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3 authors:

Martin Lazar
University of Dubrovnik

Cesare Molinari
Université de Caen Normandie

Juan Peypouquet
University of Groningen

Optimal control of parabolic equations by spectral decomposition⋆

M. Lazar†, C. Molinari‡ and J. Peypouquet‡*

† *University of Dubrovnik, Ć. Carića 4, 20000, Dubrovnik, Croatia*
‡ *Universidad Técnica Federico Santa María, Av. España 1680, Valparaíso, Chile*

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This paper deals with an optimal control problem of Bolza type for a class of parabolic equations. It consists in finding the initial datum that minimises a cost functional, which comprises an energy term on the control and a regulation term given by a distributed cost on the state, and such that the final state lies within a prescribed distance to a given target. Beyond the particularities of the chosen problem, our aim here is to propose a new methodology −based on a spectral decomposition of the operator governing the evolution of the system−, describe its implementation for this kind of problem, and finally perform some preliminary numerical experiments to assess its behavior.

Keywords: optimal control; parabolic equations; spectral decomposition; convex optimization

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1. Introduction

Let H be a Hilbert space and let $\mathcal{A} : H \to H$ be a closed, positive semidefinite, self-adjoint linear operator with dense domain $D(\mathcal{A})$ and compact resolvent. We denote by $\{\mathcal{S}_t\}_{t\geq 0}$ the strongly continuous semigroup of nonexpansive linear operators generated by $-\mathcal{A}$ (see [18, Section 1.2]). More precisely, given $u \in H$, for every $t \geq 0$, we have $\mathcal{S}_t u = y(t)$, where $y : [0, +\infty) \to H$ is the solution of the initial-value problem

$$\begin{cases} -y'(t) = \mathcal{A}y(t) & \text{for } t > 0 \\ y(0) = u. \end{cases} \tag{1}$$

The keynote example is $\mathcal{A} = -\Delta$, the Dirichlet Laplacian in $L^2(\Omega)$, for which (1) is the heat equation on the bounded domain Ω.

The system (1) is *controllable* to a *target state* $y^T \in H$ in time $T > 0$ if there is $u \in H$ such that

$$\mathcal{S}_T u = y^T. \tag{2}$$

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* Corresponding author. Email: juan.peypouquet@usm.cl

This may or may not be the case, depending on the operator $\mathcal{A}$. If $\mathcal{A}$ is the gradient of a convex function (which is the case for the Laplace operator in $L^2(\Omega)$), then $\mathcal{S}_t u \in D(\mathcal{A})$ for all $t > 0$ and all $u \in H$ (see [7, Théorème 3.2]). Therefore, no target state $y^T \in H \setminus D(\mathcal{A})$ can be attained. Even if it can, the identification of the appropriate initial condition is difficult due, in part, to the irreversible character of dissipative systems like (1). Nevertheless, it is possible to prove (using a simple orthogonality argument) that the range of $\mathcal{S}_t$ is dense in H for all $t > 0$. As a consequence, the system (1) is approximately controllable to any target state y^T in the sense that, for every $\varepsilon > 0$, there exists an initial datum $u \in H$ such that

$$\|\mathcal{S}_T u - y^T\|_H \leq \varepsilon. \tag{3}$$

Given a tolerance $\varepsilon > 0$, a control time $T > 0$, and a target state y^T, we search for an initial condition $u \in H$ such that (3) is satisfied and a given functional $J : H \to \mathbb{R}$ is minimized. More precisely, we consider the problem

$$(\mathcal{P}) \qquad \min_{u \in H} \left\{ J(u) : \|\mathcal{S}_T u - y^T\|_H \leq \varepsilon \right\},$$

where $J(u)$ is given by

$$J(u) = \frac{\alpha}{2}\|u\|_H^2 + \frac{1}{2}\int_0^T \beta(t)\, \|\mathcal{S}_t u - y^d(t)\|_H^2 \, dt, \tag{4}$$

with $\alpha > 0$, $\beta \in L^2(0, T; \mathbb{R}_+)$ and $y^d \in L^2(0, T; H)$.

The first term in the cost functional J can be interpreted as a cost that penalizes high energy initial conditions, weighted by a constant α that accounts for the relative importance of this criterion in the total value. The second term, known as a *regulation* term, is a distributed cost that penalizes deviations from the reference trajectory y^d along the interval $[0, T]$, weighted by the function β, which may have different forms depending on the applications. For instance,

- a discount factor $\beta(t) = e^{-bt}$, with discount rate $b > 0$, gives a higher importance to what happens in the near future;
- a characteristic function $\beta(t) = \chi_{\mathcal{T}}(t)$ applies a uniform regulation only on $\mathcal{T} \subset [0, T]$.

If $\beta \equiv 0$, problem $(\mathcal{P})$ can be solved by the Hilbert Uniqueness Method (HUM, see [13] and Subsection 2.3 below), which relies on the Fenchel-Rockafellar duality for convex optimisation. Numerical treatment of a penalised version of the corresponding dual problem involves the implementation of iterative schemes (see, for instance, [6]). For a general weight function $\beta(t)$ the dual problem of $(\mathcal{P})$ requires a more sophisticated treatment.

Initial condition identification is an important problem in evolution equation theory and has been addressed by different methods [4, 14, 16, 17], some including optimization and optimal control techniques [6, 10, 12, 15, 21]. See also [2], and the references therein, for other applications of optimal control in inverse problems.

The problems addressed in [6, 10, 15], for instance, are closely related to $(\mathcal{P})$, although they consider other choices for the cost or different functional frameworks. Further, the techniques employed aim at recovering the solution by applying an iterative numerical scheme, especially to the dual problem.

Classical convex optimization techniques in Hilbert spaces also provide iterative methods that can be applied to problem $(\mathcal{P})$. See [5, 19] for the general theory and examples.

There is a rich literature concerned with *distributed* optimal control problems of partial differential equations. That is, when the controls acts along the time interval $[0,T]$. This is not the focus of this research, but the interested reader may consult, among many others, [1], where semigroup methods are used, or [20] for second-order optimality conditions. Also, a quite comprehensive treatment of the topic can be found in [22].

In this paper, we follow a different approach, exploring spectral representation of the solution by eigenfunctions of $\mathcal{A}$, which eventually leads us to an explicit expression of the optimal final state $\hat{y} = S_T\hat{u}$ in terms of the given problem data. The obtained expression, combined with standard optimal control arguments enable construction of a *one-shot* algorithm providing an approximate solution.

Given $\mu \geq 0$, the resolvent operator of $\mathcal{A}$ with index μ is $R_\mu^{\mathcal{A}} := (\mu I + \mathcal{A})^{-1}$.

Clearly, if $\mathcal{A}$ is symmetric, the same is true for $R_\mu^{\mathcal{A}}$. Moreover, if $\mathcal{A}$ is closed, positive semidefinite and self-adjoint, and if $\mu > 0$, then [9, Theorem 2] and [7, Proposition 2.2] together imply that $R_\mu^{\mathcal{A}}$ is everywhere defined and nonexpansive.

The following result is a direct consequence of [8, Theorem 6.11]:

Lemma 1.1 *Let H be separable and let $\mathcal{A}$ be a positive semidefinite, self-adjoint operator with compact resolvent. Then, there exist an orthonormal basis $(\varphi_n)_{n\in\mathbb{N}}$ of H, and a nonnegative, nondecreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that each φ_n is an eigenfunction of $\mathcal{A}$ with eigenvalue λ_n, and $\lim_{n\to+\infty}\lambda_n = +\infty$.*

Observe that the Laplace operator satisfies the hypotheses of Lemma 1.1.

In the following sections, we shall present a simple methodology to recover approximate solutions for problem $(\mathcal{P})$ using spectral techniques. Roughly speaking, the idea is to use the optimality condition given by the Lagrange multiplier theory and construct the multiplier by means of the spectral decomposition of the operator $\mathcal{A}$ described in Lemma 1.1. Surprisingly enough, the whole problem is reduced to finding the unique root of a bounded, strictly increasing function defined in $\mathbb{R}$.

