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BILINEAR EMBEDDING FOR COMPLEX PARTIAL DIFFERENTIAL EQUATIONS WITH FIRST-ORDER PERTURBATIONS

Doctoral thesis

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BILINEARNA VLOŽITEV ZA KOMPLEKSNE PARCIALNE DIFERENCIALNE ENAČBE S PERTURBACIJAMI PRVEGA REDA

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Abstract

In the present dissertation we further develop the L^p-theory for elliptic partial differential operators in divergence form with complex coefficients on arbitrary open subsets of $\mathbb{R}^d$. We extend the bilinear embedding theorem of Carbonaro and Dragičević—originally established for homogeneous second-order divergence-form operators, under the so-called p-ellipticity condition imposed on the coefficients—to operators that include either first-order perturbations or else negative potentials. To handle these more general situations, we introduce two new structural conditions on the coefficients that generalize p-ellipticity in an adequate manner.

Our approach is based on a heat-flow method driven by a specifically designed Bellman function, which serves as the main analytic tool for proving the bilinear embedding. New notions of generalized convexity, which play the role of the convexity with respect to matrices introduced by Carbonaro and Dragičević in the unperturbed case, are essential for this method.

As key consequences, we establish bounded H^∞-functional calculus and parabolic maximal regularity in L^p for these operators. Under the new conditions on the coefficients, we further investigate the mapping properties of the associated semigroups, obtaining results on their contractivity and analyticity in L^p spaces that extend and unify previous findings. These properties play a fundamental role in modern analysis of partial differential equations and the study of evolution equations in Banach spaces.

Finally, we provide a new proof of the bilinear embedding in the unperturbed case by means of a novel approximation technique. We believe that this argument might prove a powerful tool for future developments, particularly in the study of trilinear embeddings for unperturbed divergence-form operators with complex coefficients, as well as bilinear embeddings for more general classes of divergence-form operators.

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Povzetek

V doktorski disertaciji nadaljujemo z razvojem L^p-teorije za parcialne diferencialne operatorje v divergenčni obliki s kompleksnimi koeficienti na poljubnih odprtih podmnožicah $\mathbb{R}^d$. Razširimo bilinearni vložitveni izrek Carbonara in Dragičevića—ki je bil sprva dokazan za homogene operatorje drugega reda v divergenčni obliki, pod tako imenovanim pogojem p-elliptičnosti na koeficiente—na operatorje, ki vključujejo bodisi perturbacije prvega reda bodisi negativne potenciale. Za obravnavo teh splošnejših situacij uvedemo dva nova strukturna pogoja na koeficiente, ki na ustrezen način posplošita p-elliptičnost.

Naš pristop temelji na metodi toplotnega toka za posebej zasnovano Bellmanova funkcija; to je naše glavno analitično orodja za dokaz bilinearne vložitve. Novi koncepti generalizirane konveksnosti, ki igrajo vlogo konveksnosti glede na matrike, vpeljane s strani Carbonara in Dragičevića v neperturbiranem primeru, so ključni za uspešno uporabo te metode.

Med temeljnimi posledicami izpeljemo omejen H^∞-funkcijski račun ter parabolično maksimalno regularnost v L^p-prostorih za omenjene operatorje. Pod novimi pogoji na koeficiente nadalje preučujemo lastnosti pripadajočih polgrup, pri čemer pridobimo rezultate o njihovi kontraktivnosti in analitičnosti v L^p-prostorih, ki razširjajo in povezujejo prejšnje ugotovitve. Te lastnosti igrajo temeljno vlogo v sodobni analizi parcialnih diferencialnih enačb in pri preučevanju evolucijskih enačb v Banachovih prostorih.

Na koncu podamo nov dokaz bilinearne vložitve v neperturbiranem primeru s pomočjo nove, izvirne aproksimacijske tehnike. Menimo, da bi ta argument lahko predstavljal močno orodje za prihodnje raziskave, zlasti pri preučevanju trilinearnih vložitev za neperturbirane operatorje v divergenčni obliki s kompleksnimi koeficienti ter bilinearnih vložitev za širše razrede operatorjev v divergenčni obliki.

