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Toward a formal theory of the measuring system

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Abstract. Measurement aims at obtaining a numerical description of objects/events/persons in real world by means of a measuring system. Measurement is widely used as a key way for obtaining high quality information from the real world, across disciplines. In the present day, there is growing consensus in holding that measurement is characterized by the use of something that qualifies as a "measuring system". Therefore, we discuss sufficient conditions for an empirical system to qualify as a measuring system.

1. Introduction

The notion of measurement (or measuring) system has been recently raised as a key issue in measurement science [3-4]. Curiously, at the dawn of measurement theory, Campbell noted that the way measurement is actually performed is of no foundational interest [1], which probably contributed to often identify the theory of measurement with the representational approach [6]. On the other hand others recognised the centrality of this issue in measurement. One of the earliest contributions in this direction was an important communication by the late Gonella, at the IMEKO World Congress in Huston [7], which unfortunately had much less resonance than it deserved. The focus on the measurement system as a way for going beyond the representational viewpoint, was well highlighted by Mari [9], although somewhat overlooking, in our opinion, the still permanent validity of the representational approach in addressing measurability issues [10]. Another curious fact is the almost total absence of such a notion in The Guide to the evaluation of uncertainty in measurement (GUM) [15], somehow reflecting, consciously or not, the initial Campbell's prejudice. On the other hand, anyone involved in measurement in real life, and/or in education in measurement, cannot fail to recognise the centrality of this concept, and, consequently, of the related modelling issues, as attested, in our experience, by all accredited textbooks on measurement, one for all being the excellent book by Bentley [8]. In consequence of this awareness, other papers have been published meanwhile on the notion of measurement (or measuring) system, including contributions by Sommer [13-14] and by Ruhm [12]. In fact, such a model is essential for measurement science to qualify as an *autonomous discipline,* and the development of a generally agreed view of measurement among disciplines [5], which constitutes the key focus of this IMEKO Joint TC1-TC7-TC13 event, strongly depends, in our humble opinion, upon that. Noteworthy, for example, the important and influential Rash model, defines *a kind of measurement system*, although of an essentially probabilistic nature [16-17]. Furthermore, from the application side, we think that one major difficulty for an extensive application of GUM's principles, very likely stems from the lack of reference, in the GUM and its related documents, to a proper modelling of the measuring system.

Therefore, as a part of such a debate, we try and contribute here by proposing a general model of the measurement process, which is based on the primitive notion of "empirical relations" which allows defining measurement in terms of a more primitive notion. As far as we know, what here proposed is totally new in the measurement-science panorama.

2. The proposed framework

2.1. The notion of cross order

To attain to a general description of the measuring system (MS), we consider an input-output description, where the input is an "object"[1] $a \in A$, in a (not directly observable) state[2] $\alpha \in A*$, in respect to some quantity[3] x, and the output is a (directly observable) indication of the MS, $b \in B$, with respect to some quantity y.

One idea we wish to convey here, is that the measuring system makes the *originally unobservable* state of the measurand *actually observable*, by transducing it into another – generally, but not necessary, different – quantity, belonging to the MS. For example, a force to be measured may be transduced into the strain of an elastic bar and then into a voltage; or a temperature into a voltage, through the thermoelectric effect, and so on. Another possibility is that, e.g., a small displacement is transformed into an easy-to-detect rotation of a needle on a gage: here a mechanical quantity is made actually observable by an amplification transformation.

Another idea to be conveyed, is that a mapping from *a scale holding for the measurand* into *a scale holding for the output quantity* of the MS, is realised.

With these ideas in mind, let us then look for appropriate properties of the MS, that fully characterise it. For the principle of parsimony, generally accepted in science, such properties should be as weak as possible.

To achieve this goal, we firstly introduce the notion of *cross-order*, that we *define* as a weak order, denoted by "$\succcurlyeq$", that holds true over the *union* of two sets, $C = A \cup B$ in our case, and that also satisfies the following additional properties:

- its restriction to B is a *strict order,* denoted by "$\succ$";
- for each $a \in A$ there exists $b \in B$ such that $a \sim b$, where "$\sim$" denotes equivalence.

Then, it is possible to show that there exist order-preserving measure functions on both A and B,

- $m_A : A \to X$, and
- $m_B : B \to Y$,

where X and Y are numerical sets, that fulfil order-representation properties. In fact, since $A \subset C$, the structure $(A, \succcurlyeq)$ is a weak order, and thus, for each $a_1, a_2 \in A$, there exists a function m_A such that [11]

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[1] Here the term "object" generically denotes the carrier of the property to be measured (this property is also called the "measurand" in the current terminology). Therefore it may be either a *material object*, such as a workpiece, whose length is measured for quality-control purposes, or an *environment*, such as an office, whose temperature, noisiness, … are measured for checking the conformance to ergonomic standards, or an *event*, such as the taking off of an airplane, where the peak loudness of the emitted noise is controlled for safety reasons, or even a *person*, whose attitude to mathematics is tested, for allowing access to a specialized high-level college.