Observe also that the problem of finding the spectral decomposition of $\mathcal{A}$ in H is independent of the optimization of the cost J. Once this decomposition is available, the optimal control problem is solved almost instantaneously (see Section 4). Therefore the spectral decomposition can be considered as a preprocessing step. Once this information has been obtained, it can be stored for future use. In particular, if some of the parameters of the problem (cost functional, target state) have to be modified or updated for any reason. Further, one can reasonably conjecture that a similar approach might be useful for second-order (wave-like) equations (as in [3]).

The paper is organised as follows: In Section 2, we provide a characterisation of the solution to problem $(\mathcal{P})$ and derive an explicit formula in a form of the Fourier series. In Section 3, we provide *a priori* bounds on truncated Fourier approximations and describe a *one-shot* algorithm for the numerical recovery of the optimal control and the final state with prescribed precision for the latter. Numerical examples based on the obtained algorithm for the case of the 1D heat equation are provided in Section 4, with and without the regulation term. Further comments and remarks are given in Section 5.

2. Characterisation and explicit formula for the solution

In this section, we obtain a characterisation for the optimal initial state $\hat{u}$ and the corresponding final state $\hat{y} = S_T \hat{u}$.

2.1. *Optimality condition*

We begin by recalling some basic notions of convex analysis (for further details, see [19]). The *subdifferential* of a proper convex function $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ at point $u_0 \in H$ is

$$\partial \Phi (u_0) = \{v \in H : \ \Phi (u) \geq \Phi (u_0) + \langle v, \ u - u_0 \rangle \quad \forall u \in H\}.$$

The *normal cone* of a nonempty closed convex set $C \subset H$ at a point $u_0 \in H$ is defined as

$$\mathcal{N}_C (u_0) = \{v \in H : \ \langle v, \ u - u_0 \rangle \leq 0 \quad \forall u \in C\},$$

and it is the subdifferential of the *indicator function* of the set C, namely:

$$I_C (y) = \begin{cases} 0 & \text{if } y \in C \\ +\infty & \text{else.} \end{cases}$$

Problem $(\mathcal{P})$ can be restated as

$$\min_{u \in H} \{J (u) + I_{\bar{B}} (\mathcal{S}_T u)\}, \tag{5}$$

where $\bar{B}$ stands for the closure of the ball $B_\varepsilon \left(y^T\right)$ of radius ε and center y^T.

Since the function $J + I_B \circ \mathcal{S}$ is proper, strongly convex and lower-semicontinuous, problem $(\mathcal{P})$ has a unique solution, that we denote by $\hat{u}$ (see, for instance, [19, Corollary 2.20]). Moreover, by Fermat's Rule [19, Theorem 3.24], along with the Moreau-Rokafellar Theorem [19, Theorem 3.30] and the Chain Rule [19, Proposition 3.28], $\hat{u}$ is the unique point of H satisfying the inclusion

$$0 \in \partial (J + I_B \circ \mathcal{S}) (\hat{u}) = \alpha \hat{u} + \int_0^T \beta (t) \mathcal{S}_t^* \left(\mathcal{S}_t \hat{u} - y^d\right) \, dt + \mathcal{S}_T^* \mathcal{N}_{\bar{B}} (\hat{y}). \tag{6}$$

Let us write

$$f = \int_0^T \beta (t) \mathcal{S}_t^* y^d (t) \, dt \tag{7}$$

and, for $u \in H$,

$$Mu = \int_0^T \beta (t) \mathcal{S}_t^* \mathcal{S}_t u \, dt. \tag{8}$$

It follows that inclusion (6) holds if, and only if, there exists an element $\hat{g}$ in $\mathcal{N}_{\bar{B}} (\hat{y})$ such that

$$\alpha \hat{u} + M \hat{u} = f - \mathcal{S}_T^* \hat{g}. \tag{9}$$

On the other hand, let $\tilde{u}$ denote the unique solution of the unconstrained problem

$$\tilde{u} = \arg\min_{u \in H} \; J(u), \tag{10}$$

and let $\tilde{y} = \mathcal{S}_T \tilde{u}$ be the corresponding final state. The control $\tilde{u}$ satisfies

$$\alpha \tilde{u} + M \tilde{u} = f, \tag{11}$$

where f and M are defined as above. More precisely, for all $v \in H$, we have

$$\alpha \langle \tilde{u}, v \rangle_H + \int_0^T \beta(t) \langle \mathcal{S}\tilde{u} - y^d, \mathcal{S}v \rangle_H \, dt = 0. \tag{12}$$

The next result provides the first characterisation for the optimal control $\hat{u}$:

Proposition 2.1 *If $\varepsilon \geq \|\tilde{y} - y^T\|_H$, then $\hat{u} = \tilde{u}$. Otherwise, the optimal final state verifies $\hat{y} \in \partial \bar{B}$, and the element $\hat{g} \in \mathcal{N}_{\bar{B}}(\hat{y})$ given in (9) has the form $\hat{g} = \hat{\lambda}\left(\hat{y} - y^T\right)$ for some $\hat{\lambda} > 0$.*

Proof. If $\varepsilon \geq \|\tilde{y} - y^T\|_H$, then $\tilde{y} \in \bar{B}$ and so $\tilde{u}$ is admissible and, clearly, optimal. Assume now that $\varepsilon < \|\tilde{y} - y^T\|_H$. We want to show that the optimal final-state lies on the boundary of the target-ball $\bar{B}$. Suppose, on the contrary, that $\hat{y} \in \mathrm{Int}B$. Take $\gamma \in \left(0, (\varepsilon - \|\hat{y} - y^T\|_H)/\|\tilde{y} - \hat{y}\|_H\right)$, and define

$$u_\gamma = \hat{u} + \gamma(\tilde{u} - \hat{u}) = (1 - \gamma)\hat{u} + \gamma \tilde{u}.$$

Since the corresponding final state $y_\gamma = \mathcal{S}_T u_\gamma$ satisfies

$$\|y_\gamma - y^T\|_H = \|\hat{y} + \gamma(\tilde{y} - \hat{y}) - y^T\|_H \leq \|\hat{y} - y^T\|_H + \gamma \|\tilde{y} - \hat{y}\|_H \leq \varepsilon,$$

it follows that u_γ is admissible. Observe that both $J(u_\gamma)$ and $J(\hat{u})$ are strictly greater than $J(\tilde{u})$.
Now compute

$$J(u_\gamma) = (1-\gamma)^2 J(\hat{u}) + \gamma^2 J(\tilde{u}) + (1-\gamma)\gamma\left(\alpha\langle \tilde{u}, \hat{u}\rangle_H + \int_0^T \beta(t) \langle \mathcal{S}\tilde{u} - y^d, \mathcal{S}\hat{u} - y^d \rangle_H \, dt\right).$$

Using (12) with $v = \hat{u}$, we obtain

$$\begin{aligned} J(u_\gamma) = J(\hat{u}) &+ \gamma^2 \left(J(\hat{u}) + J(\tilde{u}) + \int_0^T \beta(t) \langle \mathcal{S}_t \tilde{u} - y^d, y^d \rangle_H \, dt\right) \\ &- \gamma\left(2J(\hat{u}) + \int_0^T \beta(t) \langle \mathcal{S}_t \tilde{u} - y^d, y^d \rangle_H \, dt\right). \end{aligned} \tag{13}$$

Using again (12), this time with $v = \tilde{u}$, we can express the last integral as

$$\begin{aligned}\int_0^T \beta(t) \langle \mathcal{S}_t \tilde{u} - y^d, y^d \rangle_H \, dt &= -\int_0^T \beta(t) \|\mathcal{S}_t \tilde{u} - y^d\|_H^2 \, dt + \int_0^T \beta(t) \langle \mathcal{S}_t \tilde{u} - y^d, \mathcal{S}_t \tilde{u} \rangle_H \, dt \\ &= -\int_0^T \beta(t) \|\mathcal{S}_t \tilde{u} - y^d\|_H^2 \, dt + \alpha \|\tilde{u}\|_H^2 \\ &= -2J(\tilde{u})\end{aligned}$$

Thus, (13) becomes

$$J(u_\gamma) = J(\hat{u}) + \gamma(\gamma - 2)(J(\hat{u}) - J(\tilde{u})).$$

Since $J(\hat{u}) - J(\tilde{u}) > 0$, for every $\gamma < 2$, we have $J(u_\gamma) < J(\hat{u})$, which is a contradiction.

