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Analogy and invention.
Some remarks on Poincaré's Analysis situs papers

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Abstract. The primary role played by analogy in Henri Poincaré's work, and in particular in his "analysis situs" papers, is emphasized. Poincaré's "sixth example" (showing that Betti numbers do not suffice to classify 3-manifolds) and his construction of the homology sphere are discussed in detail.

In the chapter "L'avenir des mathématiques" of his *Science et méthode*,[*] Henri Poincaré famously defined mathematics as "the art of giving the same name to different things".[1] This saying is far from being a mere witticism. In the first place, it

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[*] I shall make reference to the following standard editions of Poincaré's works: *La science et l'hypothèse* [1902], préface de Jules Vuillemin, Flammarion, Paris 1968 (which reproduces the text of the second edition, published in 1907); *La valeur de la science*, préface de Jules Vuillemin, Flammarion, Paris 1970; *Science et méthode*, Flammarion, Paris 1908; *Œuvres*, 11 vv., various eds., Gauthier-Villars, Paris 1916-1956. I have used the following English translations: *Science and Hypothesis*, translated by W.J. G., with a preface by J. Larmor, The Walter Scott Publishing Co., London & Newcastle-on-Tyne 1905; *The value of Science*, translated by George Bruce Halsted, Dover, New York 1958 (originally published in 1913 by Science Press in *Foundations of Science*); *Science and Method*, translated by Francis Maitland, with a preface by B. Russell, Thomas Nelson and Sons, London, Edinburgh, Dublin & New York 1914. I shall quote from these works (recently reprinted in *The Value of Science. The Essential Writings of H. Poincaré*, The Modern Library, New York 2001) by using the acronyms *SH*, *VS*, and *SM* followed by the page number. As for the paper "Analysis situs" and its five supplements, I have used the English translation provided in H. Poincaré, *Papers in Topology*, edited by John Stillwell, American Mathematical Society Chelsea Publishing, Providence (RI) & Mathematical Society, London 2010 (with slight modifications, when necessary); only the page number of the original texts will be given.

[1] *SM,* p. 34; "la mathématique est l'art de donner le même nom à des choses différentes", *Science et méthode*, cit., p. 29. The chapter "L'avenir des mathématiques" is based on the address prepared for the 4th International Congress of Mathematicians, held in Rome, April 6-11, 1908 (the address was read by Gaston Darboux, as Poincaré lay ill at his hotel during the whole conference). The original text, longer than that published in *Science et méthode*, appeared in *Atti del IV Congresso internazionale dei matematici*, G. Castelnuovo ed., Tipografia della Reale Accademia dei Lincei, Roma 1909, pp. 167-182, and in several journals: *Rendiconti del Circolo matematico di Palermo*, 16 (1908), pp. 152-168;

aims at emphasizing the crucial importance of well-chosen words in mathematics: only a finely tuned language can enable mathematicians to establish relations between things that are different in the substance, but similar in the form, so that they can be cast in the same mold ("elles puissent […] se couler dans le même moule"). Moreover, it is well known that, accordingly to Poincaré,

> Mathematicians do not study objects, but the relations between objects; to them it is a matter of indifference if these objects are replaced by others, provided that the relations do not change. Matter does not engage their attention, they are interested by form alone.[2]

Arguably, this conception of mathematics appears to be tightly related to the ubiquitous importance that such notions as morphism (in today's parlance) or equivalence relation (ditto) have in Poincare's oeuvre. For example, the notion of isomorphism allows mathematicians to identify groups that emerge differently in different problems: indeed, as Poincaré points out, "[w]e now know that, in a group, the matter is of little interest, that the form only is of importance".[3] Analogously, two *variétés* are equivalent from the point of view of *analysis situs* when they are "homeomorphic, that it is to say, of similar form",[4] and two plane curves defined by polynomials with integer coefficients are equivalent "if one can pass from one to the other by a birational transformation with integer or rational coefficients".[5] As for equivalence relations, Poincaré had certainly been well acquainted with the procedure of constructing the orbit space associated with the action of a discrete group on a given space since the early 1880s, when he wrote his celebrated papers on Fuchsian and Kleinian groups. Indeed, already in the first *Supplément* to his essay presented for the prize competition announced by the *Académie des Sciences* on 1878, Poincaré was able to characterize the functions he (maybe, somewhat hastily) named *fuchsiennes* by the property of being invariant under the action of certain discrete subgroups of the group of isometries of the hyperbolic upper half-plane. Consistently, in the same supplement, he remarked:

> In fact, what is a geometry? It is the study of a *group of operations* formed by the displacements one can apply to a figure without deforming it. In Euclidean geometry

Revue générale des sciences pures et appliquées, 19 (1909), pp. 930-939; *Scientia. Rivista di scienza*, 2nd year, 3 (1908), pp. 1-23; *Bulletin des sciences mathématiques*, 2nd ser., 32 (1908), pp. 168-190.

[2] *SH*, p. 20; "Les mathématiciens n'étudient pas des objets, mais des relations entre les objets; il leur est donc indifférent de remplacer ces objets par des autres, pourvu que les relations ne changent pas. La matière ne leur importe pas, la forme seule les intéresse", *La science et l'hypothèse*, cit., p. 49. The passage is excerpted from the chapter "La grandeur mathématique et l'expérience", which is based on the paper "Le continu mathématique", *Revue de métaphysique et de morale*, 1 (1893), pp. 26-34.

[3] *SM*, p. 35; "Nous savons maintenant que dans un groupe la matière nous intéresse peu, que c'est la forme seule qui importe", *Science et méthode*, cit., p. 30.

[4] "*[H]oméomorphes*, c'est-à-dire de forme pareille", "Analysis situs", *Journal de l'École Polytechnique*, 1 (1895), pp. 1-121 = *Œuvres*, vol. VI, pp. 193-288; the quote is at p. 199 (Poincaré's italics).

[5] "[S]i l'on peut passer de l'une à l'autre par une transformation birationnelle, *à coefficients entiers ou rationnels*", "Sur les propriétés arithmétiques des courbes algébriques", *Journal de mathématiques pures et appliquées*, 5th ser., 7 (1901), pp. 161-233 = *Œuvres*, vol. V, pp. 483-550; the quote is at p. 484 (Poincaré's italics).

the group reduces to *rotations* and *translations*. In the pseudogeometry of Lobačevskiĭ it is more complicated.[6]

In a broader perspective, Poincaré's definition of mathematics as "the art of giving the same name to different things" reflects his conviction about the prominent role played by analogy within this discipline. Analogy is the guide that shows the way to mathematicians and steers their "groping attempts"[7] by revealing "the hidden harmony of things".[8] It lies at the very heart of the process of mathematical intuition, in a relationship of mutual dependence:

> What has taught us to know the true, profound analogies, those the eyes do not see but reason divines?
> It is the mathematical spirit, which disdains matter to cling only to pure form. This it is which has taught us to give the same name to things differing only in material, to call by the same name, for instance, the multiplication of quaternions and that of whole numbers.[9]

As usual in Poincaré, the methodological stance and the activity of "doing mathematics" remain always interrelated. This is especially apparent, as I shall try to argue, in the monumental memoir "Analysis situs" and its five *compléments*. Poincaré's extraordinary originality and creativity come along with his skillful and deliberate recourse to the tools of trade of a mathematician, above all that of analogy.

"L'inspection de la figure le demontre"

In 1896 Poincaré underwent a series of anthropometric measurements and psychological and physiological tests conducted by the French "alienist" Édouard Toulouse (1867-1947). Toulouse's research work – not without connection to the ideas developed by Cesare Lombroso in his (at that time highly reputed) books *Genio e follia* (1864) and *L'uomo di genio in rapporto alla psichiatria* (1888) – was aimed

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[6] "Qu'est-ce en effet une géométrie? C'est l'étude d'un *groupe d'opérations* formé par les déplacements que l'on peut faire subir à une figure sans la déformer. Dans la géométrie euclidienne ce groupe se réduit à des *rotations* et à des *translations*. Dans la pseudogéométrie de Lobatchewski [*sic*] il est plus compliqué.", *Trois suppléments sur la découverte des fonctions fuchsiennes,* J. Gray and S. A. Walters eds., Akademie Verlag, Berlin & Albert Blanchard, Paris 1997, p. 35 (Poincaré's italics). It is worth observing that, at that time, Poincaré was completely unaware of Felix Klein's *Erlanger programm*.

