Check for updates

Received: 3 June 2021 | Revised: 9 December 2021 | Accepted: 26 January 2022

DOI: 10.1112/plms.12485

RESEARCH ARTICLE

Proceedings of the London Mathematical Society

Limit laws for rational continued fractions and value distribution of quantum modular forms

Sandro Bettin[1] | **Sary Drappeau**[2]

[1]Dipartimento di Matematica, Università di Genova, Genova, Italy

[2]Aix Marseille Université, CNRS, UMR, Marseille, France

Correspondence
Sary Drappeau, Aix Marseille Université, CNRS, I2M UMR 7373, Marseille, France.
Email:
sary-aurelien.drappeau@univ-amu.fr

Funding information
Aix-Marseille University; INdAM

Abstract

We study the limiting distributions of Birkhoff sums of a large class of cost functions (observables) evaluated along orbits, under the Gauss map, of rational numbers in (0, 1] ordered by denominators. We show convergence to a stable law in a general setting, by proving an estimate with power-saving error term for the associated characteristic function. This extends results of Baladi and Vallée on Gaussian behaviour for costs of moderate growth. We apply our result to obtain the limiting distribution of values of several key examples of quantum modular forms. We obtain the Gaussian behaviour of central values of the Esterman function $\sum_{n\geqslant 1}\tau(n)\mathrm{e}^{2\pi inx}/\sqrt{n}$ $(x\in\mathbb{Q})$, a problem for which known approaches based on Eisenstein series have been so far ineffective. We give a new proof, based on dynamical systems, that central modular symbols associated with a holomorphic cusp form for $SL(2,\mathbb{Z})$ have a Gaussian distribution, and give the first proof of an estimate for their probabilities of large deviations. We also recover a result of Vardi on the convergence of Dedekind sums to a Cauchy law, using dynamical methods.

MSC 2020
11A55 (primary), 37C30, 11F03, 11F67, 60F05 (secondary)

© 2022 The Authors. *Proceedings of the London Mathematical Society* is copyright © London Mathematical Society. This is an open access article under the terms of the Creative Commons Attribution-NonCommercial License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited and is not used for commercial purposes.

1 | VALUE DISTRIBUTION OF QUANTUM MODULAR FORMS

Let $\Gamma \subset SL(2,\mathbb{Z})$ be a cofinite Fuchsian subgroup, which acts of functions on $\mathbb{P}^1(\mathbb{Q})$ by the weight-k 'slash operator'

$$f|_k\gamma(x) := (cx+d)^{-k} f(\gamma x) \qquad \text{if } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

where $\gamma x = \frac{ax+b}{cx+d}$ is the Möbius transformation.

In their simplest guise, quantum modular forms, introduced by Zagier [121] (see [119] for early examples), denote the set of functions

$$f : \mathbb{P}^1(\mathbb{Q}) \smallsetminus S \to \mathbb{C},$$

for some finite set S, satisfying a form of modularity, in the purposely vague sense that for all $\gamma \in \Gamma$, the function of $x \in \mathbb{P}^1(\mathbb{Q}) \setminus (S \cup \gamma^{-1}S)$ defined by

$$h_\gamma(x) := f(x) - f|_k\gamma(x), \tag{1.1}$$

has some regularity property. Part of the research effort has focused on constructing examples in interrelated ways:

- Generating series associated with combinatorial sequences: Fishburn matrices [21], unimodal sequences [19, 70], partition theory [91] (we note that in the latter, quantum modularity is actually a crucial tool for the asymptotic estimation of partition-related sequences),
- Radial limits of modular objects (mock theta function, quasi-modular forms) defined on the hyperbolic disk [22, 30, 45, 46, 120],
- Eichler integrals, periods of modular functions [20, 23]; state integrals involving the quantum dilogarithm function [37, 49, 73],
- Describing the homology of spaces of cusp forms [26, 27, 29],
- Kashaev knot invariants and Nahm sums [13, 50, 121],
- Correlations of the fractional part functions appearing as covariances in the Nyman–Beurling reformulation of the Riemann hypothesis [6, 10, 11, 77],
- Diophantine approximation and multifractal analysis [66, 100].

In the present paper, we are concerned with the following problem: given a quantum modular form f, how do the multi-sets

$$\{f(x), x \in \mathbb{Q} \cap (0,1], \mathrm{denom}(x) \leqslant Q\}, \tag{1.2}$$

appropriately normalized, distribute as $Q \to \infty$? This topic is tightly related to weak limits of partial sums of certain arithmetic functions, which goes back to Hardy–Littlewood [55], and has been since then periodically revisited: we mention in particular the works [67, 81, 115] on theta sums, and [7, 79] on cotangent sums. These works are all concerned with instances of QMF of *non-zero* weight.

Our interest in this question comes from the statistical study of additive twists central values of L functions, which as we will see are *weight-zero* QMF related to the third item described above

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

(periods of modular functions). We are aware of two occurrences of this setting in the literature. The first is a result of Vardi [114] on the existence of a limiting distribution for Dedekind sums. The second is a recent result of Petridis and Risager [96] on the distribution of modular symbols, which was motivated by conjectures of Mazur–Rubin [85] and Stein [108]. Both results exploit a close connection with Fourier analysis of the modular surface (the spectral analysis of the hyperbolic Laplacian). It is unclear how to extend these methods to more general QMF.

In the present paper, we present an approach, based on dynamical systems and the spectral properties of a family of transfer operators, which allows us to answer the question (1.2) for essentially all level 1 (*that is,* $\Gamma = SL(2,\mathbb{Z})$) and weight 0 QMF.

We denote $\sigma = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$ and $\tau = \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ the two usual generators of $SL(2,\mathbb{Z})$, so that the associated period functions (1.1) with weight $k = 0$ are given by

$$h_\tau(x) = f(x) - f(x+1),$$
$$h_\sigma(x) = f(x) - f(-1/x). \tag{1.3}$$

Our result may be stated informally as follows.

Theorem 1.1. *Let f be a weight 0 QMF for $SL(2,\mathbb{Z})$, in the sense that h_σ extends to a Hölder-continuous function on $(0,1]$ with some regular growth behaviour at 0, and $h_\tau = 0$. Then, up to a suitable renormalization, the multi-sets (1.2) become equidistributed according to a stable law, which is characterized by the growth of h_σ at 0.*

The hypothesis are stated more precisely in Theorem 3.1. The restriction to $x \in (0,1]$ and the assumption $h_\tau = 0$ are made to clarify the statement, but are inessential (cf. footnote 2, p. 8) and are natural in applications. However, the restriction to $\Gamma = SL(2,\mathbb{Z})$ is important in our argument. What is required is that the action of Γ on $\mathbb{P}^1(\mathbb{R})$ can be induced into an expanding Markov map (the Gauss map in the case $\Gamma = SL(2,\mathbb{Z})$). The restriction to weight zero QMF is also natural, as the problem for non-zero weights is of different nature and typically simpler; see [14] for more details.

We will find in practice that any bound of the shape $h_\sigma(x) = O(x^{-1/\alpha})$ as $x \to 0$ ($\alpha > 2$) ensures convergence to a Gaussian law. A bound of type $h_\sigma(x) \asymp x^{-1/\alpha+o(1)}$ for some $\alpha \in (0,2)$, will typically imply the convergence to a stable law of parameter α.

Due to the relative weakness in its hypotheses, Theorem 1.1 applies to a wide class of QMF. In the next section, using Theorem 1.1, we answer the distribution question for several arithmetic invariants. We expect further applications to follow in the future.

2 | APPLICATIONS

For all $Q \geqslant 1$, we endow the set

$$\Omega_Q := \{x = a/q,\ 1 \leqslant a \leqslant q \leqslant Q, (a,q) = 1\} \subset \mathbb{Q} \cap (0,1]$$

with the uniform probability measure $\mathbb{P}_Q$, and we denote $\mathbb{E}_Q$ the associated expectation,

$$\mathbb{E}_Q(f(x)) = \left|\Omega_Q\right|^{-1} \sum_{x \in \Omega_Q} f(x).$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

2.1 | Central values of the Estermann function

Let

$$\tau(n) = \sum_{d|n} 1 \qquad (n \geqslant 1)$$

denote the divisor function. The Estermann function, introduced in 1930 [41], is defined by

$$D(s,x) := \sum_{n\geqslant 1} \frac{\tau(n)\mathrm{e}(nx)}{n^s}$$

for $\mathrm{Re}(s) > 1$, and extended by analytic continuation otherwise. It was initially introduced in relation with the shifted divisor problem $\sum_n \tau(n)\tau(n+1)$. Its functional equation still serves as a basic tool to derive Voronoï summation formulae, which are then used in conjunction with the circle method to study moments of L-function and their arithmetics applications: we mention the proportion of critical zeroes of ζ [31], the binary divisor problems [88], and non-vanishing of central L-values [78].

We mention a further connection with moments of Dirichlet L-functions $L(s,\chi) = \sum_{n\geqslant 1} \chi(n)n^{-s}$. By [8, Theorem 5], the twisted second moment of Dirichlet L-functions satisfies

$$\begin{aligned} M(a,q) &:= \frac{1}{q^{1/2}} \sum_{\chi \,(\mathrm{mod}\, q)} \left|L\left(\tfrac{1}{2},\chi\right)\right|^2 \chi(a) \\ &= \mathrm{Re}\, D(\tfrac{1}{2},\tfrac{a}{q}) + \mathrm{Im}\, D\left(\tfrac{1}{2},\tfrac{a}{q}\right) + O(q^{-1/2}) \end{aligned} \tag{2.1}$$

for q prime and $q \nmid a$. From this expression, we see that the second moment $\sum_{a \,(\mathrm{mod}\, q)} |M(a,q)|^2$ is essentially the fourth moment of Dirichlet L-functions $\sum_{\chi \,(\mathrm{mod}\, q)} |L(\frac{1}{2},\chi)|^4$, whose full evaluation in [118] lies at the threshold of current techniques of analytic number theory (see [16, 17] for further work on this topic). This fits in the general problem of understanding the distribution of central values of L-functions and their twists, which is a fundamental topic in analytic number theory [32, 56, 99, 103, 104]. Up to now, essentially all known results have been obtained by the moments method.

Using Theorem 1.1, we obtain the following Central Limit Theorem.

Theorem 2.1. *For all $\varepsilon > 0$, $Q \geqslant 3$, and all rectangle $\mathscr{R} \subset \mathbb{C}$, we have*

$$\mathbb{P}_Q\left(\frac{D\left(\frac{1}{2},x\right)}{\sigma(\log Q)^{\frac{1}{2}}(\log\log Q)^{\frac{3}{2}}} \in \mathscr{R}\right) = \int_{v_1+iv_2\in\mathscr{R}} \frac{\mathrm{e}^{-(v_1^2+v_2^2)/2}\,\mathrm{d}v_1\,\mathrm{d}v_2}{2\pi} + O_\varepsilon\left(\frac{1}{(\log\log Q)^{1-\varepsilon}}\right),$$

where $\sigma = 1/\pi$.

We will obtain this result as an application of Theorem 1.1, using the fact, proved in [8], that $D(\frac{1}{2},x)$ is a weight-0 QMF, the associated function h_σ (1.3) being a $(\frac{1}{2}-\varepsilon)$-Hölder continuous function on (0,1].

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The simplicity with which we will deduce Theorem 2.1 from Theorem 1.1 contrasts with the fact that other methods appear to be completely ineffective with this problem. The moments method (and therefore also the approach of [92] described in the following section) cannot be applied, due to the presence of a negligible proportion of $x \in \Omega_Q$ with abnormally large continued fraction coefficients, whose contribution dominates the integer moments of $D(\frac{1}{2}, x)$. In fact, all moments of $D(\frac{1}{2}, x)$ and $M(a, q)$ have recently been computed in [9]: starting already from the second they grow faster that what is suggested by Theorem 2.1.

Another tentative approach to Theorem 2.1 consists in obtaining a limiting distribution result for $\sum_{n\geqslant 1} \tau(n)\mathrm{e}(nz)$, where $z = x + iy; x \in [0, 1]$ is chosen at random and $y \to 0^+$, and transferring these properties to its discrete counterpart $D(\frac{1}{2}, x) \approx \sum_{n\leqslant q^2} \tau(n)\mathrm{e}(nx)n^{-1/2}$, for $x = a/q \in \mathbb{Q}$, by Fourier expansion. This method is employed for the incomplete Gauss sum in [36], based on [81]. In our setting, however, connecting the continuous and the discrete averages raises several additional difficulties, among which the problem of dealing with a divergent second moment as well as those coming from considering a central value rather than an object corresponding to an L-value off the line. Also the incomplete Gauss sums are not Gaussian distributed, but rather are distributed as the push-forward measure by a theta series. This suggests a difference in nature between the two problems and, notwithstanding the technical difficulties, it suggests this method could not be adapted to our case.

It would be interesting to obtain a statement analogous to Theorem 2.1 for the values $M(a, q)$, however, the identity (2.1) as stated holds only for q prime; the corresponding identity for generic q involves a multiplicative convolution, which is not clearly accountable for using the present method. However, we still believe $M(a, q)$ is distributed according to a normal law.

Conjecture 2.2. *As $q \to \infty$ along primes, the multi-set*

$$\left\{ \frac{M(a,q)}{(\log Q)^{1/2}(\log\log Q)^{3/2}},\ 1 \leqslant a < q \right\}$$

become distributed according to a dilated centred Gaussian.

2.2 | Modular symbols

Out of many possible ways, modular symbols [80] can be seen as elements of the space of linear forms on $S_k(\Gamma_0(N))$, the vector space of cusp forms of weight $k \equiv 0 \pmod 2$ and level $N \geqslant 1$, spanned by the Shimura integrals

$$f \mapsto \langle x\rangle_{f,m} := \frac{(2\pi i)^m}{(m-1)!}\int_x^{i\infty} f(z)(z-x)^{m-1}\,\mathrm{d}z =: \langle x\rangle_{f,m}^{+} + i\langle x\rangle_{f,m}^{-}$$

for any $1 \leqslant m \leqslant k-1$, $x \in \mathbb{Q}$, $f \in S_k(\Gamma_0(N))$, and where $\langle x\rangle_{f,m}^{\pm} \in \mathbb{R}$. (In this section and Section 9.1 only, the letter f will denote a holomorphic cusp form).

Up to an explicit factor, the value $\langle x\rangle_{f,m}$ is also the special value $L(f, x, m)$ of the analytic continuation of the L-function

$$L(f, x, s) := \sum_{n=1}^{\infty} a_n \mathrm{e}(nx)n^{-s} \qquad (\mathrm{Re}\, s > k),$$

14602244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by CochraneItalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where we write $f(z) = \sum_{n\geqslant 1} a_n \mathrm{e}(nz)$ for $\mathrm{Im}(z) > 0$. Being at the intersection of the geometric, modular and arithmetic aspects of $\Gamma_0(N)$, modular symbols received a considerable amount of interest, for example, for computing with modular forms [34, 107]. The central values of the Estermann function $D(\frac{1}{2}, x)$, which was the subject of the previous section, may be interpreted as a regularized modular symbol for the derivative $\frac{\partial}{\partial s} E_2(z,s)|_{s=1/2}$ of the Eisenstein series [64, chapter 3.5].

The symbol associated with the central value $m = k/2$ plays a particular rôle and is the focus of this section. Motivated by the *abc*-conjecture, the question of the size of modular symbols was initially asked by Goldfeld [51] and later studied by Petridis and Risager [94, 95]. Interest in this question was recently revived by questions of Mazur, Rubin and Stein [85, 108], motivated in part by the growth of ranks of elliptic curves. In particular, for f a fixed form of weight $k = 2$ (and so $m = 1$) and $x \in (0,1]$ varying along rationals of reduced denominator q, with $q \to \infty$, Mazur and Rubin predicted that

$$\left\{ \frac{\langle a/q \rangle_{f,1}^{\pm}}{\sqrt{\log q}}, 1 \leqslant a \leqslant q, (a,q) = 1 \right\}$$

becomes asymptotically distributed according to a suitably dilated normal law. To our knowledge, only the first and second moments have been computed [18, Chapter 9].

The situation changes with an additional average over $q \leqslant Q$. Then the Central Limit Theorem for weight $k = 2$, level $N \geqslant 1$ forms has been proved by Petridis and Risager [96], by using the spectral analysis of twisted Eisenstein series and the location of eigenvalues of the hyperbolic Laplacian.

Using Theorem 1.1, we will prove the following Central Limit Theorem for modular symbols associated with forms of level $N = 1$ and arbitrary weight $k \geqslant 12$.

Theorem 2.3. *Let $k \geqslant 12$ be even, and $f \in S_k(SL(2,\mathbb{Z})) \setminus \{0\}$ be fixed. Then for all $Q \geqslant 2$ and rectangle $\mathscr{R} \subset \mathbb{C}$, we have*

$$\mathbb{P}_Q\left(\frac{\langle x \rangle_{f,\frac{k}{2}}}{\sigma_f \sqrt{\log Q}} \in \mathscr{R} \right) = \int_{v_1+iv_2 \in \mathscr{R}} \frac{\mathrm{e}^{-(v_1^2+v_2^2)/2}\, \mathrm{d}v_1\, \mathrm{d}v_2}{2\pi} + O\left(\frac{1}{\sqrt{\log Q}} \right), \tag{2.2}$$

with $\sigma_f^2 = \frac{3(4\pi)^k}{\pi\Gamma(k)} \|f\|_k^2$, where $\|f\|_k$ is the weight-k Petersson norm of f.

For example, this result applies for $k = 12$ with f being the discriminant modular form $\Delta(z)$. The error term is optimal and uniform with respect to $\mathscr{R}$.

Using Theorem 1.1, we will also deduce the following statement, which can be interpreted as a Large Deviations Principle.

Theorem 2.4. *Let f be fixed as in Theorem 2.3. For any $\varepsilon > 0$, we have*

$$\limsup_{Q\to\infty} \mathbb{P}_Q\left(\left| \langle x \rangle_{f,\frac{k}{2}} \right| > \varepsilon \log Q \right)^{1/\log Q} < 1. \tag{2.3}$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by CochraneItalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

We will deduce Theorems 2.3 and 2.4 from Theorem 1.1 by showing that the map $x \mapsto \langle x \rangle_{f,\frac{k}{2}}$ is a weight 0 QMF with the period function h_σ (1.3) being Hölder-continuous on [0,1]. We will estimate all small exponential moments, from which the Large Deviations Principle will follow.

Theorem 2.3 is related to three works, two of which have been carried out around the same time as the present.

- The work of Petridis and Risager [96] was concerned with the case of weight $k = 2$ forms, which is relevant to the conjectures of Mazur-Rubin and Stein (cusp forms associated with elliptic curves). Their method is based on the analysis of twisted Eisenstein series. Although the authors do not seem to mention it, their method is capable of obtaining a Large Deviations Principle. The assumption $k = 2$ is, however, crucial in their approach, since only in this case the analogues of the function h_σ in (1.3) are constant (this translates into the independence of the multiplier system defined in [96, p. 7] with respect to z_0).
- Very recently, Nordentoft [92] has obtained the Central Limit Theorem for arbitrary weight $k \geqslant 2$ and level $N \geqslant 1$ modular forms. Specializing to $N = 1$, this gives in particular an independent proof of Theorem 2.3. Similarly as [96], the spectral analysis of twisted Eisenstein series play a central rôle, however, his argument is based on a completely different construction (due to the 'independence with respect to z_0' obstruction mentioned above). As a result, it crucially relies on the consideration of moments, and falls short of establishing a Large Deviations Principle (Theorem 2.4).
- Also very recently, Lee and Sun [74] have independently obtained a proof of the main theorem of [96], for weight $k = 2$, by a method similar to the one we pursue here. This is achieved by considering a certain twisted version of the Gauss map, keeping track at each iteration of a coset in $\Gamma_0(N)\backslash SL_2(\mathbb{Z})$. On the other hand, for $k = 2$ the period functions (the analogues of h_σ in (1.3)) are constant. The technical difficulties they are faced with are of very different nature from the ones we encounter here. By mixing the methods presented here with those of [74], it is plausible that the main results of [92], for arbitrary weight and levels, could be recovered by dynamical methods.

The difference in the magnitude of the variance between Theorems 2.1 and 2.4 is noteworthy, and is due to the presence, or not, of a pole the associated L-functions ($\zeta^2(s)$ and $L(f, s)$, respectively) at $s = 1$.

We have considered only the modular symbols $\langle x \rangle_{f,m}$ at the central value $m = k/2$. The behaviour for $m \neq k/2$ is very different and simpler: for instance, when $m > k/2$, the series $L(f, x, m)$ extends to a continuous function of $x \in \mathbb{R}$, and the values $\{\langle x \rangle_{f,m}, x \in \Omega_Q\}$ become distributed according to the push-forward $(L(f, \cdot, m))_*(\mathrm{d}\nu)$ of the Lebesgue measure. We return to this question in more details in [14].

The original conjectures of Mazur–Rubin and Stein were concerned with the case of a single q. There does not seem to be an effective way, with our method or those of [74, 92, 96], to isolate rationals of fixed denominators, whence all results proven so far rely on the extra average over q.

2.3 | Kashaev invariants of the 4_1 knot and sums of continued fraction coefficients

Our next application is motivated by a question in [121]. To a knot K and an integer $n \geqslant 2$, we associate the n-coloured Jones polynomial [48, section 1], which is a Laurent polynomial

$J_{K,n}(\mathbf{q}) \in \mathbb{Z}[[\mathbf{q}]]$ defined by a combinatorial construction through a diagram representation of K. For any root of unity $\mathbf{q} \in \mathbb{C}$, we define $J_{K,0}(\mathbf{q}) := J_{K,n}(\mathbf{q})$ where $n \geqslant 1$ is the order of $\mathbf{q}$. In [89], it was shown that the sequence $(J_{K,0}(\mathrm{e}^{2\pi i/n}))_{n\geqslant 1}$ is the Kashaev invariant of K [68] (the full function can be reconstructed by the action of the Galois group). In the case of the 4_1 knot (or 'figure-eight' knot), the simplest hyperbolic knot, we have explicitly [121]

$$\mathbb{J}(x) := J_{4_1,0}(\mathrm{e}(x)) = \sum_{m=0}^{\infty} |1-\mathrm{e}^{2\pi i x}|^2 \cdots |1-\mathrm{e}^{2\pi i m x}|^2, \qquad (x \in \mathbb{Q}).$$

Note that for each given $x \in \mathbb{Q}$, the sum is finite. In this case, Zagier's modularity conjecture, stated precisely in [121], predicts that $x \mapsto \log \mathbb{J}(x)$ is a weight 0 QMF: the difference

$$h(x) := \log \mathbb{J}(-1/x) - \log \mathbb{J}(x),$$

which is depicted in [121, Figure 4] is expected to behave 'nicely' with respect to x, although not continuously. Conjecturally, we expect $h(x) \sim C/x$ as $x \to 0$, where $C = \mathrm{Vol}(4_1)/2\pi$ and $\mathrm{Vol}(4_1) = 2.02\cdots$ is the hyperbolic volume of 4_1; Kashaev's volume conjecture is the case $x = 1/n$, $n \in \mathbb{N}$, which is known in this case [2].