For the last statement, note that for any point y on the boundary of $\bar{B}$ the corresponding normal cone $\mathcal{N}_{\bar{B}}(y)$ consists of vectors pointing in the direction of $y - y^T$. □

From now on, we will suppose $\varepsilon < \|\tilde{y} - y^T\|_H$. Otherwise, the solution of problem $(\mathcal{P})$ simply coincides with the solution of the uncostrained one, determined by equation (11).

For each $c \geq \tilde{c} = J(\tilde{u})$, define

$$W_c = \{y \in H : \; y = \mathcal{S}_T u \text{ for some } u \in H \text{ with } J(u) \leq c\}. \tag{14}$$

Notice that $(W_c)_{c \geq \tilde{c}}$ is a nested family of nonempty closed convex sets centered at $\tilde{y}$, that increases with c.

If $\hat{c} = \|\hat{u}\|_H$ is the norm of the optimal control, we have

$$\hat{c} = \min\left\{c \in \mathbb{R}^+ : \; W_c \cap \overline{B_\varepsilon(y^T)} \neq \emptyset\right\} = \min\left\{c \in \mathbb{R}^+ : \; \|\Pi_c\left(y^T\right) - y^T\|_H \leq \varepsilon\right\},$$

where $\Pi_c : \; H \to H$ is the orthogonal projection operator onto the set W_c. Moreover, the optimal final state satisfies

$$\hat{y} = \Pi_{\hat{c}}\left(y^T\right). \tag{15}$$

This situation is depicted in Figure 1.

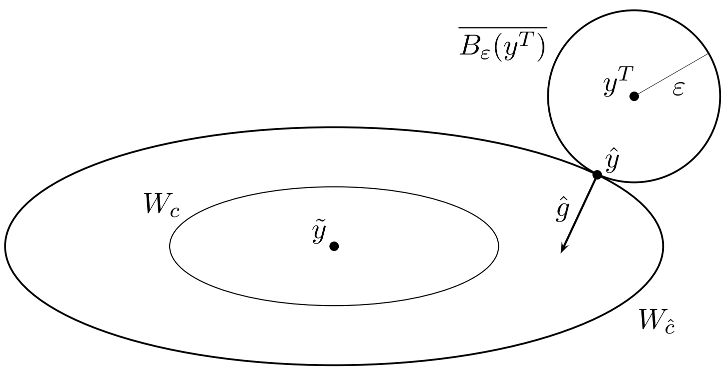

Figure 1.: Optimal final state $\hat{y} = \Pi_{\hat{c}}\left(y^T\right)$, and normal $\hat{g} = \hat{\lambda}\left(\hat{y} - y^T\right)$.

Remark 2.2 Using the previous lemma, optimality condition (6) becomes

$$\left(\alpha\mathcal{I} + M + \hat{\lambda}\mathcal{S}_T^*\mathcal{S}_T\right)\hat{u} = f + \hat{\lambda}\mathcal{S}_T^* y^T, \quad \hat{\lambda} > 0.$$

In particular, $(\mathcal{P})$ is equivalent to the unconstrained problem

$$\min_{u\in H}\left\{\frac{\alpha}{2}\|u\|_H^2 + \frac{1}{2}\int_0^T \beta(t)\,\|\mathcal{S}_t u - y^d(t)\|_H^2 dt + \frac{\hat{\lambda}}{2}\|\mathcal{S}_T u - y^T\|_H^2\right\},$$

whose optimality condition is precisely (6).

2.2. *Spectral characterisation*

In order to obtain explicit formula for the solution $\hat{u}$, we rely on the eigendecomposition of the space H induced by the operator $\mathcal{A}$ from (1). Let $(\varphi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H, where each φ_n is an eigenfunction of $\mathcal{A}$ with eigenvalue λ_n. The existence of such sequences follows from Lemma 1.1. As usual, we denote by y_n the n-th Fourier coefficient of $y \in H$ in this basis. Recall that $(y_n)_{n\in\mathbb{N}} \in \ell^2$ and $\|y\|_H^2 = \sum_{n\in\mathbb{N}} y_n^2$. In particular, we can write $\bar{B}$ as

$$\overline{B_\varepsilon(y^T)} = \left\{\sum_n y_n\varphi_n \in H : \sum_n \left(y_n - y_n^T\right)^2 \le \varepsilon^2\right\}.$$

For each $n \in \mathbb{N}$, define

$$\begin{aligned}
a_n &= \left(\frac{\alpha}{2} + \frac{1}{2}\int_0^T \beta(t)\, e^{-2\lambda_n t} dt\right) e^{2\lambda_n T};\\
b_n &= -e^{\lambda_n T}\int_0^T \beta(t)\, e^{-\lambda_n t} y_n^d(t)\, dt; \qquad (16)\\
c_n &= \frac{1}{2}\int_0^T \beta(t)\left(y_n^d(t)\right)^2 dt.
\end{aligned}$$

Observe that $a_n > 0$ and $c_n \ge 0$. With this notation, the function J can be rewritten as

$$J(u) = \sum_n \left(a_n y_n^2 + b_n y_n + c_n\right), \qquad (17)$$

where $y_n = u_n e^{-\lambda_n T}$ is the n-th Fourier coefficient of the final state $y = \mathcal{S}_T u$. The set W_c defined in (14) can be now characterised as

$$W_c = \left\{\sum_n y_n\varphi_n : \sum_n \left(a_n y_n^2 + b_n y_n + c_n\right) \le c\right\}. \qquad (18)$$

Remark 2.3 To fix the ideas, suppose that the weights of our optimal control problem are $\alpha = 2$ and $\beta \equiv 0$. In that case, W_c is the set of final states that are attained from an initial datum whose norm is less than or equal to c. The balls of controls $\sum_n u_n^2 \le c$ are deformed by the semigroup into infinite-dimensional ellipsoids W_c of final states, whose axial lengths are determined by the frequencies: the larger λ_n, the shorter the axis parallel to φ_n. Since $\lambda_n \to +\infty$, no set W_c can contain a ball around the origin.

A similar characterisation of W_c remains in general, for an arbitrary choice of the weights α and β. Due to the imposed reference trajectory y^d, the centre of corresponding ellipsoid will be displaced from the origin to the final state corresponding to the solution $\tilde{y}$ of the unconstrained problem.

Explicit expressions for its Fourier coefficients are provided by the following result:

THEOREM 2.4 *Let $\varepsilon \in \left(0, \|\tilde{y} - y^T\|_H\right)$. The Fourier coefficients of the optimal final state $\hat{y}$ are given by*

$$\hat{y}_n = \frac{y_n^T - \hat{\gamma} b_n}{1 + 2\hat{\gamma} a_n}, \tag{19}$$

where the constant $\hat{\gamma} > 0$ is the unique solution of

$$G(\gamma) := \sum_n \left(\frac{\gamma \, \left(2 a_n y_n^T + b_n\right)}{1 + 2\gamma a_n} \right)^2 = \varepsilon^2. \tag{20}$$

Proof. First, we develop a general expression for the projections of y^T onto the ellipsoids W_c, which we express as

$$W_c = \left\{ y \in H : \ \psi\left(y\right) \leq c \right\},$$

with $\psi\left(y\right) = \sum\limits_n a_n y_n^2 + b_n y_n + c_n$. Since ψ is differentiable, the Fourier coefficients of the normal direction to W_c at a point y on its boundary are given by

$$\left(\nabla \psi(y)\right)_n = 2 a_n y_n + b_n, \quad n \in \mathbb{N}.$$

Using (15), we have

$$y^T = \hat{y} + \hat{\gamma} \nabla \psi\left(\hat{y}\right)$$

for some positive constant $\hat{\gamma}$. The last two relations together imply (19).