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CHAPTER 1

Introduction

The purpose of this thesis is to provide new developments in the L^p-theory for elliptic and parabolic differential operators with complex coefficients, acting on arbitrary open subsets of $\mathbb{R}^d$. In particular, bilinear embedding theorems are established both for divergence-form operators with first-order perturbations and Schrödinger-type operators with negative potentials. As a consequence, we show that these operators admit bounded H^∞-functional calculus and parabolic maximal regularity in L^p. The main tool employed throughout is the Bellman function technique.

1.1 Bilinear embedding

Poisson-type bilinear embedding

Bilinear embedding-type theorems have become essential tools in the study of singular integral operators. Broadly speaking, these theorems typically fall into two broad categories, depending on whether they involve the Poisson semigroup or the heat semigroup. A prominent application of the former is the derivation of dimension-free bounds for Riesz transforms, which are fundamental objects in harmonic analysis. To illustrate this, we consider the prototypical case: the Euclidean Laplacian $-\Delta$ on $\mathbb{R}^d$. Let R_j denote the j-th Riesz transform $\partial_{x_j}(-\Delta)^{-1/2}$, and $(P_t)_{t>0}$ the Poisson semigroup associated with $-\Delta$. Set $\mathcal{R} = (R_1, \dots, R_d)$. For all suitable test functions $f : \mathbb{R}^d \to \mathbb{C}$ and $g : \mathbb{R}^d \to \mathbb{C}^d$, the following identity holds:

$$\langle \mathcal{R} f, g \rangle = 4 \int_0^\infty \int_{\mathbb{R}^d} \sum_{k=1}^d \partial_{x_k} P_t f(x) \cdot \overline{\partial_t P_t g_k(x)}\, t \, \mathrm{d}x \, \mathrm{d}t.$$

When combined with an appropriate bilinear embedding estimate

$$\int_0^\infty \int_{\mathbb{R}^d} |\nabla_{x,t} P_t f(x)| \, |\nabla_{x,t} P_t g(x)| \, t \, \mathrm{d}x \, \mathrm{d}t \leqslant C(p) \|f\|_{L^p(\mathbb{R}^d)} \, \|g\|_{L^q(\mathbb{R}^d;\mathbb{C}^d)},$$

this identity yields, via a duality argument, the L^p-boundedness of the Riesz transform $\mathcal{R}$. Here, $P_t g = (P_t g_1, \dots, P_t g_d)$, $q = p/(p-1)$ is the conjugate exponent of p and $\nabla_{x,t}$ denotes the gradient with respect to $(x,t) \in \mathbb{R}^d \times \mathbb{R}$.

This approach was applied in [34], where dimension-free estimates were obtained for Riesz transforms associated with both the classical and the Ornstein–Uhlenbeck Laplacians. It was subsequently extended in [36] to the setting of general Schrödinger operators with nonnegative potentials. In particular, dimension-free estimates for the Hermite–Riesz transforms on L^p spaces were established as a special case. More recently, the Riesz transform was studied in [16] on complete Riemannian manifolds whose Bakry–Émery Ricci curvature is bounded from below.

A notable feature of the above bilinear estimate is that the constant $C(p)$ is independent of the dimension and depends linearly on p. Consequently, the corresponding estimates for the Riesz transforms share these properties. In particular, the result in [16] improves upon that in [8], where Bakry proved the first dimension-free estimates, and in [60] where Li showed that in the general case the behavior of the norms with respect to p is at most quadratic as $p \to \infty$.

Heat-type bilinear embedding

Bilinear embedding techniques involving the *heat semigroup* have also proved powerful in harmonic analysis. A first notable application concerned the Ahlfors–Beurling operator $R_1^2 - R_2^2 - 2iR_1R_2$ on $\mathbb{R}^2$, for which Nazarov and Volberg obtained L^p-estimates [85]. This line of research was subsequently refined by Petermichl and Volberg, who established sharp weighted bounds [76], and by Dragičević and Volberg, who derived asymptotic estimates [33]. In the latter, in analogy with—and indeed providing the blueprint for—the Poisson case, the main idea was to combine the identity

$$\langle R_i R_j f, g\rangle = -2\int_0^\infty \int_{\mathbb{R}^2} \partial_{x_i} T_t f(x)\, \partial_{x_j} T_t g(x)\, \mathrm{d}x\, \mathrm{d}t,$$

with the bilinear embedding

$$\int_0^\infty \int_{\mathbb{R}^2} |\nabla_x T_t f(x)|\, |\nabla_x T_t g(x)|\, \mathrm{d}x\, \mathrm{d}t \leqslant C(p)\|f\|_{L^p(\mathbb{R}^2)}\, \|g\|_{L^q(\mathbb{R}^2)},$$

where $(T_t)_{t>0}$ denotes the heat semigroup associated with the Euclidean Laplacian on $\mathbb{R}^2$ and q the conjugate exponent of p.