A proof of Zagier's conjecture for the 4_1 knot has been announced by Garoufalidis and Zagier [50]. In [13], we obtained independently another proof, complemented by a reciprocity formula relative to a transformation of another kind, by which we deduced the following asymptotic estimate: for $\lambda \geqslant 0$, denote

$$\Sigma_\lambda(x) = \sum_{j=1}^{r(x)} a_j^\lambda, \qquad x \in \mathbb{Q}\cap(0,1], \quad x = [0; a_1, \dots, a_{r(x)}],\ a_{r(x)} > 1, \tag{2.4}$$

the sum of continued fraction coefficients of x. Then by [13], we have

$$\log \mathbb{J}(x) \sim C\Sigma_1(x) \qquad \text{as} \qquad \frac{\Sigma_1(x)}{r(x)} \to \infty. \tag{2.5}$$

This is in accordance with the conjectured behaviour of $h(x)$ as $x \to 0$.

The map $x \mapsto \Sigma_\lambda(x)$, suitably extended to $\mathbb{Q}$, is a weight 0 QMF with associated period function (1.3) satisfying $h_\sigma(x) = \lfloor 1/x \rfloor^\lambda$ for $x \in \mathbb{Q}\cap(0,1]$. Using Theorem 1.1, we compute in Theorem 9.4 below the distribution of Σ_λ for all $\lambda \geqslant 0$, extending work of [3, 4]. In particular, in the case $\lambda = 1$, we obtain the following result.

Theorem 2.5. *Let G_1 denote the cumulative distribution function of the stable law $S_1(\frac{6}{\pi}, 1, 0)$, that is $G_1(v) := \int_{-\infty}^{v} g_1(x)\,\mathrm{d}x$ where*

$$g_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-itx}\mathrm{e}^{-\frac{6}{\pi}|t| - \frac{12}{\pi^2} it\log|t|}\,\mathrm{d}t,$$

and let γ_0 be the Euler–Mascheroni constant. Then, as $Q \to \infty$,

$$\mathbb{P}_Q\left(\frac{\Sigma_1(x)}{\log Q} - \frac{\log\log Q - \gamma_0}{\pi^2/12} \leqslant v\right) = G_1(v) + O\left(\frac{1}{(\log Q)^{1-\varepsilon}}\right). \tag{2.6}$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Theorem 2.5 answers a question in [44], and echoes a result of Heinrich [58] which obtained a similar convergence for Lebesgue-almost all $x \in [0,1]$. In particular, (2.6) implies that the generic time complexity of the substractive algorithm for the GCD is asymptotically $(1+o(1))\frac{12}{\pi^2}(\log Q)\log\log Q$ on pairs of coprime numbers at most Q, as $Q \to \infty$. This is in sharp contrast with the average time complexity, which is $(1+o(1))\frac{6}{\pi^2}(\log Q)^2$ (the latter is known even with a single average over numerators, see [117]). Note that, contrary to the case of the Estermann function (Theorem 2.1), this discrepancy between the typical size and the average size is consistent with fact that the stable law $S_1(\frac{6}{\pi}, 1, 0)$ has a divergent first moment.

Combined with (2.5), Theorem 2.5 implies the following law of large numbers for the values of $\log \mathbb{J}(x)$.

Corollary 2.6. *As $Q \to \infty$, we have*

$$\log \mathbb{J}(x) \sim \frac{12}{\pi^2} C \log Q \log\log Q$$

for a proportion $1 - o(1)$ of fraction $x \in \mathbb{Q} \cap (0,1]$ of denominators at most Q.

The typical size of $\log \mathbb{J}(x)$ among fractions x with $\mathrm{denom}(x) \leqslant Q$ is therefore much less than that of $\log \mathbb{J}(1/Q) \sim CQ$ (by (2.5)).

We expect a convergence in law analogous to (2.6) for the values $\log \mathbb{J}(x)$, however, the error term which we obtain in (2.5) is not precise enough to carry this out. This issue is discussed more precisely in [13], along with the case of other knots.

2.4 | Dedekind sums

For all coprime integers a and q with $q \geqslant 1$, the Dedekind sum

$$s\left(\frac{a}{q}\right) := \sum_{h=1}^{q-1} \left(\!\!\left(\frac{ha}{q}\right)\!\!\right)\left(\!\!\left(\frac{h}{q}\right)\!\!\right), \qquad ((x)) := \begin{cases} \{x\} - 1/2 & (x \notin \mathbb{Z}), \\ 0 & (\text{otherwise}) \end{cases}$$

is a rational number of modulus at most $q/12 + 1/2$. They appear naturally in the multiplier system attached to the Dedekind η function; we refer to the monograph [98] for further properties and references. The value distribution of $s(x)$ on average over rational x has been studied from several points of view [25, 60, 113, 114].

From Theorem 1.1, we will deduce a short proof of the following convergence to a Cauchy law, which is the main result of [114].

Theorem 2.7 [114]. *Uniformly for $v \in \mathbb{R}$ and $Q \geqslant 2$, we have*

$$\mathbb{P}_Q\left(\frac{s(x)}{\log Q} \leqslant \frac{v}{2\pi}\right) = \frac{1}{\pi}\int_{-\infty}^{v} \frac{\mathrm{d}y}{1+y^2} + O\left(\frac{1}{(\log Q)^{1-\varepsilon}}\right). \tag{2.7}$$

This statement will easily follow from Theorem 1.1 by noting that $x \mapsto s(x)$ is a weight 0 QMF, with associated period function h_σ (1.3) being roughly $\lfloor 1/x \rfloor$.[†] This last fact is a consequence of the reciprocity formula for $s(x)$ [98, Chapter 2, Theorem 1].

[†] Note that we obtain a different distribution than in Theorem 2.5. This is due to the fact in that case the period relation is more precisely $f(x) + f(-1/x) = \lfloor 1/x \rfloor$ rather than $f(x) - f(-1/x) = \lfloor 1/x \rfloor$.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The proof of [114] builds on the close connection between $s(x)$ and the multiplier system associated to the η function, which brings the problem to an analysis of twisted Poincaré series on the modular surface, which are in turn studied by means of the Kuznetsov trace formula. By contrast, our arguments are dynamical in nature, and use little arithmetic information beyond the group structure of $SL(2,\mathbb{Z})$.

3 | OVERVIEW

3.1 | Reduction to dynamical analysis

We now overview the arguments underlying Theorem 1.1. Suppose $f : \mathbb{Q} \to \mathbb{C}$ is such that the functions defined through (1.1) satisfy $h_\tau = 0$ and h_σ extends to a continuous function on $(0,1]$. To simplify the presentation, in this section only, we assume that f is an even function.

The starting point is the remark that the action of σ and τ on $\mathbb{P}^1(\mathbb{Q})$ can be induced into an expanding Markov map on $(0,1]$, namely the Gauss map $T : x \mapsto \{1/x\}$, where the braces denote the fractional part. More precisely, by periodicity and since f is even, we have $f(x) = h_\sigma(x) + f(\{1/x\})$, which after iteration (Euclid's algorithm) yields[†]

$$f(x) = \sum_{j=1}^{r(x)} h_\sigma(T^{j-1}(x)) + f(0) \qquad (x \in (0,1]), \tag{3.1}$$

where $r(x) \geqslant 0$ is minimal subject to $T^{r(x)}(x) = 0$.

This expression relates the values of f to Birkhoff sums associated with the Gauss map T. This map is known to have good mixing properties [33, p. 174], by which we expect the sum (3.1) to behave like a sum of independent random variables. Precise limiting behaviour may be obtained through the study of spectral properties of transfer operators associated with T: this is an important theme in smooth dynamics and stationary Markov chains [1, 24, 39, 47, 62, 101, 116]. We refer in particular to [24] and to the introduction of [1] for an extensive historical account and references. Among maps of the interval, the Gauss map has been particularly studied because of its link with the analysis [82, 84, 86, 97] of geodesic flows on the surface $SL(2,\mathbb{Z})\backslash\mathfrak{h}$ (where $\mathfrak{h}$ is the upper-half plane).

A prominent example is given by [24, Theorem 8.1]: suppose $\phi : [0,1] \to \mathbb{R}$ is of bounded variations and not of the shape $c + k - k \circ T$, for some function k of bounded variations and some constant $c \in \mathbb{R}$, and let

$$S_N(\phi, x) := \sum_{j=1}^{N} \phi(T^j(x)).$$

Then for some constants $\mu_\phi \in \mathbb{R}$ and $\sigma_\phi \in \mathbb{R}_+^*$, uniformly for $v \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{S_N(\phi,x) - \mu_\phi N}{\sigma_\phi \sqrt{N}} \leqslant v \right) = \Phi(v) + O\left(\frac{1}{\sqrt{N}}\right), \qquad \Phi(v) := \int_{-\infty}^{v} \frac{\mathrm{e}^{-t^2}\,\mathrm{d}t}{\sqrt{2\pi}}$$

[†] Without the assumption $h_\tau = 0$, a similar formula holds: we have in general $f(x) = \sum_{j=1}^{r(x)} H(T^{j-1}(x)) + f(0)$, where $H(x) := h_\sigma(x) - \sum_{i=1}^{\lfloor 1/x \rfloor} h_\tau(1/x - i)$ for $x \in (0,1]$, and the ensuing analysis still applies.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

as $N \to \infty$, where $x \in (0,1)$ is chosen uniformly according to the Lebesgue measure. The implied constant may depend only on ϕ. This relies on the spectral analysis of perturbations of the Gauss–Kuzmin–Wirsing transfer operator

$$\mathbb{H}[f](x) = \sum_{n\geqslant 1} \frac{1}{(n+x)^2} f\Big(\frac{1}{n+x}\Big).$$

We are interested in the case when x is a rational chosen at random in Ω_Q, and the Birkhoff sum in (3.1) is over the full orbit, its length varying with x. Denote

$$S_\phi(x) := \sum_{j=1}^{r(x)} \phi(T^{j-1}(x)). \tag{3.2}$$

In [112], Vallée has shown that the expectations of $S_\phi(x)$ satisfy

$$\mathbb{E}_Q(S_\phi(x)) = \mu_\phi \log Q + \nu_\phi + O(Q^{-\delta})$$

for functions ϕ which are constant on each interval $(\frac{1}{n+1}, \frac{1}{n})$ $(n \geqslant 1)$ and under some growth condition at 0. Here the number $\delta > 0$ is absolute and the implied constant may depend on ϕ. The numbers μ_ϕ, ν_ϕ depend only on ϕ, and in fact

$$\mu_\phi = \frac{12\log 2}{\pi^2} \int_0^1 \phi(x)\xi(x)\,\mathrm{d}x, \qquad \xi(x) := \frac{1}{(1+x)\log 2}.$$

The quantity μ_ϕ, and the measure $\xi(x)\,\mathrm{d}x$, are essentially the projection of ϕ on the eigenspace of $\mathbb{H}$ associated to its dominant eigenvalue 1. An important point is that this question was studied within the framework of dynamical methods. The argument uses the construction of a suitable generating series involving the quasi-inverse $(\mathrm{Id} - \mathbb{H}_\tau)^{-1}$ of twisted transfer operators

$$\mathbb{H}_\tau[f] = \sum_{n\geqslant 1} \frac{1}{(n+x)^{2+i\tau}} f\Big(\frac{1}{n+x}\Big), \tag{3.3}$$

for arbitrary large $\tau \in \mathbb{R}$ (by contrast with the continuous setting when x was chosen uniformly at random in $[0,1]$, which involves perturbations of a single fixed operator). This construction crucially relies on the fact that the denominator $q(x)$ of x can be detected by means of the Birkhoff sum $\log q(x) = -\sum_{j=1}^{r(x)} \log(T^{j-1}(x))$. Earlier approaches [38, 57], restricted to $\phi = 1$, involved number-theoretic methods based on bounds on algebraic exponential sums.

The approach of [112] was developed by Baladi and Vallée [4], who proved that for ϕ constant on each interval of the shape $(\frac{1}{n+1}, \frac{1}{n})$ and under a logarithmic growth condition $\phi(x) = O(\log(2/x))$, the Laplace transform satisfies the 'quasi-powers expansion'

$$\mathbb{E}_Q(\mathrm{e}^{wS_\phi(x)}) = \exp\big\{U(w)\log Q + V(w) + O(Q^{-\delta})\big\} \tag{3.4}$$

for w in some complex neighbourhood of 0; the holomorphic functions U, V, the number $\delta > 0$ and the implied constant may depend on ϕ. By Hwang's theorem [61], this has a number of consequences in terms of the limiting distribution, and among them, an effective central limit

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

theorem: if ϕ is real, non-identically zero and as above, then for some numbers $\mu_\phi \in \mathbb{R}$, $\sigma_\phi > 0$ and all $v \in \mathbb{R}$, we have

$$\mathbb{P}_Q\left(\frac{S_\phi(x) - \mu_\phi \log Q}{\sigma_\phi (\log Q)^{1/2}} \leqslant v\right) = \int_{-\infty}^{v} \frac{e^{-t^2/2}\,dt}{\sqrt{2\pi}} + O\left(\frac{1}{\sqrt{\log Q}}\right) \qquad (t \in \mathbb{R}).$$

The power-saving error term in (3.4) depends on proving a pole-free strip for the quasi-inverse of the twists (3.3), which can be viewed as a 'quasi-Riemann hypothesis' for the generating function of interest[†]. The approach of [4] extends seminal work of Dolgopyat [40] to the case of an expanding interval map with an infinite partition (see also [5]).

Motivated by our application (3.1), we extend the methods of [4] in two directions. In the first direction, we consider cost functions $\phi : (0,1] \to \mathbb{R}$ which are not necessarily constant by parts. We will require that ϕ can be extended to a Hölder continuous function, with some uniform exponent, on each interval $[\frac{1}{n+1}, \frac{1}{n}]$.

The second direction we wish to consider is cost functions $\phi(x)$ having a possibly divergent first or second moment, say $\phi(x) = x^{-1/2}$, or $\phi(x) = \lfloor 1/x \rfloor$, with the consequence that the limit law will be stable, but not necessarily Gaussian anymore [4, p. 384]. This is a well-known theme in the theory of sums of independent random variables; see [75, Chapter VI], or [42, Chapters VI.1 and XVII.5-6]. The corresponding phenomenon for sums of continued fractions coefficients in the continuous setting (Lebesgue-almost all $x \in (0,1)$) has been investigated by elementary means by Lévy [76] (see also [58]), and later by transfer operator methods [1, 53, 54, 109]. In fact, this falls into the general 'countable Markov–Gibbs' framework of [1], where it is refered to as the 'distributional limit' problem. A large part of later work has focused on non-uniformly hyperbolic maps; we refer to the survey [52] and the references therein.

We investigate the corresponding question in the discrete setting (x at random in Ω_Q). We evaluate with effective, power-saving error terms the characteristic function

$$\mathbb{E}_Q(e^{itS_\phi(x)})$$

for t in a real neighbourhood of 0, under hypotheses which essentially reduce to the boundedness of some positive absolute moment, $\int_0^1 |\phi(x)|^{\alpha_0}\,dx < \infty$ for some $\alpha_0 > 0$.

3.2 | Statement of the main distributional result

For technical reasons, we will work in a more general setting. For a parameter $\kappa \in [0,1]$, a real interval $\mathcal{I}$ and a normed space X, let $H^\kappa(\mathcal{I}, X)$ denotes the set of functions $\mathcal{I} \to X$ such that the Hölder semi-norm

$$\|f\|_{(\kappa)} := \sup_{x,y\in\mathcal{I}, x\neq y} \frac{\|f(x) - f(y)\|}{|x-y|^\kappa} \tag{3.5}$$

is finite.

We let $\mathcal{H} := \{h : [0,1] \to \mathbb{R} \mid \exists n \geqslant 1 \text{ s.t. } h(x) = \frac{1}{n+x}\}$ be the inverse branches of the Gauss map, and $\mathcal{H}^\ell := \{h_1 \circ \ldots \circ h_\ell \mid h_j \in \mathcal{H}\}$.

[†] The same set of idea imply a zero-free strip for the Selberg zeta function of the full modular group [90].

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

We fix an integer $m \geqslant 1$, and m functions $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_m : [0,1] \to \mathbb{R}^d$. We extend this definition by periodicity, letting $\boldsymbol{\phi}_j := \boldsymbol{\phi}_{j \pmod m}$ for all $j \geqslant 1$, and we define a function on $\mathbb{Q} \cap (0,1]$ by letting $S(1) := 0$ and

$$S(x) := \sum_{j=1}^{r(x)} \boldsymbol{\phi}_j(T^{j-1}(x)), \qquad (x \in \mathbb{Q} \cap (0,1))$$

where we recall that

$$r(x) := \min\{j \geqslant 0, T^j(x) = 0\}.$$

For $x \in (0,1)$ with $r(x) \geqslant m$, we define

$$\boldsymbol{\phi}(x) := \sum_{j=1}^{m} \boldsymbol{\phi}_j(T^{j-1}(x)), \tag{3.6}$$

We make the following hypotheses.

(1) (κ_0-Hölder continuity) For each $n \geqslant 1$ and $1 \leqslant j \leqslant m$, the function $\boldsymbol{\phi}_j$ can be extended as a κ_0-Hölder continuous function on the interval $[\frac{1}{n+1}, \frac{1}{n}]$.
(2) (Norm α_0th moment) For each $1 \leqslant j \leqslant m$, we have

$$\sum_{h \in \mathcal{H}} |h'(0)| \left\| \boldsymbol{\phi}_j|_{h(I)} \right\|_\infty^{\alpha_0} < \infty. \tag{3.7}$$

(3) (Hölder λ_0th moment) For each $1 \leqslant j \leqslant m$, we have

$$\sum_{h \in \mathcal{H}} |h'(0)| \left\| \boldsymbol{\phi}_j|_{h(I)} \right\|_{(\kappa_0)}^{\lambda_0} < \infty. \tag{3.8}$$

For all $\boldsymbol{t} \in \mathbb{R}^d$, we denote $\|\boldsymbol{t}\| := \|\boldsymbol{t}\|_\infty$. Finally, let

$$\mathfrak{T}_{\boldsymbol{\phi}}(\boldsymbol{t}) := \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}(x)\rangle} - 1)\xi(x)\,\mathrm{d}x.$$

Theorem 3.1. *Let $\kappa_0, \alpha_0, \lambda_0 > 0$ be given with $\kappa_0, \lambda_0 \leqslant 1$, and $\boldsymbol{\phi} : (0,1] \to \mathbb{R}^d$ satisfying the conditions (3.7)–(3.8). There exists $t_0 > 0$, $\delta > 0$, and functions $U, V : \{\boldsymbol{t} \in \mathbb{R}^d, \|\boldsymbol{t}\| \leqslant t_0\} \to \mathbb{C}$ such that for all $\boldsymbol{t} \in \mathbb{R}^d$ with $\|\boldsymbol{t}\| \leqslant t_0$, we have*

$$\mathbb{E}_Q\left(\mathrm{e}^{i\langle \boldsymbol{t}, S(x)\rangle}\right) = \exp\left\{U(\boldsymbol{t}) \log Q + V(\boldsymbol{t}) + O(Q^{-\delta})\right\}, \tag{3.9}$$

and

$$U(\boldsymbol{t}) = \frac{12 \log 2}{m\pi^2} \mathfrak{T}_{\boldsymbol{\phi}}(\boldsymbol{t}) + O(\|\boldsymbol{t}\|^2 + \|\boldsymbol{t}\|^{2\alpha_0 - \varepsilon}), \tag{3.10}$$

$$V(\boldsymbol{t}) = O(\|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0 - \varepsilon}).$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneitalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

If moreover $\alpha_0 > 1$, then there exists a real $d \times d$ matrix $C_{\boldsymbol{\phi}}$ such that

$$U(\boldsymbol{t}) = \frac{12\log 2}{m\pi^2}\mathfrak{T}_{\boldsymbol{\phi}}(\boldsymbol{t}) + \boldsymbol{t}^T C_{\boldsymbol{\phi}}\boldsymbol{t} + O(\boldsymbol{t}^3 + |\boldsymbol{t}|^{1+\alpha_0-\varepsilon}) \tag{3.11}$$

with $\boldsymbol{t}$ interpreted as a column vector and $\boldsymbol{t}^T$ its transpose. The numbers δ, t_0 and the implied constant depend at most on $\alpha_0, \kappa_0, \lambda_0, \varepsilon$ and on an upper-bound for the left-hand sides of (3.7)–(3.8).

Remark.

- It is important to note that the actual values of κ_0 and λ_0 only affect the statement up to the value of t_0, δ and the implied constants. In particular, if $\boldsymbol{\phi}$ is C^1 on each interval $[\frac{1}{n+1}, \frac{1}{n}]$ and if there exists $C \geqslant 1$ such that $\|\mathrm{d}_x\boldsymbol{\phi}\| = O(x^{-C})$ for $x \in (0,1]$, then (3.8) is satisfied with $\kappa_0 = 1$ and any $\lambda_0 < 1/C$.
- In our applications, we will have $\|\boldsymbol{\phi}(x)\| \asymp \|\boldsymbol{\phi}(y)\|$ for all x, y in $h(\mathcal{I})$, uniformly in $h \in \mathcal{H}^m$. In this situation, (3.7) is equivalent to $\int_0^1 \|\boldsymbol{\phi}(x)\|^{\alpha_0}\,\mathrm{d}x < \infty$.
- In typical applications, the quantity $\mathfrak{T}_{\boldsymbol{\phi}}(\boldsymbol{t})$ can be evaluated by standard methods, and the results relevant for the present paper are worked out in [12].
- The results of [4] are stated in a formalism which includes the Gauss map as a special case. The most crucial assumption, at least as far as one is interested in power-saving error terms, is the 'uniform non-integrability' assumption [4, p. 357] (see [87] for a qualitative result, not using the UNI condition). In the present work, we do not use any more specific properties of the Gauss map; with suitable modifications, the arguments presented here apply to the centred and odd Euclidean algorithms [4, Figure 1] as well.
- A generalization in a different direction of Baladi–Vallée's results, for maps associated to a reduction algorithm in congruence subgroups, has very recently and independently been obtained by Lee and Sun [74][†].

The main feature of Theorem 3.1 we use is the relative weakness of the hypotheses on $\boldsymbol{\phi}$. This is important in our applications, where often little is known on $\boldsymbol{\phi}$ besides the regularity properties, and rough bounds on the Hölder norms. To obtain this uniformity, we systematically use Hölder spaces, not only because of the regularity of $\boldsymbol{\phi}$, but also in order to dampen the oscillations of $\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\rangle}$ (see (4.3)). This is the main reason why arbitrarily small values of λ_0 are admissible for Theorem 3.1.

The shape of the asymptotic estimates (3.9)–(3.11) are characteristic of infinitely divisible distribution [42, Chapter XVII.2]. The informally stated Theorem 1.1 follows, in all the cases we consider, by an evaluation of $\mathfrak{T}_{\boldsymbol{\phi}}(\boldsymbol{t})$ and the Berry–Esseen inequality [110, Theorem II.7.16], [43, Theorem IX.5].

The variety of situations we consider prevents us from making a clear and concise set of hypotheses on h_σ which would make Theorem 1.1 rigorous. Besides working out in details the applications, we restrict to illustrating the case $m = 1$ and $\alpha_0 > 2$ by the following Central Limit Theorem for the Birkhoff sums S_ϕ defined in (3.2), which recovers and extends in particular [4, Theorem 3.(a)] (see also [3, Remark 1.3]). We recall that Φ is the cumulative distribution function of the standard normal law.