To determine the constant $\hat{\gamma}$, we use the condition $\|\hat{y} - y^T\|_H = \varepsilon$ from Proposition 2.1, which implies (20) in view of (19). It remains to show that the cited equation has one and only one positive solution for every $\varepsilon \in \left(0, \|\tilde{y} - y^T\|_H\right)$. Define

$$G_N\left(\gamma\right) = \sum_{n=1}^{N} g_n\left(\gamma\right) \quad \text{for} \ \ g_n\left(\gamma\right) = \left(\frac{\gamma \, \left(2 a_n y_n^T + b_n\right)}{1 + 2\gamma a_n} \right)^2. \tag{21}$$

For every γ in $(0, +\infty)$, we have

$$\begin{aligned}
g_n(\gamma) &\leq \left(y_n^T + \frac{b_n}{2a_n}\right)^2 \\
&\leq 2\left(y_n^T\right)^2 + 2\left(\frac{b_n}{2a_n}\right)^2 \\
&= 2\left(y_n^T\right)^2 + 2\left(\frac{e^{\lambda_n T}\int_0^T \beta(t)\, e^{-\lambda_n t} y_n^d(t)\, dt}{\left(\alpha + \int_0^T \beta(t)\, e^{-2\lambda_n t} dt\right) e^{2\lambda_n T}}\right)^2 \\
&\leq 2\left(y_n^T\right)^2 + 2\frac{\left(\int_0^T \beta(t)\, e^{-\lambda_n t} y_n^d(t)\, dt\right)^2}{\alpha^2 e^{2\lambda_n T}} \\
&\leq 2\left(y_n^T\right)^2 + \frac{2}{\alpha^2}\, \|\beta\|^2_{L^2(0,T)}\, \|y_n^d\|^2_{L^2(0,T)}.
\end{aligned}$$

Since (y_n^T) and (y_n^d) belong to ℓ^2, Weierstrass M-test implies that, as $N \to +\infty$, G_N converges uniformly on $[0, +\infty)$ to its pointwise limit, which is precisely the function G defined in (20). It ensues that

$$\lim_{\gamma \to +\infty} \sum_n g_n(\gamma) = \sum_n \lim_{\gamma \to +\infty} g_n(\gamma) = \sum_n \left(y_n^T + \frac{b_n}{2a_n}\right)^2. \tag{22}$$

Using the optimality condition (11) for the unconstrained problem, we obtain

$$\alpha \tilde{u}_n + \int_0^T \beta(t)\, e^{-2\lambda_n t} \tilde{u}_n\, dt = \int_0^T \beta(t)\, e^{-\lambda_n t} y_n^d(t)\ dt.$$

It follows that

$$\frac{b_n}{2a_n} = \frac{-\int_0^T \beta(t)\, e^{-\lambda_n t} y_n^d(t)\ dt}{e^{\lambda_n T}\left(\alpha + \int_0^T \beta(t)\, e^{-2\lambda_n t}\ dt\right)} = -e^{-\lambda_n T}\tilde{u}_n = -\tilde{y}_n,$$

and so, (22) becomes

$$\lim_{\gamma \to +\infty} G(\gamma) = \sum_n \left(\tilde{y}_n - y_n^T\right)^2 = \|\tilde{y} - y^T\|_H^2.$$

We finish the proof by showing that G is strictly increasing. To this effect, compute

$$g_n'(\gamma) = \frac{2\gamma\left(2a_n\, y_n^T + b_n\right)^2}{\left(1 + 2\gamma a_n\right)^3},$$

and notice that, for every $\gamma > \gamma_0 > 0$,

$$|\, g_n'(\gamma)\,| \leq \frac{4\gamma\left(4a_n^2\left(y_n^T\right)^2 + b_n^2\right)}{4\gamma^2 a_n^2} \leq \frac{4}{\gamma_0}\left(\left(y_n^T\right)^2 + \frac{b_n^2}{4a_n^2}\right) \leq \frac{4}{\gamma_0}\left(\left(y_n^T\right)^2 + \frac{\|\beta\|^2_{L^2(0,T)}\, \|y_n^d\|^2_{L^2(0,T)}}{\alpha^2}\right),$$

where the last term is again summable. We obtain that G' is the limit of G'_N, uniformly converging in $(\gamma_0, +\infty)$ for every $\gamma_0 > 0$:

$$G'(\gamma) = \sum_n \frac{2\gamma \left(a_n\, y_n^T + b_n\right)^2}{(1 + 2\gamma a_n)^3}.$$

We conclude that $G(0) = 0$, $\lim_{\gamma\to+\infty} G(\gamma) = \|\tilde{y} - y^T\|_H^2$ and $G'(\gamma) > 0$ for every $\gamma > 0$. In particular, equation (20) has exactly one positive solution for every $\varepsilon \in \left(0, \|\tilde{y} - y^T\|_H\right)$. □

Theorem 2.4, combined with the spectral decomposition, provides an explicit formula for the solution of our optimal control problem.

Corollary 2.5 *The solution of the optimal control problem* $(\mathcal{P})$ *is given by*

$$\hat{u} = \sum_n \hat{u}_n \varphi_n = \sum_n e^{\lambda_n T} \hat{y}_n \varphi_n, \tag{23}$$

where the Fourier coefficients $(\hat{y}_n)_{n\in\mathbb{N}}$ *of the final state* $\hat{y}$ *are given by* (19).

Remark 2.6 Formula (19) can also be obtained directly from the optimality relation (9) combined with the results of Proposition 2.1. However, the approach presented in this paper, although of the same technical com- plexity, provides in addition a geometrical interpretation of the solution.

2.3. *Further remarks*

We finish this section by discussing another approach based on the Fenchel-Rockafellar duality for convex optimisation (see, for instance, [19, Section 3.6.2]), which yields the following:

Proposition 2.7 *The dual of problem* $(\mathcal{P})$ *is given by*

$$(\mathcal{P}^*) \qquad -\min_{v\in H} \left\{ -\langle \mathcal{S}_T^* v, \bar{u}\rangle_H - J(\bar{u}) + \langle v, y^T\rangle_H + \varepsilon\|v\|_H \right\},$$

where $(\alpha + M)\,\bar{u} = -\mathcal{S}_T^* v + f$.

Proof. Recall that problem $(\mathcal{P})$ is formulated as

$$\min_{u\in H} \left\{ J(u) + I_{\bar{B}}\left(\mathcal{S}_T u\right) \right\}.$$

From definition of Fenchel-Rockafellar duality (see, for instance, [19, Section 3.6.2]), the dual problem $(\mathcal{P}^*)$ is

$$-\min_{v\in H} \left\{ J^*(-\mathcal{S}_T^* v) + I_{\bar{B}}^*(v) \right\},$$

where Φ^* is the Fenchel conjugate of Φ. On the one hand, we have

$$J^*\left(-\mathcal{S}_T^* v\right) = \sup_{u\in H} \left\{ \langle -\mathcal{S}_T^* v,\ u\rangle_H - J(u) \right\} = -\langle \mathcal{S}_T^* v,\ u^*\rangle_H - J(\bar{u}),$$

where $\bar{u}$ is the solution of

$$0 = -\mathcal{S}_T^* v - \alpha \bar{u} - \int_0^T \beta(t)\, \mathcal{S}_t^* \left(\mathcal{S}_t \bar{u} - y^d(t) \right) \, dt.$$

In other words, $(\alpha + M)\, \bar{u} = -\mathcal{S}_T^* v + f$.