[7] *SM*, p. 29; "tâtonnements" *Science et méthode*, cit., p. 24.

[8] *VS*, p. 79; "l'harmonie cachée des choses", *La valeur de la science*, cit., p. 108.

[9] *VS*, p. 77; "Qui nous a appris à connaître les analogies véritables, profondes, celles que les yeux ne voient pas et que la raison devine? C'est l'esprit mathématique, qui dédaigne la matière pour ne s'attacher qu'à la forme pure. C'est lui qui nous a enseigné à nommer du même nom des êtres qui ne diffèrent que par la matière, à nommer du même nom par exemple la multiplication des quaternions et celle des nombres entiers.", *La valeur de la science*, cit., p. 106.

at "clarifying the relationships of intellectual superiority to neuropathy".[10] Among the subjects examined by Toulouse there were some leading figures of *fin de siècle* French culture: besides Poincaré, Émile Zola and Pierre Berthelot.

One of the several tests of memory devised by Toulouse consisted in observing a simple geometric figure for a few seconds and then reproducing it. The result obtained by Poincaré in this test was quite miserable: he could do not better than scribble down two clumsy attempts.[11] This is hardly surprising, since his awkwardness in drawing was almost legendary. According to Paul Appell's account, Poincaré's lack of skill in the technique called "lavis" jeopardized his own admittance to the École Polytechnique and cost him the first place in the second year final ranking.[12] Nevertheless – or rather, just because of this – figures, and in particular those "badly drawn", played an essential role in the shaping of Poincaré's geometric thinking, as he himself made it clear in the introduction to "Analysis situs":

> We know how useful geometric figures are in the theory of imaginary functions and integrals evaluated between imaginary limits, and how much we desire their assistance when we want to study, for example, functions of two complex variables.
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> If we try to account for the nature of this assistance, figures first of all make up for the infirmity of our intellect by calling on the aid of our senses; but not only this. It is worthy repeating that geometry is the art of reasoning well from badly drawn figures; however, these figures, if they are not to deceive us, must satisfy certain conditions; the proportions may be grossly altered, but the relative positions of the different parts must not be upset.
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> The use of figures is, above all, then, for the purpose of making known certain relations between the objects that we study, and these relations are those which occupy the branch of geometry that we have called *Analysis situs*, and which describes the relative situation of points and lines on surfaces, without consideration of their magnitude.[13]

Not only in his work on "analysis situs" (an area which today we would call algebraic and differential topology), but in the whole of his vast scientific production, Poincaré made an extensive use of figures both as useful aids to the study of certain

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[10] Édouard Toulouse, *Enquête médico-psychologique sur la superiorité intellectuelle: Henri Poincaré*, Flammarion, Paris 1910, p. 1.

[11] Toulouse, *op. cit.*, p. 66.

[12] Paul Appell, *Henri Poincaré*, Plon, Paris 1925, p. 28; for more details, see André Bellivier, *Henri Poincaré ou la vocation souveraine*, Gallimard, Paris 1956.

[13] "On sait quelle est l'utilité des figures géométriques dans la théorie des fonctions imaginaires et des intégrales prises entre des limites imaginaires, et comment on regrette leur concours quand on veut étudier, par exemple, les fonctions de deux variables complexes. Cherchons à nous rendre compte de la nature de ce concours; les figures suppléent d'abord à l'infirmité de notre esprit en appelant nos sens à son secours; mais ce n'est pas seulement cela. On a bien souvent répété que la Géométrie est l'art de bien raisonner sur des figures mal faites; encore ces figures, pour ne pas nous tromper, doivent-elles satisfaire à certaines conditions; les proportions peuvent être grossièrement altérées, mais les positions relatives des diverses parties ne doivent pas être bouleversées. L'emploi des figures a donc avant tout pour but de nous faire connaître certaines relations entre les objets de nos études, et ces relations sont celles dont s'occupe une branche de la Géométrie que l'on appelée *Analysis situs*, et qui décrit la situation relative des points des lignes et des surfaces, sans aucune considération de leur grandeur.", "Analysis situs", cit., p. 194.

particular cases and as key tools for proof. In this regard a paradigmatic example is provided by the *théorie des conséquents* (an early instance of the geometric construction we nowadays call "Poincaré's map") and the *théorie des cycles limites* developed in the groundbreaking "Mémoire sur les courbes définies par une équation différentielle (deuxième partie)"[14] (here, for example, one can read the phrase "l'inspection de la figure le demontre" ("the inspection of the figure proves it")[15], highly typical of Poincaré's way of reasoning).

On some occasions these two distinct functions of figures may fuse into a single one (as in the paper "Théorie des groupes fuchsiens"[16]), while in others they remain separate, if for no other reason than that the author has not yet found a way out of the maze of specific examples. This is the case of his last memoir, "Sur un théorème de géométrie",[17] that Poincaré submitted to Giovanni Battista Guccia for publication in the journal *Rendiconti del Circolo matematico di Palermo*, albeit he was sorely aware that it was *inachevé*:

> What embarrasses me is the fact that I will be forced to insert many figures, precisely because I have not yet been able to obtain a general rule, but have only accumulated the special solutions.[18]

In the "Advertissement" to his *Méchanique analytique* (1788), Lagrange notoriously declared: "On ne trouvera point de Figures dans cet Ouvrage".[19] About one century later, in Poincare's three volumes of *Les méthodes nouvelles de la mécanique céleste* (1892, 1893, 1899) figures, if not numerous,[20] are brought strategically into play in order to elucidate difficult points in a proof or to provide classification schemas of admissible behaviors. There is at least one case, however, in which the figure turns out to be impossible to visualize, even for the man who was likely to possess the most visionary geometric imagination at the time. This occurs towards the end of the third volume, where Poincaré studies the intersections of two

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14 *Journal de mathématiques pures et appliquées*, 3rd ser., 8 (1882), pp. 251-296 = *Œuvres*, vol. I, pp. 44-84.

15 *Ibid.*, p. 56.

16 *Acta mathematica*, 1 (1882), pp. 1-62 = *Œuvres*, vol. II, pp. 108-168.

17 "Ce qui m'embarrasse, c'est que je serai obligé de mettre beaucoup de figures, justement parce que je n'ai pu arriver à une règle générale, mais que j'ai seulement accumulé les solutions particulières", "Sur un théorème de géométrie", *Rendiconti del Circolo matematico di Palermo*, 33 (1912), pp. 375-407 = *Œuvres*, vol. VI, pp. 499-538. Poincaré's "last geometric theorem" (stating that any homeomorphism of the annulus into itself that is area- and orientation-preserving and rotates the outer boundary circle clockwise and the inner boundary circle anticlockwise has at least two fixed points) was proved, in full generality and through different methods, by George David Birkhoff in 1913. Though incomplete, Poincaré's proof was basically correct, as shown by C. Golé and G. R. Hall in their paper "Poincaré's proof of Poincaré's last geometric theorem", in *Twist Mappings and Their Applications*, R. McGehee & K. R. Meyers eds., Springer-Verlag, New York 1992, pp. 135-151.

18 *Le livre du centenaire de la naissance de Henri Poincaré, 1854-1954*, Gauthier-Villars, Paris 1955, p. 296.