[†] Another way to study levels $N > 1$ would be by building an expanding map out of Atkin–Lehner homographies, but our attempts to construct such a map satisfying the UNI condition were not successful.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Corollary 3.2. *Assume that the bound*

$$\sum_{n\geqslant 1}\frac{1}{n^2}\left(\sup_{x\in[\frac{1}{n+1},\frac{1}{n}]}\left|\phi(x)\right|^{\alpha_0}+\sup_{x,y\in[\frac{1}{n+1},\frac{1}{n}]}\frac{\left|\phi(x)-\phi(y)\right|^{\lambda_0}}{|x-y|^{\varkappa_0\lambda_0}}\right)<\infty.$$

holds for some $\alpha_0>2$ and $\lambda_0,\varkappa_0>0$. Suppose that ϕ is not of the form $c\log+f-f\circ T$ for a function $f:[0,1]\to\mathbb{R}$ and $c\in\mathbb{R}$. Then, for some $\sigma>0$ and

$$\mu=\frac{12}{\pi^2}\int_0^1\frac{\phi(x)\,\mathrm{d}x}{1+x},\tag{3.12}$$

we have

$$\mathbb{P}_Q\left(\frac{S_\phi(x)-\mu\log Q}{\sigma\sqrt{\log Q}}\leqslant v\right)=\Phi(v)+O\left(\frac{1}{\sqrt{\log Q}}+\frac{1}{(\log Q)^{\alpha_0/2-1-\varepsilon}}\right)\tag{3.13}$$

uniformly in $v\in\mathbb{R}$.

A variation on the argument shows that the milder hypothesis $\alpha_0=2$ implies the estimate (3.13) with a qualitative error term $o(1)$ as $Q\to\infty$. For any $k\in\mathbb{N}_{>0}$, under the condition $\alpha_0>k$, a variation on the arguments also leads to an estimate with power-saving error term for $\mathbb{E}_Q(S_\phi(x)^k)$.

Notations

For any function $f(s,t)$ of two real or complex variables, and all $k,\ell\geqslant 0$, we let $\partial_{k,\ell}f:=\frac{\partial^{k+\ell}}{\partial s^k\partial t^\ell}f$ whenever this function is defined.

We recall that the semi-norm $\|f\|_{(\varkappa)}$ is defined in (3.5). The Landau symbol $f=O(g)$ means that there is a constant $C\geqslant 0$ for which $|f|\leqslant Cg$ whenever f and g are defined. The notation $f\ll g$ means $f=O(g)$. If the constant depends on a parameter, say ε, this is indicated in subscript, for example, $f=O_\varepsilon(g)$ or $f\ll_\varepsilon g$.

4 | LEMMAS

4.1 | Hölder constants

We compile here several facts we will use on the Hölder norms $\|f\|_{(\varkappa)}$ for $f\in H^\varkappa(\boldsymbol{I},\mathbb{C})$.

(1) For $f,g\in H^\varkappa$, we have

$$\|fg\|_{(\varkappa)}\leqslant\|f\|_{(\varkappa)}\|g\|_\infty+\|f\|_\infty\|g\|_{(\varkappa)}.\tag{4.1}$$

This follows by writing $fg(x)-fg(y)=(f(x)-f(y))g(x)+f(y)(g(x)-g(y))$.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

(2) For $g \in H^1$ and $f \in H^\kappa(g(\mathcal{I}))$, we have $f \circ g \in H^\kappa$ and

$$\|f \circ g\|_{(\kappa)} \leqslant \|g\|_{(1)}^\kappa \left\|f|_{g(\mathcal{I})}\right\|_{(\kappa)}. \tag{4.2}$$

This follows by writing $\frac{|f(g(x))-f(g(y))|}{|x-y|^\kappa} = |\frac{g(x)-g(y)}{x-y}|^\kappa \frac{|f(g(x))-f(g(y))|}{|g(x)-g(y)|^\kappa}$.

(3) For $\lambda \in [\kappa, 1]$ and $f \in H^{\kappa/\lambda}$ real, we have

$$\left\|e^{if}\right\|_{(\kappa)} \leqslant \|f\|_{(\kappa/\lambda)}^\lambda. \tag{4.3}$$

This follows by writing $\frac{|e^{if(x)}-e^{if(y)}|}{|x-y|^\kappa} = \frac{|e^{i(f(x)-f(y))}-1|}{|x-y|^\kappa} \leqslant \frac{|f(x)-f(y)|^\lambda}{|x-y|^\kappa}$.

(4) For $0 < \kappa < \lambda$ and $f \in H^\lambda$, we have

$$\|f\|_{(\kappa)} \leqslant \|f\|_{(0)}^{1-\kappa/\lambda} \|f\|_{(\lambda)}^{\kappa/\lambda}. \tag{4.4}$$

This follows by writing $\frac{|f(x)-f(y)|}{|x-y|^\kappa} = |f(x)-f(y)|^{1-\kappa/\lambda} (\frac{|f(x)-f(y)|}{|x-y|^\lambda})^{\kappa/\lambda}$.

4.2 | Oscillating integrals

We will require the following analogue of van der Corput's lemma. We let

$$\mathcal{I} = [0, 1].$$

Lemma 4.1. *Let $\Delta, \kappa > 0$. Assume that $\Psi : \mathcal{I} \to \mathbb{R}$ is C^1 with $\Psi' \geqslant \Delta$ and that Ψ' is monotonous on $\mathcal{I}$. Let $g \in H^\kappa$. Then*

$$\int_0^1 g(x) e^{i\Psi(x)}\, dx \ll \frac{\|g\|_\infty}{\Delta} + \frac{\|g\|_{(\kappa)}}{\Delta^\kappa}.$$

Proof. The lemma is obtained by combining the methods of the usual van der Corput Lemma ([105, Proposition VIII.2, p. 332]), and the bound on Fourier coefficients of a Hölder continuous function ([106, example 15, p. 92]); we restrict to mentioning the main steps. We change variables and let

$$h(x) = \frac{g \circ \Psi^{-1}(x)}{\Psi' \circ \Psi^{-1}(x)}.$$

Let $R = (\Psi(\mathcal{I}) \setminus (\Psi(\mathcal{I}) - \pi)) \cup (\Psi(\mathcal{I}) \setminus (\Psi(\mathcal{I}) + \pi))$. We have

$$\begin{aligned}\int_0^1 g(x) e^{i\Psi(x)}\, dx &= \int_{\Psi(\mathcal{I})} h(x) e^{ix}\, dx \\ &= O\left(\int_R \frac{\|g\|_\infty\, dx}{\Psi' \circ \Psi^{-1}(x)}\right) - \frac{1}{2}\int_{\Psi(\mathcal{I}) \cap (\Psi(\mathcal{I})-\pi)} (h(x+\pi) - h(x)) e^{ix}\, dx.\end{aligned}$$

14602244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Now, on the one hand,

$$\begin{aligned} h(x+\pi)-h(x) &= \frac{g\circ\Psi^{-1}(x+\pi)-g\circ\Psi^{-1}(x)}{\Psi'\circ\Psi^{-1}(x)} \\ &\quad + (g\circ\Psi^{-1})(x+\pi)\left(\frac{1}{\Psi'\circ\Psi^{-1}(x+\pi)}-\frac{1}{\Psi'\circ\Psi^{-1}(x)}\right) \\ &\ll \frac{\|g\|_{(\kappa)}\|\Psi^{-1}\|_{(1)}^{\kappa}}{\Psi'\circ\Psi^{-1}(x)} + \|g\|_{\infty}\left|\frac{1}{\Psi'\circ\Psi^{-1}(x+\pi)}-\frac{1}{\Psi'\circ\Psi^{-1}(x)}\right| \end{aligned}$$

by (4.2) on the first term, and on the other hand,

$$\int_{\Psi(I)\cap(\Psi(I)-\pi)}\left|\frac{1}{\Psi'\circ\Psi^{-1}(x+\pi)}-\frac{1}{\Psi'\circ\Psi^{-1}(x)}\right|\mathrm{d}x \leqslant \int_R \frac{\mathrm{d}x}{\Psi'\circ\Psi^{-1}(x)} = O(1/\Delta)$$

by monotonicity. We conclude using $\|\Psi^{-1}\|_{(1)} \ll 1/\Delta$. □

5 | PROPERTIES OF THE TRANSFER OPERATOR

In this section and the following ones, all implied constants in the notations $O(\dots)$ and $\ll$ may depend on $\alpha_0, \kappa_0, \lambda_0, m$ and an upper-bound for the values of (3.7)–(3.8). Additional dependencies will be indicated in subscript.

Definition. Let

$$\kappa := \min\left(\tfrac{1}{3}, \tfrac{1}{2}\kappa_0\lambda_0\right), \tag{5.1}$$

where we recall that κ_0 is the Hölder exponent of $\boldsymbol{\phi}$ on each interval $(\frac{1}{n+1}, \frac{1}{n})$. For all $\boldsymbol{t}\in\mathbb{R}^d$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$, define an operator $\mathbb{H}_{s,\boldsymbol{t}}^{(j)}$ acting on $H^{\kappa}([0,1],\mathbb{R})$ by

$$\begin{aligned} \mathbb{H}_{s,\boldsymbol{t}}^{(j)}[f](x) &= \sum_{n\geqslant 1}\frac{\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}_j(1/(n+x))\rangle}}{(n+x)^s} f\left(\frac{1}{n+x}\right) \\ &= \sum_{h\in\mathcal{H}} \mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}_j\circ h(x)\rangle}\left|h'(x)\right|^{s/2}(f\circ h)(x). \end{aligned}$$

When $\boldsymbol{t}=(0,\dots,0)$, this is independent of j, in which case we drop the notation (j). We abbreviate further

$$\mathbb{H}_s := \mathbb{H}_{s,0}, \qquad \mathbb{H} := \mathbb{H}_{2,0}.$$

Define the norm and the semi-norm

$$\|f\|_0 := \left\|f/\xi\right\|_{\infty}, \qquad \|f\|_1 := \left\|f/\xi\right\|_{(\kappa)}.$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Here we recall that $\xi(x) = \frac{1}{\log 2}\frac{1}{1+x}$. We equip $H^\kappa([0,1])$ with the norms

$$\|f\|_{1,\beta} := \|f\|_0 + \beta^{-\kappa}\|f\|_1, \qquad \text{where } \beta > 0.$$

For $\boldsymbol{t} \in \mathbb{R}^d$ and $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, and $0 \leqslant j \leqslant m$, let

$$\Pi_{s,\boldsymbol{t}}^{(j)} := \mathbb{H}_{s,\boldsymbol{t}}^{(j)} \cdots \mathbb{H}_{s,\boldsymbol{t}}^{(1)},$$

with $\Pi_{s,\boldsymbol{t}}^{(0)} = \mathrm{Id}$. In what follows, we will often abbreviate

$$\Pi_{s,\boldsymbol{t}} := \Pi_{s,\boldsymbol{t}}^{(m)},$$

and define

$$\mathfrak{T}(x) := \prod_{0\leqslant j\leqslant m-1} T^j(x), \qquad g_{s,\boldsymbol{t}}(x) := \mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}(x)\rangle}\mathfrak{T}(x)^{s-2}. \tag{5.2}$$

Note that $g_{s,\boldsymbol{t}} \circ h$ belongs to $H^{\kappa_0}([0,1])$ for all $h \in \mathcal{H}^m$. Moreover, since $|T'(x)| = x^{-2}$ whenever T is differentiable at x, for all $h \in \mathcal{H}^m$, we have

$$|h'| = \prod_{j=0}^{m-1} |T' \circ T^{j-1}|^{-1} = \mathfrak{T}(x)^2,$$

and therefore

$$\Pi_{s,\boldsymbol{t}} : f \mapsto \mathbb{H}^m[g_{s,\boldsymbol{t}} f]. \tag{5.3}$$

5.1 | Properties at the central point

By [24] (Section 2.2, Proposition 4.1 and Theorem 4.2), along with the Ruelle–Perron–Frobenius theorem (see [83]), we have that the operator $\mathbb{H}_{2,0}$ acting on $H^\kappa([0,1])$ is quasi-compact. It has 1 as a simple eigenvalue, and no other eigenvalue of modulus $\geqslant 1$. The projection associated with the eigenvalue 1 is given by

$$\mathbb{P}_{2,0}[f](x) = \left(\int_{[0,1]} f \,\mathrm{d}\nu\right)\xi(x),$$

where ν is the Lebesgue measure. In particular, we have

$$\mathbb{H}_{2,0}[\xi] = \xi.$$

Since $\Pi_{2,0}^{(m)} = \mathbb{H}_{2,0}^m$, we obtain the existence of a linear operator $\mathbb{N}_{2,0}$ acting on H^κ, such that

$$\Pi_{2,0}^{(m)} = \mathbb{P}_{2,0} + \mathbb{N}_{2,0}, \tag{5.4}$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and additionally the spectral radius of $\mathbb{N}_{2,0}$ satisfies $\mathrm{srd}(\mathbb{N}_{2,0}) < 1$ and $\mathbb{P}_{2,0}\mathbb{N}_{2,0} = \mathbb{N}_{2,0}\mathbb{P}_{2,0} = 0$. We will use on several occasions that for f continuous by parts,

$$\int_{[0,1]} \mathbb{H}_{2,0}[f]\,\mathrm{d}\nu = \int_{[0,1]} f\,\mathrm{d}\mu. \tag{5.5}$$

5.2 | Dominant spectral properties

In the sequel, we will repeatedly use the following facts. We let $\mathcal{H}^* := \cup_{m\geqslant 0}\mathcal{H}^m$.

- The bounded distortion property [4, equation (3.1)]: for all $h \in \mathcal{H}^*$, we have $|h''| \ll |h'|$ with a uniform constant, in particular, independently of the depth of h. This implies that for all $x \in [0,1]$ and $h \in \mathcal{H}^*$,

$$|h'(x)| \asymp |h'(0)|. \tag{5.6}$$

- For all $q > \frac{1}{2}$, we have

$$\sum_{h\in\mathcal{H}} \|h'\|_\infty^q < \infty.$$

- The 'contracting ratio' property of the inverse branches [4, bound 2.4, Figure 1]: for some $\rho \in [0,1)$, we have the following uniform bound for all $\ell \in \mathbb{N}$ and $h \in \mathcal{H}^\ell$:

$$\|h\|_{(1)} \ll \rho^\ell. \tag{5.7}$$

Lemma 5.1.

- *For* $\mathrm{Re}(s) = \sigma > 1$, *we have*

$$\Pi_{s,t}[f] \leqslant \Pi_{\sigma,0}[|f|].$$

- *For all* $\sigma > 1$, *there exists* $A_\sigma > 0$ *such the map* $\sigma \mapsto A_\sigma$ *is Lipschitz-continuous and decreasing,* $A_2 = 1$, *and we have the bounds on operator norms*

$$\left\|\mathbb{H}_{\sigma,0}\right\|_0 \leqslant A_\sigma, \qquad \left\|\Pi_{\sigma,0}\right\|_0 \leqslant A_\sigma^m. \tag{5.8}$$

  In particular, $A_\sigma \leqslant \mathrm{e}^{O(2-\sigma)}$ *for* $\sigma \in (1,2]$.
- *For some* $\rho < 1$, *all* $k \in \mathbb{N}$ *and all* $f \in H^k$ *with* $f \geqslant 0$, *we have*

$$\left\|\Pi_{2,0}^k[f]\right\|_0 \ll \int_{[0,1]} f\,\mathrm{d}\nu + \rho^k\|f\|_0. \tag{5.9}$$

  The implied constant is absolute.

Proof.

- The first statement is trivial by the triangle inequality.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

– By a direct computation, we have

$$\|\mathbb{H}_\sigma\|_0 \leqslant A_\sigma := \sup_{x\in[0,1]} \frac{1}{\xi(x)} \sum_{n\geqslant 1} \frac{1}{(n+x)^\sigma} \xi\Big(\frac{1}{n+x}\Big).$$

The properties we require of A_σ are readily verified.

– The third item follows from (5.4) with any fixed $\rho \in (\mathrm{srd}(\mathbb{N}_{2,0}), 1)$, by the definition of the spectral radius. □

5.3 | Spectral gap at $t = 0$ and $\tau \neq 0$

Lemma 5.2. *For $\tau \neq 0$, we have $\|\mathbb{H}_{2+i\tau}\|_0 < 1$, and so similarly for $\Pi_{2+i\tau,0}$.*

Proof. This is a well-studied phenomenon; see [93, Proposition 6.1]. Our exact statement for $\mathbb{H}_{2+i\tau}$ acting on a space of holomorphic functions is proved in [111, p. 476] (see also [87, Proposition 3.2]), however, an inspection of the proof shows that it actually holds for $\mathbb{H}_{2+i\tau}$ acting on $C([0,1])$, and therefore also on H^κ. □

5.4 | Perturbation

Lemma 5.3. *For all $\varepsilon > 0$, there exists $\delta, t_0 > 0$ such that for $1 \leqslant j \leqslant m$, the following holds.*

– *For $\sigma \geqslant 2 - \delta$ and $\|\boldsymbol{t}\| \leqslant t_0$, we have*

$$\big\|\Pi_{s,\boldsymbol{t}}^{(j)} - \Pi_{s,0}^{(j)}\big\|_0 \ll_\varepsilon \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon}. \tag{5.10}$$

– *For $2 - \delta < \sigma \leqslant 3$ and all $\tau \in \mathbb{R}$, we have*

$$\big\|\Pi_{\sigma+i\tau,0}^{(j)} - \Pi_{2+i\tau,0}^{(j)}\big\|_0 \ll |\sigma - 2|. \tag{5.11}$$

– *For $\tau_1, \tau_2 \in \mathbb{R}$, we have*

$$\big\|\Pi_{2+i\tau_1,0}^{(j)} - \Pi_{2+i\tau_2,0}^{(j)}\big\|_0 \ll |\tau_1 - \tau_2|. \tag{5.12}$$

Proof. We assume $j = m$, the general case being essentially identical.

– We have

$$\begin{aligned}\|\Pi_{s,\boldsymbol{t}} - \Pi_{s,0}\|_0 &= \sup_{\|f\|_0=1} \|\mathbb{H}^m[(g_{s,\boldsymbol{t}} - g_{s,0})f]\|_0 \\ &\ll \sum_{h\in\mathcal{H}^m} |h'(0)| \|(g_{s,\boldsymbol{t}} - g_{s,0}) \circ h\|_\infty.\end{aligned}$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

However, for all $x \in (0,1]$, by the bounded distortion property (5.6), we have

$$\left|g_{s,\boldsymbol{t}}(h(x)) - g_{s,0}(h(x))\right| \ll \left|h'(0)\right|^{\sigma/2-1}\left|e^{i\langle \boldsymbol{t},\boldsymbol{\phi}(x)\rangle} - 1\right| \ll \left|h'(0)\right|^{\sigma/2-1}(\|\boldsymbol{t}\|\|\boldsymbol{\phi}(x)\|)^{\alpha}$$

with $\alpha = \min(1, \alpha_0 - \varepsilon)$. By Hölder's inequality, we deduce

$$\begin{aligned}\left\|\Pi_{s,\boldsymbol{t}} - \Pi_{s,0}\right\|_0 &\ll \|\boldsymbol{t}\|^{\alpha} \sum_{h\in\mathcal{H}^m} \left|h'(0)\right|^{\sigma/2}\|\boldsymbol{\phi}\circ h\|_{\infty}^{\alpha} \\ &\ll \|\boldsymbol{t}\|^{\alpha}\left(\sum_{h\in\mathcal{H}^m}\left|h'(0)\right|\left\|\boldsymbol{\phi}|_{h(I)}\right\|_{\infty}^{\alpha_0}\right)^{\alpha/\alpha_0}\left(\sum_{h\in\mathcal{H}^m}\left|h'(0)\right|^{q}\right)^{1-\alpha/\alpha_0} \qquad (5.13)\end{aligned}$$

with $q = \frac{\sigma\alpha_0 - 2\alpha}{2(\alpha_0-\alpha)}$. Picking $\delta = \frac{1}{2}(1 - \frac{\alpha}{\alpha_0}) = O(\varepsilon)$ ensures that $q > \frac{1}{2}$, and with our hypothesis (3.7) and the bounded distortion property (5.6), we obtain

$$\begin{aligned}\sum_{h\in\mathcal{H}^m}\left|h'(0)\right|\left\|\boldsymbol{\phi}|_{h(I)}\right\|_{\infty}^{\alpha_0} &\ll \sum_{j=1}^{m}\sum_{h=h_1\circ\ldots\circ h_m\in\mathcal{H}^m}\left|h_1'(0)\cdots h_m'(0)\right|\left\|\boldsymbol{\phi}_j\circ T^{j-1}|_{h(I)}\right\|_{\infty}^{\alpha_0} \\ &\ll \sum_{j=1}^{m}\sum_{h_j\in\mathcal{H}}\left|h_j'(0)\right|\left\|\boldsymbol{\phi}_j|_{h_j(I)}\right\|_{\infty}^{\alpha_0} < \infty. \qquad (5.14)\end{aligned}$$

Therefore, both sums in (5.13) are bounded in terms of ε and $\boldsymbol{\phi}$ only, and we deduce the claimed bound $\|\Pi_{s,\boldsymbol{t}} - \Pi_{s,0}\| \ll_{\varepsilon} \|\boldsymbol{t}\|^{\alpha}$.

– Proceeding as above, we find

$$\begin{aligned}\left\|\Pi_{\sigma+i\tau,0} - \Pi_{2+i\tau,0}\right\|_0 &\ll \sum_{h\in\mathcal{H}^m}\left|h'(0)\right|\left\|(\mathfrak{T}^{\sigma-2} - 1)\circ h\right\|_{\infty} \\ &\ll |\sigma-2|\sum_{h\in\mathcal{H}^m}(1 + |\log|h'(0)||)|h'(0)|^{\max(\sigma/2,1)}.\end{aligned}$$

For any $\sigma > 1$, the last sum is finite, so that our statement follows for any fixed $\delta \in (0,1)$.

– Once again proceeding again as above, we have

$$\left\|\Pi_{2+i\tau_1,0} - \Pi_{2+i\tau_2,0}\right\|_0 \ll \sum_{h\in\mathcal{H}^m}\left|h'(0)\right|\left\|(\mathfrak{T}^{i(\tau_1-\tau_2)} - 1)\circ h\right\|_{\infty}.$$

Letting $\tau = \tau_1 - \tau_2$, we insert the inequality $\|(\mathfrak{T}^{i\tau} - 1)\circ h\|_{\infty} \ll |\tau|(1 + |\log|h'(0)||)$. The resulting sum over h being absolutely bounded, we deduce $\|\Pi_{2+i\tau_1,0} - \Pi_{2+i\tau_2,0}\|_0 \ll |\tau|$ as required. □

5.5 | First estimate on $\|\Pi_{s,t}\|_1$

The following is a weak form of [4, Lemma 2] (which is referred to, there, as a Lasota–Yorke type inequality). We recall that the Hölder exponent $\varkappa$ was defined in (5.1).

