HOW SPATIAL STRUCTURING ABILITIES SUPPORT THE DEVELOPMENT OF LONG TERM EFFECTIVE MENTAL CALCULATION STRATEGIES

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Abstract

This paper concerns an on-going research, developed from 2013, aimed to support the development of effective mental calculation strategies along the time. This research is developed with a group of thirty teachers, both of kindergarten and primary school, and a researcher (the author). For this, it's a research starting with the development of numerical abilities (such as decomposing the number into parts) through spatial structuring abilities (such as spatial visualization of objects) already in kindergarten, in order to support effective strategies in mental calculation, in primary school. We present the analysis of results of a pilot study, carried out with children attending grade 1, and a survey, carried out with children attending the grade 3. The educational activities, which reveal a multimodal approach to numbers and arithmetical, have been presented in others research reports and they'll not discussed in this paper (Robotti, 2018). Findings of the pilot study and the survey show that the effectiveness of the calculation strategies (in terms of response time, adequacy and of correctness) seems depending on the kind of activities developed in kindergarten and primary school. Our findings indicate that the calculation strategies can be considered stable over the time and, therefore, the acquisition of adequate skills for mental calculation in kindergarten and first class of primary school (for instance decomposing numbers into parts and memorizing arithmetical facts) can be considered a sort of "heritage" for students because it can be exploited to effective calculation even after the grade 1.

Keywords: mental calculation strategies, spatial sense, number sense, kindergarten

Introduction

At the end of primary school (students aged 10), the Italian curriculum requires children to manage strategies of mental calculation, using arithmetic properties. From the latest results of the national assessment test in mathematics (year 2019) at grade 8 we can say that the percentage of students who do not reach a sufficient level is, on overage, around 42%. From these results it can be seen that the difficulties in mathematics and the calculation difficulties, seem to persist over time and the need to tackle the problem much earlier than grade 8 seems evident. Thus, I argue it is important to offer all children adequate resources for developing numerical competencies, already in kindergarten. International research has long highlighted the importance of developing both a *sense of number* and *spatial sense* to enhance calculation skills. Since the number can be represented by various signs, not necessarily symbolic, the idea of manipulating - mentally or physically - the structure of the number as a spatial configuration seems functional to the

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characterization of the number sense (van Nes & van Eerde, 2010) and, therefore, functional to (mental) calculation. In other words, the part/whole relationship applied to the number can be understood as (re)organization of the whole starting from its parts on the basis of a gestalt perception of the number - as quantity or as a set of objects, (Arcavi, 2003). This allows us to hypothesize that supporting spatial reasoning could be an effective educational strategy across time also for mental calculation. Thus, the aim of this research is investigate whether the educational approach developed in preschool-class, exerted any detectable long-term effect on these students' arithmetical performance. Findings suggest that educational approach implemented in preschool-class improves students 'long-term performance in flexibility in mental calculation. The mentioned activities implemented in preschool-class (Robotti, 2018, for an idea of this educational approach) will not described in this paper.