19 J.-L. Lagrange, *Méchanique analytique*, chez la veuve Desaint, Paris 1788, p. vi.

20 Precisely, there are 4 figures in the first, 3 in the second, and 12 in the third volume of the work.

curves belonging to the stable and instable manifolds (as we would say today) of a periodic solution, running up against a dynamics of chaotic type:

> Let us attempt to imagine the figure formed by these two curves and their intersections, which are infinitely many and each of which corresponds to a solution that is doubly asymptotic: these intersections form a sort of lattice, tissue or grid with an infinitely dense mesh; neither of the two curves must intersect itself, but must fold over itself in a very complicated way so as to intersect all of the meshes of the grid. The complexity of this figure, which I will not even attempt to draw, is striking. Nothing shows in a more compelling way the difficulty of the three body problem and, more generally, of all dynamical problems having no uniform integral or whose Bohlin series are divergent.[21]

Invariants

In his four great memoirs about "les courbes définies par une équation différentielle",[22] Poincaré adopted a "new point of view", that he himself defined "qualitative". This change in perspective enabled him to obtain results that, using a classical quantitative approach, would have been virtually impossible even to imagine. For example, in his third memoir Poincaré proved the "index theorem", according to which for any vector field X on a (closed and oriented) surface of genus g the relation

$$C - F - N = 2g - 2,$$

is satisfied, where C is the number of saddle points of X, F the number of its foci, and N the number of its nodes.[23] In the fourth and last memoir of the series Poincaré

[21] "Que l'on cherche à se représenter la figure formée par ces deux courbes et leurs intersections en nombre infini dont chacune correspond à une solution doublement asymptotique, ces intersections forment une sorte de treillis, de tissu, de réseau à mailles infiniment serrées; chacune des deux courbes ne doit jamais se recouper elle-même, mais elle doit se replier sur elle-même d'une manière très complexe pour venir recouper une infinité de fois toutes les mailles du réseau. On sera frappé de la complexité de cette figure, que je ne cherche même pas à tracer. Rien n'est plus propre à nous donner une idée de la complication du problème des trois corps et en général de tous les problèmes de Dynamique où il n'y a pas d'intégrale uniforme et où les séries de Bohlin sont divergentes.", *Les méthodes nouvelles de la mécanique céleste. Tome III. Invariants intégraux – Solutions périodiques de deuxième genre – Solutions doublements asymptotiques*, Gauthier-Villars, Paris 1899, p. 389.

[22] "Mémoire sur les courbes définies par une équation différentielle (première partie)", *Journal de mathématiques pures et appliquées*, 3rd ser., 7 (1881), pp. 375-422 = *Œuvres*, vol. I, pp. 3-44; "Mémoire sur les courbes définies par une équation différentielle (deuxième partie)", cit.; "Sur les courbes définies par les équations différentielles (première partie)", *Journal de mathématiques pures et appliquées*, 4th ser., 1 (1885), pp. 167-244 = *Œuvres*, vol. I, pp. 90-161; "Sur les courbes définies par les équations différentielles (deuxième partie)", *Journal de mathématiques pures et appliquées*, 4th ser., 2 (1886), pp. 151-217 = *Œuvres*, vol. I, pp. 167-222.

[23] "Sur les courbes définies par les équations différentielles (première partie)", cit., p. 125. This result (known today under the name of Poincaré-Hopf theorem) had previously been stated in the short research announcement "Sur les courbes définies par une équation différentielle", *Comptes rendus de l'Académie des Sciences*, 93 (1881), pp. 951-952 = *Œuvres,* vol. I, pp. 85-85. For some comments about Poincaré's qualitative approach to differential equations see C. Bartocci, "Introduzione", in H. Poincaré, *Geometria e caso. Scritti di matematica e fisica*, Bollati Boringhieri, Torino 1995 (repr. 2013), pp. VII-L.

addressed the problem of studying differential equations of higher order extending his own global theory to *n*-dimensional spaces. Commenting on the necessity of resorting to geometric objects of dimension greater than three, he remarked:

> Geometry, then, is just a language which can be more or less advantageous, but is no longer a representation talking to the senses. However, we may be led to use this language occasionally.[24]

This same qualitative strategy, combined with the handling of phase space as the geometric arena where the true dynamics takes places, allowed him, just a few years later, to open a breach – while describing the perturbations of periodic orbits – in the otherwise impregnable fortress of the three body problem.[25]

It is therefore no coincidence (and no surprise) that in the informal definition of *analysis situs* given by Poincaré in the "Analyse de ses travaux…" we find precisely the adjective "qualitative":

> *Analysis situs* is the science that allows us to know the qualitative properties of geometric figures, not only in ordinary space but is space of more than three dimensions.[26]

While *analysis situs* in three dimensions is, according to Poincaré, "une connaissance presque intuitive" ("an almost intuitive knowledge"), enormous difficulties arise in extending its concepts to higher dimensions. In order to attempt to overcome these hurdles is thus necessary "to be profoundly convinced of the great importance of this discipline"[27], a conviction that Poincaré certainly possessed in high degree:

> As for me, all of the various paths on which I was successively engaged led me to *analysis situs*. I had need of the results of this discipline to pursue my research work on curves defined by differential equations […] and to generalize it to higher order differential equations, in particular to those of the three body problem. I had need of *analysis situs* for the study of non uniform [i.e., multivalued] functions of 2 variables. I had need of it for the study of periods of multiple integrals and for the application of this study to the expansion of the perturbative function.
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> Finally, I glimpsed in *analysis situs* a tool for tackling an important problem in group theory, namely, the search for discrete or finite groups contained in a given continuous group.[28]

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[24] "La Géométrie n'est plus alors qu'un language qui peut être plus ou moins avantageux, ce n'est plus une représentation parlant aux sens. Nous pourront néanmoins être conduits à employer quelquefois ce language.", "Sur les courbes définies par les équations différentielles (deuxième partie)", cit., p. 168.

[25] "[…] ce qui nous ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici réputée inabordable", *Les méthodes nouvelles de la mécanique céleste. Tome I. Solutions périodiques – Non-existence des intégrales uniformes – Solutions asymptotiques*, Gauthier-Villars, Paris 1892, p. 82.

[26] "L'Analysis situs est la science qui nous fait connaître les propriétés qualititaves des figures géométriques non seulement dans l'espace ordinaire, mais dans l'espace à plus de trois dimensions.", "Analyse de ses travaux scientifiques faite par H. Poincaré", *Acta mathematica*, 38 (1921), pp. 36-135 (posthumously published, but written down by Poincaré in 1901); the quote is at p. 100.

[27] "[Ê]tre bien persuadé de l'extrême importance de cette science", *ibid*., p. 100.

[28] "Quant à moi, toutes les voies diverses où je m'étais engagé successivement me conduisaient à l'Analysis Sitûs. J'avais besoin des données de cette science pour poursuivre mes études sur les courbes

But what are, in concrete terms (if ever possible), the objects that analysis situs is supposed to deal with? In analogy with the definition of geometry I have quoted above, Poincaré defines analysis situs as "the Science whose object is the study of [the] group [of homeomorphisms].[29] In other words, the crux of the question lies in being able to determine, for each *variété*, suitable "quantities" (numbers, groups, vector spaces, etc.) associated with it that remain invariant when the *variété* undergoes a differentiable transformation.

In the case of surfaces in Euclidean space, the problem of invariants had been solved by August Ferdinand Möbius in 1863. In his paper "Theorie der elementaren Verwandtschaften",[30] the German mathematician sectioned any closed and oriented surface into "primitive forms" (namely, disks, cylinders or union of cylinders) by means of parallel planes. He then showed that, between two such surfaces, it is possible to establish an "elementary correlation" (*elementar Verwandtschaft*, that is, more or less, diffeomorphism in modern terminology) if and only if they belong to the same class. Each class is identified by a nonnegative integer, g, that corresponds to the number of holes of the surface and it is related to the surface's Euler characteristic χ by the formula $2 - 2g = \chi$.

Some years earlier, Bernhard Riemann, in his *Inauguraldissertation* entitled "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse",[31] had introduced a profoundly novel geometric idea – that of Riemann surface – establishing unexpected relations between complex analysis, topology and the theory of algebraic curves. In this theory, every algebraic curve can be associated with a Riemann surface, namely, the branched cover of the complex plane determined by the given curve: the genus of the curve (defined, for example, as the dimension of the space of holomorphic differentials) does coincide with the number $\frac{m-1}{2}$, where m is the "order of connectivity" (*Ordnung des Zusammenhangs*)

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définies par les équations différentielles […] et pour les étendre aux équations différentielles d'ordre supérieur et en particulier à celles du problème des trois corps. J'en avais besoin pour l'étude des fonctions non uniformes de 2 variables. J'en avais besoin pour l'étude des périodes des intégrales multiples et pour l'application de cette étude au développement de la fonction perturbatrice. Enfin j'entrevoyais dans l'Analysis Sitûs un moyen d'aborder un problème important de la théorie des groupes, la recherche des groupes discrets ou des groupes finis contenus dans un groupe continu donné.", *ibid.*, p. 101.

