

RESEARCH ARTICLE

Hilbert–Kunz multiplicity of powers of ideals in dimension two

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Abstract

We study the behavior of the Hilbert–Kunz multiplicity of powers of an ideal in a local ring. In dimension 2, we provide answers to some problems raised by Smirnov, and give a criterion to answer one of his questions in terms of a “Ratliff–Rush version” of the Hilbert–Kunz multiplicity.

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1 | INTRODUCTION

Let (R, \mathfrak{m}) be a Noetherian local ring of prime characteristic $p > 0$ and Krull dimension d . The Hilbert–Kunz multiplicity of an \mathfrak{m} -primary ideal $I \subseteq R$ is defined as

$$e_{\text{HK}}(I) = \lim_{q \rightarrow \infty} \frac{\ell_R(R/I^{[q]})}{q^d},$$

where $q = p^e$ and $I^{[q]} = (x^q \mid x \in I)$ denotes the q th Frobenius power of I . This invariant was first studied by E. Kunz [20], even though its existence as a limit was only proved several years later by P. Monsky [24]. As the name suggests, it has similarities with the classical Hilbert–Samuel multiplicity

$$e(I) = \lim_{n \rightarrow \infty} \frac{d! \ell_R(R/I^n)}{n^d}.$$

Other than being defined in a similar fashion, the two invariants share some common features. For instance, if R is formally unmixed, then $e(\mathfrak{m}) = 1$ if and only if R is regular, if and only if $e_{\text{HK}}(\mathfrak{m}) = 1$. The first equivalence is a classical result due to P. Samuel (in the equal characteristic case [35]) and M. Nagata (in the general case [25]), while the second is a much more recent result due to K.I. Watanabe and K.I. Yoshida [42, Theorem 1.5] (see also [17]). Except for some similarities, however, the behavior of the two invariants is generally quite different. For instance, $e(I)$ is always a nonnegative integer, while $e_{\text{HK}}(I)$ can be irrational [4].

As one invariant is defined in terms of Frobenius powers $I^{[q]}$, and the other in terms of ordinary powers I^n , it is quite natural to wonder how the two interplay with each other. In particular, one can study the lengths $\ell_R(R/I^{[q]n})$ as a function of both q and n . For any given q , the classical theory of Hilbert polynomials guarantees that there exists $n_0 = n_0(I, q)$ such that, for all $n \geq n_0$,

$$\ell_R(R/I^{[q]n}) = e_0(I^{[q]}) \binom{n+d-1}{d} - e_1(I^{[q]}) \binom{n+d-2}{d-1} + \dots + (-1)^d e_d(I^{[q]})$$

for some integers $e_i(I^{[q]})$, $i = 0, \dots, d$, called Hilbert coefficients of $I^{[q]}$. Note that $e_0(I^{[q]}) = e(I^q) = q^d e(I)$. In [9, Theorem 3.2], Hanes proved that

$$\ell_R(R/I^{[q]n}) = \left(\frac{e(I)}{d!} n^d + O(n^{d-1}) \right) q^d,$$

which, in particular, shows that $\lim_{n \rightarrow \infty} \frac{d! e_{\text{HK}}(I^n)}{n^d} = e(I)$ [8, Corollary II.7]. With the goal of understanding the term $O(n^{d-1})$, V. Trivedi first (for standard graded algebras and equigenerated homogeneous ideals [39]), and I. Smirnov afterward (for any \mathfrak{m} -primary ideal in a local ring [37]), show that

$$e_{\text{HK}}(I^n) = e(I) \binom{n+d-1}{d} - \lim_{q \rightarrow \infty} \frac{e_1(I^{[q]})}{q^d} \binom{n+d-2}{d-1} + o(n^{d-1}),$$

and that the limit $\lim_{q \rightarrow \infty} \frac{e_1(I^{[q]})}{q^d}$ exists. Motivated by this result, and by the shape of the Hilbert polynomials, in [37], Smirnov poses two questions and a conjecture regarding the asymptotic behavior of the function $n \mapsto e_{\text{HK}}(I^n)$:

- (Q1) $L_i(I) := \lim_{q \rightarrow \infty} \frac{e_i(I^{[q]})}{q^d}$ exists as a limit for all $i = 0, \dots, d$.
- (Q2) $e_{\text{HK}}(I^n) = \sum_{i=0}^d (-1)^i L_i(I) \binom{n+d-1-i}{d-i}$ for all $n \gg 0$.
- (C) Assume further that R is weakly F-regular, analytically unramified and Cohen–Macaulay. Then, $L_1(I) = e(I) - e_{\text{HK}}(I)$ if and only if I is stable, that is, $I^2 = IJ$ for a minimal reduction J of I .

Conjecture (C) is actually stated in [37] without the assumption that R is weakly F-regular. Without this condition, it is shown in [2] that the conjecture has actually negative answer. Note that (Q1) is trivial for $i = 0$, and it is known for $i = 1$ by [37]. Therefore, (Q1) is open for $i = 2, \dots, d$. Regarding (C), we observe that if I is stable, then $L_1(I) = e(I) - e_{\text{HK}}(I)$ is already known to hold, even without the assumption that R is weakly F-regular and analytically unramified; see [43, Theorem 1.8 (2)].

Some progress toward answering these problems was achieved in [2]. In Section 2, we present more cases in which the answers are positive: ideals whose powers are of finite projective dimension in numerically Roberts rings, and parameter ideals in generalized Cohen–Macaulay rings. In Sections 3 and 4, we study the problem in bigger generality. The novelty of our approach stems from considering an asymptotic invariant defined in terms of the Ratliff–Rush closure \widetilde{I} of an ideal I (see Section 3 for more details). If I is \mathfrak{m} -primary, we prove that the limit

$$\widetilde{e}_{\text{HK}}(I) = \lim_{q \rightarrow \infty} \frac{\ell_R(R/\widetilde{I}^{[q]})}{q^d},$$

exists (see Proposition 3.6) under mild assumptions, and we show that this invariant has strong connections with (Q1) and (Q2) for rings of dimension 2. Our main findings can be summarized as follows.

Theorem A (Theorems 4.1 and 4.9). *Let (R, \mathfrak{m}) be a 2-dimensional excellent Cohen–Macaulay reduced local ring, and I be an \mathfrak{m} -primary ideal. Then*

- (1) $L_2(I)$ exists as a limit, and therefore (Q1) has a positive answer.
- (2) For all $n \gg 0$, we have

$$\widetilde{e}_{\text{HK}}(I^n) = \sum_{i=0}^2 (-1)^i L_i(I) \binom{n+1-i}{2-i}.$$

- (3) Conjecture (C) holds true.

As an immediate consequence of (2), we deduce that (Q2) has a positive answer for an ideal I in a 2-dimensional excellent Cohen–Macaulay reduced local ring if and only if $\widetilde{e}_{\text{HK}}(I^n) = e_{\text{HK}}(I^n)$ for all $n \gg 0$. We also give a characterization of this equality in terms of tight closure, see Proposition 3.11. In general, one always has $\widetilde{e}_{\text{HK}}(I^n) \leq e_{\text{HK}}(I^n)$, and in Section 4, we give an estimate of the difference between these invariants (see Proposition 4.15).

We prove that (Q2) has a positive answer in several cases.

Theorem B (Theorems 4.3 and 4.8). *Let (R, \mathfrak{m}) be a two-dimensional Cohen–Macaulay local ring.*

- (1) *If there exists a regular local ring $S \supseteq R$ such that the inclusion is module-finite, then (Q2) has a positive answer for every \mathfrak{m} -primary ideal I of R .*

(2) If $I \subseteq R$ is an \mathfrak{m} -primary ideal such that $e_2(I) = 0$, then $L_2(I) = 0$. Moreover, for all $n \gg 0$, we have

$$e_{\text{HK}}(I^n) = e(I) \binom{n+1}{2} - L_1(I)n.$$

In particular, (Q2) has a positive answer for I .

Finally, in Section 5, we collect some evidence that (Q2) might have negative answer, in general. We exhibit a 2-dimensional hypersurface ring (R, \mathfrak{m}) for which we suspect that $e_{\text{HK}}(\mathfrak{m}^n) \neq \widehat{e}_{\text{HK}}(\mathfrak{m}^n)$ for infinitely many $n > 0$. This is supported by some computational evidence, obtained by means of Macaulay2 calculations [6].

2 | SOME POSITIVE ANSWERS

We start by analyzing some special cases where the two questions (Q1),(Q2) and the conjecture (C) have positive answer. Some of them might be already known to experts, but we decided to include them here for the sake of completeness.

2.1 | Numerically Roberts rings

We recall that a Cohen–Macaulay local ring (R, \mathfrak{m}) is called *numerically Roberts* if $e_{\text{HK}}(I) = \ell_R(R/I)$ for every \mathfrak{m} -primary ideal I of finite projective dimension.

Examples of numerically Roberts rings include complete intersections, 2-dimensional Cohen–Macaulay local rings, and 3-dimensional Gorenstein local rings. For basic properties and more examples of numerically Roberts rings, we refer the reader to [22, Theorem 1.1].