Theoretical background

In different researches, available both in cognitive psychology and mathematics educational fields, the *number sense* appears as the common main feature to gain insight into numerical relations and to develop calculation abilities. Nevertheless, there is no common interpretation of the notion of number sense across the communities of cognitive scientists. Despite this, there seems to be a certain consensus about some features of the notion of number sense: the ability to (de)compose quantities based on the part/whole relationship (van Ness & van Eerde, 2010), the ability to manage the use of fingers, called digital gnosia (Noël, 2005), the visual perception (subitizing) of quantities as the memorization of them imprinted in our visual memory. Added to number sense, a significant body of researches has discussed the development of mathematical thinking also in terms of *spatial sense* and some of them have considered the role the spatial sense plays in supporting the development of number sense. The *spatial sense* is commonly described by three components: spatial visualization, spatial orientation and shape (Clements & Sarama, 2007). Spatial visualization implies the ability to mentally imagine the movements of two-dimensional and three-dimensional spatial objects. The spatial orientation implies self-representation in space to move in it. The third component "shape" has to do with the mental manipulation of forms from a fixed perspective. This ability is the mental operation of giving an organization or form for a set of objects (for example, finger patterning and recognizing a quantity in the configuration of dots on dice). According to Arcavi (2003) the mental extraction of structures from spatial configurations (i.e., identifying a "gestalt") can support students 'counting processes: one can see how young children can also use "gestalts" to rearrange objects that are to be counted. The new spatial structure helps the child to read off the quantity and hence abbreviate the counting procedure. But the idea of manipulating - mentally or physically - the structure of the number (quantity) as a spatial configuration seems to be functional not only to support counting process, but also to support mental calculation. This reorganization of the parts, exploiting subitizing process, allows identifying small quantities without the counting process. As matter of fact, when young children are asked to determine the quantity of a randomly arranged set of objects, they initially tend to count each object. As the set grows, this procedure leads them to the difficulties of keeping track of which objects have already been counted, and of needing more time because of the larger set. Hence, the benefits of applying spatial structure to counting problems are evident when identifying a quantity, when comparing a number of objects and when extending a pattern. This is evident also in kindergarten children (van Nel & van Eerde, 2010). 'Finger gnosis' (literally "finger knowledge"), defined as the ability to differentiate one's own fingers without any visual clues when they are touched, strongly influences number sense. Noël (2005) shows that improving children's finger gnosis by training them on finger differentiation tasks increases their numerical performance. These considerations fit with the frame of embodied cognition developed by Gallese and Lakoff (2005), which has influenced also mathematics education research. According to this approach, mathematics teaching-learning processes are multimodal activities: doing, touching, moving and seeing are essential components of mathematical thinking processes – from the initial phases of the conceptual development to the most advanced learning processes. This means that, exploiting perceptual-motor components, the body becomes essential in constructing of mathematical meanings (Radford, Edwards & Arzarello, 2009). This kind of approach supports children in overcoming *naïve* calculation strategies such as counting-forward and counting-backward (Carpenter & Moser, 1984). For reasons of space, the experimental activities, implemented in kindergarten, will not described in this paper. Instead, I'll describe my current research, developed through a pilot studies and a survey, aimed to examine whether the education approach implemented in kindergarten (whose theoretical bases I described above) affects students' long-term performance in mental calculation. Thus, I try to answer the following research questions: Are the calculation strategies applied in primary school by children who took part in our path starting from Kindergarten different from those used by children who haven't took part in our path? If so, how do they differ? For example are they more effective, i.e. do they produce correct results in less time? If so, does the effectiveness last over time? Are these differences in strategies maintained across time?

Methodology and activities

A pilot study, carried out with children attending grade 1, and a survey, carried out with children attending the grade 3, took place respectively in school year 2017/18 and 2019/20. The pilot study was carried out with 97 children aged 6 (attending grade 1) divided into four groups with the following characteristics:

Group 1: 28 children who took part in the experimental activities proposed by our research group in kindergarten and in primary school (children aged 4 until 6);

Group 2: 27 children who took part in the aforementioned activities only in kindergarten (children aged 3, 4 and 5);

Group 3: 23 children who took part in the experimental activities only in grade 1 (children aged 6);

Group 4 (control group): 19 children who never took part in the experimental activities proposed by our research group. The teachers of these pupils did not know the group's activities and research. These children were belonging the same socio-economic-cultural environment (same district) of the Group 1, 2 and 3. The research team in collaboration with school leaders made the choice.

During the pilot study, children are asked to mentally calculate additions and subtractions (6 operations) available in AC-MT 6-11 test (Cornoldi, Lucangeli & Bellina, 2012), which is used in Italy to identify students aged 6 to 11 with difficulties in math. Teachers interviewed each child and they video recorded all interviews. The data have been categorized by three indexes: response time, correctness of the answers, strategies used to calculate. As far as this latest index is concerned, teachers asked each child to describe strategies s/he used to calculate. Indeed, the pilot study aims to both classify calculation strategies used by children (of all the groups) and also compare them with strategies used by control group. Moreover, the pilot study aims to relate the features of the calculation strategies used, with the spatial structuring of the numbers by highlighting if there are privileged reference spatial structuring of numbers for children and if there are privileged reference tools that allow elaborating such spatial structuring. Thus, to detect difference in calculations strategies between the control group and the other experimental groups, and also to detect changes in choice of calculation strategies across time, we assessed the students groups' mathematical performance at two moments: at the beginning of grade 1, in November (pre-test, T1), and after some months, in May (post-test, T2). In this period, G3 and G1 worked on spatial structuring of numbers in order to support calculation strategies following research group's activities. At the contrary, G2 and G4 followed a standard teaching. T1 and T2 are respectively the pre-test and the post test available in AC-MT6-11 test and addressed to the first class of primary school. In order to answer first research question, we compared scores of three mentioned indexes at T1 and T2 of the four groups. In the following, the list of operations concerning T1 and T2 test administered to the 97 children of the grade 1.