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by CochraneItalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Lemma 5.4. *For all $\delta \in (0,\kappa)$, there exists $\rho \in [0,1)$ such that for $\sigma \geqslant 2-\delta$, $\boldsymbol{t} \in \mathbb{R}^d$ with $\|\boldsymbol{t}\| \leqslant 1$, and $\tau \in \mathbb{R}$, we have*

$$\big\|\Pi_{s,\boldsymbol{t}}[f]\big\|_1 \leqslant O(1+|s|^\kappa)\|f\|_0 + \rho\|f\|_1,$$

Proof. Let $f \in H^\kappa$. We write

$$\tfrac{1}{\xi}\Pi_{s,\boldsymbol{t}}[f] = \sum_{h\in\mathcal{H}^m} \tfrac{1}{\xi}(\xi\circ h)\big|h'\big|^{s/2}\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\circ h\rangle}\left(\tfrac{f}{\xi}\circ h\right).$$

Splitting as a sum of differences, we obtain

$$\Big\|\tfrac{1}{\xi}\Pi_{s,\boldsymbol{t}}[f]\Big\|_{(\kappa)} \leqslant \|f\|_0 \sum_{h\in\mathcal{H}^m}\Big\|\tfrac{1}{\xi}(\xi\circ h)\big|h'\big|^{s/2}\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\circ h\rangle}\Big\|_{(\kappa)} + \rho_\sigma\|f\|_1,$$

where

$$\rho_\sigma := \Big\|\sum_{h\in\mathcal{H}^m}\tfrac{1}{\xi}(\xi\circ h)\big|h'\big|^{\sigma/2}w_h\Big\|_\infty, \qquad w_h(x) = \sup_{0\leqslant y\leqslant 1}\left|\frac{h(x)-h(y)}{x-y}\right|^\kappa = \big|h'(x)h'(0)\big|^{\kappa/2}.$$

This last equality follows from the fact that each $h \in \mathcal{H}^* = \cup_{m\geqslant 0}\mathcal{H}^m$ is a homography associated with an element of $GL_2(\mathbb{Z})$ with non-negative entries, $h(x) = \frac{ax+b}{cx+d}$, and so the supremum above is $\sup_y |(cx+d)(cy+d)|^{-1} = |h'(0)h'(x)|^{-1/2}$. Since $|h'(0)| \leqslant 1$ by the chain rule, we deduce

$$\sum_{h\in\mathcal{H}^m}\tfrac{1}{\xi}(\xi\circ h)\big|h'\big|^{\sigma/2}w_h \leqslant \sum_{h\in\mathcal{H}^m}\tfrac{1}{\xi}(\xi\circ h)\big|h'\big|\big|h'\big|^{(\sigma-2+\kappa)/2}.$$

Note that $\sigma-2+\kappa \geqslant \kappa-\delta > 0$ by hypothesis, and we have $|h'| \leqslant 1$ by the chain rule. Moreover, for any value of m, we may find at least one $h \in \mathcal{H}^m$ with $\|h'\|_\infty < 1$, for example, by composing repeatedly $t \mapsto \frac{1}{2+t}$. Since $\frac{1}{\xi}(\xi\circ h)|h'| > 0$, we deduce

$$\rho_{\kappa-\delta} < \Big\|\sum_{h\in\mathcal{H}^m}\tfrac{1}{\xi}(\xi\circ h)\big|h'\big|\Big\|_\infty = 1.$$

Next, by using the rules (4.2), (4.3), (4.4), the bounded distortion property $|h''| \ll |h'|$, and simple computations, we obtain successively

$$\Big\|\tfrac{1}{\xi}\Big\|_{(\kappa)} \ll 1, \qquad \Big\|\big|h'\big|^{s/2}\Big\|_{(\kappa)} \ll |s|^\kappa\big\|h'\big\|_\infty^{\sigma/2},$$

$$\big\|\xi\circ h\big\|_{(\kappa)} \leqslant \big\|h'\big\|_\infty^\kappa, \qquad \Big\|\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\circ h\rangle}\Big\|_{(\kappa)} \ll \big\|h'\big\|_\infty^\kappa\|\boldsymbol{t}\|^{\kappa/\kappa_0}\|\boldsymbol{\phi}\|_{(\kappa_0)}^{\kappa/\kappa_0}.$$

In the last line, we used the definition (5.1). Grouping these bounds using (4.1), we deduce

$$\begin{aligned}\sum_{h\in\mathcal{H}^m}\Big\|\tfrac{1}{\xi}(\xi\circ h)\big|h'\big|^{s/2}\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\circ h\rangle}\Big\|_{(\kappa)} &\ll 1+|s|^\kappa + \sum_{h\in\mathcal{H}^m}\big\|h'\big\|_\infty^{\sigma/2+\kappa}\|\boldsymbol{\phi}\|_{(\kappa_0)}^{\kappa/\kappa_0}\\ &\ll 1+|s|^\kappa\end{aligned}$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

since $\sigma/2+\kappa \geqslant 1+\kappa/2 \geqslant 1$, $\|\boldsymbol{\phi}\|_{(\kappa_0)}^{\kappa/\kappa_0} \leqslant 1+\|\boldsymbol{\phi}\|_{(\kappa_0)}^{\lambda_0}$ (by the definition (5.1)), and by our hypothesis (3.8). □

6 | MEROMORPHIC CONTINUATION

Following [112], define the generating series

$$\mathfrak{S}(\boldsymbol{t},s) := \sum_{x\in\mathbb{Q}\cap(0,1]} q(x)^{-s}\exp(i\langle \boldsymbol{t}, S_{\boldsymbol{\phi}}(x)\rangle),$$

where $q(x)$ is the reduced denominator of x.

Lemma 6.1. *For* $\mathrm{Re}(s)>2$ *and* $\boldsymbol{t}\in\mathbb{R}^d$, *we have*

$$\mathfrak{S}(\boldsymbol{t},s) = (\Pi_{s,\boldsymbol{t}}^{(0)}+\Pi_{s,\boldsymbol{t}}^{(1)}+\cdots+\Pi_{s,\boldsymbol{t}}^{(m-1)})(\mathrm{Id}-\Pi_{s,\boldsymbol{t}})^{-1}[\mathbf{1}](1).$$

Proof. This is a straightforward extension of the computations in [4, equation (2.17)]. □

The aim of this section is to show the meromorphic continuation of $\mathfrak{S}(s,t)$ to a half-plane $\mathrm{Re}(s)\geqslant 2-\delta$. This will then be used in conjunction with Mellin transformation to prodive an estimate with power-saving for $\mathbb{E}_Q(\mathrm{e}^{i\langle \boldsymbol{t},S(x)\rangle})$. We use different arguments according to the size of $\mathrm{Im}(s)$, which is referred to as the height in what follows.

6.1 | Small height

The following lemma deals with τ in a neighbourhood of 0. In this case, we use the spectral expansion of $\Pi_{s,\boldsymbol{t}}$.

Lemma 6.2. *There exists* $\delta,\tau_0,t_0>0$, *such that for all* $\sigma,\tau,t\in\mathbb{R}$ *with* $|\sigma-2|\leqslant\delta$, $|\tau|\leqslant\tau_0$ *and* $\|\boldsymbol{t}\|\leqslant t_0$, *the operator* $\Pi_{s,\boldsymbol{t}}$ *acting on* $(H^\kappa,\|\cdot\|_{1,1})$ *is quasi-compact, and for some* $\lambda(s,\boldsymbol{t})\in\mathbb{C}$, *we have*

$$\Pi_{s,\boldsymbol{t}} = \lambda(s,\boldsymbol{t})\mathbb{P}_{s,\boldsymbol{t}}+\mathbb{N}_{s,\boldsymbol{t}}$$

where $\mathbb{P}_{s,\boldsymbol{t}}$ *is of rank 1,* $\mathbb{P}_{s,\boldsymbol{t}}\mathbb{N}_{s,\boldsymbol{t}}=\mathbb{N}_{s,\boldsymbol{t}}\mathbb{P}_{s,\boldsymbol{t}}=0$, $\mathbb{P}_{s,\boldsymbol{t}}^2=\mathbb{P}_{s,\boldsymbol{t}}$, *and* $\mathrm{srd}(\mathbb{N}_{s,\boldsymbol{t}})<1-\delta$. *Moreover, for each such fixed* $\boldsymbol{t}$, *the operators* $\Pi_{s,\boldsymbol{t}}$, $\mathbb{P}_{s,\boldsymbol{t}}$, $\mathbb{N}_{s,\boldsymbol{t}}$ *and the eigenvalue* $\lambda(s,\boldsymbol{t})$ *depend analytically on* s.

Proof. This is a direct application of for example, [71, Theorem 2.3]; see also [69, chapter IV.3], and [4, p. 342]. □

6.2 | Moderate height

The following lemma is concerned with τ of bounded size and away from 0. The main tool is again perturbation theory.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Lemma 6.3. *For all $\tau_0, \tau_1 > 0$ with $\tau_0 < \tau_1$, there exists $\delta, t_0 > 0$ such that for all $t, \sigma, \tau \in \mathbb{R}$ with $\|\boldsymbol{t}\| \leqslant t_0$, $\sigma \geqslant 2 - \delta$ and $\tau_0 \leqslant |\tau| \leqslant \tau_1$, we have*

$$\left\|\Pi_{s,\boldsymbol{t}}\right\|_0 \leqslant 1 - \delta.$$

Proof. By (5.12) and the triangle inequality, the map $\tau \mapsto \|\Pi_{2+i\tau,0}\|_0$ is continuous. By Lemma 5.2, we deduce that for some number $\eta > 0$, depending on τ_0 and τ_1, we have the bound $\|\Pi_{2+i\tau,0}\|_0 \leqslant 1 - \eta$. By the perturbation bounds (5.10) and (5.11), we may pick δ, t_0 small enough so that $\|\Pi_{s,\boldsymbol{t}} - \Pi_{2+i\tau,0}\|_0 \leqslant \eta/2$, and our claim follows. □

6.3 | Large height

The following lemma deals with the case of large enough $|\tau|$.

Lemma 6.4. *For some constants $\delta, \tau_1, C > 0$, whenever $\sigma \geqslant 2 - \delta$, $\boldsymbol{t} \in \mathbb{R}^d$ with $\|\boldsymbol{t}\| \leqslant 1$, and $|\tau| \geqslant \tau_1$, the operator $\Pi_{s,\boldsymbol{t}}$ acting on H^κ has spectral radius $\operatorname{srd}(\Pi_{s,\boldsymbol{t}}) < 1$, and*

$$\sum_{j \geqslant 0} \left\|\Pi_{s,\boldsymbol{t}}^j\right\|_{1,\tau} \ll |\tau|^{C|\sigma-2|} \log|\tau|.$$

For unbounded values of τ, perturbation theory is not effective, instead we adapt the arguments of Dolgopyat [40] and Baladi–Vallée [4], which exploit the cancellation due to the varying argument of $|h'|^{i\tau}$. Compared with Baladi–Vallée's setup, we make two modifications: we work with Hölder-continuous functions, rather than C^1, and the cost function is not assumed to be constant on each interval of the partition.

6.3.1 | Sums over branches

We will require two estimates involving sums over inverse branches on T. Define, as in [4, equation (3.10)],

$$\Delta(h_1, h_2) := \inf_{x \in [0,1]} \left| \frac{h_1''(x)}{h_1'(x)} - \frac{h_2''(x)}{h_2'(x)} \right|,$$

Note that by the bounded distortion property, there exists $\Delta_+ \geqslant 1$ such that

$$\Delta(h_1, h_2) \leqslant \Delta_+ \tag{6.1}$$

for all $h_1, h_2 \in \mathcal{H}^*$. The following property is the statement that condition UNI.(a) of Baladi–Vallée [4] holds for the Gauss map.

Lemma 6.5. *For some absolute constant $\rho \in [0, 1)$, we have uniformly in $n \in \mathbb{N}$, $h_1 \in \mathcal{H}^n$ and $u \in [0, \Delta_+]$ that*

$$S(u) := \sum_{\substack{h_2 \in \mathcal{H}^n \\ \Delta(h_1,h_2) \leqslant u}} \left|h_2'(0)\right| \ll \rho^n + u.$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Proof. See [4, Lemmas 1 and 6]; the main point is the construction of a dual dynamical system ([4, Section 3.4]) which encodes naturally the quantity $\Delta(h_1, h_2)$, and satisfies the dominant spectral bound (5.9). The dual map of the Gauss map is in fact the Gauss map. □

Lemma 6.6. *Under the assumption (3.8), uniformly for all $n \in \mathbb{N}_{>0}, 0 \leqslant j \leqslant n-m$, we have*

$$\sum_{h\in\mathcal{H}^n} |h'(0)| \left\| \boldsymbol{\phi}|_{T^j} \circ h(I) \right\|_{(\kappa_0)}^{\lambda_0} \ll 1.$$

We recall that $I = [0,1]$.

Proof. We decompose $h = h_1 \circ h_2 \circ h_3$, where $h_1 \in \mathcal{H}^j$, $h_2 \in \mathcal{H}^m$ and $h_3 \in \mathcal{H}^{n-j-m}$. We have

$$|h'(0)| \ll \left|h_1'(0)h_2'(0)h_3'(0)\right|, \qquad \left\| \boldsymbol{\phi}|_{T^j} \circ h(I) \right\|_{(\kappa_0)} \leqslant \left\| \boldsymbol{\phi}|_{h_2(I)} \right\|_{(\kappa_0)},$$

so that

$$\sum_{h\in\mathcal{H}^n} |h'(0)| \left\| \boldsymbol{\phi}|_{T^j} \circ h(I) \right\|_{(\kappa_0)}^{\lambda_0} \ll \left(\sum_{h_1\in\mathcal{H}^j} |h_1'(0)| \right) \left(\sum_{h_2\in\mathcal{H}^m} |h_2'(0)| \left\| \boldsymbol{\phi}_{h_2(I)} \right\|_{(\kappa_0)}^{\lambda_0} \right) \left(\sum_{h_3\in\mathcal{H}^{n-j-m}} \left|h_3'(0)\right| \right).$$

The sums over h_1 and h_3 are uniformly bounded by (5.8). The sum over h_2 is finite by our hypothesis (3.8) and the triangle inequality (cf. 5.14). □

6.3.2 | Bound on the L^2 norm

We recall that A_σ is an upper-bound for the norm $\|H_{\sigma,0}\|_0$ provided in (5.8).

Lemma 6.7. *For some $\delta, t_0 > 0$ and $\rho \in [0,1)$, whenever $|\sigma - 2| \leqslant \delta$, $|\tau| \geqslant 1$, $\|\boldsymbol{t}\| \leqslant t_0$ and $\ell \in \mathbb{N}$, we have*

$$\left(\int_{[0,1]} \left| \Pi_{s,\boldsymbol{t}}^{\ell}[f] \right|^2 \mathrm{d}\nu \right)^{1/2} \ll A_{2\sigma-2}^{m\ell/2} \left(\left(|\tau|^{-\kappa/2} + \rho^{m\ell/4} \right) \|f\|_0 + \rho^{\kappa m\ell/2} |\tau|^{-\kappa/2} \|f\|_1 \right).$$

Proof. Changing f to $\overline{f}$, $\boldsymbol{t}$ to $-\boldsymbol{t}$ and taking conjugates if necessary, we may assume that $\tau \geqslant 0$. Define, for all $\ell \in \mathbb{N}_{>0}$, $\boldsymbol{\psi}_\ell := \sum_{0\leqslant j<\ell} \boldsymbol{\phi} \circ T^{mj}$, so that

$$\Pi_{s,\boldsymbol{t}}^{\ell}[f] = \sum_{h\in\mathcal{H}^{m\ell}} \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\psi}_\ell \circ h\rangle} |h'|^{s/2} (f \circ h).$$

We note that for all $h \in \mathcal{H}^{m\ell}$, by (4.2) and (4.3), we have

$$\begin{aligned} \left\| \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\psi}_\ell \circ h\rangle} \right\|_{(\kappa)} &\leqslant \sum_{0\leqslant j<\ell} \left\| T^{mj} \circ h \right\|_{(1)}^{\kappa} \left\| \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle}|_{T^{mj}} \circ h(I) \right\|_{(\kappa)} \\ &\ll \sum_{0\leqslant j<\ell} \rho^{m(\ell-j)\kappa} \left\| \boldsymbol{\phi}|_{T^{mj}} \circ h(I) \right\|_{(\kappa_0)}^{\kappa/\kappa_0}. \end{aligned} \tag{6.2}$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

For $h_1, h_2 \in \mathcal{H}^{m\ell}$, let

$$g_{h_1,h_2} := \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\psi}_\ell \circ h_1 - \boldsymbol{\psi}_\ell \circ h_2\rangle} \big|h_1' h_2'\big|^{\sigma/2} (f \circ h_1)\overline{(f \circ h_2)}.$$

This defines a function in H^κ. Expanding the square, we find

$$\int_{[0,1]} \left|\Pi_{s,\boldsymbol{t}}^{\ell}[f]\right|^2 \mathrm{d}\nu = \sum_{h_1,h_2 \in \mathcal{H}^{m\ell}} I(h_1,h_2), \qquad I(h_1,h_2) := \int_0^1 g_{h_1,h_2}(x) \left|\frac{h_1'(x)}{h_2'(x)}\right|^{i\tau/2} \mathrm{d}x.$$

We have, for all $h_1, h_2 \in \mathcal{H}^{m\ell}$, the trivial bound

$$|I(h_1,h_2)| \ll \left\|g_{h_1,h_2}\right\|_\infty. \tag{6.3}$$

On the other hand, for all $h_1, h_2 \in \mathcal{H}^{m\ell}$ satisfying $\Delta(h_1,h_2) > 0$, we have from Lemma 4.1 the bound

$$|I(h_1,h_2)| \ll \frac{\left\|g_{h_1,h_2}\right\|_\infty}{|\tau|\Delta(h_1,h_2)} + \frac{\left\|g_{h_1,h_2}\right\|_{(\kappa)}}{(|\tau|\Delta(h_1,h_2))^\kappa}, \tag{6.4}$$

The norms are bounded, using (5.7), (4.1), (4.2) and (6.2), by

$$\|g_{h_1,h_2}\|_\infty \ll \|f\|_\infty^2 \big|h_1'(0)h_2'(0)\big|^{\sigma/2}, \tag{6.5}$$

$$\|g_{h_1,h_2}\|_{(\kappa)} \ll \|f\|_\infty \big|h_1'(0)h_2'(0)\big|^{\sigma/2} \left\{ \left(1 + \sum_{h \in \{h_1,h_2\}} \sum_{0 \leqslant j < \ell} \rho^{m(\ell-j)\kappa} \left\|\boldsymbol{\phi}|_{T^{mj} \circ h(I)}\right\|_{(\kappa_0)}^{\kappa/\kappa_0}\right) \|f\|_\infty + \rho^{\kappa m\ell} \|f\|_{(\kappa)} \right\}.$$

We write $\|f\|_{(\kappa)}\|f\|_\infty \ll \|f\|_{(\kappa)}^2 + \|f\|_\infty^2 \ll \|f/\xi\|_{(\kappa)}^2 + \|f/\xi\|_\infty^2$, which implies the variant

$$\|g_{h_1,h_2}\|_{(\kappa)} \ll \big|h_1'(0)h_2'(0)\big|^{\sigma/2} \left\{ \left(1 + \sum_{h \in \{h_1,h_2\}} \sum_{0 \leqslant j < \ell} \rho^{m(\ell-j)\kappa} \left\|\boldsymbol{\phi}|_{T^{mj} \circ h(I)}\right\|_{(\kappa_0)}^{\kappa/\kappa_0}\right) \|f\|_0^2 + \rho^{\kappa m\ell} \|f\|_1^2 \right\}. \tag{6.6}$$

Next, for all $u \in [0, \Delta_+]$ (where we recall (6.1)), we have uniformly

$$K(u) := \max_{0 \leqslant j < \ell} \sum_{\substack{h_1,h_2 \in \mathcal{H}^{m\ell} \\ \Delta(h_1,h_2) \leqslant u}} \big|h_1'(0)h_2'(0)\big|^{\sigma/2} \left(1 + \left\|\boldsymbol{\phi}|_{T^{mj} \circ h_1(I)}\right\|_{(\kappa_0)}^{\lambda_0/2}\right)$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$\ll \max_{0\leqslant j<\ell}\left(\sum_{h_1,h_2\in\mathcal{H}^{m\ell}}|h_1'(0)|(1+\left\|\boldsymbol{\phi}|_{T^{mj}\circ h_1(I)}\right\|_{(\kappa_0)}^{\lambda_0})|h_2'(0)|^{\sigma-1}\right)^{1/2}$$
$$\times\left(\sum_{\substack{h_1,h_2\in\mathcal{H}^{m\ell}\\ \Delta(h_1,h_2)\leqslant u}}|h_1'(0)|^{\sigma-1}|h_2'(0)|\right)^{1/2}$$
$$\ll A_{2\sigma-2}^{m\ell}(\rho^{m\ell/2}+u^{1/2})$$

by Lemmas 6.5 and 6.6. Let $\eta\in(0,1]$ be a parameter. We insert the bounds (6.5) and (6.6) in (6.3), (6.4), and we sum over (h_1,h_2). When $\Delta(h_1,h_2)\leqslant\eta$, we use the trivial bound (6.3), otherwise we use (6.4). Using our bond on $K(u)$ above, the symmetry $h_1\leftrightarrow h_2$, the fact that $\kappa/\kappa_0\leqslant\lambda_0/2$, and partial summation, we find

$$A_{2\sigma-2}^{-m\ell}\sum_{h_1,h_2\in\mathcal{H}^{m\ell}}I(h_1,h_2)\ll\|f\|_0^2\left(K(\eta)+\frac{K(\Delta_+)}{\tau^\kappa}+\int_\eta^{\Delta_+}\left(\frac{1}{\tau u}+\frac{\kappa}{(\tau u)^\kappa}\right)\frac{K(u)\,\mathrm{d}u}{u}\right)$$
$$+\frac{\rho^{\kappa m\ell}\|f\|_1^2}{\tau^\kappa}\left(K(\Delta_+)+\kappa\int_\eta^{\Delta_+}\frac{K(u)\,\mathrm{d}u}{u^{\kappa+1}}\right)$$
$$\ll\|f\|_0^2\left((\rho^{m\ell/2}+\eta^{1/2})\left(1+\frac{1}{\tau\eta}+\frac{1}{(\tau\eta)^\kappa}\right)+\frac{1}{\tau^\kappa}\right)$$
$$+\|f\|_1^2\rho^{\kappa m\ell}\left(\frac{1}{\tau^\kappa}+\frac{\rho^{m\ell/2}+\eta^{1/2}}{(\tau\eta)^\kappa}\right).$$

Choosing $\eta=1/\tau$, we obtain

$$\left(\int_{[0,1]}\left|\Pi_{s,\boldsymbol{t}}^{\ell}[f]\right|^2\mathrm{d}\nu\right)^{1/2}\ll A_{2\sigma-2}^{m\ell}\left(\left(\tau^{-\kappa/2}+\rho^{m\ell/4}\right)\|f\|_0+\tau^{-\kappa/2}\rho^{\kappa m\ell/2}\|f\|_1\right)$$

as claimed. □

6.3.3 | Bound on the L^∞ norm

Next, we transfer the L^2 bound relative to the invariant measure into an L^∞ bound, following ideas of Dolgopyat [40] adapted to this context by Baladi and Vallée [4, section 3.3].