Moreover, heat-type bilinear embeddings have played a key role in establishing sharp spectral multiplier results [17, 18]. The key new idea in [17] was to study bilinear embeddings with *complex times* $te^{i\phi}$, with $\phi \in (-\pi/2, \pi/2)$, rather than at real times, as in all previous work on the subject. This led, on the one hand, to the proof that for every $1 < p < \infty$, the generator of any symmetric contraction semigroup admits a bounded holomorphic functional calculus on L^p of optimal angle $\phi_p^* = \arcsin|1 - 2/p|$, thereby resolving a question left open since the 1970s by Stein [83]. On the other hand, the complex-time heat-flow method introduced in [17] opened the way to further developments of the theory in the direction of divergence-form elliptic operators with complex coefficients [20], as will be discussed in more detail in the sections below.

Heat-type bilinear embedding for divergence-form operators

Dragičević and Volberg established a heat-type bilinear embedding for Schrödinger-type operators on $\mathbb{R}^d$ associated with *real* elliptic matrices and nonnegative potentials [35]. For clarity, we restrict the discussion at this stage to second-order operators without potentials. Specifically, let A be a real, bounded, elliptic matrix-valued function on $\mathbb{R}^d$, and consider the operator

$$\mathscr{L}u = -\mathrm{div}(A\nabla u)$$

along with the associated heat semigroup $T_t := e^{-t\mathscr{L}}$. The main result in [35] shows that, for all $f, g \in C_c^\infty(\mathbb{R}^d)$ and $p \in (1, \infty)$, the following dimension-free bilinear estimate holds:

$$\int_0^\infty \int_{\mathbb{R}^d} |\nabla_x T_t f(x)||\nabla_x T_t g(x)|\, \mathrm{d}x\, \mathrm{d}t \leqslant C(p)\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^q(\mathbb{R}^d)}, \tag{1.1}$$

where $q = p/(p-1)$ is the conjugate exponent of p. Forgetting for the moment about the dimension-independence of (1.1) of the dimension, at first glance one could deduced it from the

boundedness on L^p, for all $p \in (1,\infty)$, of the *vertical* square function operator

$$G^{\mathscr{L}} u(x) = \left(\int_0^\infty |\nabla_x T_t u(x)|^2 \, \mathrm{d}t \right)^{1/2}.$$

However, such boundedness is only known to hold (with dimension-dependent constants) for p in the interval $(1, q_+(\mathscr{L}))$, and in general fails for $p > q_+(\mathscr{L})$, where $q_+(\mathscr{L})$ denotes the supremum of the range of exponents p for which the family $(\sqrt{t}\nabla T_t)_{t>0}$ is uniformly bounded on L^p [6]. On the other hand, estimate (1.1) can still be obtained for all $p \in (1,\infty)$ by dominating the bilinear integral using suitable *conical* square functions in place of vertical ones, namely

$$\mathfrak{g}^{\mathscr{L}} u(x) = \left(\iint_{|x-y|<\sqrt{t}} |\nabla_y T_t u(y)|^2 \frac{\mathrm{d}y \, \mathrm{d}t}{t^{d/2}} \right)^{1/2},$$

which are known to be bounded on L^p for the full range $p \in (1,\infty)$ [7]. This approach, developed by Auscher, Hofmann and Martell, presents both advantages and limitations. One significant strength is its applicability to operators with complex coefficients. In fact, they proved that for complex matrices, the operator $\mathfrak{g}^{\mathscr{L}}$ is bounded on L^p for all $p \in (p_-(\mathscr{L}),\infty)$, with the lower bound being sharp. Here $(p_-(\mathscr{L}), p_+(\mathscr{L}))$ is the maximal open interval of indices p for which $(T_t)_{t>0}$ is uniformly bounded on L^p. As a consequence, the bilinear inequality (1.1) holds for $p \in (p_-(\mathscr{L}), p_+(\mathscr{L}^*))$. A drawback of this method, however, is that the resulting bilinear estimate typically depends on the dimension, unlike the approach of Dragičević and Volberg, which yields truly dimension-free bounds.