On the other hand,

$$I_{\bar{B}}^*(v) = \sup_{u \in \bar{B}} \langle v, u \rangle_H = \sup_{w \in \overline{B_1(0)}} \langle v,\ y^T + \varepsilon w \rangle_H = \langle v, y^T \rangle_H + \varepsilon \|v\|_H.$$

Summing the two functions we obtain the expression for $(\mathcal{P}^*)$. □

If $\beta \equiv 0$, then $u^* = -\frac{\mathcal{S}_T^* v}{\alpha}$, and we recover the classical problem that is subject to the Hilbert Uniqueness Method (see [6]):

$$-\min_{v \in H} \left\{ \frac{1}{\alpha} \|\mathcal{S}_T^* v\|_H^2 + \langle v,\ y^T \rangle_H + \varepsilon \|v\|_H \right\},$$

which can be treated numerically by some iterative algorithm. However, for general β, the dual problem requires a more complex numerical treatment. On the other hand, the approach presented in this section provides a direct, explicit formula for the solution in terms of the given problem data.

3. Numerical recovery of the solution

By means of Theorem 2.4 and Corollary 2.5, the optimal control $\hat{u}$ and final state $\hat{y}$ can be determined explicitly in terms of the given problem data once the equation (20) is solved for $\hat{\gamma}$.

Observe that both the explicit formulæ for $\hat{u}$ and $\hat{y}$, as well as the equation (20) itself, are given in terms of infinite series. For practical purposes, a suitable truncation procedure must be carried out in order to approximate the solution numerically.

In this section we obtain an *a priori* error estimate of the approximation, and describe in detail a numerical algorithm providing the result within a given precision.

3.1. *Error estimation*

Let us first consider the truncated version of the equation (20). To this end, recall the definition of G_N given in (21). We have the following result, which follows from the uniform convergence of the sequence $(G_N)_{N \in \mathbb{N}}$ to G:

Lemma 3.1 *For every $\varepsilon < \|\tilde{y} - y^T\|_H$ there exists an $N_\varepsilon \in \mathbb{N}$ such that for every $N > N_\varepsilon$ the equation*

$$G_N(\gamma) = \varepsilon^2 \tag{24}$$

has a unique solution, which we denote by $\hat{\gamma}_N$. Moreover, $\hat{\gamma}_N \searrow \hat{\gamma}$ as $N \to +\infty$.

We introduce the truncated approximation of the optimal final state, denoted by $\hat{y}^N$, whose Fourier coefficients are constructed by equation (19) with $\hat{\gamma}$ replaced by

$\hat{\gamma}_N$. In other words,

$$\hat{y}^N = \sum_{n=1}^{N} \hat{y}_n^N \varphi_n, \quad \text{with} \quad \hat{y}_n^N = \frac{y_n^T - \hat{\gamma}_N b_n}{1 + 2\hat{\gamma}_N a_n}. \tag{25}$$

Denoting $y^{T,N} = \sum_{n=0}^{N} y_n^T \varphi_n$ and $y^{d,N}(t) = \sum_{n=0}^{N} y_n^d(t)\varphi_n$ the truncated series representation of the target final state and the reference trajectory in the distributed cost, respectively, we obtain the following approximation estimate:

THEOREM 3.2 *Let $\varepsilon < \|\tilde{y} - y^T\|_H$, and let $\hat{y}^N$ be a truncated approximation of the optimal final state given by (25). Then the following estimate holds:*

$$\|\hat{y}^N - \hat{y}\|_H^2 \leq 4\|y^T - y^{T,N}\|_H^2 + \frac{4\|\beta\|_{L^2(0,T)}^2}{\alpha^2 e^{2\lambda_N T}} \|y^d - y^{d,N}\|_{L^2(0,T;H)}^2.$$

Proof. First, we have

$$\|\hat{y}^N - \hat{y}\|_H^2 = \sum_{n=1}^{+\infty} \left(\hat{y}_n^N - \hat{y}_n\right)^2 = \sum_{n=1}^{N} \left(\hat{y}_n^N - \hat{y}_n\right)^2 + \sum_{n=N+1}^{+\infty} \left(\hat{y}_n\right)^2. \tag{26}$$

In view of Theorem 2.4, for the last term in (26), we have

$$\begin{aligned}
\sum_{n=N+1}^{+\infty} (\hat{y}_n)^2 &= \sum_{n=N+1}^{+\infty} \frac{\left(y_n^T - \hat{\gamma} b_n\right)^2}{\left(1 + 2\hat{\gamma} a_n\right)^2} \\
&\leq \sum_{n=N+1}^{+\infty} \left[\frac{2\left(y_n^T\right)^2}{\left(1 + 2\hat{\gamma} a_n\right)^2} + \frac{2\left(\hat{\gamma} b_n\right)^2}{\left(1 + 2\hat{\gamma} a_n\right)^2}\right] \\
&\leq 2 \sum_{n=N+1}^{+\infty} \left[\left(y_n^T\right)^2 + \left(\frac{b_n}{2a_n}\right)^2\right] \\
&= 2\|y^T - y^{T,N}\|_H^2 + 2 \sum_{n=N+1}^{+\infty} \left[\frac{\int_0^T \beta(t) e^{-\lambda_n t} y_n^d(t)\ dt}{e^{\lambda_n T}\left(\alpha + \int_0^T \beta(t) e^{-2\lambda_n t}\ dt\right)}\right]^2 \\
&\leq 2\|y^T - y^{T,N}\|_H^2 + 2 \sum_{n=N+1}^{+\infty} \frac{\|\beta\|_{L^2(0,T)}^2\ \|y_n^d\|_{L^2(0,T)}^2}{\alpha^2 e^{2\lambda_n T}} \\
&\leq 2\|y^T - y^{T,N}\|_H^2 + 2\frac{\|\beta\|_{L^2(0,T)}^2}{\alpha^2 e^{2\lambda_N T}} \|y^d - y^{d,N}\|_{L^2(0,T;H)}^2.
\end{aligned} \tag{27}$$

In order to estimate the first sum on the right-hand side of (26), note that from $G_N(\hat{\gamma}_N) = \varepsilon^2 = G(\hat{\gamma})$, it follows that

$$\sum_{n=1}^{N} \left[\left(\frac{\hat{\gamma}_N\left(2a_n y_n^T + b_n\right)}{1 + 2\hat{\gamma}_N a_n}\right)^2 - \left(\frac{\hat{\gamma}\left(2a_n y_n^T + b_n\right)}{1 + 2\hat{\gamma} a_n}\right)^2\right] = \sum_{n=N+1}^{+\infty} \left(\frac{\hat{\gamma}\left(2a_n y_n^T + b_n\right)}{1 + 2\hat{\gamma} a_n}\right)^2.$$

Manipulating the left-hand side and estimating the right-hand one, we have

$$(\hat{\gamma}_N - \hat{\gamma}) \sum_{n=1}^{N} [\hat{\gamma}_N + \hat{\gamma} + 4a_n\hat{\gamma}_N\hat{\gamma}] \left[\frac{2a_n y_n^T + b_n}{(1+2\hat{\gamma}a_n)(1+2\hat{\gamma}_N a_n)}\right]^2 \leq \sum_{n=N+1}^{+\infty} \left(y_n^T + \frac{b_n}{2a_n}\right)^2$$

which implies

$$\sum_{n=1}^{N} \left[\frac{\left(2a_n y_n^T + b_n\right)(\hat{\gamma}_N - \hat{\gamma})}{(1+2\hat{\gamma}_N a_n)(1+2\hat{\gamma}a_n)}\right]^2 \leq \sum_{n=N+1}^{+\infty} \left(y_n^T + \frac{b_n}{2a_n}\right)^2, \tag{28}$$

in view of the trivial inequality $\hat{\gamma}_N^* - \hat{\gamma} \leq \hat{\gamma}_N^* + \hat{\gamma} + 4a_n\hat{\gamma}_N^*$. Using (28), we obtain

$$\begin{aligned}
\sum_{n=1}^{N} \left(\hat{y}_n^N - \hat{y}_n\right)^2 &= \sum_{n=1}^{N} \left[\frac{\left(2a_n y_n^T + b_n\right)(\hat{\gamma}_N - \hat{\gamma})}{(1+2\hat{\gamma}_N a_n)(1+2\hat{\gamma}a_n)}\right]^2 \\
&\leq \sum_{n=N+1}^{+\infty} \left(y_n^T + \frac{b_n}{2a_n}\right)^2 \\
&\leq 2\|y^T - y^{T,N}\|_H^2 + 2\frac{\|\beta\|_{L^2(0,T)}^2}{\alpha^2 e^{2\lambda_N T}} \|y^d - y^{d,N}\|_{L^2(0,T;H)}^2, \qquad (29)
\end{aligned}$$

where the last inequality follows from the same procedure as in (27). We conclude by summing (27) and (29). □