Proposition 2.1. *Let (R, \mathfrak{m}) be a numerically Roberts ring of dimension d , and let $I \subseteq R$ be an \mathfrak{m} -primary ideal such that $\text{pd}_R(I^n) < \infty$ for every $n > 0$. Then, $L_i(I) = e_i(I)$ for every $i = 0, \dots, d$, and $e_{\text{HK}}(I^n) = \sum_{i=0}^d (-1)^i L_i(I) \binom{n+d-1-i}{d-i}$. Moreover, $L_1(I) = e(I) - e_{\text{HK}}(I)$ if and only if I is stable.*

Proof. Let $\sigma(I)$ denote the postulation number of I , that is, the largest integer for which Hilbert function and Hilbert polynomial of I do not coincide. By a result of C. Peskine and L. Szpiro [28], if I^n has finite projective dimension, then so does $(I^n)^{[q]}$ for any $q = p^e$. For $n > \sigma(I)$, we then have

$$\begin{aligned} \ell_R(R/(I^n)^{[q]}) &= e_{\text{HK}}((I^n)^{[q]}) \\ &= q^d e_{\text{HK}}(I^n) \\ &= q^d \ell_R(R/I^n) \\ &= q^d \left[\sum_{i=0}^d (-1)^i e_i(I) \binom{n+d-1-i}{d-i} \right] \\ &= \sum_{i=0}^d (-1)^i (q^d e_i(I)) \binom{n+d-1-i}{d-i}. \end{aligned}$$

Since, for any fixed q , this happens for all $n > \sigma(I)$, we have that $e_i(I^{[q]}) = q^d e_i(I)$ for every $i = 0, \dots, d$. Thus, clearly, $L_i(I) = e_i(I)$ for every $i = 0, \dots, d$. Dividing the above expression by q^d and taking limits $q \rightarrow \infty$ gives that

$$e_{\text{HK}}(I^n) = \sum_{i=0}^d (-1)^i L_i(I) \binom{n+d-1-i}{d-i}.$$

Finally, since $L_1(I) = e_1(I)$ and $e_{\text{HK}}(I) = \ell_R(R/I)$, the fact that $L_1(I) = e(I) - e_{\text{HK}}(I)$ is equivalent to I being stable follows from the result of C. Huneke [15] and A. Ooishi [27]. □

Two immediate corollaries are as follows.

Corollary 2.2. *If (R, \mathfrak{m}) is a regular local ring, then (Q1),(Q2),(C) hold true.*

Corollary 2.3. *Let (R, \mathfrak{m}, k) be a d -dimensional complete numerically Roberts ring, and let x_1, \dots, x_d be a system of parameters. Let $A = k[[x_1, \dots, x_d]] \subseteq R$, and $J \subseteq A$ be a (x_1, \dots, x_d) -primary ideal. Then (Q1),(Q2),(C) hold true for the ideal $I = J \cdot R$.*

Proof. It suffices to observe that, since R is Cohen–Macaulay, it is a finitely generated free A -module. It follows that $\text{pd}_R(I^n) = \text{pd}_A(J^n) < \infty$ for all n . □

2.2 | Parameter ideals in generalized Cohen–Macaulay rings

We say that a local ring (R, \mathfrak{m}) of dimension d is generalized Cohen–Macaulay if $\ell_R(H_{\mathfrak{m}}^i(R)) < \infty$ for every $i = 0, \dots, d - 1$. Note that, if R is equidimensional, it is generalized Cohen–Macaulay if and only if $R_{\mathfrak{p}}$ is Cohen–Macaulay for every $\mathfrak{p} \in \text{Spec}(R) \setminus \{\mathfrak{m}\}$.

Proposition 2.4. *Let (R, \mathfrak{m}) be a generalized Cohen–Macaulay local ring of dimension d , and $J \subseteq R$ be a parameter ideal. We have $L_i(J) = 0$ for all $i = 1, \dots, d$. In particular, questions (Q1) and (Q2) have positive answer for J .*

Proof. If $d = 0$ the conclusion is void, hence true. Assume that $d \geq 1$. Since R is generalized Cohen–Macaulay, there exists an integer $N > 0$ such that $e_i(J)$ is a function of $\ell_R(H_{\mathfrak{m}}^0(R)), \dots, \ell_R(H_{\mathfrak{m}}^{d-1}(R))$ for any $i = 1, \dots, d$ and any parameter ideal $J \subseteq \mathfrak{m}^N$ [36]. In particular, if J is a parameter ideal, then there exists q_0 such that $e_i(J^{[q]})$ is independent of q for any $q \geq q_0$.

It follows that

$$L_i(J) = \lim_{q \rightarrow \infty} \frac{e_i(J^{[q]})}{q^d} = 0$$

for every $i = 1, \dots, d$. Finally, by [43, Lemma 1.3], we have that

$$e_{\text{HK}}(J^n) = e(J) \binom{n+d-1}{d}$$

for every integer n . □

Remark 2.5. The special case of Proposition 2.4 in which R is Buchsbaum was already covered by [2, Theorem 1.3]. Also, note that Conjecture (C) is trivial for parameter ideals.

3 | PRELIMINARIES AND GENERAL METHODS

Let (R, \mathfrak{m}) be a d -dimensional Cohen–Macaulay local ring, and I be an \mathfrak{m} -primary ideal. A collection of ideals $\mathbb{F} = \{F_n\}$ such that $R = F_0 \supsetneq F_1 \supseteq \dots$ is called an I -filtration if $IF_n \subseteq F_{n+1}$ for all $n \geq 0$. Note that F_1 is an \mathfrak{m} -primary ideal, since it contains I . Also, note that we do not assume that $F_m \cdot F_n \subseteq F_{n+m}$ for $n, m \geq 0$, as we will not need to impose this condition. We say that \mathbb{F} is an I -good filtration if $IF_n = F_{n+1}$ for all $n \gg 0$. Let J be a minimal reduction of \mathbb{F} , that is, an ideal minimal with respect to containment in I such that $F_{n+1} = JF_n$ for $n \gg 0$. If R/\mathfrak{m} is infinite, we may assume that J is generated by a regular sequence x_1, \dots, x_d . Let $r_J(\mathbb{F})$ be the reduction number of \mathbb{F} with respect to J , that is, the smallest n such that $F_{n+1} = JF_n$. For an integer $n \geq 0$, let $v_{\mathbb{F}}(n) = \ell_R(F_n/JF_{n-1})$. Let $H_{\mathbb{F}}(n) = \ell_R(R/F_n)$ be the Hilbert–Samuel function of \mathbb{F} , which in our assumptions for $n \gg 0$ coincides with the Hilbert Polynomial $P_{\mathbb{F}}(n) = \sum_{i=0}^d (-1)^i e_i(\mathbb{F}) \binom{n+d-1-i}{d-i}$. Given a numerical function $H : \mathbb{Z} \rightarrow \mathbb{N}$ satisfying $H(n) = 0$ for $n \ll 0$, we will denote by $\Delta H : \mathbb{Z} \rightarrow \mathbb{N}$ the function defined as $\Delta H(n) = H(n) - H(n-1)$. For $j \geq 1$, we inductively define $\Delta^j = \Delta(\Delta^{j-1})$. If \mathbb{F} is a filtration as above, we will use the convention that $F_n = R$ and $\ell_R(R/F_n) = 0$ for $n < 0$. In particular, we will regard $H_{\mathbb{F}}$ as a function defined over \mathbb{Z} by setting $H_{\mathbb{F}}(n) = 0$ for $n < 0$.

Let $G(\mathbb{F}) = \bigoplus_{n \geq 0} F_n/F_{n+1}$. Our assumptions guarantee that $G(\mathbb{F})$ is a graded module over the associated graded ring $G(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$. We call $G(\mathbb{F})$ the associated graded module of the filtration \mathbb{F} . If we assume that $\text{depth}(G(\mathbb{F})) \geq d-1$, then we have some useful relations between the invariants we have defined, see [7, 14] and [33, Theorem 2.5] for the case of filtrations of modules.

Proposition 3.1. *Let (R, \mathfrak{m}) be a d -dimensional Cohen–Macaulay local ring, I be an \mathfrak{m} -primary ideal, and \mathbb{F} be an I -good filtration. Assume that $\text{depth}(G(\mathbb{F})) \geq d-1$. Then,*

- (1) $\Delta^d(P_{\mathbb{F}}(n) - H_{\mathbb{F}}(n)) = v_{\mathbb{F}}(n)$.
- (2) $\Delta^d(P_{\mathbb{F}}(n)) = e_0(\mathbb{F}) = e_0(I)$.
- (3) $\Delta^d(H_{\mathbb{F}}(n)) = \sum_{i=0}^d (-1)^i \binom{d}{i} \ell_R(R/F_{n-i})$.
- (4) $e_i(\mathbb{F}) = \sum_{n \geq i} \binom{n-1}{i-1} v_{\mathbb{F}}(n)$ for $i = 1, \dots, d$.

We recall the definition of Ratliff–Rush closure of an ideal.

Definition 3.2. Let R be a Noetherian ring, and let $I \subseteq R$ be a regular ideal, that is, an ideal containing a regular element. The Ratliff–Rush closure of I is defined as $\widetilde{I} = \bigcup_{n \geq 1} (I^{n+1} : I^n)$.

As pointed out in [32], $(\widetilde{})$ is not a closure “in the usual sense;” for instance, it is not always true that $I \subseteq J$ implies that $\widetilde{I} \subseteq \widetilde{J}$. Since R is Noetherian, for any regular ideal I , there exists $n \gg 0$ such that $\widetilde{I} = I^{n+1} : I^n$. Moreover, $\widetilde{I^N} = I^N$ for any $N \gg 0$, possibly depending on I (see [30]). We refer the interested reader to [32] for more details on this operation.

Now assume that (R, \mathfrak{m}) has characteristic $p > 0$. For any $q = p^e$, consider the filtration $\widetilde{\mathbb{F}}_q = \{\widetilde{\mathbb{F}}_{q,n}\} = \left\{ \widetilde{(I^{[q]})^n} \right\}$, which can be proved to be $I^{[q]}$ -good. We also let $\mathbb{F}_q = \{\mathbb{F}_{q,n}\}$ with $\mathbb{F}_{q,n} = (I^{[q]})^n$. In the case of the latter, we will write $P_{I^{[q]}}$, $H_{I^{[q]}}$, and $e_i(I^{[q]})$ in place of $P_{\mathbb{F}_q}$, $H_{\mathbb{F}_q}$, and $e_i(\mathbb{F}_q)$, respectively. We now collect some basic facts about $\widetilde{\mathbb{F}}_q$.

Proposition 3.3. *With the above notation, for any $q = p^e$, we have that*

- (1) $\text{depth}(G(\widetilde{\mathbb{F}}_q)) \geq 1$.
- (2) $r(\widetilde{\mathbb{F}}_q) \leq r$, where $r = r_J(I)$ is the reduction number of the filtration $\{I^n\}$ with respect to a minimal reduction J .
- (3) If $\dim(R) = 2$, then for all $n \geq r - 1$, we have $H_{\widetilde{\mathbb{F}}_q}(n) = P_{\widetilde{\mathbb{F}}_q}(n)$.