Lemma 6.8. *For some $\rho\in[0,1)$ and $c_0,\delta,\tau_0>0$, depending on η and κ at most, whenever*

$$\sigma\geqslant 2-\delta,\qquad \tau\geqslant\tau_0,\qquad \|\boldsymbol{t}\|\leqslant 1,$$

then letting $n=\lfloor c_0\log\tau\rfloor$, we have

$$\left\|\Pi_{s,\boldsymbol{t}}^{n}[f]\right\|_0\leqslant\rho^n\|f\|_{1,\tau}.$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Proof. By using the Cauchy–Shwarz inequality, as in [4, Lemma 1], for all $x \in [0,1]$ and $f \in H^\kappa$, we have

$$\frac{\left|\Pi_{s,\boldsymbol{t}}^k[f](x)\right|}{\xi(x)} \leqslant \left(\sum_{h\in\mathcal{H}^{mk}} |h'(x)|^{\sigma-1}\right)^{1/2}\left(\sum_{h\in\mathcal{H}^{mk}} |h'(x)|(|f|^2\circ h)(x)\right)^{1/2}$$
$$\ll A_{2\sigma-2}^{mk/2}\left(\int_0^1 |f|^2\,\mathrm{d}\nu + \rho_1^{mk}\|f\|_0^2\right)^{1/2}$$

for some $\rho_1 \in [0,1)$ independent of k. We use this with f replaced by $\Pi_{s,\boldsymbol{t}}^\ell[f]$, with k, ℓ being any choice with $k+\ell = n$ and $\ell = k + O(1)$. By Lemma 6.7 and the bound (5.8), we deduce that for all small enough $\delta \geqslant 0$, if $\sigma \geqslant 2-\delta$, then

$$\left\|\Pi_{s,\boldsymbol{t}}^n[f]\right\|_0 \ll \mathrm{e}^{O(\delta n)}\left((\tau^{-\kappa/2} + \rho_1^{mn/8})\|f\|_0 + \rho_1^{mn\kappa/4}\tau^{-\kappa/2}\|f\|_1\right).$$

By choosing $n = c\log|\tau| + O(1)$ with $c = 4(m|\log\rho_1|)^{-1}$, and then τ_0 large enough and δ small enough in terms of κ, we may ensure that

$$\left\|\Pi_{s,\boldsymbol{t}}^n[f]\right\|_0 \leqslant \rho^n\|f\|_{1,\tau},$$

with $\rho = \rho_1^{m\kappa/10} < 1$ and as claimed. □

6.3.4 | Proof of Lemma 6.4

Iterating the bound of Lemma 5.4, and using (5.8), we have for δ small enough and all $n \geqslant 0$, the bound

$$\left\|\Pi_{s,\boldsymbol{t}}^n[f]\right\|_1 \ll \mathrm{e}^{O(\delta n)}|\tau|^\kappa\|f\|_0 + \rho^n\|f\|_1,$$

for some $\rho \in [0,1)$. We replace f by $\Pi_{s,\boldsymbol{t}}^n[f]$ and use Lemmas 6.8 and 5.4. We find that for some constants $\tau_0 \geqslant 0$, $c_0 > 0$ and $\rho \in [0,1)$, if δ is small enough and $n = \lfloor c_0\log\tau\rfloor$, then

$$\left\|\Pi_{s,\boldsymbol{t}}^{2n}f\right\|_{1,\tau} \ll (\rho^n + \mathrm{e}^{O(\delta n)})\left\|\Pi_{s,\boldsymbol{t}}^n f\right\|_0 + |\tau|^{-\kappa}\rho^n\left\|\Pi_{s,\boldsymbol{t}}^n f\right\|_1$$
$$\ll (\rho^{2n} + \mathrm{e}^{O(\delta n)}\rho^n)\|f\|_0 + |\tau|^{-\kappa}(\rho^{2n} + \mathrm{e}^{O(\delta n)}\rho^n)\|f\|_1.$$

At the cost of choosing c_0 large enough and δ small enough in terms of the implied constants, we obtain

$$\left\|\Pi_{s,\boldsymbol{t}}^{2n}f\right\|_{1,\tau} \leqslant \rho^{n/2}\|f\|_{1,\tau}.$$

By iterating, this bounds also holds for $n = k\lfloor c_0\log\tau\rfloor$, $k \in \mathbb{N}$, from which we deduce by Gelfand's inequality that $\mathrm{srd}(\Pi_{s,\boldsymbol{t}}) \leqslant \rho^{1/4}$, and $\|(\mathrm{Id} - \Pi_{s,\boldsymbol{t}}^{2n})^{-1}\|_{1,\tau} \ll 1$. Finally, from the bounds

$$\left\|(\mathrm{Id} - \Pi_{s,\boldsymbol{t}})^{-1}\right\|_{1,\tau} \leqslant \left\|(\mathrm{Id} - \Pi_{s,\boldsymbol{t}}^{2n})^{-1}\right\|_{1,\tau}\sum_{0\leqslant j<2n}\left\|\Pi_{s,\boldsymbol{t}}^j\right\|_{1,\tau}$$

and $\|\Pi_{s,\boldsymbol{t}}^j\|_{1,\tau} \ll \mathrm{e}^{O(|\sigma-2|j)}$, we get the claimed result.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

6.4 | Deduction of the meromorphic continuation

Proposition 6.9. *For some $\tau_0, t_0, \delta > 0$, and all $\|\boldsymbol{t}\| \leqslant t_0$, the function $s \mapsto \mathfrak{S}(s, \boldsymbol{t})$, initially only defined for $\mathrm{Re}(s) > 2$, has a meromorphic continuation to the set*

$$H := \{s \in \mathbb{C},\ s = \sigma + i\tau,\ \sigma \geqslant 2 - \delta\},$$

with possible poles occurring only for $|\tau| < \tau_0$ and $\lambda(s,t) = 1$. The meromorphic continuation of $s \mapsto \mathfrak{S}(s, \boldsymbol{t})$ is bounded uniformly in $\mathrm{Re}(s) \geqslant 2 - \delta$ and $|\tau| \geqslant \tau_0$ by

$$\left|\mathfrak{S}(s, \boldsymbol{t})\right| \ll |\tau|^{O(\max(0, 2-\sigma))} \log(|\tau| + 2). \tag{6.7}$$

More precisely, for $|\tau| \leqslant \tau_0$, the function

$$s \mapsto \mathfrak{S}(s, \boldsymbol{t}) - \frac{\lambda(s, \boldsymbol{t})}{1 - \lambda(s, \boldsymbol{t})}(\Pi_{s,\boldsymbol{t}}^{(0)} + \Pi_{s,\boldsymbol{t}}^{(1)} + \cdots + \Pi_{s,\boldsymbol{t}}^{(m-1)})\mathbb{P}_{s,\boldsymbol{t}}[\mathbf{1}](1) \tag{6.8}$$

has an analytic continuation to $\sigma \geqslant 2 - \delta$ and $|\tau| \leqslant \tau_0$, and is uniformly bounded there.

Proof. By Lemma 6.1, we have for $\mathrm{Re}(s) > 1$

$$\mathfrak{S}(s, \boldsymbol{t}) = \left(\Pi_{s,\boldsymbol{t}}^{(0)} + \cdots + \Pi_{s,\boldsymbol{t}}^{(m-1)}\right) \sum_{j \geqslant 0} \Pi_{s,\boldsymbol{t}}[\mathbf{1}](0). \tag{6.9}$$

Then Lemma 6.2 yields the analytic continuation of (6.8) for $|\tau| \leqslant \tau_0$ for some $\tau_0 > 0$. Then Lemma 6.4 ensures the existence of $\tau_1 > 0$ such that the sum over j in (6.8) converges uniformly over compacts in the region $\sigma \geqslant 2 - \delta$ and $|\tau| \geqslant \tau_1$, and yields the bound (6.7). Finally, applying Lemma 6.3 with the values of τ_0 and τ_1 gives the same conclusion for $\tau_0 \leqslant |\tau| \leqslant \tau_1$. The conjunction of these three cases covers the whole half-plane H. □

7 | ASYMPTOTIC BEHAVIOUR OF THE LEADING EIGENVALUE

In this section, we study the behaviour, for small t and s close to 2, of the leading eigenvalue $\lambda(s, \boldsymbol{t})$. The estimates in this section will reduce the problem to the estimation as $\boldsymbol{t} \to 0$ of the integral

$$\int_0^1 \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}(x)\rangle} \xi(x)\, \mathrm{d}x,$$

where we recall the notation (3.6). We recall the hypotheses (3.7), (3.8).

7.1 | Perturbation theory and existence

Let

$$\mathfrak{d} := -m \int_0^1 \log(x)\xi(x)\, \mathrm{d}x = \frac{m\pi^2}{12 \log 2}. \tag{7.1}$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Lemma 7.1. *For all small enough $\varepsilon > 0$, there exists $t_0 > 0$ such that whenever $|s-2| \leqslant \varepsilon$ and $\|\boldsymbol{t}\| \leqslant t_0$, we have*

$$\partial_{10}\lambda(s,\boldsymbol{t}) = -\mathfrak{d} + O(\varepsilon), \tag{7.2}$$

and

$$\lambda(s,\boldsymbol{t}) - 1 = (-\mathfrak{d} + O(\varepsilon))(s-2) + O(\varepsilon). \tag{7.3}$$

Proof. Let $f_{s,\boldsymbol{t}} = \mathbb{P}_{s,\boldsymbol{t}}[\xi]$ denote an eigenfunction of $\Pi_{s,\boldsymbol{t}}$ associated with the eigenvalue $\lambda(s,\boldsymbol{t})$. By Lemma 5.3 and [71, Theorem 2.6, estimation of P_L], we have

$$\left\|f_{s,\boldsymbol{t}} - \xi\right\|_\infty \ll_\varepsilon |s-2| + \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon}. \tag{7.4}$$

On the other hand, differentiating the eigenvalue equation $\mathbb{H}_{2,0}[g_{s,\boldsymbol{t}} f] = \lambda(s,\boldsymbol{t}) f_{s,\boldsymbol{t}}$ and integrating with respect to the Lebesgue measure, we get

$$\partial_{10}\lambda(s,\boldsymbol{t}) \int_{[0,1]} f_{s,\boldsymbol{t}}\,\mathrm{d}\nu = \int_{[0,1]} \left((\log \mathfrak{T}) g_{s,\boldsymbol{t}} f_{s,\boldsymbol{t}} + (g_{s,\boldsymbol{t}} - \lambda(s,\boldsymbol{t}))\partial_{10} f_{s,\boldsymbol{t}}\right)\mathrm{d}\nu.$$

Here we recall the notation (5.2). Setting $(s,\boldsymbol{t}) = (2,\mathbf{0})$, with $f_{2,\mathbf{0}} = \xi$ and $g_{2,\mathbf{0}} = 1$, gives $\partial_{10}\lambda(2,\mathbf{0}) = -\mathfrak{d}$. Using the bound (7.4), we get the approximation (7.2). □

Lemma 7.2. *For all $\eta > 0$, there exists $t_0 > 0$ and a unique function $s_0 : [-t_0,t_0]^d \to \mathbb{C}$ such that $s_0(0) = 2$ and, for $\|\boldsymbol{t}\| \leqslant t_0$,*

$$\left|s_0(\boldsymbol{t}) - 2\right| \leqslant \eta, \qquad \lambda(s_0(\boldsymbol{t}),\boldsymbol{t}) = 1.$$

Proof. This follows from a general form of the implicit functions theorem, for example, [72, Theorem 1.1], whose hypotheses are satisfied by virtue of Lemma 7.1. □

In what follows we will not discuss the regularity of $s_0(\boldsymbol{t})$ at each $\boldsymbol{t}$: we are only interested about its asymptotic behaviour around $\boldsymbol{t} = 0$. We will use results on effective perturbation theory of linear operators, which have been worked out recently in [71].

7.2 | The sub-CLT case

We first focus on the case where we do not aim at extracting a term of order 2 in the asymptotic expansion.

Lemma 7.3. *For $\|\boldsymbol{t}\| \leqslant t_0$, we have*

$$s_0(\boldsymbol{t}) - 2 = \frac{1}{\mathfrak{d}} \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}(x)\rangle} - 1)\xi(x)\,\mathrm{d}x + O_\varepsilon(\|\boldsymbol{t}\|^2 + \|\boldsymbol{t}\|^{2\alpha_0-\varepsilon}).$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Proof. By [71, Theorem 2.6], we have

$$\begin{aligned}\lambda(s,\boldsymbol{t}) &= \int_0^1 \Pi_{s,\boldsymbol{t}}\xi(x)\,\mathrm{d}x + O\left(\left\|\Pi_{s,\boldsymbol{t}} - \Pi_{2,0}\right\|^2\right)\\ &= (s-2)\int_0^1 \left[\frac{\partial}{\partial s}\Pi_{s,0}\xi\right]_{s=2}(x)\,\mathrm{d}x + \int_0^1 \Pi_{2,\boldsymbol{t}}\xi(x)\,\mathrm{d}x + O_\varepsilon(|s-2|^2 + \|\boldsymbol{t}\|^2 + \|\boldsymbol{t}\|^{2\alpha_0-\varepsilon})\\ &= (s-2)\int_0^1 \left[\frac{\partial}{\partial s}\mathbb{H}^m[g_{s,0}\xi]\right]_{s=2}(x)\,\mathrm{d}x + \int_0^1 \mathbb{H}^m[\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\rangle}\xi](x)\,\mathrm{d}x\\ &\quad + O_\varepsilon(|s-2|^2 + \|\boldsymbol{t}\|^2 + \|\boldsymbol{t}\|^{2\alpha_0-\varepsilon}) \qquad (7.5)\end{aligned}$$

by (5.2). Since $\int \mathbb{H}[f]\,\mathrm{d}\nu = \int f\,\mathrm{d}\nu$ and $\mathbb{H}[\xi] = \xi$, the first integral is $m\int_0^1 \log(x)\xi(x)\,\mathrm{d}x = -\mathfrak{d}$. The second equals $\int_0^1 \mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}(x)\rangle}\xi(x)\,\mathrm{d}x$. For $\alpha = \min(\alpha_0, 1)$, we have

$$\left|\int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t},\boldsymbol{\phi}\rangle} - 1)\xi\,\mathrm{d}\nu\right| \leqslant \|\boldsymbol{t}\|^\alpha \int_0^1 \|\boldsymbol{\phi}\|^\alpha \xi\,\mathrm{d}\nu = \|\boldsymbol{t}\|^\alpha \int_0^1 \mathbb{H}[\|\boldsymbol{\phi}\|^\alpha \xi]\,\mathrm{d}\nu \ll \|\boldsymbol{t}\|^\alpha \qquad (7.6)$$

by our hypothesis (3.7) and the triangle inequality (cf. (5.14)). Setting $s = s_0(\boldsymbol{t})$, we deduce $s_0(\boldsymbol{t}) - 2 = O(\|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0})$ by combining (7.5) and (7.6). Then another use of (7.5) yields our claimed estimate. □

7.3 | The CLT case

Our next goal is to extract the term of order $\|\boldsymbol{t}\|^2$ in the analysis above. We assume throughout that $\alpha_0 > 1$. In order to describe the order 2 coefficients, we introduce the following notation. Recalling (5.2), let

$$\begin{aligned}\mu_{\boldsymbol{\phi}} &:= \frac{1}{\mathfrak{d}}\int_0^1 \boldsymbol{\phi}(x)\xi(x)\,\mathrm{d}x,\\ \mathbb{K}[f] &:= \frac{1}{\xi}(\mathrm{Id} - \mathbb{H}^m)^{-1}\mathbb{N}^m[f\xi], \qquad (7.7)\\ \boldsymbol{\psi} &:= \boldsymbol{\phi} + \mu_{\boldsymbol{\phi}}\log\mathfrak{T},\\ \boldsymbol{\chi} &:= \mathbb{K}\boldsymbol{\psi}. \qquad (7.8)\end{aligned}$$

The well-definedness of $\mu_{\boldsymbol{\phi}}$ follows from our hypothesis $\alpha_0 > 1$. Note that $\boldsymbol{\chi}$ is bounded on $[0,1]$, because

$$\begin{aligned}\|\mathbb{N}^m[\boldsymbol{\psi}\xi]\|_\infty &\ll \int_{[0,1]} \|\boldsymbol{\psi}\|\,\mathrm{d}\nu + \|\mathbb{H}^m[\boldsymbol{\psi}\xi]\|_\infty\\ &\ll 1 + \int_{[0,1]} \|\boldsymbol{\phi}\|\,\mathrm{d}\nu + \sum_{h\in\mathcal{H}^m} |h'(0)|\,\|\boldsymbol{\phi}|_{h(I)}\|_\infty,\end{aligned}$$

which is finite by (3.7).

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Lemma 7.4. *If $\alpha_0 > 1$, then*

$$s_0(\boldsymbol{t}) - 2 = \frac{1}{\mathfrak{d}} \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}(x)\rangle} - 1)\xi(x)\,\mathrm{d}x - \boldsymbol{t}^T C_{\boldsymbol{\phi}} \boldsymbol{t} + O_\varepsilon(\|\boldsymbol{t}\|^3 + \|\boldsymbol{t}\|^{\alpha_0+1-\varepsilon}), \tag{7.9}$$

with

$$C_{\boldsymbol{\phi}} = \frac{1}{\mathfrak{d}} \int_0^1 \Big(\frac{1}{2}(\boldsymbol{\psi} - \boldsymbol{\phi}) \cdot (\boldsymbol{\psi} - \boldsymbol{\phi})^T + \boldsymbol{\phi} \cdot (\boldsymbol{\psi} - \boldsymbol{\phi})^T + \boldsymbol{\psi} \cdot \boldsymbol{\chi}^T \Big) \xi \,\mathrm{d}\nu. \tag{7.10}$$

Moreover, if $\alpha_0 > 2$, then

$$s_0(\boldsymbol{t}) - 2 = i\langle \boldsymbol{t}, \mu_{\boldsymbol{\phi}} \rangle - \tfrac{1}{2} \boldsymbol{t}^T \Sigma_{\boldsymbol{\phi}} \boldsymbol{t} + O_\varepsilon(\|\boldsymbol{t}\|^3 + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon}), \tag{7.11}$$

with

$$\Sigma_{\boldsymbol{\phi}} := \frac{1}{\mathfrak{d}} \int_0^1 (\boldsymbol{\psi} + \boldsymbol{\chi} - \boldsymbol{\chi} \circ T^m) \cdot (\boldsymbol{\psi} + \boldsymbol{\chi} - \boldsymbol{\chi} \circ T^m)^T \xi \mathrm{d}\nu. \tag{7.12}$$

Remark. With the definition (7.12), it is clear that the matrix $\Sigma_{\boldsymbol{\phi}}$ is symmetric, positive semi-definite. It is definite if and only if the vectors $\{(\boldsymbol{\psi} + \boldsymbol{\chi} - \boldsymbol{\chi} \circ T^m)(x), x \in (0,1)\}$ span the whole space $\mathbb{R}^d$. By our hypothesis (3.7), we have $\int_0^1 \|\boldsymbol{\phi}\| |\log \mathfrak{T}|\,\mathrm{d}\nu < \infty$ whenever $\alpha_0 > 1$, so that the matrix $C_{\boldsymbol{\phi}}$ is well defined in this case. Similarly, the matrix $\Sigma_{\boldsymbol{\phi}}$ is well defined whenever $\alpha_0 > 2$, since in that case $\int_{[0,1]} \|\boldsymbol{\psi} \cdot \boldsymbol{\psi}^T\|\,\mathrm{d}\nu \ll 1 + \int_{[0,1]} \|\boldsymbol{\phi}\|^2\,\mathrm{d}\nu < \infty$.

Proof. We extend the computations of Lemma 7.3, using our hypothesis on α_0 to expand the quantity $g_{s,\boldsymbol{t}} = \mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle} \mathfrak{T}^{s-2}$ to order 2 at $s = 2$ and order 1 at $\boldsymbol{t} = 0$. Let

$$\Delta_{s,\boldsymbol{t}} = \Pi_{s,\boldsymbol{t}} - \Pi_{2,0}.$$

We write $\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle} = 1 + i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle + O((\|\boldsymbol{\phi}\|\|\boldsymbol{t}\|)^{\min(2,\alpha_0-\varepsilon)})$. Letting $s = s_0(\boldsymbol{t}) = 2 + O(\|\boldsymbol{t}\|)$, we obtain

$$\begin{aligned}
\int_0^1 \Delta_{s,\boldsymbol{t}} \xi \,\mathrm{d}\nu &= \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle} - 1)\xi \,\mathrm{d}\nu - \mathfrak{d}(s-2) + i(s-2) \int_0^1 \langle \boldsymbol{t}, \boldsymbol{\phi}\rangle (\log \mathfrak{T}) \xi \,\mathrm{d}\nu \\
&\quad + \frac{1}{2}(s-2)^2 \int_0^1 (\log \mathfrak{T})^2 \xi \,\mathrm{d}\nu + O(\|\boldsymbol{t}\|^3 + \|\boldsymbol{t}\|^{1+\alpha_0-\varepsilon}) \\
&= \int_0^1 (\mathrm{e}^{i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle} - 1)\xi \,\mathrm{d}\nu - \mathfrak{d}(s-2) - \boldsymbol{t}^T C_{1,\boldsymbol{\phi}} \boldsymbol{t} + O(\|\boldsymbol{t}\|^3 + \|\boldsymbol{t}\|^{1+\alpha_0-\varepsilon}),
\end{aligned}$$

where $C_{1,\boldsymbol{\phi}} := \int_0^1 \boldsymbol{\phi} \cdot (\boldsymbol{\psi} - \boldsymbol{\phi})^T \xi \,\mathrm{d}\nu + \frac{1}{2} \int_0^1 (\boldsymbol{\psi} - \boldsymbol{\phi}) \cdot (\boldsymbol{\psi} - \boldsymbol{\phi})^T \xi \,\mathrm{d}\nu$. We use again [71, Theorem 2.6], getting

$$\lambda(s, \boldsymbol{t}) = 1 + \int_0^1 \Delta_{s,\boldsymbol{t}} \xi \,\mathrm{d}\nu + \int_{[0,1]} \Delta_{s,\boldsymbol{t}} (\mathrm{Id} - \mathbb{H}^m)^{-1} (\mathrm{Id} - \mathbb{P}) \Delta_{s,\boldsymbol{t}} \xi \,\mathrm{d}\nu + O\left(\|\Delta_{s,\boldsymbol{t}}\|^3 \right).$$

By computations similar to (5.10), we have

$$\left\| \Delta_{s,\boldsymbol{t}}[f] - i\mathbb{H}^m[\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle f] - i\mathfrak{d}^{-1} \left(\int_0^1 \langle \boldsymbol{t}, \boldsymbol{\phi}\rangle \xi \,\mathrm{d}\nu \right) \mathbb{H}^m[\log \mathfrak{T} f] \right\|_0 \ll_\varepsilon (\|\boldsymbol{t}\|^{\alpha_0-\varepsilon} + \|\boldsymbol{t}\|^2) \|f\|_0.$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneitalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Note that the left-hand side can be written $\|\Delta_{s,t}[f] - i\langle \boldsymbol{t}, \mathbb{H}^m[\boldsymbol{\psi} f]\rangle\|_0$. We deduce

$$\int_{[0,1]} \Delta_{s,\boldsymbol{t}}(\mathrm{Id} - \mathbb{H}^m)^{-1}(\mathrm{Id} - \mathbb{P})\Delta_{s,\boldsymbol{t}}\xi \, \mathrm{d}\nu$$
$$= -\int_0^1 \langle \boldsymbol{t}, \boldsymbol{\psi}\rangle(\mathrm{Id} - \mathbb{H}^m)^{-1}(\mathrm{Id} - \mathbb{P})\mathbb{H}^m[\langle \boldsymbol{t}, \boldsymbol{\psi}\rangle\xi] \, \mathrm{d}\nu + O_\varepsilon(\|\boldsymbol{t}\|^3 + \|\boldsymbol{t}\|^{1+\alpha_0-\varepsilon}).$$

Since $\mathbb{P}\mathbb{H}^m = \mathbb{P}$ and $\mathbb{H}^m = \mathbb{P} + \mathbb{N}^m$, we have $(\mathrm{Id} - \mathbb{H}^m)^{-1}(\mathrm{Id} - \mathbb{P})\mathbb{H}^m[f\xi] = \xi\mathbb{K}[f]$ where we recall the definition (7.7). Therefore, we have

$$-\int_0^1 \langle \boldsymbol{t}, \boldsymbol{\psi}\rangle(\mathrm{Id} - \mathbb{H}^m)^{-1}(\mathrm{Id} - \mathbb{P})\mathbb{H}^m[\langle \boldsymbol{t}, \boldsymbol{\psi}\rangle\xi] \, \mathrm{d}\nu = -\boldsymbol{t}^T C_{2,\boldsymbol{\phi}}\boldsymbol{t},$$

where $\mathbf{u}^T$ denotes the transpose of the column vector $\mathbf{u}$, and $C_{2,\boldsymbol{\phi}} = \int_{[0,1]}(\boldsymbol{\psi} \cdot \mathbb{K}[\boldsymbol{\psi}]^T)\xi \, \mathrm{d}\nu$. This proves (7.9) with $C_{\boldsymbol{\phi}} = C_{1,\boldsymbol{\phi}} + C_{2,\boldsymbol{\phi}}$ as claimed.