Remark 3.3 The relationship between the Fourier coefficients of the control and corresponding final state, namely $u_n = e^{\lambda_n T} y_n$, gives a similar approximation estimate for the optimal control. More precisely,

$$\hat{u}^N = \sum_{n=1}^{N} \hat{u}_n^N \varphi_n = \sum_{n=1}^{N} e^{\lambda_n T} y_n^N \varphi_n \tag{30}$$

is the initial state corresponding to the truncated approximation $\hat{y}^N$ given by (25). Then, clearly,

$$\|\hat{y}^N - \hat{y}\|_H \leq e^{-\lambda_N T} \|\hat{u}^N - \hat{u}\|_H.$$

Moreover, if the target y^T is reachable, then

$$\|\hat{u}^N - \hat{u}\|_H^2 \leq 4\|u^T - u^{T,N}\|_H^2 + \frac{4\|\beta\|_{L^2(0,T)}^2}{\alpha^2} \|y^d - y^{d,N}\|_{L^2(0,T;H)}^2,$$

where u^T is the initial state corresponding to the given target y^T.

Using Theorem 3.2, we can determine the number N of Fourier coefficients needed to obtain the desired precision for the truncated optimal final state, in terms of the approximations for the data y^T and y^d.

3.2. *Description of the algorithm*

Recall that the problem formulation comprises the following data:

- A target final state $y^T \in H$, along with a tolerance $\varepsilon > 0$;
- A weight $\alpha > 0$ for the energy cost of the control; and
- A reference trajectory for the distributed cost $y^d \in L^2(0,T;H)$, and a corresponding nonnegative weight function $\beta \in L^2(0,T)$.

We choose a precision $\rho > 0$ for the approximation error. The following procedure produces the approximate optimal final state $\hat{y}^N$, with precision ρ:

Step 1. Determine N such that

$$\max\left\{ \|y^T - y^{T,N}\|_H^2, \frac{\|\beta\|_{L^2(0,T)}^2}{\alpha^2 e^{2\lambda_N T}} \|y^d - y^{d,N}\|_{L^2(0,T;H)}^2 \right\} \leq \frac{\rho^2}{8}.$$

Step 2. Compute $\varepsilon_N := \lim_{\gamma\to\infty} G_N(\gamma)$.

Step 3. For $\varepsilon > \varepsilon_N$ the approximative solution is $\hat{u}^N = \tilde{u}^N = \sum_{n=1}^N -\frac{b_n}{2a_n}\varphi_n$. Otherwise, proceed to Step 4.

Step 4. Solve equation (24) numerically to find $\hat{\gamma}_N$ (bisection method or other).

Step 5. Compute the final state $\hat{y}^N$ using (25), and the optimal control $\hat{u}^N$ by (30).

4. Numerical examples

In order to illustrate this method, we study the classical heat equation on the one-dimensional domain $\Omega = (0,\pi)$ with homogeneous Dirichlet boundary condition and final time $T = 0.01$. More precisely, $\mathcal{S}_t u = y(t)$, where y is the solution of the partial differential equation

$$\begin{cases} \frac{d}{dt}y - \Delta y = 0 & (0,\pi)\times(0,T) \\ y = 0 & \{0,\pi\}\times(0,T) \\ y(0) = u & (0,\pi). \end{cases} \tag{31}$$

In both simulations, we used the eigenfunctions of the Dirichlet Laplacian

$$\varphi_k(x) = \sqrt{\frac{2}{\pi}}\ \sin(kx), \qquad k \geq 1,$$

with correspondent eigenvalues $\lambda_k = k^2$, as an orthonormal basis for $H = L^2(0,\pi)$. Computations were performed on an equispaced grid with 10^4 elements.

4.1. *Energy minimization*

This first example is concerned with the following problem:

$$(\mathcal{P}), \qquad \hat{u} \in \operatorname*{arg\,min}_{u\in L^2(0,\pi)} \left\{ \frac{\alpha}{2}\|u\|_{L^2}^2 : \mathcal{S}_T u \in \overline{B_\varepsilon(y^T)} \right\}.$$

with no prescribed reference trajectory.

The coefficients (16) in this case take the form $a_n = \frac{\alpha}{2} e^{2\lambda_n T}$ and $b_n = c_n = 0$, while solutions of the unconstrained problem are trivial, i.e. $\tilde{y} = \tilde{u} = 0$. In particular,

$$\lim_{\gamma \to +\infty} G(\gamma) = \|y^T\|_{L^2}^2.$$

The setting of the experiment is the following:

- $\alpha = 1$ ($\beta \equiv 0$);
- $N = 100$ Fourier coefficients.

Given a reachable final target y^T, we want to investigate the differences between the initial datum u^T that generates the target and the solution of the optimal control problem for various values of ε.

We introduce

$$u = \left(e^{-x^2} - 1\right)\left(e^{-(x-L)^2} - 1\right) \sin\left(2x^3\right),$$

and choose $u^T = u^N$, the Fourier representation of u by first $N = 100$ coefficients. Accordingly, the final target, given as $y^T = S_T u^T$, has a finite series representation.

All three functions, u, u^T, y^T are represented in Fig. 2. Note that representation of u by $N = 100$ Fourier coefficients provides a perfect match of the corresponding plots.

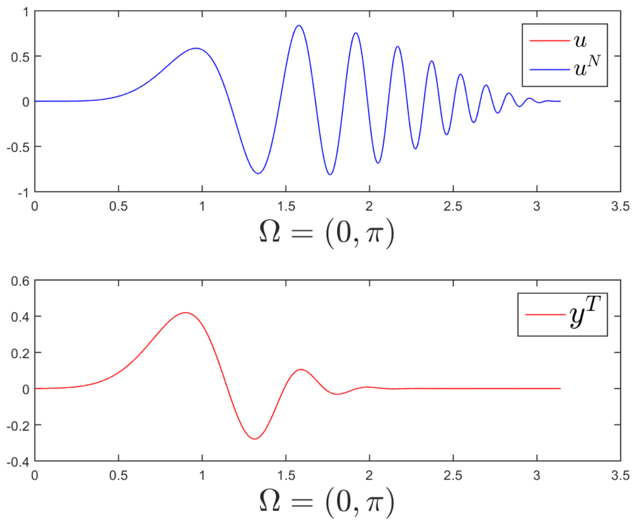

Figure 2.: TOP: comparison between u and its approximation by N Fourier coefficients (indistinguishable). BOTTOM: final target $y^T = \mathcal{S}_T u^T$.

We select three different values of ε (and consequently, different values of $\hat{\gamma}_N$; see Fig. 3), namely

$$\varepsilon^2 = \left[10^{-6}, 0.5, 0.99\right] \cdot \|y^T\|_{L^2}^2,$$

to cover intermediate and extreme situations. For all the three values of ε, the execution time was 0.4 seconds.

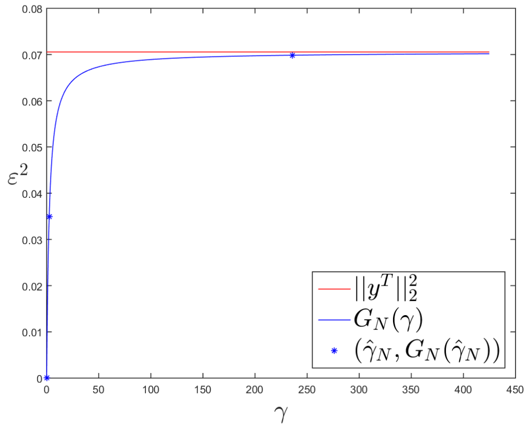

Figure 3.: Function $G_N(\gamma)$ with asymptotic value $\|y^T\|_{L^2}^2$ for $\gamma \to +\infty$. Marked with stars, the solutions $\hat{\gamma}_N$ corresponding to the three values of ε chosen.