Proof. (1) follows from [30, (2.3.1)] and [10, (1.2)], and (3) is an immediate consequence of (2) and Proposition 3.1 (1). For (2), first observe that $J^{[q]}$ is a minimal reduction of $I^{[q]}$. Moreover, if $I^{r+1} = JI^r$, then by taking Frobenius powers, we obtain that $(I^{[q]})^{r+1} = J^{[q]}(I^{[q]})^r$, and thus, $r(I^{[q]}) \leq r$ for all q . We conclude by [32, Proposition 4.5] and the main result of [23]. \square

In what follows, $\mathcal{R} = R[It]$ will denote the Rees algebra of I , and $\mathcal{R}' = R[It, t^{-1}]$ the extended Rees algebra. For any q , we let $\mathcal{R}'_q = R[I^{[q]}t, t^{-1}]$. Let $J = (x_1, \dots, x_d)$ be a minimal reduction of I , and set $J_q = (x_1^q t, \dots, x_d^q t)$. By an analog of the Grothendieck–Serre’s formula for the Rees algebra (see [18] and [3]), we have that for all $n \in \mathbb{Z}$ and all $q = p^e$,

$$\begin{aligned} P_{I^{[q]}}(n) - H_{I^{[q]}}(n) &= \sum_{i=0}^d (-1)^i e_i(I^{[q]}) \binom{n + d - 1 - i}{d - i} - \ell_R(R/(I^{[q]})^n) \\ &= \sum_{i=0}^d (-1)^i \ell_R(H_{J_q}^i(\mathcal{R}'_q)_n). \end{aligned}$$

Given a \mathbb{Z} -graded module M , we let $a(M) = \sup\{n \in \mathbb{Z} \mid M_n \neq 0\}$.

Proposition 3.4. *Let (R, \mathfrak{m}) be a d -dimensional local ring. Then, $a(H_{J_q}^d(\mathcal{R}'_q)) \leq a(H_{J_1}^d(\mathcal{R}'))$ for all $q = p^e$.*

Proof. Let x_1, \dots, x_d be a system of parameters in R , and let $n \in \mathbb{Z}$ be such that $H_{J_1}^d(\mathcal{R}')_n = 0$. Now let $\eta \in H_{J_q}^d(\mathcal{R}'_q)_n$. From the description of local cohomology in terms of the Čech complex, we can write η as the equivalence class of an element $\frac{\alpha t^{n+ds}}{(x_1^q t \cdots x_d^q t)^s}$ modulo the image of the localization map:

$$\bigoplus_{i=1}^d (\mathcal{R}'_q)_{(x_1^q t) \cdots (\widehat{x_i^q t}) \cdots (x_d^q t)} \longrightarrow (\mathcal{R}'_q)_{(x_1^q t) \cdots (x_d^q t)},$$

and the fact that η has degree n means that $\alpha \in (I^{[q]})^{n+ds}$. Thus, we can write $\alpha = \sum_{i=1}^m r_i \alpha_i^q$ for some $\alpha_i \in I^{n+ds}$. Using again the Čech complex description, we can see each $\eta_i = \left[\frac{\alpha_i t^{n+ds}}{(x_1 t \cdots x_d t)^s} \right]$ as an element of $H_{J_1}^d(\mathcal{R}')_n$, and thus $\eta_i = 0$ by assumption. This means that there exist integers N_i such that

$$(x_1 \cdots x_d)^{N_i} \alpha_i \in (x_1^{N_i+s}, \dots, x_d^{N_i+s}) I^{(d-1)(N_i+s)+n}.$$

Setting $N = \max\{N_i \mid i = 1, \dots, m\}$ and raising the above relations to the q th Frobenius power, we obtain that

$$(x_1 \cdots x_d)^{Nq} \alpha_i^q \in (x_1^{q(N+s)}, \dots, x_d^{q(N+s)}) (I^{[q]})^{(d-1)(N+s)+n}$$

for all $i = 1, \dots, m$. In particular, the same relation holds for α :

$$(x_1 \cdots x_d)^{Nq} \alpha \in \left(x_1^{q(N+s)}, \dots, x_d^{q(N+s)}\right) (I^{[q]})^{(d-1)(N+s)+n},$$

and this precisely means that $\eta = 0$. □

In what follows, $F_*^e(R)$ will denote R viewed as a module over itself via the e th iterate of the Frobenius map $F^e : R \rightarrow R$, $F(r) = r^{p^e}$, for any integer $e \geq 1$. In case $e = 1$, we will write F in place of F^1 . The ring R is said to be F -finite if $F_*(R)$ is a finitely generated R -module. This condition is equivalent to $F_*^e(R)$ being a finitely generated R -module for some $e \geq 1$, and also equivalent to $F_*^e(R)$ being a finitely generated R -module for all $e \geq 1$.

The next lemma is a minor modification of [29, Theorem 4.3 (i)].

Lemma 3.5. *Let (R, \mathfrak{m}) be an F -finite reduced local ring of dimension d , and $\{I_e\}_{e \in \mathbb{N}}$ be a sequence of ideals satisfying:*

- (a) *There exists an \mathfrak{m} -primary ideal I such that $I^{[p^e]} \subseteq I_e$ for all e , and*
- (b) *$I_e^{[p]} \subseteq I_{e+1}$.*

Then, $\lim_{e \rightarrow \infty} \frac{\ell_R(R/I_e)}{p^{ed}} = \eta$ exists, and there exists a constant C independent of the sequence $\{I_e\}$ such that $\eta - \frac{\ell_R(R/I_e)}{p^{ed}} \leq \frac{C}{p^e}$ for all $e \in \mathbb{N}$.

Proof. Let $k = R/\mathfrak{m}$. Let $\text{Assh}(R) = \{\mathfrak{q} \in \text{Ass}(R) \mid \dim(R/\mathfrak{q}) = d\}$. Since R reduced, we have that $R_{\mathfrak{q}}$ is a field for every $\mathfrak{q} \in \text{Assh}(R)$. Moreover, since R is F -finite, for every $\mathfrak{q} \in \text{Assh}(R)$ by [21, Proposition 2.3], we have that $F_*(R)_{\mathfrak{q}} \cong F_*(R_{\mathfrak{q}}) \cong R_{\mathfrak{q}}^{\oplus p^\gamma}$, where $\gamma = d + \log_p([F_*(k) : k])$. Since the modules $R^{\oplus p^\gamma}$ and $F_*(R)$ agree when localized at $W = R \setminus \bigcup_{\mathfrak{q} \in \text{Assh}(R)} \mathfrak{q}$, there is an exact sequence

$$0 \longrightarrow C \longrightarrow R^{\oplus p^\gamma} \xrightarrow{\Phi} F_*(R) \longrightarrow T \longrightarrow 0$$

such that $C_W = T_W = 0$. The assumption $I_e^{[p]} \subseteq I_{e+1}$ gives that $\Phi(I_e^{\oplus p^\gamma}) \subseteq I_e F_*(R) \subseteq F_*(I_{e+1})$, and thus, there is an induced exact sequence

$$\left(\frac{R}{I_e}\right)^{\oplus p^\gamma} \xrightarrow{\bar{\Phi}} F_*\left(\frac{R}{I_{e+1}}\right) \longrightarrow \text{coker}(\bar{\Phi}) \longrightarrow 0.$$

The assumption that $I^{[p^{e+1}]} \subseteq I_{e+1}$ gives $I^{[p^e]} F_*(R) \subseteq F_*(I_{e+1})$, and thus $I^{[p^e]} \subseteq \text{ann}_R(\text{coker}(\bar{\Phi}))$. Since $\text{coker}(\bar{\Phi})$ is a quotient of T , we conclude that

$$\ell_R(F_*(R/I_{e+1})) \leq p^\gamma \ell_R(R/I_e) + \ell_R(\text{coker}(\bar{\Phi})) \leq p^\gamma \ell_R(R/I_e) + \ell_R(T/I^{[p^e]}T).$$

Note that $\ell_R(F_*(R/I_{e+1})) = p^{\gamma-d} \ell_R(R/I_{e+1})$, therefore dividing by $p^{\gamma+ed}$ the above estimates, we get

$$\frac{\ell_R(R/I_{e+1})}{p^{(e+1)d}} \leq \frac{\ell_R(R/I_e)}{p^{ed}} + \frac{\ell_R(T/I^{[p^e]}T)}{p^{\gamma+ed}}.$$

Now, if I is generated by μ elements, then $I^{\mu p^e} \subseteq I^{[p^e]}$, and thus,

$$\ell_R(T/I^{[p^e]}T) \leq \ell_R(T/I^{\mu p^e}T) \leq D\mu^{\dim(T)} p^{e \dim(T)}$$

for some D that only depends on T and I , since $\ell_R(T/I^{\mu p^e}T)$ is eventually a polynomial of degree $\dim(T)$ in the variable μp^e . Note that, since $T_{\mathfrak{q}} = 0$ for all $\mathfrak{q} \in \text{Assh}(R)$, we have that $\dim(T) \leq d - 1$. Set $C = D\mu^{d-1}p^{-\gamma}$, and note that it is independent of the sequence $\{I_e\}$. From the above estimates, we conclude that

$$\frac{\ell_R(R/I_{e+1})}{p^{(e+1)d}} \leq \frac{\ell_R(R/I_e)}{p^{ed}} + \frac{C}{p^e},$$

and the lemma follows from [29, Lemma 3.5 (i)]. □

As a consequence of Lemma 3.5, we obtain the existence of the limit of Ratliff–Rush closures of Frobenius powers of an ideal.