To prove (7.11), we note that by the hypothesis $\alpha_0 > 2$, the quantity (7.12) is well defined. In order to expand it, we first note that, with the definition (7.8), that by construction

$$\mathbb{P}[\boldsymbol{\psi}\xi] = \mathbb{P}[\boldsymbol{\phi}\xi + \mu_{\boldsymbol{\phi}}\xi \log \mathfrak{T}] = \left(\int \boldsymbol{\phi}\xi + \mathfrak{d}^{-1}\left(\int \boldsymbol{\phi}\xi \, \mathrm{d}\nu\right)\left(\int (\log \mathfrak{T})\xi \, \mathrm{d}\nu\right)\right)\xi \, \mathrm{d}\nu = 0$$

since $\int (\log \mathfrak{T})\xi = -\mathfrak{d}$. Therefore

$$\boldsymbol{\chi}\xi = (\mathrm{Id} - \mathbb{H}^m)^{-1}\mathbb{N}^m[\boldsymbol{\psi}\xi] = \sum_{j \geqslant 1} \mathbb{H}^{jm}[\boldsymbol{\psi}\xi] = \mathbb{H}^m[(\boldsymbol{\psi} + \boldsymbol{\chi})\xi].$$

By the property $\int f(g \circ T^m) \, \mathrm{d}\nu = \int \mathbb{H}^m[f]g \, \mathrm{d}\nu$, we have

$$\int (\boldsymbol{\psi} + \boldsymbol{\chi}) \cdot (\boldsymbol{\chi} \circ T^m)^T \xi \, \mathrm{d}\nu = \int \mathbb{H}^m[(\boldsymbol{\psi} + \boldsymbol{\chi})\xi] \cdot \boldsymbol{\chi}^T \, \mathrm{d}\nu = \int \boldsymbol{\chi} \cdot \boldsymbol{\chi}^T \xi \, \mathrm{d}\nu.$$

Similarly, we have

$$\int (\boldsymbol{\chi} \circ T^m) \cdot (\boldsymbol{\chi} \circ T^m)^T \xi \, \mathrm{d}\nu = \int \mathbb{H}^m[(\boldsymbol{\chi} \circ T^m)\xi] \cdot \boldsymbol{\chi}^T \, \mathrm{d}\nu = \int \boldsymbol{\chi} \cdot \boldsymbol{\chi}^T \xi \, \mathrm{d}\nu.$$

We deduce that, with the definition (7.12), we have

$$\Sigma_{\boldsymbol{\phi}} = \int (\boldsymbol{\psi} + \boldsymbol{\chi}) \cdot (\boldsymbol{\psi} + \boldsymbol{\chi})^T \xi \, \mathrm{d}\nu - 2\int (\boldsymbol{\psi} + \boldsymbol{\chi})(\boldsymbol{\chi} \circ T^m) \, \mathrm{d}\nu + \int (\boldsymbol{\chi} \circ T^m) \cdot (\boldsymbol{\chi} \circ T^m)^T \xi \, \mathrm{d}\nu$$
$$= \int \boldsymbol{\psi} \cdot \boldsymbol{\psi}^T \xi \, \mathrm{d}\nu + 2\int \boldsymbol{\psi} \cdot \boldsymbol{\chi}^T \xi \, \mathrm{d}\nu.$$

On the other hand, expanding the squares in (7.10), we find

$$2C_{\boldsymbol{\phi}} = \int \boldsymbol{\psi} \cdot \boldsymbol{\psi}^T \xi \, \mathrm{d}\nu - \int \boldsymbol{\phi} \cdot \boldsymbol{\phi}^T \xi \, \mathrm{d}\nu + 2\int \boldsymbol{\psi} \cdot \boldsymbol{\chi}^T \xi \, \mathrm{d}\nu.$$

The claimed formula (7.11) follows by the Taylor expansion $\mathrm{e}^{iu} = 1 + u + \frac{1}{2}u^2 + O(u^{\alpha_0-\varepsilon})$ with $u = i\langle \boldsymbol{t}, \boldsymbol{\phi}\rangle$. □

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

8 | PROOF OF THEOREM 3.1

Recall that Ω_Q consists of the rationals in (0,1] of denominators at most Q, and let

$$\chi_Q(\boldsymbol{t}) := \sum_{x\in\Omega_Q} \exp(\langle i\boldsymbol{t}, S_{\boldsymbol{\phi}}(x)\rangle). \tag{8.1}$$

Proposition 8.1. *For all $\varepsilon > 0$, there exists $\delta, t_0 > 0$ such that for $\|\boldsymbol{t}\| \leqslant t_0$, we have*

$$\mathbb{E}_Q(e^{i\langle \boldsymbol{t}, S_{\boldsymbol{\phi}}(x)\rangle}) = Q^{s_0(\boldsymbol{t})-2}\big\{1 + O_\varepsilon(Q^{-\delta} + \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon})\big\}.$$

Proof. Recall that $\chi_Q(\boldsymbol{t})$ was defined in (8.1), so that $\mathbb{E}_Q(e^{i\langle \boldsymbol{t}, S_{\boldsymbol{\phi}}(x)\rangle}) = \chi_Q(\boldsymbol{t})/\chi_Q(0)$. Let $\Omega \geqslant 1$ be a parameter, and $w : \mathbb{R}_+ \to [0,1]$ be a smooth function satisfying

$$\mathbf{1}_{[0,1]} \leqslant w \leqslant \mathbf{1}_{[0,1+\Omega^{-1}]}, \qquad \|w^{(j)}\|_\infty \ll_j \Omega^j. \tag{8.2}$$

Then by a trivial bound on the contribution of $q \in [Q, Q(1+\Omega^{-1})]$, we have

$$\chi_Q(\boldsymbol{t}) = O(\Omega^{-1}Q^2) + \widetilde{\chi}_Q(\boldsymbol{t}), \qquad \text{where} \quad \widetilde{\chi}_Q(\boldsymbol{t}) := \sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} e^{i\langle \boldsymbol{t}, S_{\boldsymbol{\phi}}(a/q)\rangle} w\Big(\frac{q}{Q}\Big). \tag{8.3}$$

By Perron's formula, we have

$$\widetilde{\chi}_Q(\boldsymbol{t}) = \frac{1}{2\pi i}\int_{3-i\infty}^{3+i\infty} Q^s \mathfrak{S}(s,\boldsymbol{t})\widehat{w}(s)\,\mathrm{d}s,$$

where the Mellin transform $\widehat{w}(s) = \int_0^\infty w(u)u^{s-1}\,\mathrm{d}u$ is defined for $\mathrm{Re}(s) > 0$. We move the contour to the line $\mathrm{Re}(s) = 2-\delta$. If $t_0, \delta > 0$ are small enough, then by Proposition 6.9, we encounter exactly one pole, at $s = s_0(\boldsymbol{t})$. By Cauchy's theorem, we deduce

$$\widetilde{\chi}_Q(\boldsymbol{t}) = \operatorname*{Res}_{s=s_0(\boldsymbol{t})}(Q^s\mathfrak{S}(s,\boldsymbol{t})\widehat{w}(s)) + \frac{1}{2\pi i}\int_{2-\delta-i\infty}^{2-\delta+i\infty} Q^s\mathfrak{S}(s,\boldsymbol{t})\widehat{w}(s)\,\mathrm{d}s. \tag{8.4}$$

For some absolute constant $C \geqslant 1$, Proposition 6.9 yields the bound

$$|\mathfrak{S}(s,\boldsymbol{t})| \ll (|\tau|+1)^C.$$

On the other hand, we have $|\widehat{w}(s)| \ll_C \Omega^{C+2}|s|^{-C-2}$ for $\mathrm{Re}(s) \in [1/2, 3]$ by integration by parts and (8.2), so that

$$\left|\int_{2-\delta-i\infty}^{2-\delta+i\infty} Q^s\mathfrak{S}(s,\boldsymbol{t})\widehat{w}(s)\,\mathrm{d}s\right| \ll \Omega^{C+2}Q^{2-\delta}. \tag{8.5}$$

Finally, we have by (6.8)

$$\operatorname*{Res}_{s=s_0(\boldsymbol{t})}(Q^s\mathfrak{S}(s,\boldsymbol{t})\widehat{w}(s)) = -\frac{Q^{s_0}\widehat{w}(s_0)}{\partial_{10}\lambda(s_0,\boldsymbol{t})}\sum_{0\leqslant j<m}\Pi^{(j)}_{s_0,\boldsymbol{t}}\mathbb{P}_{s_0,\boldsymbol{t}}[\mathbf{1}](1), \tag{8.6}$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where we abbreviated $s_0 = s_0(\boldsymbol{t})$ in the right-hand side. Again by [71, Theorem 1.6, estimation of P_L] and Lemma 5.3, we have $\|P_{s_0,\boldsymbol{t}} - \mathbb{P}_{2,0}\|_0 \ll \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon}$, and therefore

$$\Pi^{(j)}_{s_0,\boldsymbol{t}}\mathbb{P}_{s_0,\boldsymbol{t}}[\mathbf{1}](1) = \frac{1}{2\log 2}\{1 + O_\varepsilon(\|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon})\}$$

for $0 \leqslant j < m$. The quantity $\partial_{10}\lambda(s_0(\boldsymbol{t}), \boldsymbol{t})$ was estimated in (7.2). Finally, we have by (8.2)

$$\widehat{w}(s_0) = \frac{1}{2}\{1 + O_\varepsilon(\Omega^{-1} + \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon})\}.$$

Inserting these estimates in (8.6), we deduce

$$\operatorname*{Res}_{s=s_0(\boldsymbol{t})}(Q^s\mathfrak{S}(s,\boldsymbol{t})\widehat{w}(s)) = \frac{3}{\pi^2}Q^{s_0(\boldsymbol{t})}\big\{1 + O_\varepsilon(\Omega^{-1} + \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon})\big\}.$$

Grouping this (8.3)–(8.5), we conclude

$$\chi_Q(\boldsymbol{t}) = \frac{3}{\pi^2}Q^{s_0(\boldsymbol{t})}\big\{1 + O_\varepsilon(\Omega^{-1} + \Omega^{C+2}Q^{-\delta} + \|\boldsymbol{t}\| + \|\boldsymbol{t}\|^{\alpha_0-\varepsilon})\big\}.$$

Our claim follows by optimizing $\Omega = Q^{\delta/(C+3)}$ and dividing by $\chi_Q(0)$. □

Proof of Theorem 3.1. We use Proposition 8.1 along with Lemmas 7.3–7.4 and the value (7.1). □

9 | APPLICATIONS

When using the Berry–Esseen inequality, we will require a separate treatment of very small values of t in order to handle the error term $O(Q^{-\delta})$ in Theorem 3.1 (the argument described in [52, Remark 3.8] is not readily adapted since part of this term originates from counting pairwise coprime numbers).

Lemma 9.1. *Suppose that the function $\boldsymbol{\phi}$ satisfies* (3.7) *and* (3.8). *Then we have*

$$\mathbb{E}_Q(\mathrm{e}^{i\langle \boldsymbol{t}, S_{\boldsymbol{\phi}}(x)\rangle}) = 1 + O(\|\boldsymbol{t}\|^{\alpha_0/3}\log Q). \tag{9.1}$$

Proof. For all $n \geqslant 1$, define $c(n) := \sup_{x\in(\frac{1}{n+1},\frac{1}{n})}\|\boldsymbol{\phi}(x)\|^{\alpha_0/3}$. Then with the terminology of [3, p. 750], the function c has strong moments to order 3. Setting $a_j(x) = \lfloor 1/T^{j-1}(x)\rfloor$, we deduce $\mathbb{E}_Q(\sum_{j=1}^r c(a_j)) \ll \log Q$ by [3, Remark 1.2]. We conclude by taking expectations in the bound $|\mathrm{e}^{i\langle \boldsymbol{t}, S_{\boldsymbol{\phi}}(x)\rangle} - 1| = O(\sum_{j=1}^r(\|\boldsymbol{t}\|\|\boldsymbol{\phi}(T^{j-1}(x))\|)^{\alpha_0/3})$. □

Proof of Corollary 3.2. In (3.11), we estimate the integral $\mathfrak{T}_\phi(t)$ using [12, Proposition 2.1], which amounts here to integrating the Taylor expansion of $\mathrm{e}^{it\phi(x)}$ at $t = 0$. From (7.12), and with the notation (3.12), we obtain

$$U(t) = it\mu - \frac{t^2}{2}\sigma^2 + O_\varepsilon(|t|^{\min(3,\alpha_0-\varepsilon)}), \tag{9.2}$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by CochraneItalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

where, with $\psi(x) = \phi(x) + \mu \log x$,

$$\sigma = \frac{12 \log 2}{\pi^2} \int_0^1 (\psi(x) + \chi(x) - \chi(T(x)))^2 \frac{\mathrm{d}x}{1+x}.$$

We recall that χ is related to ψ by (7.8) (with $m = 1$). It is clear that $\sigma \geqslant 0$. If $\sigma = 0$, then the integrand vanishes identically, and we would conclude that $\phi = -\mu \log - \chi + \chi \circ T$, contradicting our hypothesis. We use the Berry–Esseen theorem [110, theorem II.7.16], [42, equation XVI.(3.13)] with $T = c(\log Q)^{1/2}$ for some parameter $c > 0$ to be chosen. Let

$$f(\tau) := \mathbb{E}_Q\left(\exp\left(i\tau \frac{S_\phi(x) - \mu \log Q}{\sigma\sqrt{\log Q}}\right)\right), \qquad g(\tau) := \mathrm{e}^{-\tau^2/2}.$$

When $|\tau| \leqslant Q^{-\delta}$, we use Lemma 9.1 and a Taylor bound, getting

$$f(\tau) = 1 + O(|\tau|^{\alpha_0/3}(\log Q)^{1-\alpha_0/6} + |\tau|(\log Q)^{1/2}).$$

When $Q^{-\delta} < |\tau| \leqslant (\log Q)^\varepsilon$, we use Theorem 3.1, getting

$$\begin{aligned} f(\tau) &= \exp\left\{-\frac{\tau^2}{2} + O\left(\frac{|\tau|}{(\log Q)^{1/2}} + \frac{|\tau|^{\alpha_0-\varepsilon}}{(\log Q)^{\alpha_0/2-1-\varepsilon/2}} + \frac{1}{Q^\delta}\right)\right\} \\ &= \mathrm{e}^{-\tau^2/2}\left(1 + O\left(\frac{|\tau|}{(\log Q)^{1/2}} + \frac{|\tau|^{\alpha_0-\varepsilon}}{(\log Q)^{\alpha_0/2-1-\varepsilon/2}} + \frac{1}{Q^\delta}\right)\right). \end{aligned}$$

Finally, when $(\log Q)^\varepsilon < |\tau| \leqslant T$, we get

$$\begin{aligned} f(\tau) &= \exp\left\{-\frac{\tau^2}{2}\left(1 + O\left(\frac{1}{|\tau|(\log Q)^{1/2}} + \frac{|\tau|^{\alpha_0-2-\varepsilon}}{(\log Q)^{\alpha_0/2-1-\varepsilon/2}} + \frac{1}{\tau^2 Q^\delta}\right)\right)\right\} \\ &= \exp\left\{-\frac{\tau^2}{2}(1 + O(c^{\alpha_0-2-\varepsilon})\right\} \end{aligned}$$

and so, for some small enough choice of $c > 0$, we have $|f(\tau)| \leqslant \mathrm{e}^{-\tau^2/3}$.

We deduce

$$\begin{aligned} &\int_{-T}^{T} \left|\frac{f(\tau) - g(\tau)}{\tau}\right| \mathrm{d}\tau \\ &\ll \int_0^{Q^{-\delta}} (\tau^{\alpha_0/3-1}(\log Q)^{1-\alpha_0/6} + (\log Q)^{1/2} + \tau)\,\mathrm{d}t \\ &\quad + \int_{Q^{-\delta}}^{(\log Q)^\varepsilon} \mathrm{e}^{-\tau^2/2}((\log Q)^{-1/2} + \tau^{\alpha_0-1-\varepsilon}(\log Q)^{-\alpha/2+1+\varepsilon/2} + \tau^{-1}Q^{-\delta})\,\mathrm{d}\tau \\ &\quad + \int_{(\log Q)^\varepsilon}^{T} \tau^{-1}\mathrm{e}^{-\tau^2/3}\,\mathrm{d}\tau \\ &\ll (\log Q)^{-1/2} + (\log Q)^{-\alpha_0/2+1+\varepsilon}, \end{aligned}$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and therefore by the Berry–Esseen inequality

$$\sup_{v\in\mathbb{R}}\left|\mathbb{P}_Q\left(\frac{S_\phi(x)-\mu\log Q}{\sigma\sqrt{\log Q}}\leqslant v\right)-\Phi(v)\right|\ll\frac{1}{T}+\int_{-T}^{T}\left|\frac{f(\tau)-g(\tau)}{\tau}\right|\mathrm{d}\tau\ll\frac{1}{(\log Q)^{\min(1/2,\alpha_0/2-1-\varepsilon)}}$$

as claimed. □

9.1 | Central modular symbols

Let $f(z)=\sum_{n\geqslant 1}a_n\mathrm{e}(nz)$ be a non-zero primitive Hecke eigencuspform of weight k for $SL(2,\mathbb{Z})$ with trivial multiplier. Note that k is necessarily even and $k\geqslant 12$.

Define, for all integer $1\leqslant m\leqslant k-1$ and all $x\in\mathbb{Q}$, the modular symbol

$$\langle x\rangle_{f,m}:=\frac{(2\pi i)^m}{(m-1)!}\int_x^{i\infty}f(z)(z-x)^{m-1}\,\mathrm{d}z. \tag{9.3}$$

Lemma 9.2. *For $m>k/2$, the function $x\mapsto\langle x\rangle_{f,m}$, initially defined over $\mathbb{Q}$, can be extended to a bounded function in $H^{1-\varepsilon}(\mathbb{R})$ for any $\varepsilon\in(0,1)$.*

Proof. By Deligne's bound [35], we have $|a_n|\ll_{\varepsilon,f}n^{(k-1)/2+\varepsilon}$. Therefore, the sum $\sum_{n\geqslant 1}|a_n|/n^m$ is finite, and we deduce by Fubini's theorem that for $x\in\mathbb{Q}$,

$$\langle x\rangle_{f,m}=(-1)^m\sum_{n\geqslant 1}\frac{a_n\mathrm{e}(nx)}{n^m},$$

and the left-hand side is now defined for $x\in\mathbb{R}$. By [63, Theorem 5.3], we have

$$\sum_{n\leqslant t}a_n\mathrm{e}(nx)\ll_f t^{k/2}\log(2t)\qquad(t\geqslant 1) \tag{9.4}$$

uniformly in $x\in\mathbb{R}$. Let $x,x'\in\mathbb{R}$, $\delta=|x-x'|$, and for $t\geqslant 1$, $S(t):=\sum_{n\leqslant t}a_n(\mathrm{e}(nx)-\mathrm{e}(nx'))$. Then using (9.4) and partial summation, we obtain $|S(t)|\ll_{\varepsilon,f}t^{k/2+\varepsilon}\min(1,\delta t)$, and so

$$\begin{aligned}\left|\langle x\rangle_{f,m}-\langle x'\rangle_{f,m}\right|&\ll_{\varepsilon,f}\int_1^{\infty}t^{-m-1+k/2+\varepsilon}\min(1,\delta t)\,\mathrm{d}t\\&\ll_{\varepsilon,f}\delta+\delta^{m-k/2-\varepsilon}\end{aligned}$$

as claimed, since $m\geqslant k/2+1$. □

Lemma 9.3. *For any $\varepsilon\in(0,1)$, some function $\phi_f\in H^{1-\varepsilon}([0,1],\mathbb{C})$, and all $x\in\mathbb{Q}\setminus\{0\}$, we have*

$$\langle x\rangle_{f,k/2}=\left\langle\frac{-1}{x}\right\rangle_{f,k/2}+\phi_f(x).$$

Moreover, we have $\|\phi_f\circ h\|_{(1-\varepsilon)}\ll 1$ uniformly for $h\in\mathcal{H}^*$.

We will actually only require the last bound for $h\in\mathcal{H}$.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Proof. For $\mathrm{Im}(z) > 0$, define

$$\begin{aligned}\widehat{f}(z) &:= \sum_{n\geqslant 1} \frac{a_n}{n^{k-1}} \mathrm{e}(nz) \\ &= \sum_{n\geqslant 1} a_n \mathrm{e}(nz) \frac{(-2\pi i)^{k-1}}{(k-1)!} \int_0^{i\infty} \tau^{k-2} \mathrm{e}^{2\pi i n\tau}\, \mathrm{d}\tau \\ &= \frac{(-2\pi i)^{k-1}}{(k-1)!} \int_z^{i\infty} (\tau - z)^{k-2} f(\tau)\, \mathrm{d}\tau.\end{aligned}$$

Using the modularity relation $f(-1/z) = z^k f(z)$, and changing variables $\tau \to -1/\tau$, we obtain

$$\begin{aligned}\frac{(k-1)!}{(-2\pi i)^{k-1}} z^{k-2} \widehat{f}(-1/z) &= z^{k-2} \int_{-1/z}^{i\infty} (\tau + 1/z)^{k-2} f(\tau)\, \mathrm{d}\tau = \int_z^0 (-1/\tau + 1/z)^{k-2} (z\tau)^{k-2} f(\tau)\, \mathrm{d}\tau \\ &= -\int_0^z (\tau - z)^{k-2} f(\tau)\, \mathrm{d}\tau\end{aligned}$$

and so the period polynomial of f

$$r_f(z) := \widehat{f}(z) - z^{k-2}\widehat{f}(-1/z) = \frac{(-2\pi i)^{k-1}}{(k-1)!} \int_0^{i\infty} (\tau - z)^{k-2} f(\tau)\, \mathrm{d}\tau, \qquad (\mathrm{Im}(z) > 0),$$

is indeed a polynomial in z of degree at most $k-2$.