For the smallest value of ε, one should expect the optimal control $\hat{u}$ to be close to u^T, but with a smaller L^2 norm. On the other hand, for ε close to the critical value $\|y^T\|_{L^2}$, the corresponding $\hat{u}$ should be close to 0, the solution of the unconstrained problem.

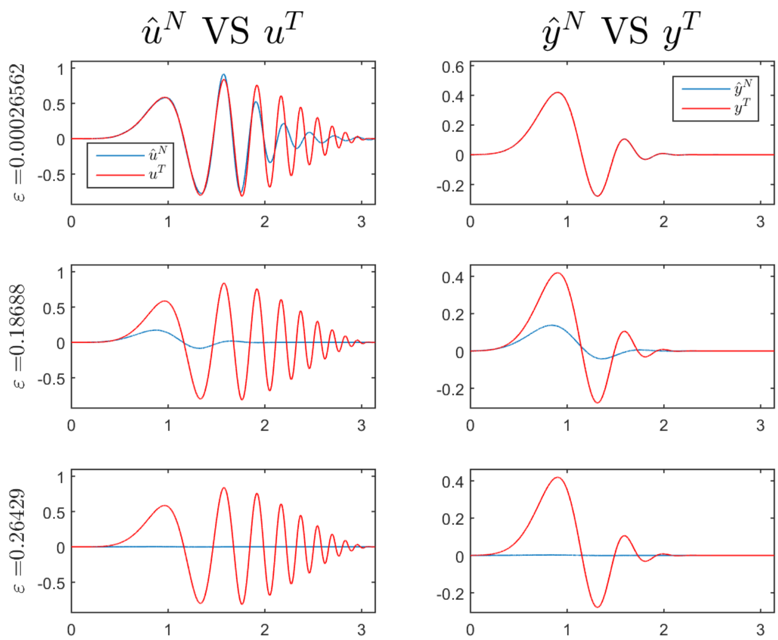

Figure 4.: LEFT: for the three values of ε, comparison between the data that generates the target u^T and the solution of the optimal control problem $\hat{u}^N$. RIGHT: same for the final states (y^T and $\hat{y}^N$).

This prediction is confirmed by the numerical simulations (Fig. 4). For small ε we note slight discrepancies between $\hat{u}^N$ and u^T, which, however, disappear with time, and the corresponding solutions become indistinguishable when they reach the final state. A higher tolerance increases the initial discrepancies, which also become evident at the final time, since, according to Proposition 2.1, $\|\hat{y} - y^T\|_{L^2} = \varepsilon$. Finally, as the tolerance approaches the critical value $\lim_\gamma G(\gamma) = \|y^T\|^2_{L^2}$, the solution becomes indistinguishable from the solution $\tilde{u}$ of the unconstrained problem.

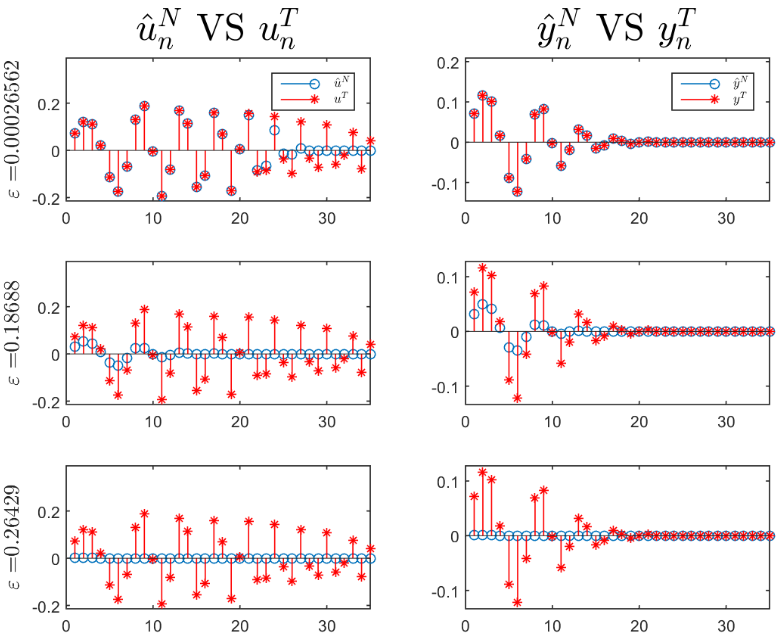

Figure 5.: LEFT: for the three values of ε, comparison between the first 35 Fourier coefficients of the data that generates the target u^T and the solution of the optimal control problem $\hat{u}^N$. RIGHT: same for the corresponding final states (y^T and $\hat{y}^N$).

The analysed behaviour can be also checked at the level of Fourier spectral decomposition: Fig. 5 represents the first 35 coefficients of u^T and the target y^T, in comparison with the coefficients of the initial and final solution (whose subsequent coefficients are essentially zero). When ε is small, coefficients $\hat{u}_n^N$ and u_n^T coincide for low n, while discrepancies emerge for higher frequencies. However, at the final state level this distinctions vanish due to the strong dissipation effects of the heat equation for high frequencies. As the tolerance ε increases, the solution coefficients converge to the zero value, corresponding to the solution $\tilde{u} \equiv 0$ of the unconstrained problem. This is confirmed by the decreasing values of the energy functional (Table 1). The first row of the table corresponds to the norm of the initial datum u^T.

ε^2	$J\left(\hat{u}^N\right)$
0	0.2235
$10^{-6} \cdot \|y^T\|^2_{L^2}$	0.1772
$0.5 \cdot \|y^T\|^2_{L^2}$	0.0057
$0.99 \cdot \|y^T\|^2_{L^2}$	$1.486e^{-6}$

Table 1.: Values of the energy functional J for the solution corresponding to different tolerances ε.

4.2. *Energy minimization and regulation*

For the second numerical experiment, we include the regulation term that forces the state towards a reference trajectory on a subinterval $[t_1, t_2]$ of the time horizon $[0, T]$. In particular, we set Ω, T as in the previous experiment and

- $\alpha = 10^{-4}$;
- $\beta(t) = \mathbb{1}_{[t_1,t_2]}(t)$, with $t_1 = T/3$ and $t_2 = 2T/3$;
- $y^d(x,t)$ as a smoothing regularization (through classical mollifier) of the function $x \longmapsto \mathbb{1}_{[x_1,x_2]}(x)$, with $x_1 = \pi/5$ and $x_2 = 2\pi/5$;
- $y^T(x)$ as a smoothing regularization (again, through mollifier) of the function $x \longmapsto \mathbb{1}_{[x_3,x_4]}(x)$, with $x_3 = 3\pi/5$ and $x_4 = 4\pi/5$.

The trajectory target and the final-state one are illustrated in Fig. 6.

We can compute explicitly the coefficients associated to the ellipsoids W_c in (18):

$$a_n = \left(\frac{\alpha}{2} + \frac{1}{2}\int_0^T \beta(t)\, e^{-2\lambda_n t} dt\right) e^{2\lambda_n T} = \frac{\alpha}{2} e^{2\lambda_n T} + \frac{1}{4\lambda_n}\left(e^{2\lambda_n (T-t_1)} - e^{2\lambda_n (T-t_2)}\right);$$

$$b_n = -e^{\lambda_n T}\int_0^T \beta(t)\, e^{-\lambda_n t} y_n^d(t)\, dt = \frac{y_n^d}{\lambda_n}\left(e^{\lambda_n (T-t_2)} - e^{\lambda_n (T-t_1)}\right);$$

$$c_n = \frac{1}{2}\int_0^T \beta(t)\, y_n^d(t)^2\, dt = \frac{\left(y_n^d\right)^2}{2}(t_2 - t_1).$$

To guarantee a priori fixed precision of $\rho = 10^{-3}$ on $\|\hat{y}_N - \hat{y}\|_{L^2}$, due to Theorem 3.2, we need to use $N = 185$ Fourier coefficients.