Proposition 3.6. *Let (R, \mathfrak{m}) be an excellent reduced local ring of dimension $d > 0$. If $I \subseteq R$ is an \mathfrak{m} -primary ideal, then the following limit exists:*

$$\widehat{e}_{\text{HK}}(I; R) := \lim_{q \rightarrow \infty} \frac{\ell_R(R/I^{[q]})}{q^d}.$$

Proof. Since R is excellent, its completion is reduced. Moreover, we have that $\widetilde{I}\widehat{R} = \widehat{I}\widehat{R}$ since $R \rightarrow \widehat{R}$ is faithfully flat [10, (1.7)]. Thus, we may assume that R is complete. By [13, Lemma 6.13], there exists an F -finite reduced ring S such that $R \rightarrow S$ is faithfully flat, purely inseparable, and $\mathfrak{m}S$ is the maximal ideal of S . Such a ring is obtained from R via the “so-called” Γ construction. Since $R \rightarrow S$ is faithfully flat, if J is an ideal of R containing a regular element, we have that $\widetilde{J}S = \widetilde{J}\widetilde{S}$ [10, (1.7)]. Moreover, the condition that $\mathfrak{m}S$ is the maximal ideal of S guarantees that $\ell_R(R/J) = \ell_S(S/JS)$ for any \mathfrak{m} -primary ideal $J \subseteq R$. Since R is reduced, $(I^{[q]})^n$ contains a regular element for every $q = p^e$ and every $n > 0$. From all these facts, it follows that

$$\ell_R\left(\frac{R}{(I^{[q]})^n}\right) = \ell_S\left(\frac{S}{((I^{[q]})^n)_S}\right) = \ell_S\left(\frac{S}{((IS)^{[q]})^n}\right).$$

By replacing R with S , we may therefore assume that R is reduced and F -finite. It now suffices to show that the sequence $\{I_e\}$ with $I_e = \widetilde{I}^{[p^e]}$ satisfies the assumptions of Lemma 3.5: if $t \in \mathbb{N}$ is such that $\widetilde{I}^{[p^e]} = (I^{[p^e]})^{t+1} :_R (I^{[p^e]})^t$, then

$$\widetilde{I}^{[p^e]}^{[p]} = \left[(I^{[p^e]})^{t+1} :_R (I^{[p^e]})^t \right]^{[p]} \subseteq (I^{[p^{e+1}]})^{t+1} :_R (I^{[p^{e+1}]})^t \subseteq \widetilde{I}^{[p^{e+1}]}.$$

Moreover, $I^{[p^e]} \subseteq \widetilde{I}^{[p^e]}$ for all e . □

If no confusion on the ambient ring may arise, we will denote $\widehat{e}_{\text{HK}}(I; R)$ simply by $\widehat{e}_{\text{HK}}(I)$.

Remark 3.7. We point out that $\widetilde{e}_{\text{HK}}(I)$ may differ from $e_{\text{HK}}(I)$ in general. In fact, let (R, \mathfrak{m}) be a regular local ring of dimension d , and $I \subseteq R$ be an \mathfrak{m} -primary ideal such that $I \neq \widetilde{I}$. Let $t > 0$ be such that $\widetilde{I} = I^{t+1} : I^t$. By flatness of Frobenius on R , for all $q = p^e$, we have that

$$\widetilde{I^{[q]}} \supseteq (I^{[q]})^{t+1} : (I^{[q]})^t = (I^{t+1} : I^t)^{[q]} = \widetilde{I}^{[q]}.$$

It follows that

$$\ell_R\left(\frac{R}{\widetilde{I^{[q]}}}\right) \leq \ell_R\left(\frac{R}{\widetilde{I}^{[q]}}\right) = q^d \ell_R\left(\frac{R}{\widetilde{I}}\right) < q^d \ell_R\left(\frac{R}{I}\right) = \ell_R\left(\frac{R}{I^{[q]}}\right)$$

and, in particular, $\widetilde{e}_{\text{HK}}(I) < e_{\text{HK}}(I)$. An explicit example of such an ideal in dimension 2 is $R = k[[x, y]]$, with k a field, and $I = (x^4, x^3y, xy^3, y^4)$. In fact, in this case, $x^2y^2 \in \widetilde{I} \setminus I$.

An alternative proof of the existence of $\widetilde{e}_{\text{HK}}(I)$ for excellent reduced local rings can be obtained using uniform estimates as in [40, Theorem 3.6] or [38, Corollary 5.5]. However, with the approach of Lemma 3.5 based on [29], one can get further information on the convergence of $\ell_R(R/I^{[q]})/q^d$ to its limit $\widetilde{e}_{\text{HK}}(I)$. To this end, we record another slight variation of a result of Polstra and Tucker.

Proposition 3.8. *Let (R, \mathfrak{m}) be an F -finite reduced local ring of dimension d , and $\{I_{t,e}\}_{t,e \in \mathbb{N}}$ be a sequence of ideals such that:*

- (a) $I_{t,e}^{[p]} \subseteq I_{t,e+1}$ for all t and e ,
- (b) $I_{t,e} \subseteq I_{t+1,e}$ for all t and e ,
- (c) there exists an \mathfrak{m} -primary ideal I such that $I^{[p^e]} \subseteq I_{t,e}$ for all t and e .

If we set $I_e = \sum_{t \in \mathbb{N}} I_{t,e}$, then $\lim_{e \rightarrow \infty} \frac{\ell_R(R/I_e)}{p^{ed}}$ exists as a limit, and

$$\lim_{t \rightarrow \infty} \lim_{e \rightarrow \infty} \frac{\ell_R(R/I_{t,e})}{p^{ed}} = \lim_{e \rightarrow \infty} \frac{\ell_R(R/I_e)}{p^{ed}} = \lim_{e \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{\ell_R(R/I_{t,e})}{p^{ed}}.$$

Proof. The proof is analogous to that of [29, Theorem 6.3], using Lemma 3.5 in place of [29, Theorem 4.3 (i)]. □

Corollary 3.9. *Let (R, \mathfrak{m}) be an F -finite reduced local ring of dimension $d > 0$, and $I \subseteq R$ be an \mathfrak{m} -primary ideal. Then,*

$$\widetilde{e}_{\text{HK}}(I) = \lim_{t \rightarrow \infty} \lim_{e \rightarrow \infty} \frac{\ell_R(R/I_{t,e})}{p^{ed}},$$

where $I_{t,e} = (I^{[p^e]})^{t+1} :_R (I^{[p^e]})^t$.

We show that a transformation rule along finite map holds for $\widetilde{e}_{\text{HK}}$.

Proposition 3.10. *Let $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$ be a finite extension of excellent local domains of dimension $d > 0$, and $I \subseteq R$ be an \mathfrak{m} -primary ideal. Then,*

$$\widetilde{e}_{\text{HK}}(I; R) = \frac{[S/\mathfrak{n} : R/\mathfrak{m}]}{\text{rank}_R(S)} \lim_{q \rightarrow \infty} \frac{\ell_S(S/I^{[q]}S)}{q^d}.$$

In addition, if S is flat over R , then

$$\widetilde{e}_{\text{HK}}(I; R) = \frac{[S/\mathfrak{m} : R/\mathfrak{m}]}{\text{rank}_R(S)} \widetilde{e}_{\text{HK}}(IS; S).$$

Proof. Let $r = \text{rank}_R(S)$, and $s = [S/\mathfrak{m} : R/\mathfrak{m}]$. Since S , as an R -module, is isomorphic to $R^{\oplus r}$ when localized at $R \setminus \{0\}$, we have short exact sequences $0 \rightarrow R^{\oplus r} \rightarrow S \rightarrow C_1 \rightarrow 0$ and $0 \rightarrow S \rightarrow R^{\oplus r} \rightarrow C_2 \rightarrow 0$, with $\dim(C_1) < d$ and $\dim(C_2) < d$. The exactness on the left is due to the fact that both $R^{\oplus r}$ and S are torsion-free R -modules. Tensoring the first sequence with $R/\widetilde{I}^{[q]}$ and counting lengths, we obtain

$$\begin{aligned} \ell_R\left(S/\widetilde{I}^{[q]}S\right) - r\ell_R\left(R/\widetilde{I}^{[q]}\right) &\leq \ell_R(C_1/\widetilde{I}^{[q]}C_1) \\ &\leq \ell_R(C_1/I^{[q]}C_1) \\ &\leq \ell_R(C_1/I^{\mu(I)q}C_1), \end{aligned}$$

where we use that $I^{\mu(I)q} \subseteq I^{[q]}$ by the pigeonhole principle and that $I^{[q]} \subseteq \widetilde{I}^{[q]}$. Since the function $n \mapsto \ell_R(C_1/I^nC_1)$ is eventually a polynomial of degree $\dim(C_1)$, we can find $D_1 > 0$ such that $\ell_R(C_1/I^{\mu(I)q}C_1) \leq D_1(\mu(I)q)^{\dim(C_1)} \leq E_1q^{d-1}$, where $E_1 := D_1\mu(I)^{d-1}$ is a constant independent of q . A similar argument applied to the second short exact sequence gives the existence of a constant $E_2 > 0$ independent of q such that

$$r\ell_R\left(R/\widetilde{I}^{[q]}\right) - \ell_R\left(S/\widetilde{I}^{[q]}S\right) \leq E_2q^{d-1}.$$

By setting $E := \max\{E_1, E_2\}$, we therefore get that

$$\left| \ell_R\left(S/\widetilde{I}^{[q]}S\right) - r\ell_R\left(R/\widetilde{I}^{[q]}\right) \right| \leq Eq^{d-1}$$

for some constant $E > 0$ independent of q . Dividing by q^d and taking limits gives that

$$r\widetilde{e}_{\text{HK}}(I; R) = \lim_{q \rightarrow \infty} \frac{\ell_R(S/\widetilde{I}^{[q]}S)}{q^d}.$$

Finally, note that $\ell_R(S/\widetilde{I}^{[q]}S) = s\ell_S(S/\widetilde{I}^{[q]}S)$, and the first claimed equality is now proved. If $R \rightarrow S$ is flat, we have that $\widetilde{I}^{[q]}S = \widetilde{I}^{[q]}S = (\widetilde{IS})^{[q]}$ by [10, (1.7)]. Note that I contains a regular element since R is a domain of positive dimension and I is \mathfrak{m} -primary. The second claimed equality now follows as well. \square

By definition, we have $\widetilde{e}_{\text{HK}}(I) \leq e_{\text{HK}}(I)$ for any \mathfrak{m} -primary ideal I . We now give a characterization of the equality $\widetilde{e}_{\text{HK}}(I) = e_{\text{HK}}(I)$ in terms of tight closure, in the spirit of [12, Theorem 8.17]. Let R° be the complement of the union of the minimal primes of R . We recall that, given an ideal $I \subseteq R$, its tight closure is

$$I^* = \left\{ x \in R \mid \text{there exists } c \in R^\circ \text{ such that } cx^q \in I^{[q]} \text{ for all } q = p^e \gg 0 \right\}.$$

Recall that a local ring (R, \mathfrak{m}) is said to be formally equidimensional if its \mathfrak{m} -adic completion \widehat{R} is equidimensional.