- Pre-test T1 (end of November 2017): 1 + 2, 3 + 4, 2 + 6, 3 – 1, 8 – 5, 7 – 3;
- Post-test T2 (end of may 2018): 4 + 5, 10 + 3, 8 + 1, 9 – 3, 12 – 4, 8 – 5.

Although the tests don't give a complete picture of students' knowledge in mental calculation strategies, I consider them enough to describe the basic level in such calculation strategies for children aged 6, because the operations are conceived to concern the fundamental arithmetic properties. This allows us to identify different kinds of strategies put in place by children of the different groups in performance required in T1 and T2.

The second part of this research involved a survey on some children of G1 group. The survey was carried out in the school year 2019/20 with 18 children belonging the group G1 (those were the only children of G1 group still attending in that school). They are aged 8 and they attend grade 3. The children ‘performance was assessed using some operations available in AC-MT 6-11test for children aged 8: 12 + 8, 14 + 7, 21 + 6, 13 – 9, 15 – 8, 19 – 6, 12 – 4. These operations are judged more complex because they concern bigger numbers. As mentioned above, the survey aimed to investigate if the effectiveness of strategies implemented by children in pilot study lasts over time.

Results and discussion

In order to identify more easily each group performances in the following data collection, I associated a color to each group: G1-red, G2-rose, G3-green and G4-blue. In order to answer the first research question, concerning effectiveness of calculation strategies (in terms of correctness and response time), we need a quantitative analysis of data (response time, number of correct and incorrect answers). Instead, in order to identify the kinds of strategies used by children (second research question), we need a qualitative analysis of data (verbal descriptions of the used strategies). Now, I discuss about data concerning both the response time (of each group for T1 and T2 tests and for additions and subtractions) and the correctness of the answers (of each group for both T1 and T2 test).

As far as response time in the *additions* (left part of the Figure 1) is concerned, we may note that the G4 (blue control group) needs more time both in T1 and T2 test than the others groups even if in T1 the difference in response times among groups is not very relevant.

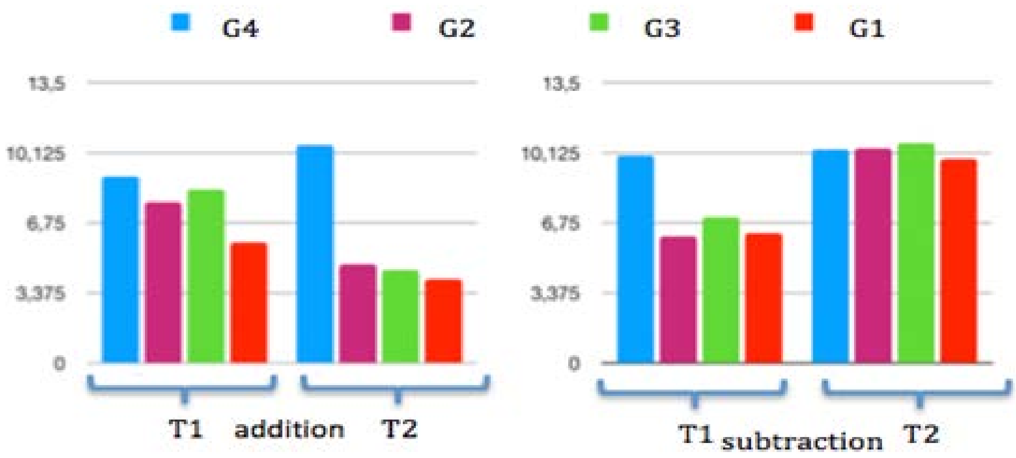

Figure 1: On vertical axis the average response time of each group in additions and subtractions tasks both in T1 and T2 test

As I'll show, the strategies of addition performed by G4 are very trivial (counting-on, counting-all, Carpenter& Moser, 1984) but they do not require a significantly longer time than strategies more complex such as: "get to 5" or "get to10", retrieval arithmetical facts, derived facts and application of arithmetic properties (commutative, associative) used by G1, G2 and G3. This is probably due because additions required in T1 are very simple, also for children of grade 1, and they can be performed in short time.

