}{3\eta\mu + 2\delta\nu_1}\sum_{t-N}^{t-1}\eta^{t-1-i}\left|v_i\right|^2 \\ & + \frac{12\eta c_x\omega}{3\eta\mu + 2\delta\nu_1}\sum_{i=t-N}^{t-1}\eta^{t-1-i}\left|w_i\right|^2 \\ & \frac{12\kappa\eta c_x}{3\eta\mu + 2\delta\nu_1}\sum_{i=t-N}^{t-1}\eta^{t-1-i}\left|\rho_i - \bar{\rho}_i\right|^2. \end{aligned} \quad (29)$$

Finally, since $\delta > 0$, then by imposing

$$\frac{12\eta\mu c_x}{3\eta\mu + 2\delta\nu_1} < 1$$

for appropriate choices of μ and ν_1, we deduce (15) by propagating (29), using feedforward substitution. Of course, to get (15), we take into account the bounds found in Lemma 1 for $t = 1, ..., N-1$. Hence the use of the "max" in (15). This completes the proof of Theorem 1.

4. APPLICATION TO VEHICLE TRACKING

This section focuses on a nonlinear observer for an autonomous vehicle to estimate motion variables of other vehicles on a road, based on measurements from an onboard radar sensor (Rajamani et al. (2020)). The estimation algorithm uses a vehicle tracking algorithm based on a single model to represent all possible vehicle motions involving both longitudinal and lateral maneuvers.

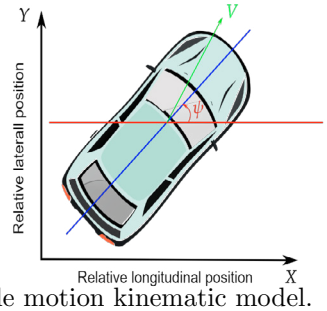

Fig. 1. Vehicle motion kinematic model.

A bicycle model represented in Figure 1 is used for each tracked vehicle with X being its relative longitudinal

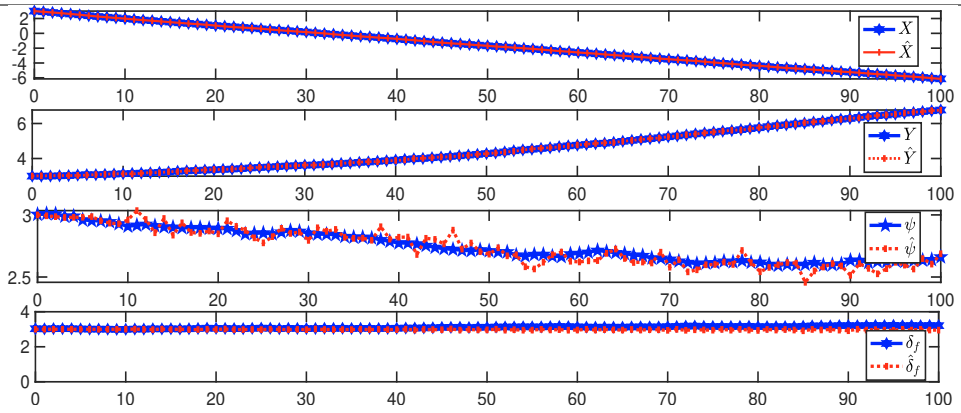

Fig. 2. The system states and their estimates.

position, Y its relative lateral position, and ψ the yaw angle of the vehicle.

The positions X and Y are measured by a radar tracking sensor. We define the vehicle motion model state, x, as $x = [\, X \;\; Y \;\; \psi \;\; \delta_f \,]^\top$. Under the assumption that the derivative of the steering angle is zero (Rajamani et al., 2020), the process dynamics are

$$\dot{x} = \begin{bmatrix} V\cos(\psi) \\ V\sin(\psi) \\ \frac{V}{l_f+l_r}\tan(\delta_f) \\ 0 \end{bmatrix} \tag{30}$$

where

- l_f and l_r are distances to the front and rear tires from c.g. of the vehicle;
- δ_f is the steering angle of front wheels;
- V is the total velocity at c.g. of the vehicle.

The output of the system is then given by:

$$y_t = Cx_t, \quad \text{with} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

As in Rajamani et al. (2020), for this application, we assume that the total velocity V is constant or slowly varying. The corresponding quasi-LPV discrete-time system form to (30) is obtained by using Euler first-order approximation for the derivative with a sampling time $T_e = 0.01$s, and by using the differential mean value theorem.

$$x_{t+1} = \begin{bmatrix} X_t + V - VT_e\,\rho_1\psi_t \\ Y_t + VT_e\,\rho_2\psi_t \\ \psi_t + \frac{V}{l_f+l_r}T_e\rho_3\delta_f \\ \delta_f \end{bmatrix}, \rho_t = \begin{bmatrix} \sin(\theta_1) \\ \cos(\theta_2) \\ 1+\tan^2(\theta_3) \end{bmatrix}.$$

The minimization of the cost function can be carried out by means of a descent method. The optimization was performed by using the general-purpose Matlab routine *fmincon* with a cost function with the parameters $\mu = 0.01$, $\nu = 1$, and $\eta = 0.9$, for different values of N and tolerance in the stopping criterion. The initial and estimated states given by $[1\ 1\ 1\ 1]^\top$ and $[-1\ -1\ -1\ -1]^\top$, respectively. Figure 2 illustrates the results obtained in simulation runs with system and measurement noises generated according to zero-mean Gaussian distributions with covariances equal to 0.01. The estimated LPV parameters are not depicted here due to lack of space and their useless for this application since all the vehicle states are recovered.

5. CONCLUSION AND PROSPECTS

In this paper, we provided a novel robust stability analysis for MHE for quasi-LPV systems satisfying an exponential i-EIOSS condition jointly with the observability condition. Further, a more general result is proposed by considering a novel prediction equation, which allows for overcoming the limitation related to the observability assumption. As prospects, we aim at improving the stability conditions of the MHE by developing a novel approach for stability analysis based only on the i-EIOSqS condition. We also aim to make a link between LMI-based LPV/nonlinear observer design and MHE by exploring the novel prediction equation.

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