[29] "[L]a Science dont l'objet est l'étude [du] groupe [des homéomorphismes]", "Analysis situs", cit., p. 198. As for the meaning of the words *variété* and *homéomorphisme*", see note 41.

[30] *Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physische Classe*, 17 (1863), pp. 31-68 = *Gesammelte Werke*, vol. II, herausgegeben von F. Klein, Hirzel, Leipzig 1886, pp. 433-471.

[31] Reprinted in B. Riemann, *Gesammelte Mathematische Werke, Wissenschaftlicher Nachlass und Nachträge/Collected Papers*, nach der Ausgabe von H. Weber und R Dedekind, neu herausgegeben von Raghavan Narasimhan, Springer-Verlag, Berlin-Heidelberg – Teubner, Leipzig 1990, pp. 35-77.

of the surface (defined, instead, in topological terms).[32] Had Möbius been aware of Riemann's results,[33] he would not have failed to realize that his classifying number, multiplied by 2 and then added to 1, corresponded exactly to Riemann's "order of connectivity".

Classification theorems for surfaces "equivalent" to Möbius's result were obtained, following distinct and innovative proof strategies, by Camille Jordan[34] and, successively, by William Kingdon Clifford.[35] Along a different line of research, Enrico Betti, profoundly influenced by Riemann's ideas, generalized (and modified) the notion of order of connectivity to spaces of higher dimensions: in his memoir "Sopra gli spazi di un numero qualunque di dimensioni",[36] he defined, for every n-dimensional space, $n-1$ "orders of connectivity" $p_1, \dots, p_{n-1}$ (integer numbers), which later mathematicians would call "Betti numbers". Are Betti numbers "invariants" in the same sense in which the genus of a Riemann surface is? If so, can they be used to obtain classification theorems for higher-dimensional geometric spaces analogous to the classification theorem for surfaces? These two issues – absolutely natural in the theoretical context of geometry in the early 1880s – were pinpointed with remarkable lucidity, in 1884, by Walther Dyck for the case of dimension 3:

> The object is to determine certain characteristical [*sic*] numbers for closed threedimensional spaces, analogous to those introduced by Riemann in the theory of his surfaces, so that their identity shows the possibility of its [*sic*] "one-to-one correspondence".[37]

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[32] This result, valid for any compact Riemann surface, was proved by Riemann in his pioneering (and somewhat cryptic) paper "Theorie der Abel'schen Functionen", *Journal für die reine und angewandte Mathematik*, 54 (1857), pp. 115-155 = *Gesammelte Mathematische Werke*…, cit., pp. 88-142. For a detailed account of Riemann's geometric function theory see U. Bottazzini & J. Gray, *Hidden Harmony – Geometric Fantasies. The Rise of Complex Function Theory*, Springer, New York 2013, chap. 5. The word "genus" was introduced by Rudolf Friedrich Alfred Clebsch in his paper "Über die Anwendung der Abelschen Functionen in der Geometrie", *Journal für die reine und angewandte Mathematik* , 63 (1864), pp. 189–243.

[33] Jean-Claude Pont (*La topologie algébrique des origines à Poincaré*, Presses Universitaires de France, Paris 1974, p. 97) argues that this was not the case.

[34] "Des contours tracés sur les surfaces", *Journal de mathématiques pures et appliquées*, 2nd ser., 11 (1866), pp. 110-130.

[35] "On the canonical form and dissection of a Riemann's surface", *Proceedings of the London Mathematical Society*, 8 (1877), pp. 292-304 = *Mathematical Papers*, edited by R. Tucker, Macmillan, London 1882 (reprinted AMS Chelsea Publishing, Providence (RI) 2007), pp. 241-254. It may be of some interest to note that, in this paper, Clifford proved what is now known as the Riemann-Hurwitz formula (the formula in itself being usually attributed to Hurwitz, who obtained it in 1893): *"an n-sheeted Riemann's surfaces with w branch-points may be transformed, without tearing, into the surface of a body with* $p, = \frac{1}{2}w - n + 1$*, holes in it"* (Clifford's italics), op. cit., p. 251.

[36] *Annali di matematica pura e applicata*, 2nd ser., 4 (1871), pp. 140-158 = *Opere matematiche*, vol. II, Hoepli, Milano 1913, pp. 273-290.

[37] "On the 'Analysis situs' of threedimensional spaces", *Report of the Fifty-fourth Meeting of the British Association for the Advancement of Science held at Montreal in August and September 1884*, John Murray, London 1885, p. 648.

A similar question would be formulated, eight years later, by Poincaré in his first work explicitly devoted to the subject of *analysis situs*, a note barely four pages long presented to the *Académie des Sciences*:

> One may ask whether the Betti numbers suffice to determine a closed surface from the viewpoint of *analysis situs*; that is, whether, given two closed surfaces with || the same Betti numbers, it is possible to pass from one to the other by a continuous transformation.[38]

Unlike Dyck, Poincaré also had the answer: no. He claimed, in fact, to be able to construct a family of three-dimensional *surfaces* (the word *variété* is not employed yet) whose first Betti number can only assume values less or equal to 4, and which nonetheless contain an infinite number of non diffeomorphic surfaces. In order to prove this fact Poincaré needed to invent a new invariant: the fundamental group. The analogy with certain constructions used in the theory of Fuchsian groups provided him the guiding principle to devise the appropriate counterexample.

Poincaré's sixth example

The rudimentary ideas outlined in the 1892 *Comptes rendus* note were developed by Poincaré in a memoir of more than a hundred pages, "Analysis situs", published in 1895 in *Journal de l'École Polytechnique*, a masterwork that strikes even the most blasé reader with its amazing originality. Jean Dieudonné, certainly not one well-disposed toward a mathematician who represented the antithesis of the Bourbakist ideal, called it a "fascinating and exasperating paper".[39] Here I will not give a detailed account of "Analysis situs" and its five *compléments*,[40] but will rather confine myself to highlighting some key issues related to Poincaré's quest for understanding the import of invariants, with special attention paid to the role of analogy in the creative process.

After providing the definition of *variété* and *homémorphisme*[41] (henceforth, these words will be translated as "manifold" and "diffeomorphism") and introducing the

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[38] "On peut se demander si les nombres de Betti suffisent pour déterminer une surface fermée au point de vue de l'Analysis situs, c'est-à-dire si, étant données deux surfaces fermées qui possèdent || mêmes nombres de Betti, on peut toujours passer de l'une à l'autre par voie de déformation continue.", "Sur l'analysis situs", *Comptes rendus de l'Académie des Sciences*, 115 (1892), pp. 603-606 = *Œuvres*, vol. VI, pp. 189-192 (the quote is at pp. 189-190).

[39] *A History of Algebraic and Differential Topology*, Birkhäuser, Basel-Boston 1989, p. 28.

[40] Accurate accounts, each focusing on different aspects of Poincaré's work, are provided in: Jean Dieudonné, *A History of Algebraic and Differential Topology*, cit., chap. I; Erhard Scholz, *Geschichte des Mannigfaltigkeitsbegriffs von Riemann bis Poincaré*, Birkhäuser, Boston-Basel-Stuttgart 1980, chap. VII; K.S. Sarkaria, "The topological work of Henri Poincaré", in *History of Topology*, edited by I.M. James, Elsevier, Amsterdam 1999, pp. 121-167; Klaus Volkert, *Das Homöomorphismusproblem insbesondere der 3-Mannigfaltigkeiten, in der Topologie 1892-1935*, Kimé, Paris 2002; Jeremy Gray, *Henri Poincaré. A Scientific Biography*, Princeton University Press, Princeton (NJ) and London 2013, chap. 8.