Again, for the numerical experiments, we used three different values of ε; the smallest is a value close to zero, the second an intermediate one and the biggest is a value close to the limit of $G_N(\gamma)$ for $\gamma \to +\infty$ (see Fig. 7, where we represent also the function G_N). The elapsed time, for all the three values of ε, is 0.3 seconds.

From Figures 8 and 9, we see that the results confirm what intuition suggests: for small ε, the final solution stays close to the target final state; so most of the mass is concentrated on the right side of the domain. On the opposite, when ε is large (more freedom in the constraint), the optimisation of the energy requires the mass to be concentrated on the left. This to guarantee that the solution will be close to the reference trajectory y^d for the distributed cost between times t_1 and t_2. Finally, the intermediate value of ε is a compromise. A trade-off is necessary because of term $\alpha\|u\|_{L^2}^2$ in the functional J, that induces the minimization of the energy of the initial datum.

ε^2	$J(\hat{u}_N)$
$0.1^2 \cdot \varepsilon_N^2$	0.0032
$0.4^2 \cdot \varepsilon_N^2$	0.0010
ε_N^2	0.0001

Table 2.: Values of the energy functional J computed in the solutions corresponding to different tolerances ε, where $\varepsilon_N = \lim_{\gamma\to+\infty} G_N(\gamma)$.

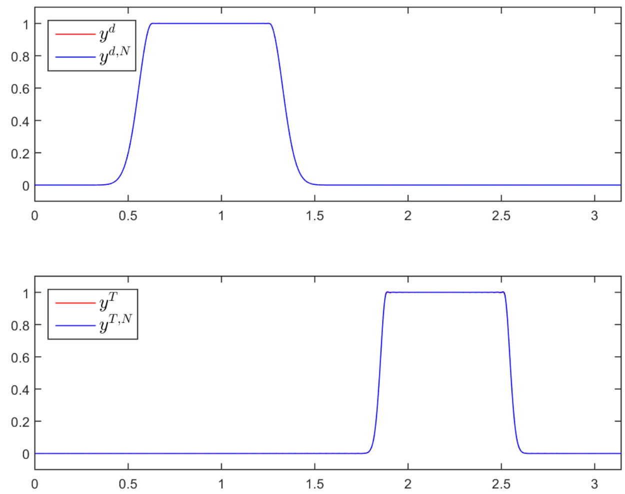

Figure 6.: Top: reference trajectory y^d for the distributed cost. Bottom: target final state y^T, in comparison with their reconstructions after Fourier decomposition (indistinguishable).

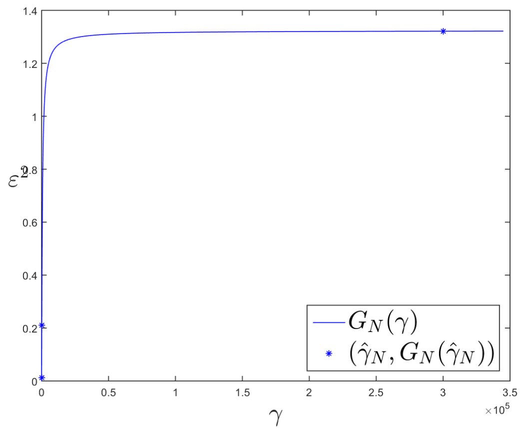

Figure 7.: The function G_N. The chosen values of ε, and the corresponding $\hat{\gamma}_N$, are marked with stars.

5. Conclusions

We have presented a numerical method for the finding the initial datum that minimises the cost functional J, which comprises an energy term on the control and a regulation term given by a distributed cost on the state, and such that the final state lies within a prescribed distance to a given target. The new methodology we have developed is based on a spectral decomposition of the operator $\mathcal{A}$ governing the evolution (1) of the system. It consists in a one-shot (not iterative) algorithm that reduces the problem to finding the unique zero of a bounded, strictly increasing function in $\mathbb{R}$. As a consequence, once this decomposition is available −which can be done off-line and saved for future use−, the optimal control problem is solved almost instantaneously. The preliminary numerical experiments provided here prove the method to be fast and effective, and deserve future study. Also, in

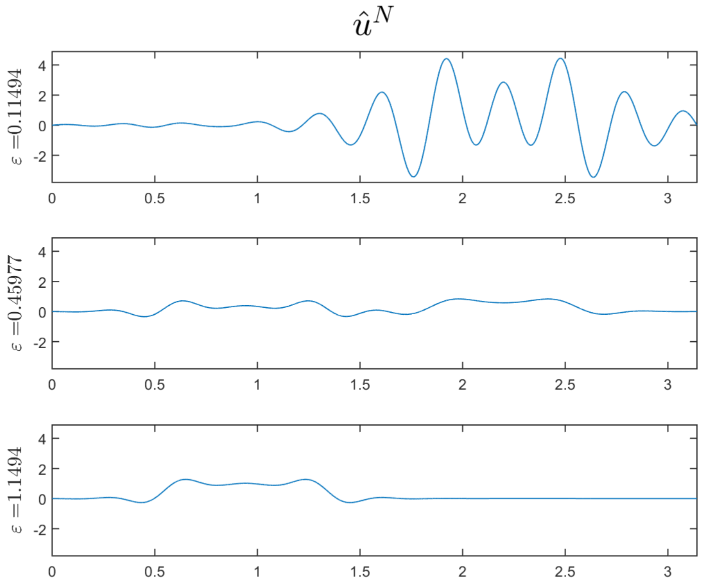

Figure 8.: Solution $\hat{u}$ of the control problem obtained by the proposed algorithm for the three values of ε.

view of applications, it would be interesting to include a L^1 term in the cost, in order to induce sparsity of the initial condition. The geometric analysis carried out in Subsection 2.1 is the same. The nature of the method also suggests that other kinds of evolution equations, in particular, of higher order, can be dealt with using this procedure.

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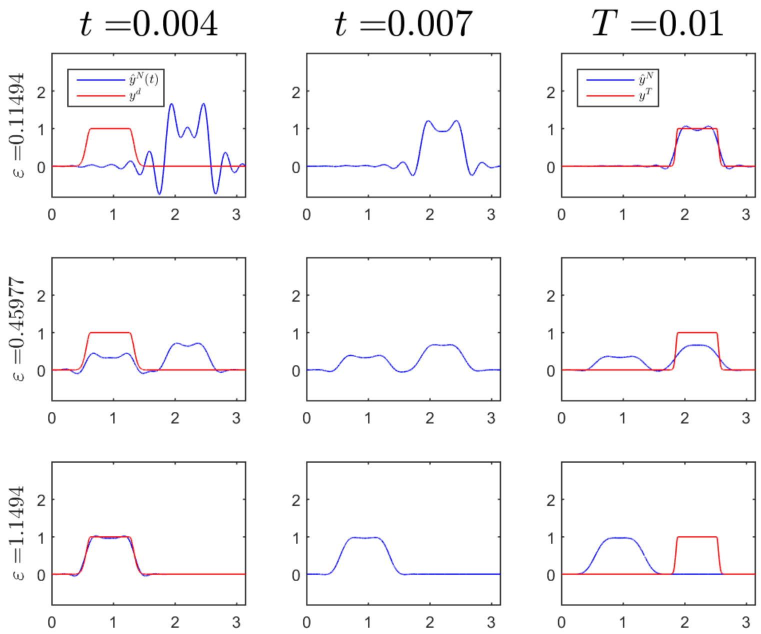

Figure 9.: For the three values of ε, LEFT: comparison between the computed solution $\mathcal{S}_t\hat{u}^N$ and the reference trajectory for the distributed cost $y^d(t)$ at time $t = 0.004$. MIDDLE: computed solution $\mathcal{S}_t\hat{u}^N$ at time $t = 0.007$. RIGHT: comparison between the computed final state $\hat{y} = \mathcal{S}_T\hat{u}^N$ and the target y^T.

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