[2] The notion of "state" here considered corresponds to the notion of "quantity value", of the current terminology, at least in the interpretation recently provided by Mari and Giordani [4]. In a few words, the state of a quantity is a non-numeric entity that expresses the way in which an object manifests some (measurable, in our case) property.

[3] Here quantity means "measurable property".

$$a_1 \succcurlyeq a_2 \leftrightarrow m_A(a_1) \geq m_A(a_2). \tag{1}$$

Furthermore, since $(B, \succ)$ is a strict order, for each $b_1, b_2 \in B$, there exists a function m_B such that

$$b_1 \succ b_2 \leftrightarrow m_B(b_1) > m_B(b_2). \tag{2}$$

2.2. Transduction or observation

As already noted, one of the main ideas we want to express is that, in the input/output model we are considering, the output, y, constitutes an *observable entity*, whilst the input, x, *does not*. This idea can be expressed by regarding m_B as a *known* function, which somewhat implies that we assume to be able to express instrument indication by numbers, whilst m_A is *known to exist* but it *is otherwise unknown*: this is indeed the reason why *we need a measuring system for actually performing measurement*.

Regarding m_B as a *known* function, is equivalent to assume that we are always able to express instrument indications by numbers. Let us then briefly discuss this point. One could argue that y is also a quantity, and therefore it also needs measuring, which requires an additional transduction step associated to y. But this would lead to a *regressio ad infinitum*, since for the transduction of y we would also have an output stage needing measurement, and so on. Therefore, for the measurement procedure to be really applicable – as it actually is – we should require *that it ends with a transduction where the indication may be directly expressed by numbers*. As an example, in the case of temperature measurement by liquid-in-glass thermometers, we can assume that a expresses the thermal state of the measurand, and b the height of the liquid column in the thermometer. Such states (heights) of the column may be revealed by traits on the external surface of the glass tube containing the thermometric liquid, and a number may be associated to each trait. Now, although the number we *directly* read on the glass is linked to the height of the column through intermediate transformations – for example if we associate the level of the column to its representative trait by visual inspection, we may have an error due to misalignment, and so on – but the very final step, that is the association of the number to the traits, may be reasonably assumed as a direct, uncertainty-free step: the hypothesis that we may associate a greater number to a trait that precedes, rather than following, a trait with a lower number, would be an unreasonable speculation! Furthermore, for the reasoning above, this possibility of finally associating a number to an indication, *must* always be the case. Lastly, note now that it is immaterial how many transduction steps we have before this last one: we may thus always consider, without loss of generality, an overall transduction that corresponds to the chaining of all such transduction steps, followed by the final conversion-to-numbers step. This is exactly what we propose in this model.

After this premise, let us now go back to the characterisation of what we have called the "transduction" or, more generally, the *observation* process. This process may be described, *at different discourse levels*, by the following functions, *all of which may be called observation or transduction functions*, to avoid an unnecessary multiplication of terms that would be confusing.

- A function, φ', form "things" to "things", that maps objects into states[4] of the output quantity of the MS; this function may be formally defined and characterised as follows: $\varphi' : A \rightarrow B$ such that

$$b = \varphi'(a) \leftrightarrow a \sim b. \tag{3}$$

Note that such a function it is well defined, since the order on B is strict.

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[4] The attentive reader may have noted that we describe the output by "indications" or by "states of the output". In fact, since a strict order is assumed for the output quantity, states and indications are mutually isomorphic.

- A function, φ, form "things" to "numbers", that maps objects into indications of the MS; that is, $\varphi : A \rightarrow Y$ such that

$$y = \varphi(a) \leftrightarrow y = m_B\big(\varphi'(a)\big). \tag{4}$$

- Lastly, a function, f, from "numbers" to "numbers", $f : X \rightarrow Y$ such that

$$y = f(x) \leftrightarrow y = m_B\big(\varphi'(a)\big) \wedge x = m_A(x). \tag{5}$$

In the application of this model to the description of real systems, under certain circumstances, f may be called "calibration function" (whilst a general acceptance of this definition may create problems, as has been discussed elsewhere [18]).

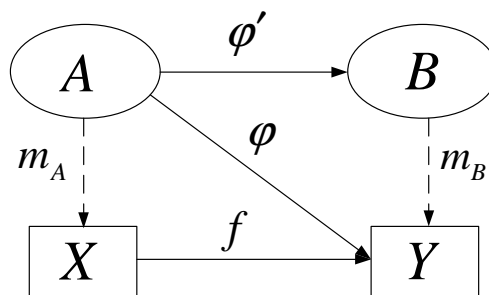An overview of these transformations is provided in Fig. 1.