[41] In actual fact, Poincaré gives two different definitions of *variété*. According to the first definition, a *variété* is a subspace of $\mathbb{R}^n$ determined by a system of equations (satisfying suitable conditions of

notion of homologie (as, in modern parlance, an equivalent relation on the free abelian group generated by closed submanifolds), Poincaré spells out what *he* means by "order of connectivity":

> We say that the manifolds
>
> $$v_1, v_2, \ldots, v_\lambda$$
>
> which have the same number of dimensions and form part of V are *linearly independent* if they are not connected by any homology with integral coefficients.
>
> If there exist P_{m-1} *closed* manifolds of m dimensions which are linearly independent and form part of V, but not more than P_{m-1}, then we shall say that the order of connectivity of V with respect to manifolds of m dimensions is equal to P_m.[42]

In spite of the fact that these numbers are called "Betti numbers", they do not coincide – as it would emerge from the critical observations of Poul Heegaard – with the orders of connectivity defined by the Italian mathematician in his 1871 paper.

In §7 of "Analysis situs", Poincaré, relying on his previous paper "Sur les résidus des intégrales doubles",[43] gives an interpretation of *his* Betti numbers as the maximum number of "periods" of multiple integrals that satisfy certain integrability conditions. This fundamental result is presented in a rather cursory way and not used in the rest of the memoir: in fact, it would remain dormant for more than thirty years, until Élie Cartan would reformulate it in the language of differential forms.[44] A rigorous (and almost complete) proof would be provided by Georges de Rham in his thesis of 1931.[45]

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regularity) and inequalities; in modern terminology, one would call it an immersed C^1submanifold of $\mathbb{R}^n$, possibly with boundary and with some mild singularities. According to the second broader definition, instead, the spaces to be studied are *chaînes continues* of *variétés* of the same dimension n pairwise glued together along a common smooth part of dimension *n*, or, more generally, *réseaux continus* of *variétés*. A *réseau continu* of *variétés* corresponds more or less to what today is called a "C^1manifold" (with or without boundary), possibly with some singularities (for example, some of the spaces discussed by Poincaré in §15 are orbifolds). Poincaré's notion of *homéomorphisme* corresponds to that of C^1diffeomorphism.

[42] "Nous dirons que les varietiés $v_1, v_2, \ldots, v_\lambda$ d'un même nombre de dimensions et faisant partie de V, sont *linéairement indépendentes*, si elles ne sont liées par aucune homologie à coéfficients entiers. S'il existe P_{m-1} variétés *fermées* à m dimensions faisant partie de V et linéairement indépendentes et s'il n'en existe que P_{m-1}, nous dirons que l'ordre de connexion de V par rapport aux variétés à m dimensions est égal à P_m.", "Analyis situs", cit., p. 207 (Poincaré's italics).

[43] *Acta mathematica*, 9 (1887), pp. 321-380 = *Œuvres*, vol. III, pp. 440-489.

[44] É. Cartan, "Sur le nombres de Betti des espaces de groupes clos", *Comptes rendus de l'Académie des Sciences*, 187 (1928), pp. 196-198.

[45] G. de Rham, "Sur l'analysis situs de variétés à n dimensions", *Journal de mathématiques pures et appliquées*, 9th ser., 10 (1931), pp. 115-200; the essential new tool in de Rham's proof is the notion of current. The theorem proved by de Rham can be stated in the following way: for every closed and orientable differentiable manifold of dimension n, the dimension (over $\mathbb{Z}$) of the k-th homology group with integer coefficients is equal to the dimension (over $\mathbb{R}$) of the vector space consisting of degree k closed differential forms modulo degree k exact differential forms, for all $k = 0, \ldots n$. The latter vector spaces are now (rather preposterously) called "de Rham cohomology groups". De Rham's proof cannot be considered "complete" beacause it takes as an assumption that every closed manifold admits a cellular decomposition. Poincaré himself had attempted to prove this fact in his "Complément à l''analysis situs'", *Rendiconti del Circolo matematico di Palermo*, 13 (1899), pp. 285-343 = *Œuvres*, vol. VI, pp. 290-337; a proof was first provided by John Henry C. Whitehead in his paper "On C^1-complexes", *Annals of Mathematics*, 2nd ser., 41 (1940), pp. 809–824.

The famous homology duality theorem is stated and “proved” in §9:

> Consequently, for any closed manifold, the Betti numbers equally distant from the extremes are equal.[46]

However, Poincaré’s argument, which is based on the notion of the “intersection number” of two submanifolds that intersect (transversally) in a finite number of points, is convoluted and unclear, even “totally unconvincing”, in the words of Dieudonné.[47] As we shall see below, it did not fail to draw the harsh criticism of Heegaard.

The key ingredient to prove the claim contained in the 1892 *Comptes rendus* note is supplied by the generalization of a geometric technique that Poincaré had first employed many years earlier. In the already mentioned memoir “Théorie des groupes fuchsiens”, he had constructed closed surfaces by “gluing together”[48] in a suitable way the sides of curvilinear polygons. In §§10-11 of “Analysis situs” this procedure is extended to three-dimensional manifolds:

> There is a manner of visualizing manifolds of three dimensions situated in a space of four dimensions which considerably facilitates their study.[49]

This “way of visualizing three-dimensional manifolds” consists in realizing them as quotient spaces (as we would say today) of convex polyhedra modulo equivalence relations. To begin with, Poincaré examines in detail – almost with the eye of a taxonomist of geometric forms – a series of five examples: the first is the three-dimensional torus obtained by identifying, without twisting or reflecting, the opposite faces of cube, and the last is the real projective space (inexplicably not given any name at all) constructed as quotient of a regular octahedron.

In §11 Poincaré goes on to describe another way of representing three-dimensional manifolds that “can be applied in certain cases”. This construction consists in gluing together the faces of a polyhedron by using the equivalence relation induced by the action of a “properly discontinuous group”. As Poincaré laconically remarks,

> The analogy with the theory of Fuchsian groups is too evident to need stressing; I shall confine myself to a single example.[50]

This single example provided by Poincaré is the same which he alluded to in the note of 1892. Let us consider the transformations:

$$(x, y, z) \longrightarrow (x + 1, y, z)$$
$$(x, y, z) \longrightarrow (x, y + 1, z)$$
$$(x, y, z) \longrightarrow (\alpha x + \beta y, \gamma x + \delta y, z + 1),$$

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46 “Par conséquent, pour une variété fermée, les nombres de Betti également distants des extrêmes sont égaux.”, “Analysis situs”, cit., p. 228.

47 *History of Algebraic and Differential Topology*, cit., p. 221.

48 Poincaré himself uses the verb *coller* (“Analysis situs”, cit., p. 237).

49 “Il y a une manière de se représenter les variétés à trois dimensions situées dans l’espace à quatre dimensions, manière qui en facilite singulièrement l’étude .”, “Analysis situs”, cit., p. 229.

50 “L’analogie avec la théorie des groupes fuchsiens est trop évidente pour qu’il soit nécessiare d’insister; je me bornerai à un seul exemple”, *ibid.*, p. 237

where $\alpha, \beta, \gamma, \delta$ are integers such that $\alpha\delta - \beta\gamma = 1$. This amounts to say that the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

belongs to the modular group $\mathrm{SL}(2,\mathbb{Z})$, a group that Poincaré knows well and since long time, precisely for the reason that it plays a role of primary importance in the theory of automorphic functions. The previous transformations can be used to glue together the faces of the cube in $\mathbb{R}^3$ having one vertex at the origin $(0,0,0)$ and another at the point $(1,1,1)$. Through the identifications prescribed by the first two *substitutions* (which are independent of z) we obtain a family of tori parameterized by the segment $0 \leq z \leq 1$, namely, a three-dimensional cylinder whose sections are two-dimensional tori. The third *substitution* identifies the two bases of this cylinder by means of the linear transformation defined by the parameters $\alpha, \beta, \gamma, \delta$. The resulting three-dimensional manifold (Poincaré's sixth-example) can be thought of as a family of two-dimensional tori parameterized by the circle (in other words, more technically, it is a torus fibration over S^1).