The goal of [20] was to extend both (1.1) and the aforementioned corollary of Auscher, Hofmann and Martell, aiming to establish a *dimension-free* bilinear embedding for *complex* elliptic matrices. In addition, the authors sought to explore a more general form of embedding: whereas in earlier results the same semigroup—associated with either Laplace-type or divergence-form operators—was applied to both f and g, in [20] they initiated the study of embeddings involving *different* semigroups, corresponding to completely independent matrices. Unlike in (1.1), the dimension-free bilinear embedding does not hold across the full range $(1,\infty)$ for arbitrary complex elliptic matrices. Instead, the admissible range of exponents is determined by a structural condition on the matrices introduced in [20], known as *p-ellipticity*. This is an algebraic condition that generalizes the classical notion of ellipticity to exponents $p \neq 2$, and interpolates real and complex elliptic matrices. Independently and simultaneously, the same condition was introduced by Dindoš and Pipher in the context of boundary value problems [30], where it provided regularity estimates for local solutions of $\mathscr{L}u = 0$ with complex coefficients, thus serving as a substitute for the De Giorgi–Nash–Moser theory available only in the real case.

So far, all the operators considered acted on the whole space $\mathbb{R}^d$. The extension of the bilinear embedding to arbitrary open domains $\Omega \subseteq \mathbb{R}^d$ was carried out in [19]. In this framework, Carbonaro and Dragičević considered divergence-form operators with either Dirichlet, Neumann, or mixed boundary conditions. Working on arbitrary open sets is substantially more challenging than working in the whole space $\mathbb{R}^d$, as many favorable properties of the operators and the generated semigroups no longer hold. The main difficulty addressed in [19] was justifying an integration by parts within the so-called *heat-flow method*, which was resolved through the development of a novel approximation technique.

Subsequently, this line of research was further developed in [21], where Carbonaro and Dragičević considered Schrödinger-type operators with nonnegative potentials, of the form

$$\mathscr{L} = -\mathrm{div}(A\nabla u) + Vu,$$

still defined on arbitrary open subsets Ω of $\mathbb{R}^d$, and proved the following bilinear estimate for all $f, g \in (L^p \cap L^q)(\Omega)$,

$$\int_0^\infty \int_\Omega \sqrt{\left|\nabla_x T_t^{A,V} f(x)\right|^2 + V(x)\left|T_t^{A,V} f(x)\right|^2}\sqrt{\left|\nabla_x T_t^{B,W} g(x)\right|^2 + W(x)\left|T_t^{B,W} g(x)\right|^2}\,\mathrm{d}x\,\mathrm{d}t \leqslant C(p)\|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}, \tag{1.2}$$

where $q = p/(p-1)$ is the conjugate exponent of p. Here, $(T_t^{A,V})_{t>0}$ and $(T_t^{B,W})_{t>0}$ denote the semigroups generated by the operators associated with the coefficients (A, V) and (B, W), respectively. To the best of our knowledge, inequality (1.2) is the most recent bilinear embedding estimate involving divergence-form operators with *static* boundary conditions. Recently, by adapting the heat-flow method from [19], Böhnlein, Egert and Rehberg proved in [12] a bilinear embedding for elliptic operators in divergence form without potentials, subject to *dynamical* boundary conditions, provided that the coefficient matrices are p-elliptic.

Let us now turn to a more general class of operators of the form

$$\mathscr{L}u = -\mathrm{div}(A\nabla u) + \left\langle \nabla u, \bar{b}\right\rangle - \mathrm{div}(cu) + Vu,$$

where b, c are vector-valued functions with bounded and complex coefficients. When $b = c = V = 0$, we will often refer to $\mathscr{L}$ as the *unperturbed* operator. A natural question arises as to whether bilinear embedding estimates analogous to (1.2) remain valid in this broader context, and under which assumptions. In this thesis, we treat separately two types of perturbations involving first- and zero-order terms. Specifically, we assume that either:

- b, c might be *not identically zero* and V is a nonnegative locally integrable function on Ω; or (⋆)
- $b = c = 0$ and V is a real locally integrable function on Ω, possibly with nontrivial *negative* part.