Let now $x \in \mathbb{Q}_{>0}$. As $\delta \to 0$ with $\delta > 0$, we have

$$\begin{aligned}\widehat{f}(x(1+i\delta)) &= \frac{(-2\pi i)^{k-1}}{(k-1)!} \int_{x(1+i\delta)}^{i\infty} (\tau - x - ix\delta)^{k-2} f(\tau)\, \mathrm{d}\tau \\ &= \frac{(-2\pi i)^{k-1}}{(k-1)!} \left(\int_x^{i\infty} - \int_x^{x(1+i\delta)} \right) (\tau - x - ix\delta)^{k-2} f(\tau)\, \mathrm{d}\tau.\end{aligned}$$

The second integral is $O_{M,x,f}(\delta^M)$ as $\delta \to 0$ for any fixed $M > 0$, since f is a cusp form. Thus, by the binomial formula, we obtain

$$\widehat{f}(x(1+i\delta)) = \frac{(-1)^{k-1}}{(k-1)} \sum_{\ell=0}^{k-2} \frac{(2\pi x\delta)^\ell}{\ell!} \langle x \rangle_{f,k-1-\ell} + o_{x,f}(\delta^{k-2}).$$

In the same way, since $-\frac{1}{x(1+i\delta)} = -\frac{1}{x}(1 - i\delta')$ with $\delta' = \delta/(1+i\delta)$, so that $\mathrm{Re}(\delta) > 0$, we have

$$\begin{aligned}&(x(1+i\delta))^{k-2} \widehat{f}\left(-\frac{1}{x(1+i\delta)} \right) \\ &= \frac{(-1)^{k-1}}{(k-1)} \sum_{\ell=0}^{k-2} (i\delta)^\ell \sum_{j=0}^{\ell} \frac{(-2\pi i)^j}{j!} \binom{k-2-j}{\ell-j} x^{k-2-j} \left\langle \frac{-1}{x} \right\rangle_{f,k-1-j} + o_{x,f}(\delta^{k-2}).\end{aligned}$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by CochraneItalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

With $m := k/2-1$, reading the coefficients of δ^m on each side of the definition of $r_f(x(1+i\delta))$, and since k is even, we deduce

$$\langle x\rangle_{f,1+m} - \sum_{j=0}^{m} c_{j,k}x^j\left\langle\frac{-1}{x}\right\rangle_{f,1+m+j} = -(k-1)(2\pi)^{m+1}i^m r_f^{(m)}(x),$$

$$c_{j,k} := j!\binom{m}{j}\binom{m+j}{j}(-2\pi i)^{-j}.$$

We single out the term $j=0$. The function

$$\phi_f(x) := \sum_{j=1}^{m} c_{j,k}x^j\left\langle\frac{-1}{x}\right\rangle_{f,1+m+j} - (k-1)(2\pi)^{m+1}i^m r_f^{(m)}(x),$$

defines, by Lemma 9.2, a function in $H^{1-\varepsilon}([\frac{1}{n+1},\frac{1}{n}])$ for all $\varepsilon\in(0,1)$ and $n\geqslant 1$. The value $c_{0,k}=1$ proves our claimed formula. Finally, for $h\in\mathcal{H}$, by the rules (4.1), (4.2), (4.4) and 1-periodicity of $x\mapsto\langle x\rangle_{f,1+m+j}$, we have

$$\begin{aligned}\left\|\phi_f\circ h\right\|_{(1-\varepsilon)} &\ll_f \sum_{j=1}^{m}\left\|x\mapsto h(x)^j\langle x\rangle_{f,1+m+j}\right\|_{(1-\varepsilon)} + \left\|r_f^{(m)}\circ h\right\|_{(1-\varepsilon)}\\ &\ll_f \sum_{j=1}^{m}\left(\left\|(h^j)'\right\|_\infty^{\varepsilon}\left\|h^j\right\|_\infty^{1-\varepsilon} + \left\|h^j\right\|_\infty\right) + \|h'\|_\infty^{1-\varepsilon}\\ &\ll_f 1.\end{aligned}$$

By the rule (4.2) again, and since $\|h\|_{(1)}\leqslant\|h'\|_\infty\ll 1$ for $h\in\mathcal{H}^*$, we deduce that the same bound $\|\phi_f\circ h\|_{(1-\varepsilon)}\ll 1$ holds for $h\in\mathcal{H}^*$. □

Proof of Theorems 2.3 and 2.4. Iterating Lemma 9.3, we have for all $x\in\mathbb{Q}\cap(0,1]$,

$$\langle x\rangle_{f,k/2} = \sum_{j=1}^{r}\phi((-1)^{j-1}T^{j-1}(x)) + \langle 0\rangle_{f,k/2}. \tag{9.5}$$

Note that changing the coordinates of the set $\mathscr{R}\subset\mathbb{R}^2$ by an amount $O(1/\sqrt{\log Q})$ in (2.2) does not alter the right-hand side, so that we may replace $\langle x\rangle_{f,k/2}$ by $\langle x\rangle_{f,k/2}-\langle 0\rangle_{f,k/2}$. For all $\boldsymbol{t}\in\mathbb{R}^2$, identifying $\mathbb{C}\simeq\mathbb{R}^2$ with basis $(1,i)$, we let

$$\chi(t) := \mathbb{E}_Q\left(\exp\left\{\frac{i\langle\boldsymbol{t},\langle x\rangle_{f,k/2}-\langle 0\rangle_{f,k/2}\rangle}{\sigma_f\sqrt{\log Q}}\right\}\right),$$

where we recall that σ_f was defined in Theorem 2.3. We apply Theorem 3.1 with $m=1$ and $d=2$. The hypothesis (3.7) is satisfied with $\alpha_0=4$, since the function ϕ_f is continuous on [0,1] by Lemma 9.3 and therefore bounded. The hypothesis (3.8) is satisfied for any $\lambda_0<\frac{1}{2-2\varepsilon}$, by using Lemma 9.3 and noting that $\|\phi_f|_{h(I)}\|_{(1-\varepsilon)}\ll|h'(0)|^{-1+\varepsilon}\|\phi_f\circ h\|_{(1-\varepsilon)}$. Using (3.11) along with the

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

expressions (7.11) and (7.12), we obtain for some $\mu_f \in \mathbb{R}^2$ and real 2×2 matrix Σ_f the estimate

$$\chi(\boldsymbol{t}) = \exp\left\{ i\langle \boldsymbol{t}, \mu_f\rangle - \tfrac{1}{2}\boldsymbol{t}^T\Sigma_f\boldsymbol{t} + O\left(\frac{\|\boldsymbol{t}\| + \|\boldsymbol{t}\|^3}{\sqrt{\log Q}} + \frac{1}{Q^\delta}\right)\right\}.$$

To compute the variance, we appeal to the bound

$$\mathbb{E}_Q(|\langle x\rangle_{f,m} - \mu_f \log Q|^4) \ll (\log Q)^2. \tag{9.6}$$

This can be proved by shifting to the setting of [4], where the variable t is extended to a complex neighbourhood of the origin; the functions U, V defined in (3.10) are, in this case, analytic in t near the origin, by boundedness of $\boldsymbol{\phi}_j$. Then by, for example, [15, theorem 25.12] and (9.6), we find

$$\mu_f = \lim_{Q\to\infty} \frac{\mathbb{E}_Q(\langle x\rangle_{f,k/2})}{\log Q}, \qquad \Sigma_f = \lim_{Q\to\infty} \frac{\mathbb{E}_Q(\langle x\rangle_{f,k/2}\langle x\rangle_{f,k/2}^T)}{\sigma_f^2 \log Q}.$$

On the other hand, as $Q\to\infty$, we have the following asymptotic formulae, where now $\langle x\rangle_{f,k/2}$ is interpreted as a complex number:

$$\mathbb{E}_Q(\langle x\rangle_{f,m}) = o(\log Q), \tag{9.7}$$

$$\mathbb{E}_Q(\langle x\rangle_{f,m}^2) = o(\log Q), \tag{9.8}$$

$$\mathbb{E}_Q(|\langle x\rangle_{f,m}|^2) \sim 2\sigma_f^2 \log Q. \tag{9.9}$$

These statements can be proven (in a stronger form) by standard methods, using orthogonality of additive characters, the approximate functional equation [65, Theorem 5.3] and Rankin–Selberg theory [63, Chapter 13.6]. The value σ_f appears as $\sigma_f^2 = \mathrm{Res}_{s=1} L(f\times\overline{f}, 1)$, which is evaluated in [63, equation (13.52)]. Note that the analogues of (9.8) and (9.9) with a single average over numerator have recently been computed in [18]; in their result as stated, however, the denominator is assumed to be prime.

The equality (9.7) shows that $\mu_f = 0$. The equality (9.8) shows that the matrix Σ_f is a multiple of the identity, and the equality (9.9) then shows that $\Sigma_f = \mathrm{Id}$. Using the Berry–Esseen inequality, along with Lemma 9.1, concludes the proof of Theorem 2.3.

To justify Theorem 2.4, we use again the analyticity of U, V defined in (3.10) in a neighbourhood of the origin. Hypothesis (4) of [61] is therefore satisfied in our case with $\phi(n)$ replaced by $\log Q$, and (2.3) follows by [61, Theorem 1], taking $t = \varepsilon\sqrt{\log Q}$. □

9.2 | Central value of the Estermann function

Proof of Theorem 2.1. The Estermann function D corresponds to D_0 in notation of [8]. For $x \in \mathbb{Q}$, it is the analytic continuation of

$$D(s,x) = \sum_{n\geqslant 1} \frac{\mathrm{e}(nx)\tau(n)}{n^s},$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

initially defined for $\mathrm{Re}(s) > 1$, evaluated at $s = \frac{1}{2}$. We recall that $\tau(n)$ is the number of divisors of n. We use [8, Lemma 10], noting that the quantity v_{j-1}/v_j corresponds to $T^{j-1}(x)$. Therefore

$$D\left(\tfrac{1}{2}, x\right) = \zeta\left(\tfrac{1}{2}\right)^2 + \sum_{j=1}^{r} \phi_j(T^{j-1}(x)),$$

where

$$\phi_j(x) = \tfrac{1}{2}x^{-1/2}\left(\log(1/x) + \gamma_0 - \log(8\pi) - \tfrac{\pi}{2}\right)$$
$$+ (-1)^{j-1} i \tfrac{1}{2} x^{-1/2}\left(\log(1/x) + \gamma_0 - \log(8\pi) + \tfrac{\pi}{2}\right) + \zeta\left(\tfrac{1}{2}\right)^2 + \mathcal{E}((-1)^j x),$$

and $\mathcal{E}$, which corresponds to $\mathcal{E}(0, \cdot)$ in the notation of [8, p. 6900], is bounded and continuous. By comparing the cases $N = 0$ and $N = 1$ of [8, equation (3.17)], we have

$$\mathcal{E}(x) = \mathcal{E}_1(x) + \sum_{j\in\{1,2\}} \frac{(-1)^j}{j\pi}\Gamma\left(\tfrac{1}{2}+j\right)^2\left(\frac{x}{2\pi i}\right)^j\left(D\left(\tfrac{1}{2}+j, -1/x\right) + \zeta\left(\tfrac{1}{2}+j\right)^2/j\right), \tag{9.10}$$

where $\mathcal{E}_1 \in C^1([0,1])$. For $j \geqslant 1$, the function $D(\frac{1}{2}+j, \cdot)$ belongs to $H^{1/2-\varepsilon}([0,1], \mathbb{C})$ for all $\varepsilon \in (0, 1/2)$. We deduce that the right-hand side of (9.10) defines, for all $n \geqslant 1$, a function in $H^{1/2-\varepsilon}([\frac{1}{n+1}, \frac{1}{n}], \mathbb{C})$. By an argument identical to Lemma 9.3, we also have the bound $\|\mathcal{E} \circ h\|_{(1/2-\varepsilon)} \ll 1$ for all $h \in \mathcal{H}^2$, and similarly for $x \mapsto \mathcal{E}(-x)$. This validates the hypothesis (3.8) with any $\lambda_0 < 1$.

On the other hand, setting $\alpha_0 = 2 - \varepsilon$, we have

$$\sum_{h\in\mathcal{H}} |h'(0)| \|\phi_j|_{h(I)}\|_\infty^{\alpha_0} \ll \sum_{n\geqslant 1} n^{-2}\left|n^{1/2}\log(2n)\right|^{\alpha_0} < \infty,$$

and so the hypothesis (3.7) holds. We apply Theorem 3.1 with $m = 2$, $d = 2$, and the above given value $\alpha = 2 - \varepsilon$. In the estimate (3.11), we evaluate the integrals by appealing to [12, Corollary 3.1]. We deduce that there is a constant $\mu \in \mathbb{C}$ such that, letting $\sigma = 1/\pi$ and

$$\chi(t) := \mathbb{E}_Q\left(\exp\left\{i\,\mathrm{Re}\left(t\frac{D(\frac{1}{2}, x) - \mu\log Q}{\sigma(\log Q)^{1/2}(\log\log Q)^{3/2}}\right)\right\}\right), \qquad (t \in \mathbb{C}),$$

we have

$$\chi(t) = \exp\left\{-\frac{|t|^2}{2} + O\left(\frac{1}{Q^\delta} + \frac{|t| + |t|^2}{(\log\log Q)^{1-\varepsilon}}\right)\right\}. \tag{9.11}$$

We then obtain, by the Berry–Essen inequalities and the bound (9.1) for small frequencies, the statement of Theorem 2.1 up to the value of the expectation. We compute the expectation from the initial object, using the expression [65, equation (3.2)] for Ramanujan sums. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have

$$\sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} D(s, a/q) = \zeta(s)^2 \sum_{\ell|q} \mu\left(\frac{q}{\ell}\right)\frac{\tau(\ell)}{\ell^{s-1}} \prod_{p^\nu\|\ell}\left(1 - \tfrac{\nu}{\nu+1}p^{-s}\right),$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and so by analytic continuation

$$\sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} D\big(\tfrac{1}{2},a/q\big)=\zeta\big(\tfrac{1}{2}\big)^2\sum_{\ell\mid q}\mu\Big(\frac{q}{\ell}\Big)\tau(\ell)\ell^{1/2}\prod_{p^\nu\parallel\ell}\Big(1-\tfrac{\nu}{\nu+1}p^{-1/2}\Big)=O_\varepsilon(q^{1/2+\varepsilon}).$$

We deduce that

$$\mathbb{E}_Q\big(D\big(\tfrac{1}{2},x\big)\big)\ll_\varepsilon Q^{-1/2+\varepsilon}. \tag{9.12}$$

On the other hand, by (9.11), we have

$$\chi(t)=1+O_\varepsilon(Q^{-\delta}+(\log\log Q)^{-1/2+\varepsilon}t+t^2),$$

whereas expanding the exponential as $\mathrm{e}^{iu}=1+iu+O(u^{3/2})$ in the definition of $\chi(t)$, by (9.12), we get

$$\chi(t)=1+it\mu(\log Q)^{1/2}(\log\log Q)^{-3/2}+O(Q^{-1/3}+|t|^{3/2}((\log Q)^{1/2}+\mathbb{E}_Q(|D(\tfrac{1}{2},x)|^{3/2})).$$

Using the trivial bound $|D(\frac{1}{2},x)|\ll\sum_{j=1}^{r(x)}a_j(x)^{2/3}$ and Hölder's inequality, we find

$$\begin{aligned}\mathbb{E}_Q(|D(\tfrac{1}{2},x)|^{3/2})&\ll\mathbb{E}_Q\left(\left(\sum_{j=1}^{r(x)}1\right)^{1/2}\left(\sum_{j=1}^{r(x)}a_j(x)\right)\right)\\&\ll(\log Q)^{5/2}\end{aligned}$$

by the bound $r(x)\ll\log(\mathrm{denom}(x)+1)$ and [117, Theorem]. Setting $t=Q^{-\delta/2}$, we obtain

$$\mu(\log Q)^{1/2}(\log\log Q)^{-3/2}=O_\varepsilon((\log\log Q)^{-1/2+\varepsilon})$$

and so $\mu=0$, by letting $Q\to\infty$. □

9.3 | Large moments of continued fractions expansions

For all $\lambda\geqslant 0$ and $x\in\mathbb{Q}\cap(0,1)$, we recall that $\Sigma_\lambda(x)$ was defined in (2.4). For $0<\alpha<2$, define

$$c_\alpha=\left(\frac{\Gamma(1-\alpha)\cos(\frac{\pi\alpha}{2})}{\pi^2/12}\right)^{1/\alpha}$$

and by continuity $c_1=\frac{6}{\pi}$. Let

$$g_\alpha(x):=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\mathrm{e}^{-itx-(c_\alpha|t|)^\alpha(1-i\,\mathrm{sgn}(t)\tan(\frac{\pi\alpha}{2}))}\,\mathrm{d}t, & (\alpha\neq 1)\\ \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\mathrm{e}^{-itx-c_1|t|(1+i\frac{2}{\pi}\,\mathrm{sgn}(t)\log|t|)}\,\mathrm{d}t, & (\alpha=1)\end{cases}$$

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

be the probability distribution function of a stable law $S_\alpha(c_\alpha, 1, 0)$ (see [102]), and

$$G_\alpha(v) := \int_{-\infty}^{v} g_\alpha(x)\,\mathrm{d}x.$$

Theorem 9.4. *Let $\lambda \geqslant 0$ and $v \in \mathbb{R}$, and for $\lambda < 1$ define $\mu_\lambda = \frac{12}{\pi^2}\sum_{n\geqslant 1} n^\lambda \log(\frac{(n+1)^2}{n(n+2)})$.*

(1) *If $\lambda < 1/2$, then with for some $\sigma_\lambda > 0$, we have*

$$\mathbb{P}_Q\left(\frac{\Sigma_{1/2}(x) - \mu_\lambda \log Q}{\sigma_\lambda \sqrt{\log Q}} \leqslant v\right) = \Phi(v) + O_\varepsilon\left(\frac{1}{(\log Q)^{\min(1/2,1/(2\lambda)-1-\varepsilon)}}\right). \tag{9.13}$$

(2) *If $\lambda = 1/2$, then with $\sigma = (\pi^2/6)^{-1/2}$, we have*

$$\mathbb{P}_Q\left(\frac{\Sigma_{1/2}(x) - \mu_{1/2} \log Q}{\sigma\sqrt{\log Q \log\log Q}} \leqslant v\right) = \Phi(v) + O\left(\frac{1}{(\log\log Q)^{1-\varepsilon}}\right). \tag{9.14}$$

(3) *If $1/2 < \lambda < 1$, then*

$$\mathbb{P}_Q\left(\frac{\Sigma_\lambda(x) - \mu_\lambda \log Q}{(\log Q)^\lambda} \leqslant v\right) = G_{1/\lambda}(v) + O\left(\frac{1}{(\log\log Q)^{1-\varepsilon}}\right). \tag{9.15}$$

(4) *If $\lambda = 1$, then letting γ_0 denote the Euler constant,*

$$\mathbb{P}_Q\left(\frac{\Sigma_1(x)}{\log Q} - \frac{\log\log Q - \gamma_0}{\pi^2/12} \leqslant v\right) = G_1(v) + O\left(\frac{1}{(\log Q)^{1-\varepsilon}}\right). \tag{9.16}$$

(5) *If $\lambda > 1$, then*

$$\mathbb{P}_Q\left(\frac{\Sigma_\lambda(x)}{(\log Q)^\lambda} \leqslant v\right) = G_{1/\lambda}(v) + O\left(\frac{1}{(\log\log Q)^{1-\varepsilon}}\right). \tag{9.17}$$

In all four cases, the implied constant depends at most on ε and λ.

Except for (9.15), we expect the error terms to be optimal up to an exponent ε. The estimate (9.17) is in accordance with results on the statistical distribution of $\max_{1\leqslant j\leqslant r} a_j$ [28, 59].

To prove Theorem 9.4, we note that $\Sigma_\lambda(x) = S_{\phi_\lambda}(x)$ with $\phi_\lambda(x) := \lfloor 1/x\rfloor^\lambda$. The function ϕ_λ satisfies the hypotheses of Theorem 3.1 with $d = m = 1$, $\alpha_0 = 1/\lambda - \varepsilon$, $\kappa_0 = 1$, and all small enough exponents $\lambda_0 > 0$.

9.3.1 | Case $\lambda < 1/2$

We wish to apply Corollary 3.2. To proceed, we need to show that ϕ_λ is not of the shape $c\log + f - f\circ T$. Suppose that it were, let $n \geqslant 1$ and $x \in (0,1)$ solve $x = \frac{1}{n+x}$. Evaluating at x yields $n^\lambda \sim -c\log n$ as $n\to\infty$, a contradiction. Corollary 3.2 may be applied and yields (9.13).

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

9.3.2 | Case $\lambda = 1/2$

The estimates (3.9)–(3.11) hold, and the integral is evaluated in [12, Corollary 3.2]. With the notation $\sigma = (\pi^2/6)^{-1/2}$ and

$$\chi_{1/2,Q}(t) := \mathbb{E}_Q\left(\exp\left\{it\frac{\Sigma_{1/2}(x) - \mu_{1/2}\log Q}{\sigma\sqrt{\log Q \log\log Q}}\right\}\right),$$

we find that for $|t| \leqslant \log\log Q$,

$$\chi_{1/2,Q}(t) = \exp\left\{-\frac{3t^2}{\pi^2} + O\left(\frac{1}{Q^\delta} + \frac{|t|}{(\log Q)^{1/2-\varepsilon}} + \frac{t^2}{(\log\log Q)^{1-\varepsilon}}\right)\right\}.$$

On the other hand, by (9.1), we have $\chi_{1/2,Q}(t) = 1 + O(|t|^{1/2}\log Q)$. Inserting these two bounds in the Berry–Esseen theorem [110, theorem II.7.16] yields the claimed conclusion (9.14).

9.3.3 | Case $\lambda = 1$

We use the estimates (3.9)–(3.10). Define

$$\chi_{1,Q}(t) := \mathbb{E}_Q\left(\exp\left\{it\left(\frac{\Sigma_1(x)}{\log Q} - \frac{\log\log Q - \gamma_0}{\pi^2/12}\right)\right\}\right).$$

Then for $0 < t \leqslant (\log Q)^{1-\varepsilon}$, we obtain by [12, Corollary 3.3]

$$\chi_{1,Q}(t) = \exp\left\{-\frac{it}{\pi^2/12}(|\log t| - \pi i) + O\left(\frac{1}{Q^\delta} + \frac{|t|^{1-\varepsilon} + |t|^{2-\varepsilon}}{(\log Q)^{1-\varepsilon}}\right)\right\},$$

and we may again conclude by the Berry–Esseen inequality.