Proposition 3.11. *Let (R, \mathfrak{m}) be an excellent, formally equidimensional, reduced local ring of dimension $d > 0$, and I be an \mathfrak{m} -primary ideal. Then, $\widehat{e}_{\text{HK}}(I) = e_{\text{HK}}(I)$ if and only if $\widetilde{I^{[q]}} \subseteq (I^{[q]})^*$ for all $q = p^e$.*

Proof. First assume that $\widetilde{I^{[q]}} \subseteq (I^{[q]})^*$ for all $q = p^e$. By [13, Theorem 6.1], there exists a test element $c \in R^\circ$, that is, an element c such that $\dim(R/(c)) < d$ and $cJ^* \subseteq J$ for any ideal J of R . Consider the exact sequence

$$0 \longrightarrow \frac{I^{[q]} :_R (c)}{I^{[q]}} \longrightarrow \frac{R}{I^{[q]}} \longrightarrow \frac{R}{(I^{[q]}, c)} \longrightarrow 0.$$

Since all modules involved have finite length, we have that the first and last ones have the same length. Moreover, we have that $c\widetilde{I^{[q]}} \subseteq c(I^{[q]})^* \subseteq I^{[q]}$, so that

$$0 \leq \ell_R\left(\frac{\widetilde{I^{[q]}}}{I^{[q]}}\right) \leq \ell_R\left(\frac{I^{[q]} :_R c}{I^{[q]}}\right) = \ell_R\left(\frac{R}{(I^{[q]}, c)}\right) \leq Dq^{d-1}$$

for a constant $D > 0$ that only depends on I and c . Dividing by q^d and taking limits gives that $\widehat{e}_{\text{HK}}(I) = e_{\text{HK}}(I)$.

Conversely, suppose that there exists q_0 such that $\widetilde{I^{[q_0]}} \not\subseteq (I^{[q_0]})^*$. Since the first ideal always contains $I^{[q_0]}$, by [12, Theorem 8.17], we have that $e_{\text{HK}}(I^{[q_0]}) > e_{\text{HK}}(\widetilde{I^{[q_0]}})$. But then, since $(\widetilde{I^{[q_0]}})^{[q]} \subseteq \widetilde{I^{[q_0q]}}$ holds for all q , we conclude that

$$0 < \lim_{q \rightarrow \infty} \frac{\ell_R\left(\frac{(\widetilde{I^{[q_0]}})^{[q]}}{I^{[q_0q]}}\right)}{q^d} \leq \lim_{q \rightarrow \infty} \frac{\ell_R\left(\frac{\widetilde{I^{[q_0q]}}}{I^{[q_0q]}}\right)}{q^d} = q_0^d (e_{\text{HK}}(I) - \widehat{e}_{\text{HK}}(I)). \quad \square$$

4 | RESULTS IN DIMENSION 2

Throughout this section, unless otherwise stated, we will assume that (R, \mathfrak{m}) is a 2-dimensional Cohen–Macaulay local ring. Without loss of generality, by replacing R with $R[x]_{\mathfrak{m}R[x]}$ if needed, we may also assume that R/\mathfrak{m} is infinite. Throughout, we will use the convention that any nonpositive power of an ideal is the unit ideal. The following is our main result.

Theorem 4.1. *Let (R, \mathfrak{m}) be a 2-dimensional excellent Cohen–Macaulay reduced local ring, and I be an \mathfrak{m} -primary ideal. Let $r = r_J(I)$ be the reduction number of I with respect to a minimal reduction J . With the above notation, we have*

- (1) $L_1(I) = r e(I) + \widehat{e}_{\text{HK}}(I^{r-1}) - \widehat{e}_{\text{HK}}(I^r)$.
- (2) $L_2(I) = \binom{r}{2} e(I) + r \widehat{e}_{\text{HK}}(I^{r-1}) - (r-1) \widehat{e}_{\text{HK}}(I^r)$. In particular, $L_2(I)$ exists as a limit and (Q1) has a positive answer.

(3) The equality $\widetilde{e}_{\text{HK}}(I^n) = \sum_{i=0}^2 (-1)^i L_i(I) \binom{n+1-i}{2-i}$ holds for all $n \geq r - 1$. In particular, (Q2) has a positive answer for I if and only if $\widetilde{e}_{\text{HK}}(I^n) = e_{\text{HK}}(I^n)$ for all $n \gg 0$.

Proof. Let $\widetilde{\mathbb{F}}_q = \{(\overline{I^{[q]})^n}\}$, which is an $I^{[q]}$ -good filtration as already observed. We use the notation introduced in Section 3. By Proposition 3.1 and [30, Theorem 2.1], we have

$$e_i(I^{[q]}) = e_i(\widetilde{\mathbb{F}}_q) = \sum_{n=i}^r \binom{n-1}{i-1} v_{\widetilde{\mathbb{F}}_q}(n)$$

for $i = 1, 2$. Moreover, we get

$$\begin{aligned} v_{\widetilde{\mathbb{F}}_q}(n) &= \Delta^2(P_{\widetilde{\mathbb{F}}_q}(n) - H_{\widetilde{\mathbb{F}}_q}(n)) \\ &= e(I^{[q]}) - \Delta^2(H_{\widetilde{\mathbb{F}}_q}(n)) \\ &= q^2 e(I) - \left[\ell_R\left(\frac{R}{(I^{[q]})^n}\right) - 2\ell_R\left(\frac{R}{(I^{[q]})^{n-1}}\right) + \ell_R\left(\frac{R}{(I^{[q]})^{n-2}}\right) \right]. \end{aligned}$$

Replacing in the above equality, for $i = 1$, we get that

$$e_1(I^{[q]}) = r q^2 e(I) + \ell_R\left(\frac{R}{(I^{[q]})^{r-1}}\right) - \ell_R\left(\frac{R}{(I^{[q]})^r}\right).$$

Dividing by q^2 , taking $\lim_{q \rightarrow \infty}$ and using Proposition 3.6, we get the desired statement for $L_1(I)$. For $i = 2$, the proof is analogous; in particular, the limit $L_2(I)$ exists thanks to Proposition 3.6.

For (3): by Proposition 3.3, we have that $r(\widetilde{\mathbb{F}}_q) \leq r$, and for all $n \geq r - 1$, we have

$$\ell_R\left(\frac{R}{(I^{[q]})^n}\right) = H_{\widetilde{\mathbb{F}}_q}(n) = P_{\widetilde{\mathbb{F}}_q}(n) = \sum_{i=0}^2 (-1)^i e_i(I^{[q]}) \binom{n+1-i}{2-i}.$$

Dividing by q^2 , taking limits and Proposition 3.6 gives the desired statement. □

We record an immediate consequence of Theorem 4.1 and Proposition 3.11.

Corollary 4.2. *Let (R, \mathfrak{m}) be a 2-dimensional excellent Cohen–Macaulay reduced local ring, and I be an \mathfrak{m} -primary ideal. The following are equivalent:*

- (1) (Q2) has a positive answer for I ;
- (2) $e_{\text{HK}}(I^n) = \widetilde{e}_{\text{HK}}(I^n)$ for all $n \gg 0$;
- (3) $(I^{[q]})^n \subseteq ((I^{[q]})^n)^*$ for all $q = p^e$ and all $n \gg 0$.

Thanks to these equivalences, we obtain that (Q2) has positive answer for a large class of rings: finite subrings of 2-dimensional regular local rings. The argument is the outcome of some useful discussions with Smirnov. We thank him for allowing us to include this result here.

Theorem 4.3. *Let (R, \mathfrak{m}) be a 2-dimensional excellent Cohen–Macaulay local ring, and suppose that there exists a regular local ring $S \supseteq R$ such that the inclusion is module-finite. Then, (Q2) has a positive answer for any \mathfrak{m} -primary ideal I of R .*

Proof. Let $I \subseteq R$ be an \mathfrak{m} -primary ideal. Let $J = IS$, and choose n_0 such that $\widetilde{J}^n = J^n$ for all $n \geq n_0$. Let $n \geq n_0$ and $q = p^e$. Since S is regular, for $t \gg 0$, we have

$$\widetilde{(J^{[q]})^n} = (J^{[q]})^{n+t} : (J^{[q]})^t = (J^{n+t} : J^t)^{[q]} \subseteq (\widetilde{J}^n)^{[q]} = (J^n)^{[q]} = (J^{[q]})^n.$$

In particular, we get that

$$\left(\widetilde{(I^{[q]})^n}\right)S \subseteq \widetilde{(I^{[q]})^n}S = \widetilde{(J^{[q]})^n} = (J^{[q]})^n = ((I^{[q]})^n)S.$$

Thus, we have $\left(\widetilde{(I^{[q]})^n}\right)S = ((I^{[q]})^n)S$ for all $q = p^e$, and by Proposition 3.10, we then get that

$$\widetilde{e}_{\text{HK}}(I^n; R) = \frac{[S/\mathfrak{n} : R/\mathfrak{m}]}{\text{rank}_R(S)} \lim_{q \rightarrow \infty} \frac{\ell_S(S/(J^{[q]})^n)}{q^d} = \frac{[S/\mathfrak{n} : R/\mathfrak{m}]}{\text{rank}_R(S)} e_{\text{HK}}(J^n; S).$$

On the other hand, the transformation rule for Hilbert–Kunz multiplicities along finite maps (for instance, see [16, Theorem 3.16]) guarantees that

$$e_{\text{HK}}(I^n; R) = \frac{[S/\mathfrak{n} : R/\mathfrak{m}]}{\text{rank}_R(S)} e_{\text{HK}}(J^n; S),$$

and it follows that $e_{\text{HK}}(I^n; R) = \widetilde{e}_{\text{HK}}(I^n; R)$ for all $n \geq n_0$. Since R is a subring of a regular local ring, it is reduced. Thus, we conclude thanks to Corollary 4.2. \square

We recall the definition of F-signature for a ring of any dimension. Let $\alpha = \log_p([F_*(k) : k])$, where $k = R/\mathfrak{m}$.