The strategic significance of the sixth example it is not immediately clear, but emerges gradually throughout the three sections that follow, §§12, 13, 14, which in themselves occupy some good twenty pages of the paper. In §12 Poincaré introduces the notion of fundamental group (*groupe fondamental*) of a manifold: this group is defined as the set consisting of closed paths (*contours fermés*) based at an initial point M_0 modulo the equivalence relation that identifies to zero paths that can be continuously deformed to a trivial *lacet* around M_0, equipped with the natural operations of composition and inverse. It should be remarked that all necessary ingredients for Poincaré's definition were already present in Jordan's paper "Des contours tracés sur les surfaces"[51] (where the expression "contour fermé" also appeared). Of course the fundamental group is not conceived by Poincaré as an abstract group, but as a group of *substitutions* acting on a certain set of functions F that are not supposed to be "uniform" on the given manifold:

> When the point M leaves its initial position M_0 and returns to that position after || traversing an arbitrary path, it may happen that the functions F do not return to their initial values.[52]

In today's terminology, one would say that the functions F are not defined on the manifold itself, but on its universal cover.

Poincaré comes back to his sixth example in §§13-14. After some computations involving the so-called *contours fondamentaux* (i.e. the generators of the fundamental group, which, in this case, is $\mathrm{SL}(2,\mathbb{Z})$) and a quite tortuous argument, he manages to

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[51] Cit..

[52] "Lorsque le point M, partant de sa position initiale M_0, reviendra à cette position, après avoir || parcoru un chemin quelconque, il pourra se faire que les fonctions F ne reviennent pas à leurs valeurs primitives.", "Analysis situs", cit., pp. 239-240.

prove that it is possible to find an infinite number of matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $\mathrm{SL}(2,\mathbb{Z})$) for which the associated quotient manifolds have fundamental groups that are not isomorphic, so that the manifolds themselves are not diffeomorphic. However, his computations show that the first Betti number of all these manifolds can only assume values less or equal to 4. Poincaré therefore concludes:

> *Thus for two closed manifolds to be homeomorphic, it does not suffice for them to have the same Betti numbers.*[53]

The homology sphere

No doubt a difficult paper to digest, "Analysis situs" found nonetheless an attentive reader in the Danish mathematician Poul Heegaard (1871-1948). In his thesis, presented at the University of Copenhagen in 1898,[54] Heegaard provided an original description of 3-dimensional manifolds in terms of certain diagrams[55] and showed himself critical of the results obtained by Poincaré:

> [...] the theory of Riemann and Betti regarding the order of connectivity has many shortcomings and is difficult to use in the case of manifolds of dimension greater than two. Poincaré [...] attempted to complete it but, we believe, he did not succeed.[56]

In particular, Heegaard's criticisms were directed at the duality theorem:

> not only is the theorem not proved; *it must be incorrect*.[57]

By March 1899 Poincaré had already been able to look through the "travail très remarquable" ("very remarkable work") of his younger Danish colleague and partially agreed with the objections raised by him:

> These criticisms are partly well founded; the theorem does not hold true for the Betti numbers as defined by Betti; this follows from an example provided by Heegaard; this followed as well by an example that I had described in my Memoir. However, the theorem is true for the Betti numbers as defined by myself; I found a new proof based on the study of n-dimensional polyhedra, which I shall expound, in the next future, in a more extended memoir.[58]

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[53] "*Pour que deux variétés fermées soient homéomorphes, il ne suffit donc pas qu'elles aient mêmes nombres de Betti.*", *ibid.*, p. 257 (Poincaré's italics).

[54] *Forstudier til en topologisk Teori for de algebraiske Fladers Sammenhaeng*, Det Nordiske Forlag, København 1898; French translation revised by the author, "Sur l''Analysis situs'", *Bulletin de la Société mathématique de France*, 44 (1916), pp. 161-242.

[55] The "Heegaard diagrams" are still in use today.

[56] *Forstudier til en topologisk Teori for de algebraiske Fladers Sammenhaeng*, cit., p. 5.

[57] *Ibid.*, p. 72 (Heegaard's italics).

[58] "Ces critiques sont en partie fondées; le théorème n'est pas vrai des nombres de Betti *tels que Betti les définit*; c'est ce qui résulte d'un exemple cité par M. Heegaard; c'est ce qui résultait d'un exemple que j'avais moi-même rencontré dans mon Mémoire.[58] Le théorème est vrai, au contraire, des nombres de Betti tels que je les définis; j'en trouvé une démonstration qui est fondée sur la considération des polyèdres a n dimensions et que je développerai prochainement dans un Mémoire plus étendu.", "Sur

In his "Complément à l''analysis situs'" Poincaré provided such a proof, based on the assumption (which he also attempted to prove) that every manifold admits a cellular decomposition).[59] On a more fundamental level, one year later, in his "Second complément à l''analysis situs'"[60] Poincaré introduced the key concepts of *variétés sans torsion* and *variétés à torsion*, whose distinction is made by the inspection of the associated *tableaux d'incidence*.[61] I will pass over these developments and focus on the theorem stated by Poincaré in the very last page of his "Second complément":

> *Each polyhedron which has all its Betti numbers equal to 1 and all tableaux T_q orientable is simply connected, i.e., homeomorphic to the hypersphere.*[62]

This is the first version of the famous Poincaré conjecture: in today's language, it claims that any 3-manifold having the same homology groups with integer coefficients as the 3-sphere is homeomorphic to the 3-sphere. As matter of fact, the statement is false: the zigzag path through which Poincaré would come up with stating an "improved" conjecture appears to fit in quite well with the Lakatosian pattern of the heuristic of "proofs and refutations".[63]

Neither Poincaré nor anyone else worried about proving or disproving the truth of the concluding "theorem" of the "Second complément à l''analysis situs'" over the course of the three/four years that followed. The third and fourth supplements to "Analysis situs"[64], published in 1902, were primarily concerned with questions related to complex analysis and algebraic geometry. Here it will be enough to recall that Poincaré took up the strategy (adopted also by his friend Émile Picard[65]) of describing an algebraic surface as the total space of a family of hyperplane sections

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les nombres de Betti", *Comptes rendus de l'Académie des Sciences*, 128 (1899), pp. 629-630 = *Œuvres*, vol. VI, p. 289.

[59] See note 45.

[60] *Proceedings of the London Mathematical Society*, 32 (1900), pp. 277-308 = *Œuvres*, vol. VI, pp. 338-370.

[61] A *variété à torsion* is, in modern terminology, a manifold whose homology groups with integer coefficients contain torsion elements. Given a polyhedral decomposition of a manifold M, the associated *tableau d'incidence* T_q $(0 < q < \dim M)$ can be tought of as a matrix whose columns correspond to the oriented q-polyhedra of the decomposition and whose rows correspond to the $(q-1)$-polyhedra; at each intersection of a row and column one places either 0, 1, or -1, depending whether, respectively, the $(q-1)$-polyhedron is not a face of the q-polyhedron, is a face and has the same orientation, or is a face and has the opposite orientation.

[62] "*Tout polyhèdre qui a tous ses nombres de Betti égaux à 1 et tous ses tableaux T_q bilatères est simplement connexe, c'est-à-dire homéomorphe à l'hypersphère*" (Poincaré's italics). If a polyhedron has all its "tableaux T_q orientable (*bilatères*)", then it has no torsion coefficients in its homology groups ("Second complément à l''analysis situs'", cit., p. 307); in Poincaré's terminology, a manifold is "simplement connexe" if its fundamental group is trivial.

[63] Cf. Imre Lakatos, *Proofs and Refutations. The Logic of Mathematical Discovery*, edited by J. Worrall and E. Zahar, Cambridge University Press, Cambridge 1976 (see, in particular, p. 42).

[64] "Sur certaines surfaces algébriques. Troisième complément à l''analysis situs'", *Bulletin de la Société mathématique de France*, 30 (1902), pp. 49-70 = *Œuvres*, vol. VI, pp. 373-392; "Sur les cycles des surfaces algébriques. Quatrième complément à l''analysis situs'", *Journal de mathématiques pures et appliquées*, 5th ser., 8 (1902), pp. 169-214 = *Œuvres*, vol. VI, pp. 397-434.

[65] See Émile Picard & Georges Simart, *Théorie des fonctions algébriques de deux variables complexes*, tome I, Gauthier-Villars, Paris 1897 (in particular, chap. IV).

(i.e., as a Lefschetz pencil, in today's language).[66] This approach – transposed by analogy to the realm of topology – led him, in the fifth supplement to "Analysis situs",[67] to develop a new and powerful method for studying the geometry of differentiable manifolds.