Fig. 1 Topology of the measuring system. Ellipses denote sets of "things", rectangles represent sets of numbers, solid-line arrows indicate real transformations, described at different discourse levels, and dashed-lines arrows express purely numerical correspondences, *which do not correspond to real transformations but nonetheless do contribute to understand the properties of some real transformations.*

Now, the key property of MS is that *the observation transformation is order preserving*, and *this holds true at all the description levels just considered.*

In formal terms, for each $a_1, a_2 \in A$,

$$a_1 \succcurlyeq a_2 \leftrightarrow \varphi'(a_1) \succcurlyeq \varphi'(a_2) \leftrightarrow \varphi(a_1) \geq \varphi(a_2) \leftrightarrow f(a_1) \geq f(a_2). \tag{6}$$

2.3. The overall measurement process

Let us now show why property (6) is so important: in fact *it ensures that the MS is capable of performing proper, consistent, measurements.*

To show this, let us introduce the *measurement function*

$$\gamma : A \rightarrow \hat{X}, \tag{7}$$

where $\hat{X}$ is the set of the possible measurement values[5]. Obviously, such a set coincides with the set of measure values, X, we already encountered. This simply means that if $x \in X$ is one of the possible measure values, it will also be one of the possible measurement values, that is $x \in \hat{X}$, and vice versa.

Then, interestingly enough, if the MS satisfies property (6), it also satisfies the following representational property, that holds true between objects and measurement values:

$$a \succcurlyeq b \leftrightarrow \gamma(a) \geq \gamma(b). \tag{8}$$

In fact such a property may be satisfied by taking, for each $a \in A$,

$$\hat{x} = \gamma(a) = f^{-1}(\varphi(a)) = f^{-1}(y), \tag{9}$$

which makes sense, since under the above assumptions, f turns out to be invertible.

Therefore, γ may be implemented, in the current practice of measurement, by

- Inputting the measurand to the MS and acquiring the (numerical) indication y: this is the *observation* phase;
- Applying the transformation f^{-1} to the indication, where f is known, typically by calibration, and is invertible, thanks to the above assumptions, which constitutes the *restitution* phase.

3. Conclusion

The notion of measuring system is gaining increasing interest in measurement science. Here we have presented a formal model of the MS, in terms of empirical relations among objects to be measured and the measuring device. We hope that this may help fill a gap in these studies.

By now, we have limited our investigation to order structures and related ordinal quantities, but we plan to extend this treatment to all the structures of metrological interest, as a further step of this study. Another possible development may consist in searching for a suited probabilistic counterpart of this model, which does not look like an easy task, at the present stage.

References
[1]   N.R. Campbell, Physics – the Elements (1920). Reprinted as Foundations of Science, Dover, New York, 1957.
[2]   G. B. Rossi, F. Crenna, A first-order probabilistic logic with application to measurement representations, Measurement 79 (2016) 251-259
[3]   A. Frigerio, A. Giordani, L. Mari, Outline of a general model of measurement, Synthese (2010) 123-149
[4]   L.Mari, A.Giordani, Quantity and quantity value, Metrologia 49 (2012) 756–764
[5]   A. Maul, D. Torres Irribarra, M. Wilson, On the philosophical foundations of psychological measurement, Measurement 79 (2016) 311–320
[6]   J. Michell, The logic of measurement: A realist overview, Measurement 38 (2005) 285–294
[7]   Gonella, L. (1988). Measuring instruments and theory of measurement. Paper presented at the XI IMEKO World Congress, Houston.
[8]   J. P. Bentley, Principles of measurement systems, Pearson-Prentice Hall, Harlow, 2005.

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[5] This set is often assumed to be the set of the real numbers. In fact this very strong assumption, in our opinion, is not really needed in measurement science. This is why we leave it undefined in this presentation. See Ref. [11], or even Ref. [9], for additional ideas on this very important point, that cannot anyway be treated here.

[9]   L. Mari, Beyond the representational viewpoint: a new formalization of measurement, Measurement 27 (2000) 71-84

[10] G. B. Rossi, Measurability, Measurement 40 (2007) 545-562

[11] G. B. Rossi, Measurement and probability, Springer, Dordrecht, 2014.

[12] K.H. Ruhm, Dynamics and stability – A proposal for related terms in Metrology from a mathematical point of view, Measurement 79 (2016) 276-284

[13] K.D. Sommer, Modelling of measurements, system theory and uncertainty evaluation. In: F. Pavese, A. Forbes (Eds), Data Modeling for Metrology and Testing in Measurement Science. Birkhäuser, Boston, 2009 (ISBN 978-0-8176-4592-2).

[14] K.D. Sommer, U. Hanebeck, M. Krystek, F. Sawo, Modelling of dynamic measurement systems for uncertainty analysis employing discretised state-space forms. In: F. Pavese, M. Bar, A.B. Forbes, J.M. Linares, C. Perruchet, N. F. Zhang (Eds), Advanced Mathematical & Computational Tools in Metrology & Testing VIII, World Scientific, Singapore, 2009, pp. 309-318.

[15] BIPM, IEC, IFCC, ISO, IUPAC, IUPAP; OIML, Guide to the expression of uncertainty in measurement, ISO, Geneva, (1993).

[16] L. Mari, M. Wilson, An introduction to the Rasch measurement approach for metrologists, Measurement 51 (2014) 315-327

[17] T. Bond, C.M. Fox, Applying the Rasch model: Fundamental measurement in the human sciences, Taylor and Francis, New York, 2015.

[18] G. B. Rossi, Toward a unified framework for static and dynamic measurements, 2015 IEEE International Instrumentation and Measurement Technology Conference (I2MTC) Proceedings, Pisa, 2015, pp. 1516-1521.