Definition 4.4. Let (R, \mathfrak{m}) be an F-finite local ring of dimension d . For every $e > 0$, write $F_*^e(R) = R^{\oplus a_e} \oplus M_e$, where M_e is an R -module with no free summands. The F-signature of R is

$$s(R) = \lim_{e \rightarrow \infty} \frac{a_e}{p^{e(d+\alpha)}}.$$

The existence of the limit was proved in full generality by K. Tucker [40].

For a 2-dimensional local ring R , we let $\underline{L}_2(I) = \liminf_{q \rightarrow \infty} \frac{e_2(I^{[q]})}{q^2}$ and $\overline{L}_2(I) = \limsup_{q \rightarrow \infty} \frac{e_2(I^{[q]})}{q^2}$.

Proposition 4.5. *Let (R, \mathfrak{m}) be a 2-dimensional F-finite Cohen–Macaulay local ring. For all $q = p^e$, we have $\frac{a_e}{q^\alpha} e_2(I) \leq e_2(I^{[q]}) \leq e_2(I) \ell_R(R/\mathfrak{m}^{[q]})$. In particular,*

$$e_2(I) s(R) \leq \underline{L}_2(I) \leq \overline{L}_2(I) \leq e_2(I) e_{\text{HK}}(\mathfrak{m}).$$

Proof. Let $J = (x, y)$ be a minimal reduction of I , and for $s \in \mathbb{N}$ set $A_s = I^{2s}$ and $B_s = J^{[s]}I^s$, where $J^{[s]} := (x^s, y^s)$. As in Section 3, we denote by \mathcal{R}'_q the extended Rees algebra of $I^{[q]}$, and $J_q = (x^qt, y^qt)$. For every $q = p^e$, by the Grothendieck–Serre’s formula evaluated at $n = 0$ (see [3, 18]) and by [34, Lemma 2.2], we have that

$$e_2(I^{[q]}) = \ell_R(H_{J_q}^2(\mathcal{R}'_q)_0) = \ell_R\left(\frac{A_s^{[q]}}{B_s^{[q]}}\right)$$

for all $s \gg 0$. For a fixed $q = p^e$, choose $s \gg 0$ so that $\ell_R(A_s/B_s) = e_2(I)$ and $\ell_R(A_s^{[q]}/B_s^{[q]}) = e_2(I^{[q]})$. Then,

$$e_2(I^{[q]}) = \ell_R\left(\frac{A_s^{[q]}}{B_s^{[q]}}\right) = \frac{1}{q^\alpha} \ell_R\left(\frac{A_s F_*^\alpha(R)}{B_s F_*^\alpha(R)}\right) \geq \frac{a_e}{q^\alpha} \ell_R\left(\frac{A_s}{B_s}\right) = \frac{a_e}{q^\alpha} e_2(I).$$

Dividing by q^2 and taking \liminf gives

$$\underline{L}_2(I) \geq e_2(I) \lim_{e \rightarrow \infty} \frac{a_e}{q^{2+\alpha}} = e_2(I) s(R).$$

For the upper bound, as above let s be such that $\ell_R(A_s/B_s) = e_2(I)$ and $\ell_R(A_s^{[q]}/B_s^{[q]}) = e_2(I^{[q]})$. Let $B_s = L_0 \subsetneq L_1 \subsetneq \dots \subsetneq L_n = A_s$ with $L_{j+1}/L_j \cong R/\mathfrak{m}$ be a composition series of A_s/B_s . Since $\mathfrak{m}L_{j+1} \subseteq L_j$ we have that $\mathfrak{m}^{[q]}L_{j+1}^{[q]} \subseteq L_j^{[q]}$, that is, we have a surjection $R/\mathfrak{m}^{[q]} \twoheadrightarrow L_{j+1}^{[q]}/L_j^{[q]}$. It follows that

$$e_2(I^{[q]}) = \ell_R\left(\frac{A_s^{[q]}}{B_s^{[q]}}\right) \leq \ell_R\left(\frac{A_s}{B_s}\right) \cdot \ell_R\left(\frac{R}{\mathfrak{m}^{[q]}}\right) = e_2(I) \ell_R\left(\frac{R}{\mathfrak{m}^{[q]}}\right).$$

Dividing by q^2 and taking \limsup gives the desired statement. □

Proposition 4.6. *Let (R, \mathfrak{m}) be a 2-dimensional F -finite Cohen–Macaulay local ring, and consider the following conditions:*

- (1) $e_2(I) = 0$.
- (2) $e_2(I^{[q]}) = 0$ for all $q = p^e$.
- (3) $L_2(I)$ exists as a limit, and $L_2(I) = 0$.
- (4) $\underline{L}_2(I) = 0$

Then, (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4). In addition, (1) \iff (2) if R is F -pure, and they are all equivalent if R is weakly F -regular.

Proof. First, recall $e_2(I) \geq 0$ by [26]. Moreover, recall that R is F -pure if and only if $a_e > 0$ for some (equivalently, for all) $e > 0$. In dimension 2, R is weakly F -regular if and only if it is strongly F -regular [44], and the latter is equivalent to $s(R) > 0$ in any dimension by [1]. The statement is now an immediate consequence of Proposition 4.5. □

Remark 4.7. If (R, \mathfrak{m}) is a 2-dimensional local ring, and I is an \mathfrak{m} -primary ideal, then by [11], there exists n_0 such that for $n \geq n_0$, the reduction number of I^n is independent of the choice of

a minimal reduction, and it equals either 1 or 2. If R is Cohen–Macaulay, it can be shown that $e_2(I) = 0$ if and only if such a value equals 1.

Theorem 4.8. *Let (R, \mathfrak{m}) be a 2-dimensional Cohen–Macaulay local ring. Assume that $e_2(I) = 0$. Then, for all $n \gg 0$, we have*

$$e_{\text{HK}}(I^n) = e(I) \binom{n+1}{2} - L_1(I)n.$$

In particular, (Q2) has a positive answer for I .

Proof. First, we reduce to the case in which R is F-finite. Let k be the residue field of R . By Cohen’s structure theorem, the completion \widehat{R} of R at \mathfrak{m} contains an isomorphic copy of k . Let \overline{k} denote an algebraic closure of k , and let S denote the completion of $\widehat{R} \otimes_k \overline{k}$ at the ideal obtained by extending \mathfrak{m} . Then, (S, \mathfrak{n}) is a complete local ring with residue field $S/\mathfrak{n} \cong \overline{k}$. In particular, it is an F-finite ring. Moreover, the map $(R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$ is faithfully flat with $\mathfrak{m}S = \mathfrak{n}$, therefore base changing to S does not affect calculations of length, multiplicity or Hilbert coefficients. Thus, after replacing R with S , we may directly assume that R is F-finite. Let $J = (x, y)$ be a minimal reduction of I . For all $n > 0$, let $J^{[n]} = (x^n, y^n)$. By [34, Lemma 2.2], the condition $e_2(I) = 0$ gives that $I^{2n} = J^{[n]}I^n$ for all $n \geq n_0$, for some n_0 . In particular, I^n is a stable ideal and, as already pointed out, this gives that $L_1(I^n) = e(I^n) - e_{\text{HK}}(I^n)$ for all $n \geq n_0$ (see [43, Theorem 1.8 (2)], or [37]). For any ideal K , one has $e_1(K^n) = n e_1(K) + \binom{n}{2} e(K)$, and it follows that $L_1(I^n) = nL_1(I) + \binom{n}{2} e(I)$. Since $e(I^n) = n^2 e(I)$, a direct calculation shows that $e_{\text{HK}}(I^n) = e(I) \binom{n+1}{2} - L_1(I)n$ for all $n \geq n_0$. To conclude the proof, it suffices to note that $L_2(I) = 0$ by Proposition 4.6. \square

We now turn our attention to Conjecture (C). We prove that it holds true for local rings of dimension 2.

Theorem 4.9. *Let (R, \mathfrak{m}) be an excellent 2-dimensional analytically unramified weakly F-regular local ring of characteristic $p > 0$. Then, (C) holds true.*

Proof. We only need to show that $L_1(I) = e(I) - e_{\text{HK}}(I)$ implies that I is stable. By Proposition 3.4, there exists n_0 such that $H_{J_q}^2(\mathcal{R}'_q)_n = 0$ for all $n \geq n_0$ and all $q = p^e$. For any such n and all q , we have that

$$e(I^{[q]}) \binom{n+1}{2} - e_1(I^{[q]})n + e_2(I^{[q]}) - \ell_R(R/(I^{[q]})^n) = -\ell_R(H_{J_q}^1(\mathcal{R}'_q)_n) \leq 0.$$

Since R is reduced, dividing by q^2 and taking limits, we conclude that $e_{\text{HK}}(I^n) \geq e(I) \binom{n+1}{2} - L_1(I)n + L_2(I)$ by Theorem 4.1. Since $e_2(K) \geq 0$ for any \mathfrak{m} -primary ideal K [26], we have that $L_2(I) \geq 0$. By [43, Theorem 1.8 (1)], we have that $e_{\text{HK}}(I^n) \leq e(I) \binom{n+1}{2} - (e(I) - e_{\text{HK}}(I))n$. Putting these facts together and canceling out terms, we get that $L_1(I)n - L_2(I) \geq (e(I) - e_{\text{HK}}(I))n$. If we assume that $L_1(I) = e(I) - e_{\text{HK}}(I)$, then from $L_2(I) \geq 0$, we conclude that $L_2(I) = 0$, and that $e_{\text{HK}}(I^n) = e(I) \binom{n+1}{2} - (e(I) - e_{\text{HK}}(I))n$. The claim now follows from [43, Theorem 1.8 (4)]. \square

Remark 4.10. We note that the assumptions of Theorem 4.9 guarantee that R is Cohen–Macaulay. In fact, by [19], we have that excellent rings are the homomorphic image of a Cohen–Macaulay

local ring. This condition, together with the fact that R is weakly F-regular, implies that R is Cohen–Macaulay by [13, Theorem (3.4) (c)].