Poincaré's idea – essentially the same which lies at the basis of Morse theory – consists in "slicing" a given manifold V of dimension m immersed in $\mathbb{R}^k$ by means of a one-parameter family of $(k-1)$-dimensional hypersurfaces $\varphi(x_1, x_2, \dots, x_k) = t$. In this way, one obtains a "system" $W(t)$ composed (generally) of "a certain number of $(m-1)$-dimensional manifolds $w_1(t), w_2(t), \dots, w_p(t)$":

> When t varies continuously from $-\infty$ to $+\infty$, the system $W(t)$ varies continuously and *generates* the manifold V. If the manifold V is closed, the manifolds $w_1(t), w_2(t), \dots, w_p(t)$ are likewise.[68]

Next, each manifold $w_i(t)$ is associated with a point in the Euclidean 3-space; as t varies, the moving points will produce a "sort of network of lines", that Poincaré calls the *squelette* (*skeleton*) of the manifold M:

> Under these circumstances, when t varies in a continuous manner, the points representing the p manifolds $w_1(t), w_2(t), \dots, w_p(t)$ generate p continuous lines $L_1, L_2, \dots, L_p$, at least as long as the number p does not change. But this number can change at $t = t_0$, if one of the manifolds decomposes into two, or if, on the contrary, two manifolds merge into one. In the first case one of the lines L bifurcates, in the second case two of the lines L combine into one.[69]

It is perhaps worth recalling that Poincaré had extensively dealt with the phenomenon of bifurcation in his research work about the equilibrium figures of rotating fluid masses.[70] So, his notion of *squelette* of a manifold seems to emerge as the combined outcome of a dual analogy process: on the one hand, the slicing procedure draws inspiration from the the theory of algebraic surfaces, on the other, the bifurcating lines are reminiscent of the bifurcating continuous families of equilibrium ellipsoids.

The crucial observation, at this point, is the following:

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[66] Cf. Simon K. Donaldson, "One hundred years of manifolds topology", in *History of topology*, cit., p. 436.

[67] "Cinquième complément à l''analysis situs'", *Rendiconti del Circolo matematico di Palermo*, 18 (1904), pp. 45-110 = *Œuvres*, vol. VI, pp. 435-498.

[68] "Quand t variera d'une manière continue de $-\infty$ à $+\infty$, le système $W(t)$ variera d'une manière continue et *engendra* la variété V. Si la variété V est fermée, les variétés $w_1(t), w_2(t), \dots, w_p(t)$ le seront également.", *ibid.*, p. 436 (Poincaré's italics).

[69] "Dans ces conditions, quand t variera d'une manière continue, les points représentatifs des p variétés $w_1(t), w_2(t), \dots, w_p(t)$ engendreront p lignes continues $L_1, L_2, \dots, L_p$; du moins tant que le nombre p ne varie pas. Mais ce nombre peut varier pour $t = t_0$, si l'une des variétés se décompose en deux, ou si, au contraire, deux variétés se réunissent en une seule. Dans le premier cas l'une des lignes L se bifurque, dans le second deux des lignes L se réunissent en une seule.", *ibid.*, p. 437.

[70] See, for example, "Les formes d'équilibre d'une masse fluide en rotation", *Revue générale des sciences pures et appliquées*, 3 (1892), pp. 809-815 = *Œuvres*, vol. VII, pp. 529-537; "Sur l'équilibre d'un fluide en rotation", *Bulletin astronomique*, 16 (1899), pp. 161-169 = *Œuvres*, vol. VII, pp. 151-158; "Sur la stabilité de l'équilibre des figures piriformes affectées par une masse fluide en rotation", *Philosophical Transactions*, 198 A (1901), pp. 333-373 = *Œuvres*, vol. VII, pp. 161-162.

> If we follow one of these lines, L_1, for example, described by the point representing $w_1(t)$, we see that this manifold remains homeomorphic to itself (and in such a way that two manifolds $w_1(t)$ and $w_1(t)+\varepsilon$ corresponding to neighboring points differ very little from each other) *as long as we do not pass through a value* t *such that* $w_1(t)$ *has a singular point.*
> We must then mark the lines of our network at points where the corresponding manifolds $w(t)$ have singular points. These will be the points of division which cut our lines into sections, but as long as we remain on one of these sections the corresponding manifold $w(t)$ will remain homeomorphic to itself.[71]

It is thus necessary "to study these singular points", and Poincaré does not retreat. First, he states (with no proof) the result that is called the "Morse lemma" in today's textbooks of differential topology,[72] namely that in a neighborhood of a point p in $\mathbb{R}^{m+1}$ where all first order partial derivates of an analytic functions φ vanish, there are coordinates $(y_1, y_2, \dots, y_{m+1})$ such that

$$\varphi = \varphi(p) + \sum A_i y_i^2 - \sum B_k y_k^2$$

where the coefficients A_i, B_k are positive.[73] Next, he proves that

> if a two-dimensional V is orientable, its skeleton will not have singular points other than *culs-de-sac* and bifurcations[74]

and exploits this fact (using as well a good deal of hyperbolic geometry) to classify closed and orientable surfaces.[75] Finally, in a real tour de force, he applies his new method to three-dimensional manifolds.

As paradoxical as it may sound, the crowning achievement of the "Cinquième complément" is the disproving of the theorem stated at the end of the "Deuxième complément". Indeed, Poincaré provides an example (or rather a counterexample) of a 3-manifold, "all Betti numbers and torsion coefficients of which equal 1, but which is not simply connected".[76] The procedure followed by Poincaré consists in taking two

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[71] "Si nous suivons l'une de ces lignes L_1, par exemple, décrite par le point représentitif de $w_1(t)$, nous voyons que cette variété $w_1(t)$ reste constamment homéomorphe à elle-même (et cela de telle façon que sur les deux variétés trè voisines $w_1(t)$ et $w_1(t)+\varepsilon$, deux points correspondants diffèrent très peu l'un de l'autre) *tant que l'on ne passe pas par une valeur de* t *telle que* $w_1(t)$ *ait un point singulier.* Nous devons donc marquer sur les lignes de notre réseau les points qui correspondent aux variétés $w(t)$ qui ont des points singuliers. Ce seront des points de division qui partegeront nos lignes en tronçons, mais tant qu'on suivra l'un de ces tronçons, la variété $w(t)$ correspondante restera homéomorphe à elle-même.", "Cinquième complément à l''analysis situs'", cit., p. 437 (Poincaré's italics).

[72] See, for example, John Milnor, *Morse Theory*, based on lectures notes by M. Spivak and R. Wells, Princeton University Press, Princeton (NJ) 1969, p. 6.

[73] See "Cinquième complément à l''analysis situs'", cit., p. 439.

[74] "Si […] V a deux dimensions et est bilatère, son squelette n'aura d'autre point singulier que les culs de sac et les bifurcations", *ibid.*, p. 443.

[75] Poincaré's argument follows more or less the same guidelines as the "modern" Morse theoretic proof of the classification theorem for topological surfaces; cf. M. W. Hirsch, *Differential Topology*, Springer-Verlag, New York 1976, chap. 9.

[76] "[D]ont tous les nombres de Betti et les coéfficients de torsion sont égaux à 1, et qui pourtant n'est pas simplement connexe", "Cinquième complément à l''analysis situs'", cit., p. 436.