One of the main objective of this thesis is to extend the bilinear embedding (1.2) to operators of the above form, by identifying new structural conditions on the coefficients that generalize the notion of p-ellipticity. Roughly speaking, the terms b, c and the negative part of V will be assumed to be "small" enough with respect to A and the positive part V_+, so that their contribution remains controlled and does not affect the validity of the estimate. As in the case of p-ellipticity, the genesis of these conditions is rooted in the *Bellman–heat method*.

The Bellman-heat method

The central analytical tool underlying the dimension-free bilinear embedding is the *Bellman–heat method*, also referred to as the *heat–flow method*. Among the earliest works in which this approach was developed are those by Nazarov–Treil [67], Petermichl–Volberg [76] and Nazarov–Volberg [85]. The specific version of the method employed in this thesis originates from Nazarov and Treil [67] and was subsequently refined in a series of later papers, including [16–18, 34–36].

The core idea of the Bellman-heat method is to study the monotonicity of a suitable flow associated with an explicit *Bellman function* $\mathcal{Q}$, introduced by Nazarov and Treil. Within this framework, the desired inequality is obtained by proving that a certain energy functional involving $\mathcal{Q}$ evolves in a nonincreasing way over the time. This monotonicity property ultimately relies on specific convexity features of $\mathcal{Q}$. In particular, in [36], a detailed analysis of the Bellman function revealed delicate convexity properties that turned out to be crucial for establishing the bilinear embedding theorem. These findings also hinted at the existence of a deeper structural convexity principle underlying the method—a hypothesis that was later confirmed and

systematically developed in a series of subsequent works [16–18], culminating in [20]. There, the focus shifted from the Bellman function $\mathcal{Q}$ itself to its *elementary building blocks*, namely the power functions $|\zeta|^p$, whose convexity was studied with respect to a matrix of coefficients A. This investigation led to the definition of *generalized convexity with respect to matrices*, which coincides with the classical convexity when A is the identity. Within this framework, the notion of p-ellipticity naturally emerged as the condition ensuring uniform convexity of the power functions relative to A.

In this thesis, we further develop this perspective by extending the concept of generalized convexity to encompass the presence of first- and zero-order perturbations. In particular, the new conditions that we will introduce may be interpreted as generalized forms of p-ellipticity—capturing uniform convexity of power functions not only with respect to matrices A, but also in relation to additional vector and potential terms. These conditions play a central role in establishing bilinear embedding estimates for the broader class of perturbed operators introduced above.

1.2 Semigroup contractivity

As previously noted, in the case of real coefficients, the dimension-free bilinear embedding for (unperturbed) divergence-form operators is valid for all $p \in (1, \infty)$, which exactly matches the full range of L^p-contractivity of the generated semigroup. Similarly, when complex coefficients are allowed, the range of validity of the bilinear embedding obtained by Auscher, Hofmann, and Martell is closely tied to the uniform boundedness on L^p of the associated semigroup. This indicates a deep interplay between bilinear embeddings and the boundedness properties of the semigroup.

As already discussed, Carbonaro and Dragičević established a dimension-free bilinear embedding valid in the sharp range

$$\{p \in (1, \infty) : \ A \text{ is } p\text{-elliptic}\}.$$

This naturally raises the following question:

Is p-ellipticity linked with the contractivity of the semigroup $(T_t^A)_{t>0}$ on L^p?

A precise answer was given in [20], where the authors showed that the semigroup is contractive on $L^p(\Omega)$ under a condition weaker than p-ellipticity. Remarkably, this result holds not only on $\mathbb{R}^d$, but also on *arbitrary open subsets* $\Omega \subseteq \mathbb{R}^d$, and it remains valid under several boundary conditions, including Dirichlet, Neumann, and mixed types [19, 41]. This provides an explicit and verifiable condition that connects the algebraic structure of the coefficient matrix A with the analytic behavior of the associated semigroup on L^p.

In the special case of Dirichlet boundary conditions on arbitrary open subsets $\Omega \subseteq \mathbb{R}^d$, Carbonaro and Dragičević further characterized the L^p-contractivity of $(T_t^A)_{t>0}$ when the distributional divergence of every column of the antisymmetric part of the imaginary part of A is zero [20]. This result generalizes a previous theorem by Cialdea and Maz'ya, who had established it under additional regularity and symmetry assumptions on A and geometric assumptions on the domain Ω [24].