Remark 4.11. For normal ideals in a pseudorational 2-dimensional local ring, (Q1),(Q2) and (C) trivially hold true. Indeed, in this case, I is stable by [31]. Note that by Proposition 4.6 and Remark 4.7, we have that $L_2(I) = 0$ in this case.

Recall that, from [37], for $n \gg 0$, we have that $e_{\text{HK}}(I^n) = e(I) \binom{n+1}{2} - L_1 n + f(n)$ for some function $f(n)$ such that $\lim_{n \rightarrow \infty} \frac{f(n)}{n} = 0$. Our next goal is to show that $f(n)$ is eventually nonincreasing.

Lemma 4.12. *Let (R, \mathfrak{m}) be a 2-dimensional Cohen–Macaulay local ring, and $J = (x, y)$ be a parameter ideal. For any $t > 0$ and any ideal $I \subseteq R$, we have an exact sequence*

$$0 \longrightarrow \text{Tor}_1^R(R/I, J^t) \longrightarrow \left(\frac{R}{I}\right)^{\oplus t} \xrightarrow{\varphi} \left(\frac{R}{I}\right)^{\oplus(t+1)} \longrightarrow \frac{J^t}{IJ^t} \longrightarrow 0,$$

where $\text{Im}(\varphi) \subseteq J(R/I)^{\oplus(t+1)}$. In particular, $((I : J)/I)^{\oplus t} \subseteq \ker(\varphi)$.

Proof. As x, y is a regular sequence, a minimal free resolution of J^t is given by the Eagon–Northcott complex (or by Hilbert–Burch, in this case):

$$0 \longrightarrow R^{\oplus t} \xrightarrow{\begin{bmatrix} x & 0 & 0 \dots & 0 \\ -y & x & 0 \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}} R^{\oplus(t+1)} \longrightarrow J^t \longrightarrow 0.$$

Tensoring the above with R/I gives the desired exact sequence, together with the containment $\text{Im}(\varphi) \subseteq J(R/I)^{\oplus(t+1)}$. It is therefore clear that $((I : J)/I)^{\oplus t} \subseteq \ker(\varphi)$. □

Corollary 4.13. *Let (R, \mathfrak{m}) be a 2-dimensional Cohen–Macaulay local ring, and J be a parameter ideal. For any ideal $I \subseteq R$, we have a short exact sequence*

$$0 \longrightarrow \frac{R}{I : J} \xrightarrow{\varphi} \left(\frac{R}{I}\right)^{\oplus 2} \longrightarrow \frac{J}{IJ} \longrightarrow 0.$$

Proof. Apply Lemma 4.12 with $t = 1$, and observe that

$$\text{Tor}_1^R(R/I, J) = \ker(\varphi) = \{z \in R/I \mid zJ \subseteq I\} = (I : J)/I. \quad \square$$

Corollary 4.14. *Let (R, \mathfrak{m}) be a 2-dimensional Cohen–Macaulay local ring, and J be a parameter ideal. For any $t > 0$ and any ideal $I \subseteq R$ containing J , we have an isomorphism*

$$\left(\frac{R}{I}\right)^{\oplus(t+1)} \cong \frac{J^t}{IJ^t}.$$

Proof. In the notation of Lemma 4.12, it suffices to observe that, since $J \subseteq I$, we have $\text{Im}(\varphi) \subseteq J(R/I)^{\oplus t} = 0$. \square

Proposition 4.15. *Let (R, \mathfrak{m}) be a 2-dimensional excellent Cohen–Macaulay reduced local ring, and I be an \mathfrak{m} -primary ideal. With the notation introduced above, we have that $f(n - 1) \geq f(n) \geq L_2(I)$ for all $n \gg 0$.*

Proof. Let J be a minimal reduction of I . From Lemma 4.12, for every n, t and $q = p^e$, we have an exact sequence

$$0 \longrightarrow \text{Tor}_1^R \left(\frac{R}{(I[q])^n}, (J[q])^t \right) \longrightarrow \left(\frac{R}{(I[q])^n} \right)^{\oplus t} \xrightarrow{\varphi} \left(\frac{R}{(I[q])^n} \right)^{\oplus(t+1)} \longrightarrow \frac{(J[q])^t}{(I[q])^n(J[q])^t} \longrightarrow 0.$$

Again by Lemma 4.12, we have that

$$\left(\frac{(I[q])^{n-1}}{(I[q])^n} \right)^{\oplus t} \subseteq \left(\frac{(I[q])^n : J[q]}{(I[q])^n} \right)^{\oplus t} \subseteq \ker(\varphi),$$

so that we have an exact sequence

$$\left(\frac{R}{(I[q])^{n-1}} \right)^{\oplus t} \xrightarrow{\varphi} \left(\frac{R}{(I[q])^n} \right)^{\oplus(t+1)} \longrightarrow \frac{(J[q])^t}{(I[q])^n(J[q])^t} \longrightarrow 0.$$

Choose $n \geq r(I)$. Since $r(I[q]) \leq r(I)$, we have that $(I[q])^n(J[q])^t = (I[q])^{n+t}$. Counting lengths, we therefore get

$$(t + 1)\ell_R \left(\frac{R}{(I[q])^n} \right) - t\ell_R \left(\frac{R}{(I[q])^{n-1}} \right) \leq \ell_R \left(\frac{R}{(I[q])^{n+t}} \right) - \ell_R \left(\frac{R}{(J[q])^t} \right). \tag{1}$$

For the moment assume that q is fixed, and choose $t \gg 0$ so that $\ell_R(R/(I[q])^{n+t})$ coincides with its Hilbert polynomial. As a function of t , we can write

$$\begin{aligned} \ell_R \left(\frac{R}{(I[q])^{n+t}} \right) &= e_0(I[q]) \binom{n + t + 1}{2} - e_1(I[q])(n + t) + e_2(I[q]) \\ &= \frac{q^2 e(I)}{2} t^2 + \left[nq^2 e(I) + \frac{q^2 e(I)}{2} - e_1(I[q]) \right] t + o(t). \end{aligned}$$

Since

$$\ell_R \left(\frac{R}{(J[q])^t} \right) = \binom{t + 1}{2} \ell_R \left(\frac{R}{J[q]} \right) = \frac{q^2 e(I)}{2} t^2 + \frac{q^2 e(I)}{2} t,$$

the inequality (1) becomes

$$(t + 1)\ell_R \left(\frac{R}{(I[q])^n} \right) - t\ell_R \left(\frac{R}{(I[q])^{n-1}} \right) \leq \left[nq^2 e(I) - e_1(I[q]) \right] t + o(t).$$

Dividing by t and taking $\lim_{t \rightarrow \infty}$ gives

$$\ell_R\left(\frac{R}{(I^{[q]})^n}\right) - \ell_R\left(\frac{R}{(I^{[q]})^{n-1}}\right) \leq nq^2 e(I) - e_1(I^{[q]}).$$

Now divide by q^2 and take $\lim_{q \rightarrow \infty}$ to get

$$e_{\text{HK}}(I^n) - e_{\text{HK}}(I^{n-1}) \leq n e(I) - L_1(I).$$

A direct calculation shows that the quantity on the left-hand side for $n \gg 0$ is $n e(I) - L_1(I) + f(n) - f(n - 1)$, so that we get $f(n) \leq f(n - 1)$. For the last claim, it suffices to recall that for all $n \gg 0$, we have

$$e(I)\binom{n+1}{2} - L_1(I)n + f(n) = e_{\text{HK}}(I^n) \geq \widetilde{e}_{\text{HK}}(I^n) = e(I)\binom{n+1}{2} - L_1(I)n + L_2(I),$$

which gives $f(n) \geq L_2(I)$. □

Remark 4.16. In the previous proof, we use the trivial containment $(I^{[q]})^{n-1} \subseteq (I^{[q]})^n : J^{[q]}$. If $(I^{[q]})^{n-1} \subseteq (I^{[q]})^n : J^{[q]}$ holds true for all $q \gg 0$, then with the same calculations, one gets that $f(n) \leq L_2(I)$. Since the other inequality is always true, equality holds. In particular, if $(I^{[q]})^{n-1} \subseteq (I^{[q]})^n : J^{[q]}$ holds true for all $q, n \gg 0$, then (Q2) has positive answer for I .

5 | A POSSIBLE NEGATIVE ANSWER TO (Q2)

We suspect that (Q2) has negative answer, even in dimension 2. Even if at the moment, we are not able to fully support this claim, we present a ring that seems to contradict the expectations of (Q2), at least after some experimental calculations performed with Macaulay 2 [6]. Before, we recall the notion of test ideal. Let R be a ring of characteristic $p > 0$. The (finitistic) test ideal of R is

$$\tau(R) = \bigcap_I (I :_R I^*),$$

where the intersection runs over all ideals I of R . If R is F-finite, and $c \in R^\circ$ is such that R_c is regular, then c has a power that belongs to $\tau(R)$; such an element is called a test element. Thus, if J is an ideal which defines the singular locus of R , we have that $J \subseteq \sqrt{\tau(R)}$. In particular, if R has an isolated singularity at a maximal ideal \mathfrak{m} , then $\tau(R)$ is either the whole ring or it is an \mathfrak{m} -primary ideal. If, in addition, R is F-pure, then $\tau(R)$ is a radical ideal; for instance, see [5, 41]. As a consequence, if R is an F-finite and F-pure isolated singularity at \mathfrak{m} , then $\tau(R) \supseteq \mathfrak{m}$. In addition, if R is not weakly F-regular (that is, there exists an ideal that is not tightly closed), then $\tau(R) = \mathfrak{m}$.