3-manifolds V', V'' whose boundary is a surface W of genus 2 (i.e., in today's parlance, two handle-bodies of genus) and pasting them together by identifying the boundary surfaces via a suitable homeomorphism, so to obtain a 3-manifold V. Since "every cycle of V is equivalent (i.e., homotopic) to a cycle of W"[77] (an essential lemma that Poincaré establishes after a lengthy argument), in order to determine "the homologies" and the fundamental group it suffices to choose a basis C_1, C_2, C_3, C_4 for the homology of W and then take into account the relations induced by the gluing homeomorphism. To this purpose, Poincaré identifies the "fundamental cycles" K_1', K_2' of V' with C_1, C_3; one has $C_1 \equiv 0$ and $C_3 \equiv 0$. Making reference to Figure 1, the cycle C_1 is represented by the "conjugate circles" $-A, +A$, while C_2 is represented by the "conjugate circles" $-B, +B$. The "fundamental cycles" K_1'', K_2'' of $V''$are represented by the arcs of curves running between labeled points on the perimeter of the figure; more precisely,

> The arcs which represent K_1'' are shown as unbroken lines; those which represent K_2'' are dotted. The arrows indicate the sense in which they are traversed.[78]

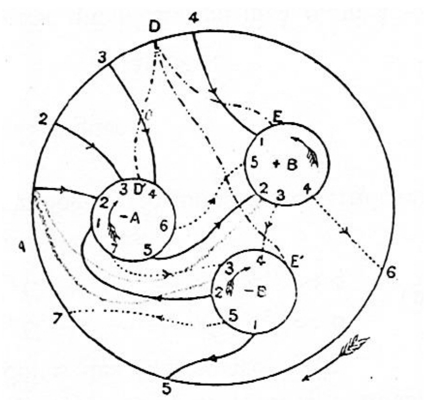

Figure 1 (Fig. 4, "Cinquième complément à l''analysis situs'", cit., p. 494)

By expressing the cycles K_1'', K_2'' in terms of the cycles C_1, C_2, C_3, C_4, Poincaré, after some computations, gets the relations

$$-C_2 + C_4 - C_2 + C_4 \equiv 0; \quad 5C_2 \equiv 0; \quad 3C_4$$

(note that, in this case, the fundamental group is not commutative). These relations – he immediately remarks – are "the relations of the structure in which the substitutions C_2 and C_4 generate the icosahedral group".[79] He can therefore conclude that

> the fundamental group of V cannot reduce to the identical substitution, since it contains the icosahedral group as a subgroup.[80]

[77] *Ibid.*, p. 490.

[78] "Les arcs qui représentent K_1'' sont en trait plein; ceux qui représentent K_2'' sont en trait pointillé. Une pointe de flêche placée sur le trait lui-même indique dans quel sens ce trait doit être parcouru.", *ibid.*, p. 494.

[79] "[L]es relations de structure qui ont lieu entre les deux substitutions C_2 et C_4 qui engendrent le *groupe icosaédrique*", *ibid.*, p. 498 (see notes 81 and 82).

[80] "Le groupe fondamental de V ne saurait se réduire à la substitution identique, puisqu'il contient comme sous-groupe le groupe icosaédrique.", *ibid.*

On the other hand, Poincaré shows that the first Betti of V is equal to 0, so that V is what we today call a "homology sphere".

How did Poincaré arrive at producing his example? This is, of course, an unanswerable question: we can only hazard some guesses. We can safely assume that Poincaré was not unaware of Felix Klein's book *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade* (Teubner, Leipzig 1884), where the icosahedral group was described in detail.[81] However, there is evidence that Poincaré had no inkling of what today is called the Hurewicz theorem (namely, the fact that the abelianization of the fundamental group is the first homology group with integer coefficients): in fact, he did not exploit the property of the icosahedral group of being a perfect group,[82] and instead explicitly computed both the "homology equivalences" and the "homotopy equivalences" of the cycles C_1, C_2, C_3, C_4. Moreover, it seems certain that he believed that his example was only one among many possible others ("[…] nous nous bornerons à donner un example"), while we know that he stumbled over the *only possible* example.[83] In conclusion, we can plausibly suppose that Poincaré essentially proceeded by trial and error, perhaps keeping in his mind (or in the back of his mind) that the icosahedral group could have been a workable algebraic object. Figure 1 above is drwan well (at least in its printed

[81] Klein gives a description of the icosahedral group as the group generated by the operation S, T with the relations $S^5 = 1$, $T^2 = 1$, $(ST)^3 = 1$ (in multiplicative notation; see *Vorlesungen über das Ikosaeder…*, cit., p.41); by letting $C_2 = S$ and $C_4 = ST$, one obtains Poincaré's relations (in additive notation). The same relations had been previously derived by William Rowan Hamilton in his papers "Memorandum respecting a new system of roots of unity", *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science*, 4th ser., 12 (1856), p. 446 and "Account of the icosian calculus", *Proceedings of the Royal Irish Academy*, 6 (1858), pp. 415–416.

[82] Let us briefly review a possible realization of Poincaré's homology sphere using the language and tools of differential topology. The icosahedral group I is the symmetry group of the icosahedron; it is isomorphic to the group of even permutations of 5 elements, i.e., the alternating group A_5. The group I is of order 60, simple (i.e., it does not contain any proper normal subgroup) and perfect (i.e., $[I, I] = I$); it admits the presentation $\langle x, y |\, (xy)^2 = x^3 = y^5 = 1\rangle$. Since the isometries of the icosahedron are proper rotations, there is a natural immersion $I \subset \mathrm{SO}(3)$. Let us consider the double cover $\mathrm{SU}(2) \longrightarrow \mathrm{SO}(3)$ and denote by I^* the inverse image of I. The group I^* – called the binary icosahedral group – is of order 120 and perfect, but no longer simple (its center is the group of the two elements $\{1, -1\}$); it admits the presentation $\langle x, y |\, (xy)^2 = x^3 = y^5\rangle$. Now $\mathrm{SU}(2)$ is homeomorphic to the three-dimensional sphere S^3 and it can be shown that the quotient space $\mathrm{SU}(2)/I^*$ is smooth. The fundamental group of $\mathrm{SU}(2)/I^*$ is, of course, I^*; by the Hurewicz theorem, its first homology group is the abelianization of I^*, namely $I^*/[I^*, I^*] = 0$; moreover, one can prove that $H_2(\mathrm{SU}(2)/I^*, \mathbb{Z}) = 0$ (this follows from the fact that I^* is superperfect). In conclusion, the manifold $\mathrm{SU}(2)/I^*$ is an integral homology sphere, actually homeomorphic to Poincaré's example V. For further details see R.C. Kirby & M.G. Scharlemann, "Eight faces of the Poincaré homology sphere", in *Geometric Topology. Proceedings of the 1977 Georgia Topology Conference*, edited by J.C. Cantrell, Academic Press, New York 1979, pp. 113-146; N. Saviliev, *Invariants for Homology* 3*-Spheres*, Encyclopedia of Mathematical Sciences, vol. 140, Springer-Verlag, Berlin-Heidelberg 2002.

[83] This result was proved by Michel A. Kervaire in his paper "Smooth homology spheres and their fundamental groups", *Transactions of the American Mathematical Society*, 144 (1969), pp. 67-72: if M is a 3-dimensional manifolds such that $H_*(M) = H_*(\mathrm{S}^3)$ and its fundamental group Γ is finite, then either $\Gamma = \{1\}$ or else $\Gamma = I^*$.

version), but behind it lie, most probably, dozens of geometric experiments on figures drawn badly.[84]

As it almost universally known, the "Cinquième complement" ends with the statement of what will be later called the "Poincaré conjecture", which is but a mere question in its original formulation:

> Is it possible for the fundamental group of V to reduce to the identity without V being simply connected?[85]

Even a cursory review of the various attempts to solve this problem it until the solution provided by Grigori Y. Perelman in 2003[86] "nous entraînerait – I can say in Poincaré's words – trop loin".

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[84] "It is clear that in order to arrive at his example of a nonsimply-connected homology sphere […] Poincaré must have done a good deal of experimentation with Heegaard diagrams, of genus 2, and presumably of higher order too", C.McA. Gordon, "3-dimensional topoogy", in *History of Topology*, cit., pp. 449-489 (the quote is at p. 462).

[85] "Est-il possible que le groupe fondamental de V se réduise à la substitution identique, et que pourtant V ne soit pas simplement connexe?", "Cinquième complément à l''analysis situs'", cit., p. 498.

[86] See, for example, J. Stillwell, "Poincaré and the early history of 3-manifolds", *Bulletin of the American Mathematical Society*, new ser., 49 (2012), pp. 555-576; C. Rourke, "La congettura di Poincaré", in *La matematica. II. Problemi e teoremi*, edited by C. Bartocci, Einaudi, Torino 2008, pp. 731-763. It should be pointed out that Perelman solved a more general problem than the Poincaré conjecture, namely Thusrston's geometrization conjecture.

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