At the contrary, response time in T2 is very different: G4 needs more time rather than the other groups. Moreover, if G4 increases its response time in T2, the other groups decrease their response time with respect to T1. The teachers and researcher's hypothesis to explain this data is that G4 maintains the same strategy of counting-on used in T1, which is more and more demanding in terms of working memory since bigger numbers. Instead, the other groups decreases their response time since they improve addition strategies based on arithmetic properties and arithmetical facts, significantly.

As far as response time in *subtractions* (right part of the Figure 1) is concerned, we may note that for G4 remains almost unchanged in T1 and T2. This could suggest that children didn't modify the strategies to calculate in T1 and T2 tests. Instead, we can note that response time from T1 test to T2 test of the other groups increases. This could be due to new strategies more demanding in terms of cognitive load in grade 1 (but more effective in future). This will be shown in the description of the survey. Figure 2 shows the ranges of variability of data (time in seconds) in each group.

Ranges describing variability of data (time in seconds)			
G1/T1:0,8-20,8;	G1/T2: 0,5-15;	G1/T1: 1,3-27;	G1/T2: 1,8-23,8;
G3/T1: 1,7-11,8;	G3/T2: 1,4-13,2;	G3/T1: 1,9-13,3;	G3/T2: 2,9- 21;
G2/T1: 1,1- 18,5;	G2/T2: 1,5-19;	G2/T1: 1,5-16,6;	G2/T2: 2,5-30;
G4/T1: 1-37.	G4/T2: 1-28.	G4/T1: 2- 33.	G4/T2: 4-30.

Figure 2: Ranges describing variability of data (time in seconds)

As far as correctness in the additions in T1 and T2 tests (Figure 3) is concerned, we can note that in G4 the correctness of answers remains almost unchanged in T1 and T2 (from 82% to 84%) as in the groups G1 and G3 (respectively from 96% to 99% and from 92% to 97%). However, we can see a significant difference in the amount: in G4 is at around 80% and in G1 and G3 is at around 90%. In G2 correct answers increase significantly in T2 test (from 76% to 91%). This can be interpreted as the development of more effective additive strategies or as the strengthening of more complex strategies.

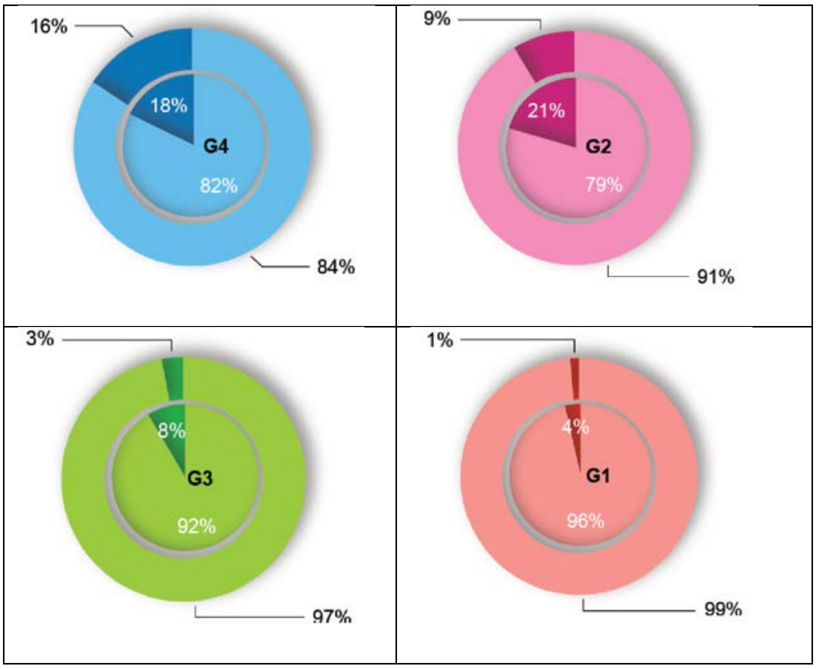

Figure 3: Results of Addition. Light color: correct answers; dark color: incorrect answers; annulus: T2; inner part of circle: T1

As far as correctness of answers in subtractions (Figure 4) is concerned, we can observe that G4 increase correct answers from T1 to T2 but still the half of children in T2 test produce incorrect answers. We can note a similar trend for the other groups even if with a minor extend. Thus, even if the strategies in subtractions used by the groups G1, G2 and G3 increase the response time (Figure 1), I consider them effective strategies because they get a large percentage of correct answers.