9.3.4 | Case $\lambda \notin \{1/2, 1\}$

Assume first $\lambda > 1$. Then we use the estimates (3.9)–(3.10). Define

$$\chi_{\lambda,Q}(t) := \mathbb{E}_Q\left(\exp\left\{it\frac{\Sigma_\lambda(x)}{(\log Q)^\lambda}\right\}\right).$$

Then, by [12, Corollary 3.2], for $0 \leqslant t \leqslant \log\log Q$, we obtain

$$\chi_{\lambda,Q}(t) = \exp\left\{-(c_{1/\lambda}t)^{1/\lambda}\left(1 - i\tan\left(\tfrac{\pi}{2\lambda}\right)\right) + O\left(\frac{1}{Q^\delta} + \frac{t^{1/\lambda-\varepsilon}}{(\log Q)^{1-\varepsilon}} + \frac{t^{1/\lambda}}{(\log\log Q)^{1-\varepsilon}}\right)\right\}$$

and we may again conclude by the Berry–Esseen inequality and Lemma 9.1.

The case $\lambda \in (1/2, 1)$ follows by identical computations, the shift by $\mu_\lambda \log Q$ being accounted for by the linear term in the asymptotic evaluation of (3.11), as performed in [12, Corollary 3.2]

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by CochraneItalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

9.4 | Dedekind sums

Proof of Theorem 2.7. By [60, Theorem 1], we have for $x \in \mathbb{Q} \cap (0,1)$ the equality

$$s(x) = \delta_x + \frac{1}{12}\sum_{j=1}^{r} \phi_j(T^{j-1}(x)),$$

where $|\delta_x| \leqslant \frac{5}{12}$ and $\phi_j(x) := (-1)^{j-1}\lfloor 1/x \rfloor$. Note that ϕ_j depends only on the parity of j. Since changing v by an amount $O(1/\log Q)$ does not affect the right-hand side of (2.7), we may replace $s(x)$ by $s(x) - \delta_x$. Let

$$\chi(t) := \mathbb{E}_Q\left(\exp\left\{it\frac{s(x)-\delta_x}{\log Q}\right\}\right).$$

Then Theorem 3.1 applies with $d = 1$, $m = 2$ and the functions ϕ_j defined above, with $\alpha_0 = 1 - \varepsilon$. We use the expression (3.10) and refer to [12, Corollary 3.4] for the evaluation of the integral, obtaining

$$\chi(t) = \exp\left\{-\frac{|t|}{2\pi} + O\left(\frac{1}{Q^\delta} + \frac{|t| + |t|^{1-\varepsilon}}{(\log Q)^{1-\varepsilon}}\right)\right\},$$

and we conclude again by the Berry–Esseen inequality. □

ACKNOWLEDGEMENTS

This paper was partially written during a visit of S. Bettin at the Aix-Marseille University and a visit of S. Drappeau at the University of Genova. The authors thank both Institution for the hospitality and Aix-Marseille University and INdAM for the financial support for these visits. The authors thank B. Vallée for help regarding the references, J. Marklof and B. Borda for discussions, and the anonymous referees for her or his remarks.

S. Bettin is member of the INdAM group GNAMPA and his work is partially supported by PRIN 2017 'Geometric, algebraic and analytic methods in arithmetic'.

JOURNAL INFORMATION

The *Proceedings of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. J. Aaronson and M. Denker, *Local limit theorems for partial sums of stationary sequences generated by Gibbs-Markov maps*, Stochastics Dyn. **1** (2001), no. 2, 193–237.
2. J. E. Andersen and S. K. Hansen, *Asymptotics of the quantum invariants for surgeries on the figure 8 knot*, J. Knot Theory Ramifications **15** (2006), no. 4, 479–548.
3. V. Baladi and A. Hachemi, *A local limit theorem with speed of convergence for Euclidean algorithms and diophantine costs*, Ann. Inst. H. Poincaré Probab. Statist. **44** (2008), no. 4, 749–770.
4. V. Baladi and B. Vallée, *Euclidean algorithms are Gaussian*, J. Number Theory **110** (2005), no. 2, 331–386.
5. V. Baladi and B. Vallée, *Exponential decay of correlations for surface semi-flows without finite Markov partitions*, Proc. Amer. Math. Soc. **133** (2005), no. 3, 865–874.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by Cochraneltalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

6. M. Balazard and B. Martin, *Sur une équation fonctionnelle approchée due à J. R. Wilton*, Mosc. Math. J. **15** (2015), no. 4, 629–652.
7. S. Bettin, *On the distribution of a cotangent sum*, Int. Math. Res. Not. (2015), no. 21, 11419–11432.
8. S. Bettin, *On the reciprocity law for the twisted second moment of Dirichlet L-functions*, Trans. Amer. Math. Soc. **368** (2016), no. 10, 6887–6914.
9. S. Bettin, *High moments of the Estermann function*, Algebra Number Theory **13** (2019), no. 2, 251–300.
10. S. Bettin and J. B. Conrey, *Period functions and cotangent sums*, Algebra Number Theory **7** (2013), no. 1, 215–242.
11. S. Bettin and J. B. Conrey, *A reciprocity formula for a cotangent sum*, Int. Math. Res. Not. (2013), no. 24, 5709–5726.
12. S. Bettin and S. Drappeau, *Asymptotic expansions of oscillatory integrals*, Ramanujan J. **57** (2022), 849–861.
13. S. Bettin and S. Drappeau, *Modularity and value distribution of quantum invariants of hyperbolic knot*, Math. Ann., **382** (2022), no. 3–4, 1631–1679.
14. S. Bettin and S. Drappeau, *On the distribution of non-zero weight quantum modular forms*, Preprint.
15. P. Billingsley, *Probability and measure*, 3rd edn., Wiley Series Probab. Math. Stat., John Wiley & Sons, Inc., New York, N.Y., 1995.
16. V. Blomer, É. Fouvry, E. Kowalski, Ph. Michel, and D. Milićević, *On moments of twisted L-functions*, Amer. J. Math. **139** (2017), no. 3, 707–768.
17. V. Blomer, É. Fouvry, E. Kowalski, Ph. Michel, and D. Milićević, *Some applications of smooth bilinear forms with Kloosterman sums*, Tr. Mat. Inst. Steklova **296** (2017), 24–35; English version published in Proc. Steklov Inst. Math. **2**96 (2017), no. 1, 18–29.
18. V. Blomer, É. Fouvry, E. Kowalski, Ph. Michel, D. Milićević, and W. Sawin, *The second moment theory of families of L-functions*, Mem. Amer. Math. Soc., to appear.
19. K. Bringmann, A. Folsom, and R. C. Rhoades, *Unimodal sequences and "strange" functions: a family of quantum modular forms*, Pacific J. Math. **274** (2015), no. 1, 1–25.
20. K. Bringmann, J. Kaszian, and A. Milas, *Vector-valued higher depth quantum modular forms and higher Mordell integrals*, J. Math. Anal. Appl. **480** (2019), no. 2, 123397, 22.
21. K. Bringmann, Y. Li, and R. C. Rhoades, *Asymptotics for the number of row-Fishburn matrices*, European J. Combin. **41** (2014), 183–196.
22. K. Bringmann, J. Lovejoy, and L. Rolen, *On some special families of q-hypergeometric Maass forms*, Int. Math. Res. Not. (2018), no. 18, 5537–5561.
23. K. Bringmann and L. Rolen, *Half-integral weight Eichler integrals and quantum modular forms*, J. Number Theory **161** (2016), 240–254.
24. A. Broise, *Transformations dilatantes de l'intervalle et théorèmes limites*, Astérisque (1996), no. 238, 1–109.
25. R. Bruggeman, *Dedekind sums and Fourier coefficients of modular forms*, J. Number Theory **36** (1990), no. 3, 289–321.
26. R. Bruggeman, Y. Choie, and N. Diamantis, *Holomorphic automorphic forms and cohomology*, Mem. Amer. Math. Soc. **253** (2018), no. 1212, vii+167.
27. R. Bruggeman, J. Lewis, and D. Zagier, *Period functions for Maass wave forms and cohomology*, Mem. Amer. Math. Soc. **237** (2015), no. 1118, v+128.
28. E. Cesaratto and B. Vallée, *Small quotients in Euclidean algorithms*, Ramanujan J. **24** (2011), no. 2, 183–218.
29. D. Choi and S. Lim, *Finite-dimensional period spaces for the spaces of cusp forms*, Israel J. Math. **216** (2016), no. 2, 507–543.
30. D. Choi, S. Lim, and R. C. Rhoades, *Mock modular forms and quantum modular forms*, Proc. Amer. Math. Soc. **144** (2016), no. 6, 2337–2349.
31. J. B. Conrey, *More than two fifths of the zeros of the Riemann zeta function are on the critical line*, J. reine angew. Math. **399** (1989), 1–26.
32. J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, *Integral moments of L-functions*, Proc. Lond. Math. Soc. (3) **91** (2005), no. 1, 33–104.
33. I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinaĭ, *Ergodic theory*, Grundlehren der Mathematischen Wissenschaften, vol. 245, Springer, New York, N.Y., 1982. (Translated from the Russian by A. B. Sosinskiĭ.)
34. J. E. Cremona, *Algorithms for modular elliptic curves*, 2nd edn., Cambridge Univ. Press, Cambridge, 1997.
35. P. Deligne, *La conjecture de Weil. I*, Publ. Math. Inst. Hautes Études Sci. (1974), no. 43, 273–307.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by CochraneItalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

36. E. Demirci Akarsu, *Short incomplete Gauss sums and rational points on metaplectic horocycles*, Int. J. Number Theory **10** (2014), no. 6, 1553–1576.
37. T. Dimofte, S. Gukov, J. Lenells, and D. Zagier, *Exact results for perturbative Chern-Simons theory with complex gauge group*, Commun. Number Theory Phys. **3** (2009), no. 2, 363–443.
38. J. D. Dixon, *The number of steps in the Euclidean algorithm*, J. Number Theory **2** (1970), 414–422.
39. W. Doeblin, *Remarques sur la théorie métrique des fractions continues*, Compos. Math. **7** (1940), 353–371.
40. D. Dolgopyat, *On Decay of correlations in Anosov flows*, Ann. of Math. (2) **147** (1998), no. 2, 357–390.
41. T. Estermann, *On the representations of a number as the sum of two product*, J. Lond. Math. Soc. (2) **s1-5** (1930), no. 2, 131–137.
42. W. Feller, *An introduction to probability theory and its applications*, vol. II, 2nd edn., John Wiley & Sons, Inc., New York–London–Sydney, 1971.
43. Ph. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge Univ. Press, Cambridge, 2009.
44. Ph. Flajolet, B. Vallée, and I. Vardi, *Continued fractions from Euclid to the present day*, unpublished.
45. A. Folsom, *Mock modular forms and d-distinct partitions*, Adv. Math. **254** (2014), 682–705.
46. A. Folsom, K. Ono, and R. C. Rhoades, *Mock theta functions and quantum modular forms*, Forum Math. Pi **1** (2013), e2.
47. R. Fortet, *Sur une suite egalement répartie*, Studia Math. **9** (1940), 54–70.
48. S. Garoufalidis, *Quantum knot invariants*, Res. Math. Sci. **5** (2018), no. 1, 11.
49. S. Garoufalidis and R. Kashaev, *Evaluation of state integrals at rational points*, Commun. Number Theory Phys. **9** (2015), no. 3, 549–582.
50. S. Garoufalidis and D. Zagier, *Quantum modularity of the Kashaev invariant*, Preprint, arXiv:1511.05628 [math.GT], 2015.
51. D. Goldfeld, *The distribution of modular symbols*, *Number theory in progress*, vol. 2, de Gruyter, Berlin, 1999, pp. 849–865.
52. S. Gouëzel, *Limit theorems in dynamical systems using the spectral method*, Hyperbolic dynamics, fluctuations and large deviations, Proc. Sympos. Pure Math., vol. 89, Amer. Math. Soc., Providence, R.I., 2015, pp. 161–193.
53. Y. Guivarc'h and J. Hardy, *Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov*, Ann. Inst. H. Poincaré Probab. Statist. **24** (1988), no. 1, 73–98.
54. Y. Guivarc'h and Y. Le Jan, *Asymptotic winding of the geodesic flow on modular surfaces and continued fractions*, Ann. Sci. Éc. Norm. Supér. (4) **26** (1993), no. 1, 23–50.
55. G. H. Hardy and J. E. Littlewood, *Some problems of diophantine approximation*, Acta Math. **37** (1914), no. 1, 193–239.
56. A. J. Harper, *Sharp conditional bounds for moments of the Riemann zeta function*, Preprint, 2013.
57. H. Heilbronn, *On the average length of a class of finite continued fractions*, Number theory and analysis (Papers in Honor of Edmund Landau), Plenum, New York, N.Y., 1969, pp. 87–96.
58. L. Heinrich, *Rates of convergence in stable limit theorems for sums of exponentially ψ-mixing random variables with an application to metric theory of continued fractions*, Math. Nachr. **131** (1987), 149–165.
59. D. Hensley, *The largest digit in the continued fraction expansion of a rational number*, Pacific J. Math. **151** (1991), no. 2, 237–255.
60. D. Hickerson, *Continued fractions and density results for Dedekind sums*, J. reine angew. Math. **290** (1977), 113–116.
61. H.-K. Hwang, *Large deviations for combinatorial distributions. I. Central limit theorems*, Ann. Appl. Probab. **6** (1996), no. 1, 297–319.
62. C. T. Ionescu Tulcea and G. Marinescu, *Théorie ergodique pour des classes d'opérations non complètement continues*, Ann. of Math. (2) **52** (1950), 140–147.
63. H. Iwaniec, *Topics in classical automorphic forms*, Grad. Stud. Math., vol. 17, Amer. Math. Soc., Providence, R.I., 1997.
64. H. Iwaniec, *Spectral methods of automorphic forms*, 2nd edn., Grad. Stud. Math., vol. 53, Amer. Math. Soc., Providence, R.I.; Revista Matemática Iberoamericana, Madrid, 2002.
65. H. Iwaniec and E. Kowalski, *Analytic number theory*, vol. 53, Cambridge Univ. Press, Cambridge, 2004.
66. S. Jaffard and B. Martin, *Multifractal analysis of the Brjuno function*, Invent. Math. **212** (2018), no. 1, 109–132.
67. W. B. Jurkat and J. W. Van Horne, *The uniform central limit theorem for theta sums*, Duke Math. J. **50** (1983), no. 3, 649–666.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by CochraneItalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

68. R. M. Kashaev, *A link invariant from quantum dilogarithm*, Modern Phys. Lett. A **10** (1995), no. 19, 1409–1418.
69. T. Kato, *Perturbation theory for linear operators*, Classics Math., Springer, Berlin, 1995. (Reprint of the 1980 edition.)
70. B. Kim, S. Lim, and J. Lovejoy, *Odd-balanced unimodal sequences and related functions: parity, mock modularity and quantum modularity*, Proc. Amer. Math. Soc. **144** (2016), no. 9, 3687–3700.
71. B. R. Kloeckner, *Effective perturbation theory for simple isolated eigenvalues of linear operators*, J. Operator Theory **81** (2019), no. 1, 175–194.
72. S. Kumagai, *Technical comment to: "An implicit function theorem" by K. Jittorntrum*, J. Optim. Theory Appl. **31** (1980), no. 2, 285–288.
73. R. Lawrence and D. Zagier, *Modular forms and quantum invariants of 3-manifolds*, Asian J. Math **3** (1999), no. 1, 93–107. (Sir Michael Atiyah: a great mathematician of the twentieth century.)
74. J. Lee and H.-S. Sun, *Dynamics of continued fractions and distribution of modular symbols*, Preprint, 2019.
75. P. Lévy, *Calcul des probabilités*, VIII + 352 S. Paris, Gauthier-Villars (1925), 1925.
76. P. Lévy, *Fractions continues aléatoires*, Rend. Circ. Mat. Palermo (2) **1** (1952), 170–208.
77. J. Lewis and D. Zagier, *Cotangent sums, quantum modular forms, and the generalized Riemann hypothesis*, Res. Math. Sci. **6** (2019), no. 1, 4.
78. W. Luo, *Nonvanishing of the central L-values with large weight*, Adv. Math. **285** (2015), 220–234.
79. H. Maier and M. Th. Rassias, *Generalizations of a cotangent sum associated to the Estermann zeta function*, Commun. Contemp. Math. **18** (2016), no. 1, 1550078, 89.
80. Y. I. Manin, *Lectures on modular symbols*, Arithmetic geometry, Clay Math. Proc., vol. 8, Amer. Math. Soc., Providence, R.I., 2009, pp. 137–152.
81. J. Marklof, *Limit theorems for theta sums*, Duke Math. J. **97** (1999), no. 1, 127–153.
82. D. H. Mayer, *On a ζ function related to the continued fraction transformation*, Bull. Soc. Math. France **104** (1976), no. 2, 195–203.
83. D. H. Mayer, *Continued fractions and related transformations*, Ergodic theory, symbolic dynamics, and hyperbolic spaces (Trieste, 1989), Oxford Sci. Publ., Oxford Univ. Press, New York, N.Y., 1991, pp. 175–222.
84. D. H. Mayer, *The thermodynamic formalism approach to Selberg's zeta function for* PSL(2, **Z**), Bull. Amer. Math. Soc. (N.S.) **25** (1991), no. 1, 55–60.
85. B. Mazur and K. Rubin, *The statistical behavior of modular symbols and arithmetic conjectures*, Presentation at Toronto, November 2016, http://www.math.harvard.edu/~mazur/papers/heuristics.Toronto.12.pdf
86. A. Momeni and A. B. Venkov, *Mayer's transfer operator approach to Selberg's zeta function*, Algebra i Analiz **24** (2012), no. 4, 1–33.
87. I. D. Morris, *A short proof that the number of division steps in the Euclidean algorithm is normally distributed*, Preprint, 2015.
88. Y. Motohashi, *The binary additive divisor problem*, Ann. Sci. Éc. Norm. Supér. (4) **27** (1994), no. 5, 529–572.
89. H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186** (2001), no. 1, 85–104.
90. F. Naud, *Selberg's zeta function and Dolgopyat's estimates for the modular surface*, Based on lectures given at IHP Paris, July 2005, https://tinyurl.com/y7f2uxqb.
91. H. T. Ngo and R. C. Rhoades, *Integer partitions, probabilities and quantum modular forms*, Res. Math. Sci. **4** (2017), Paper No. 17, 36.
92. A. C. Nordentoft, *Central values of additive twists of modular L-functions*, J. reine angew. Math. **776** (2021), 255–293.
93. W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque (1990), no. 187-188, 268.
94. Y. N. Petridis, *Spectral deformations and Eisenstein series associated with modular symbols*, Int. Math. Res. Not. (2002), no. 19, 991–1006.
95. Y. N. Petridis and M. S. Risager, *Modular symbols have a normal distribution*, Geom. Funct. Anal. **14** (2004), no. 5, 1013–1043.
96. Y. N. Petridis and M. S. Risager, *Arithmetic statistics of modular symbols*, Invent. Math. **212** (2018), 1–57.
97. M. Pollicott, *Distribution of closed geodesics on the modular surface and quadratic irrationals*, Bull. Soc. Math. France **114** (1986), no. 4, 431–446.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/plms.12485 by CochraneItalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

98. H. Rademacher and E. Grosswald, *Dedekind sums*, The Carus Math. Monographs, No. 16, The Mathematical Association of America, Washington, D.C., 1972.
99. M. Radziwiłł and K. Soundararajan, *Moments and distribution of central L-values of quadratic twists of elliptic curves*, Invent. Math. **202** (2015), no. 3, 1029–1068.
100. T. Rivoal and J. Roques, *Convergence and modular type properties of a twisted Riemann series*, Unif. Distrib. Theory **8** (2013), no. 1, 97–119.
101. J. Rousseau-Egele, *Un théorème de la limite locale pour une classe de transformations dilatantes et monotones par morceaux*, Ann. Probab. **11** (1983), no. 3, 772–788.
102. G. Samorodnitsky and M. S. Taqqu, *Stable non-Gaussian random processes: Stochastic models with infinite variance*, Chapman & Hall, New York, N.Y., 1994.
103. A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), Univ. Salerno, Salerno, 1992, pp. 367–385.
104. K. Soundararajan, *Moments of the Riemann zeta function*, Ann. of Math. (2) **170** (2009), no. 2, 981–993.
105. E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. Series, vol. 43, Princeton Univ. Press, Princeton, N.J., 1993.
106. E. M. Stein and R. Shakarchi, *Fourier analysis*, Princeton Lect. Anal., vol. 1, Princeton Univ. Press, Princeton, N.J., 2003.
107. W. Stein, *Modular forms, a computational approach*, Grad. Stud. Math., vol. 79, Amer. Math. Soc., Providence, R.I., 2007. (With an appendix by Paul E. Gunnells.)
108. W. Stein, *Modular symbol statistics*, Beamer presentation, 2015.
109. Z. S. Szewczak, *On limit theorems for continued fractions*, J. Theoret. Probab. **22** (2009), no. 1, 239–255.
110. G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, 3rd edn., Grad. Stud. Math., vol. 163, Amer. Math. Soc., Providence, R.I., 2015.
111. B. Vallée, *Dynamical analysis of a class of Euclidean algorithms*, Theoret. Comput. Sci. **297** (2003), no. 1-3, 447–486.
112. B. Vallée, *Digits and continuants in Euclidean algorithms. Ergodic versus Tauberian theorems*, J. Théor. Nombres Bordeaux **12** (2000), no. 2, 531–570, Colloque International de Théorie des Nombres (Talence, 1999).
113. I. Vardi, *A relation between Dedekind sums and Kloosterman sums*, Duke Math. J. **55** (1987), no. 1, 189–197.
114. I. Vardi, *Dedekind sums have a limiting distribution*, Int. Math. Res. Not. (1993), no. 1, 1–12.
115. J. R. Wilton, *An approximate functional equation with applications to a problem of Diophantine approximation*, J. reine angew. Math. **169** (1933), 219–237.
116. E. Wirsing, *On the theorem of Gauss-Kusmin-Lévy and a Frobenius-type theorem for function spaces*, Acta Arith. **24** (1973/1974), 507–528. (Collection of articles dedicated to Carl Ludwig Siegel on the occasion of his seventy-fifth birthday, V.)
117. A. C. Yao and D. E. Knuth, *Analysis of the subtractive algorithm for greatest common divisors*, Proc. Natl. Acad. Sci. U.S.A **72** (1975), no. 12, 4720–4722.
118. M. P. Young, *The fourth moment of Dirichlet L-functions*, Ann. of Math. (2) **173** (2011), no. 1, 1–50.
119. D. Zagier, *From quadratic functions to modular functions*, Number theory in progress, vol. 2 (Zakopane-Kościelisko, 1997), de Gruyter, Berlin, 1999, pp. 1147–1178.
120. D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function*, Topology **40** (2001), no. 5, 945–960.
121. D. Zagier, *Quantum modular forms*, Quanta of maths, Clay Math. Proc., vol. 11, Amer. Math. Soc., Providence, R.I., 2010, pp. 659–675.

1460244x, 2022, 6, Downloaded from https://londmathsoc.onlinelibrary.wiley.com/doi/10.1112/jlms.12485 by CochraneItalia, Wiley Online Library on [21/09/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License