Example 5.1. Let k be an F-finite field of characteristic $p \neq 3$, let $R = k[[x, y, z]]/(x^3 + y^3 + z^3)$, and $\mathfrak{m} = (x, y, z)R$. Note that R has an isolated singularity at \mathfrak{m} , it is F-finite and not weakly F-regular. If $p \equiv 1 \pmod 3$, then it is also F-pure, and in this case, we then have $\tau(R) = \mathfrak{m}$. In this case, if $J \subseteq I$ are two ideals, in order to show that $I \not\subseteq J^*$, one only has to prove that $\mathfrak{m}I \not\subseteq J$. For $p = 7$,

calculations performed with Macaulay2 seem to suggest that, for all n , there exists $q = 7^e \gg 0$ such that

$$\mathfrak{m} \cdot \left((\mathfrak{m}^{[q]})^{n+1} :_R \mathfrak{m}^{[q]} \right) \not\subseteq (\mathfrak{m}^{[q]})^n.$$

When this is the case, we have that $\widehat{(\mathfrak{m}^{[q]})^n} \not\subseteq ((\mathfrak{m}^{[q]})^n)^*$, and by Proposition 3.11, we conclude that $e_{\text{HK}}(\mathfrak{m}^n) \neq \widehat{e}_{\text{HK}}(\mathfrak{m}^n)$ for such n . If this was indeed true for all n (or even just for infinitely many n), then (Q2) would have negative answer for this ring with respect to the ideal \mathfrak{m} .

Remark 5.2. If $R = k[[x, y, z]]/(x^3 + y^3 + z^3)$, and $\text{char}(k) = 2$ or 5 , then one can still show that $\tau(R) = \mathfrak{m}$ even if R is not F-pure. These choices of smaller characteristics could be helpful for calculations, and perhaps still provide a negative answer to (Q2).

In [32], it is proved that if (R, \mathfrak{m}) is 2-dimensional Cohen-Macaulay, I is an \mathfrak{m} -primary ideal, and $J = (x, y)$ is a minimal reduction of I such that $I^{m+1} = JI^m$ and $I^{m+1} : x = I^m$, then $\widehat{I}^n = I^n$ for all $n \geq m$. Rossi and Swanson also mention that they do not know whether $I^{m+1} = JI^m$ is sufficient to imply that $\widehat{I}^m = I^m$. We show, with the aid of Macaulay 2, that this is indeed not the case.

Example 5.3. Let $R = \mathbb{F}_2[x, y, z]/(x^3 + y^3 + z^3)$, let $\mathfrak{m} = (x, y, z)R$ and $J = (y, z)$, which is a minimal reduction of \mathfrak{m} . Note that $r_f(\mathfrak{m}) = 2$, and therefore $(\mathfrak{m}^{[q]})^3 = J^{[q]}(\mathfrak{m}^{[q]})^2$ for all $q = 2^e$. Calculations performed with Macaulay2 give that the element $x^2y^4z^{13}$ belongs to $(\mathfrak{m}^{[8]})^4 : (\mathfrak{m}^{[8]})^2$, and hence to $\widehat{(\mathfrak{m}^{[8]})^2}$, but not to $(\mathfrak{m}^{[8]})^2$. To have a local example, note that all these facts still hold in the localization $R_{\mathfrak{m}}$.

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REFERENCES

1. I. M. Aberbach and G. J. Leuschke, *The F-signature and strong F-regularity*, Math. Res. Lett. **10** (2003), no. 1, 51–56.
2. A. Banerjee, K. Goel, and J. K. Verma, *On the Hilbert-Samuel coefficients of Frobenius powers of an ideal*, Proc. Amer. Math. Soc. **152** (2024), no. 4, 1501–1515.
3. C. Blanford, *On Hilbert functions and cohomology*, J. Algebra **192** (1997), no. 1, 439–459.
4. H. Brenner, *Irrational Hilbert-Kunz multiplicities*, arXiv: 1305.5873, 2013.
5. R. Fedder and K.-i. Watanabe, *A characterization of F-regularity in terms of F-purity*, Commutative algebra (Berkeley, CA, 1987), vol. 15, Math. Sci. Res. Inst. Publ., Springer, New York, 1989, pp. 227–245.
6. D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at <http://www2.macaulay2.com>.
7. A. Guerrieri and M. E. Rossi, *Hilbert coefficients of Hilbert filtrations*, J. Algebra **199** (1998), no. 1, 40–61.
8. D. Hanes, *Bounds on multiplicities of local rings*, Comm. Algebra **30** (2002), no. 8, 3789–3812.
9. D. Hanes, *Notes on the Hilbert-Kunz function*, J. Algebra **265** (2003), no. 2, 619–630.
10. W. Heinzer, D. Lantz, and K. Shah, *The Ratliff-Rush ideals in a Noetherian ring*, Comm. Algebra **20** (1992), no. 2, 591–622.
11. L. T. Hoa, *Reduction numbers and Rees algebras of powers of an ideal*, Proc. Amer. Math. Soc. **119** (1993), no. 2, 415–422.
12. M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), no. 1, 31–116.
13. M. Hochster and C. Huneke, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), no. 1, 1–62.
14. S. Huckaba and T. Marley, *Hilbert coefficients and the depths of associated graded rings*, J. London Math. Soc. (2) **56** (1997), no. 1, 64–76.
15. C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987), no. 2, 293–318.
16. C. Huneke, *Hilbert-Kunz multiplicity and the F-signature*, Commutative algebra, Springer, New York, 2013, pp. 485–525.
17. C. Huneke and Y. Yao, *Unmixed local rings with minimal Hilbert-Kunz multiplicity are regular*, Proc. Amer. Math. Soc. **130** (2002), no. 3, 661–665.
18. B. Johnston and J. Verma, *Local cohomology of Rees algebras and Hilbert functions*, Proc. Amer. Math. Soc. **123** (1995), no. 1, 1–10.
19. T. Kawasaki, *On arithmetic Macaulayfication of Noetherian rings*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 123–149.
20. E. Kunz, *Characterizations of regular local rings of characteristic p*, Amer. J. Math. **91** (1969), 772–784.
21. E. Kunz, *On Noetherian rings of characteristic p*, Amer. J. Math. **98** (1976), no. 4, 999–1013.
22. K. Kurano, *On Roberts rings*, J. Math. Soc. Japan **53** (2001), no. 2, 333–355.
23. A. Mafi, *Ratliff-Rush ideal and reduction numbers*, Comm. Algebra **46** (2018), no. 3, 1272–1276.
24. P. Monsky, *The Hilbert-Kunz function*, Math. Ann. **263** (1983), no. 1, 43–49.
25. M. Nagata, *Local rings*, Robert E. Krieger Publishing Co., Huntington, NY, 1975. Corrected reprint.
26. M. Narita, *A note on the coefficients of Hilbert characteristic functions in semi-regular local rings*, Proc. Cambridge Philos. Soc. **59** (1963), 269–275.
27. A. Ooishi, *Genera and arithmetic genera of commutative rings*, Hiroshima Math. J. **17** (1987), no. 1, 47–66.
28. C. Peskine and L. Szpiro, *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Inst. Hautes Études Sci. Publ. Math. **42** (1973), 47–119.
29. T. Polstra and K. Tucker, *F-signature and Hilbert-Kunz multiplicity: a combined approach and comparison*, Algebra Number Theory **12** (2018), no. 1, 61–97.
30. L. J. Ratliff, Jr. and D. E. Rush, *Two notes on reductions of ideals*, Indiana Univ. Math. J. **27** (1978), no. 6, 929–934.
31. D. Rees, *Hilbert functions and pseudo-rational local rings of dimension two*, J. London Math. Soc. (2) **24** (1981), no. 3, 467–479.

32. M. E. Rossi and I. Swanson, *Notes on the behavior of the Ratliff-Rush filtration*, Commutative algebra (Grenoble/Lyon, 2001), vol. 331, Contemp. Math., Amer. Math. Soc., Providence, RI, 2003, pp. 313–328.
33. M. E. Rossi and G. Valla, *Hilbert functions of filtered modules*, vol. 9, Lecture Notes of the Unione Matematica Italiana, Springer, Berlin; UMI, Bologna, 2010.
34. J. D. Sally, *Ideals whose Hilbert function and Hilbert polynomial agree at $n = 1$* , J. Algebra **157** (1993), no. 2, 534–547.
35. P. Samuel, *La notion de multiplicité en algèbre et en géométrie algébrique*, J. Math. Pures Appl. (9) **30** (1951), 159–205.
36. P. Schenzel, *Multiplizitäten in verallgemeinerten Cohen-Macaulay-Moduln*, Math. Nachr. **88** (1979), 295–306.
37. I. Smirnov, *Hilbert-Kunz multiplicity of the powers of an ideal*, Proc. Amer. Math. Soc. **147** (2019), no. 8, 3331–3338.
38. I. Smirnov, *An invitation to equimultiplicity of F -invariants*, arXiv: 2309.04322, 2023.
39. V. Trivedi, *Asymptotic Hilbert-Kunz multiplicity*, J. Algebra **492** (2017), 498–523.
40. K. Tucker, *F -signature exists*, Invent. Math. **190** (2012), no. 3, 743–765.
41. J. C. Vassilev, *Test ideals in quotients of F -finite regular local rings*, Trans. Amer. Math. Soc. **350** (1998), no. 10, 4041–4051.
42. K.-i. Watanabe and K.-i. Yoshida, *Hilbert-Kunz multiplicity and an inequality between multiplicity and colength*, J. Algebra **230** (2000), no. 1, 295–317.
43. K.-i. Watanabe and K.-i. Yoshida, *Hilbert-Kunz multiplicity of two-dimensional local rings*, Nagoya Math. J. **162** (2001), 87–110.
44. L. J. Williams, *Uniform stability of kernels of Koszul cohomology indexed by the Frobenius endomorphism*, J. Algebra **172** (1995), no. 3, 721–743.