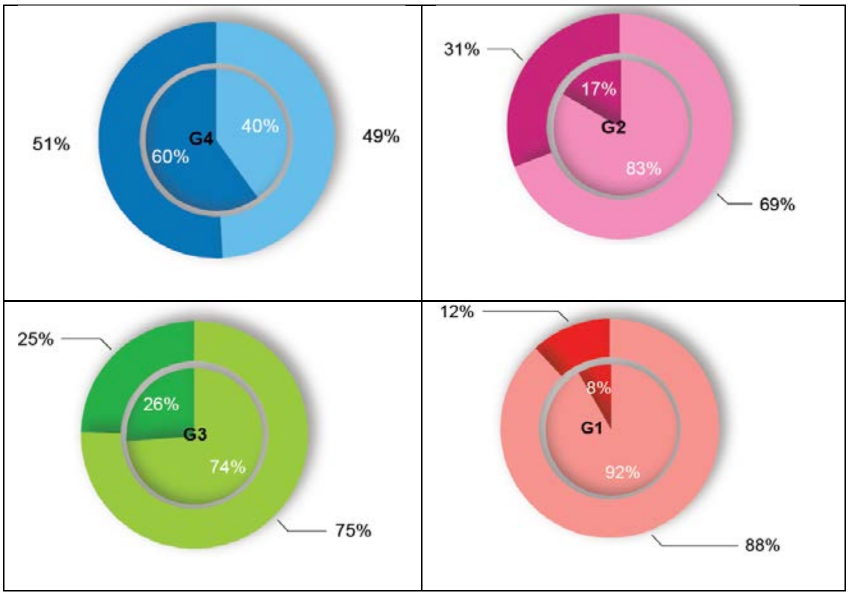

Figure 4: Results of Subtraction. Light color: correct answers; dark color: incorrect answers; annulus: T2; inner part of circle: T1

In the following, I present the categorization, developed by the research group, of the different strategies used by children for the addition and the subtraction asked in T2 test. As said above, teachers interviewed children and they video-recorded all the interviews. Through the analysis of both the children's verbal descriptions and gestures, we identified two main categories: children using hands (and fingers) to perform the required operations, and children acting on a "mental model" (for instance, the model of hands that they use mentally moving fingers, as they did in their previously educational experiences) or other models referring to a disposition of quantities on the space (for instance, dots on the dice face or ten-frame). In the first case (whit hands) we identified 6 different strategies; in the second case (without hands) we identified 7 different strategies. Moreover, in each category, different levels of complexities are present. I mean, naïve strategies such as counting-on for additions or take-away for subtraction, or strategies more complexes exploiting commutative and associative properties, and arithmetical facts as well. For reasons of space, in the following I list only some significant additive and subtractive strategies with and without the use of hands. *With the use of hands*: child raises fingers that correspond to one of the addends, then continue raising the fingers one by one (strategy for addition); child raise the fingers that correspond to the minuend, and then s/he puts the fingers that correspond to subtrahend down (strategy for subtraction). *Without the use of hands*: child decomposes an addend to get to 5 or to 10 with the other addend. Then, s/he recomposes numbers. Ex: $4 + 3 = 4 + 1 + 2 = 5 + 2$. This strategy exploits associative property of addition, sometime commutative property and arithmetical facts; child pronounce the minuend and then s/he counts down (subtraction); *complementary addition:* child answers through complementary operation (Ex. 8-5=? since $5 + 3 = 8$, then $8-5 = 3$) (subtraction); *removing 5 by imaging an hand* (Ex. 8-5=?, child takes off one hand to the representation of 8 as $5 + 3$, so s/he remains with 3 fingers). The analysis of data shows different children's behaviors of G4 and of the other groups. For example, 16% of children in G1, 10% of children in G2 and 13% in G3 but no children in G4 uses the complementary addition strategy. The strategy "takes away 5", that is take away one hand, is mostly used by the children of G3 group (40%), instead only 8% in G1 use it, and 10% in G2, but no children in G4. A very interesting strategy that is performed exploiting "spatial sense" is "assemble and disassemble the numbers”. As said above, it concerns in visualizing a number as a whole that can be decomposed into parts. This can be done moving into the mind dots on a card or managing fingers of the hands or cells in ten-frame. Children described these images during their interviews. This strategy is used by the 18% of children in G1 and by 10% in G2 but nobody in G3 or G4. So, the question is: which kinds of strategies prefer to use children of G4? Data suggest that they used mostly "counting forwards" in the additions and “counting backward” in subtractions (all children in G4 group choose this strategy in order to calculate subtractions). The interviews provided interesting findings: G4 uses more frequently naïve strategies than the others groups. Children in G1, G2, and G3 manage the "more complex strategies" through visuo-spatial image of objects into the space (dots, fingers of the hands, ten-frame...). They claim managing mental model that they used in previous time (in kindergarten or in primary school).

The survey allowed us to identify different calculation strategies after three school years. One of the most interesting strategies used in the addition consists in decomposing numbers into parts in order to act through arithmetical facts (additive arithmetical facts, getting to 10 or to 5, or multiplicative arithmetical fact, such as the multiplication table). For instance: the addition $12 + 8$ is calculated decomposing 12 into $10 + 2$, then adding 2 with 8 ($2 + 8 = 10$, it's an arithmetical fact), then $10 + 10 = 20$ is another arithmetical fact. One child claims that she used the multiplication by 7 in the addition $14 + 7$ (identifying the multiplication as repeated additions: $7 \rightarrow 14 \rightarrow 21$). As far as subtractions are concerned, three main strategies can be identified: decomposing numbers into parts in order to act through arithmetical facts. For instance, the subtraction $13 – 9$ is calculated decomposing 13 into $10+ 3$, then subtracting 3 to 9 to obtain 6 so that the subtraction is now $10 – 6$. At this point some children recall the arithmetical fact $10 – 6 = 4$, some others take-away 3 to 10 ($10 – 3 = 7$ arithmetical fact) and then, by recalling arithmetical fact $7 – 3 = 4$, they get to the result. Adding a number to the minuend or subtrahend in order to get to 10 or 5 and then, take-away the number added. For instance, in order to calculate $13 – 9$, adding 1 to 9 so to have $13 – 10 = 3$ and then adding 1to the result. Adding the same number to the minuend and the subtrahend in order to get to 10 or 5. For instance, the subtraction $13 – 9$ is calculated adding 1 to both 13 and to 9 in order to have $14 – 10 = 4$ by recalling the arithmetical fact ($10 + n – 10 = n$).

These strategies recall the properties of the operations described in the pilot study and refer to a flexible use of the number intended as composed into parts. Videos allowed us to identify some children's gestures recalling use of fingers or other well-known tools. In other videos, children refer explicitly to the use of fingers or tools (dots, ten-frame…). All children produce correct answers and the response time is, on average, 3 sec. Interesting aspect concerns a child with low achievement in math who claims to have performed $12 - 4$ using a model made up of a 10×2 table mentally (Ten-Frames artifact has been used in math teaching[1]): at 12 cells, visualized as $10 + 2$, he first removes 2 cells and, from the 10 remaining cells, he removes another 2 cells. His response time is 10 sec. Teachers highlighted the difficulties to manage the working memory in calculation tasks especially for low achieving students. The use of fingers and the mentally use of the table 10×2 as well, allowed them to perform mentally calculations. We can observe that the strategies performed by children who took part in the survey are very effective in the response time and in the correctness as

[1] To have an idea of how to use a ten frame: https://www.youtube.com/watch?v=N-v6HVSso70

well. Moreover, since the students' times got shorter in comparison with times in grade 1, this suggests us that calculation strategies are at the moment more stable and consolidated.

In conclusion, findings show us the need of a future long-term and systematic research study to investigate how "spatial sense" can be exploited in effective mental calculation. We underline how manipulating - mentally or physically - the structure of the number as a spatial configuration seems to be functional to develop effective mental calculation strategies. Children use "gestalts" to rearrange numbers as new spatial structure which helps to read off the quantity and hence abbreviate the calculation procedure managing in more efficient way working memory. Frequently, they rearrange numbers by reasoning with imagined tools such as ten-frame, fingers or other tools used in the educational experience (also past experience). Thus, findings suggest to consider approach to "spatial sense" of number as educational aim already in kindergarten to rethink the educational approach to numbers and calculation much more in term of visuo-spatial approach than exclusively in term of calculation procedure.

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