A central objective of this thesis is to explore whether the new structural conditions—introduced to generalize the notion of p-ellipticity and imposed on the coefficients falling under the scenarios described in (⋆)—are also sufficient to guarantee L^p-contractivity of the semigroup. We answer this question affirmatively: in the same spirit as the unperturbed setting, we demonstrate that conditions weaker than those required for dimension-free bilinear embeddings still ensure contractivity of the semigroup.

In particular, for divergence-form operators perturbed by first-order terms, the sufficient condition for L^p-contractivity that we establish is slightly stronger than the one provided by Cialdea and Maz'ya for L^p-dissipativity [24]. This difference stems from the weaker assumptions we impose on the coefficients, as well as from the fact that we do not restrict our attention solely to Dirichlet boundary conditions.

On the other hand, for Schrödinger-type operators with negative potentials, the condition we identify is consistent with the classical results of Belyi and Semenov [10] and Perelmuter [74]. In these works, a nontrivial interval of L^p-contractivity was established for the Schrödinger operator $-\Delta + V$, where V is form-bounded and $\Omega = \mathbb{R}^d$. In our more general setting, involving operators of the form $-\mathrm{div}(A\nabla\cdot) + V$, the condition we derive reproduces the same contractivity interval when A is the identity matrix, thereby confirming consistency with the classical results of Belyi–Semenov and Perelmuter. Moreover, our framework applies to arbitrary open sets $\Omega \subseteq \mathbb{R}^d$ and accommodates a broad class of boundary conditions.

1.3 Application: H^∞-functional calculus and L^p-maximal regularity

The bilinear embedding for divergence-form operators has proven to be a powerful tool for establishing various L^p-properties of such operators, including the boundedness of the H^∞-functional calculus and L^p-maximal parabolic regularity. In the unperturbed setting, Carbonaro and Dragičević showed that divergence-form operators with a p-elliptic coefficient matrix admit a bounded H^∞-functional calculus and, as a byproduct, also enjoy maximal regularity on L^p. The key idea relies on a general result by Cowling, Doust, McIntosh, and Yagi [26], which states that if a sectorial operator $\mathscr{L}$ on a Banach space $\mathcal{X}$ satisfies the bilinear estimate

$$\int_0^\infty \left| \langle \mathscr{L} e^{-te^{\pm i\vartheta}\mathscr{L}} f, g \rangle \right| \, \mathrm{d}t \lesssim \|f\|_{\mathcal{X}} \|g\|_{\mathcal{X}^*}, \tag{1.3}$$

for every $f \in \mathcal{X}$, $g \in \mathcal{X}^*$, and for some $\vartheta > 0$, then $\mathscr{L}$ admits a bounded H^∞-functional calculus on $\mathcal{X}$ of angle $\pi/2 - \vartheta$. The bounded H^∞-functional calculus, in turn, implies L^p-maximal regularity by means of the Dore–Venni theorem [32, 79].

In the specific case where $\mathscr{L}$ is a divergence-form operator (without lower-order perturbations), the strategy in [19] was to dominate the left-hand side of (12) by a bilinear integral—of the same type as the bilinear embedding—associated with the pair $(e^{\pm i\vartheta}A, e^{\mp i\vartheta}A^*)$, and then to apply the bilinear embedding theorem. This was possible due to the fact that p-ellipticity is preserved under small complex rotations and adjoints: if A is p-elliptic, then so are A^* and $e^{\pm i\vartheta}A$ for small ϑ. These invariance properties allow one to reduce the problem to the application of the bilinear embedding under the same p-ellipticity assumption.

Since the results of Cowling, Doust, McIntosh, and Yagi and Dore–Venni are purely functional-analytic and depend only on the sectorial nature of the operator—without depending on its specific form—this scheme naturally extends to the two classes of operators considered in this thesis: divergence-form operators with first-order perturbations and Schrödinger-type operators with negative potentials. Therefore, to apply the strategy used in [19], it suffices to verify that the new structural conditions preserve the key invariance properties of p-ellipticity needed for the bilinear embedding to hold for complex rotations of the original coefficients and their "adjoint" counterparts. And indeed, this is precisely the case.

This broader perspective reveals a consistent picture: the range of validity of bilinear embeddings, the development of functional calculus, and the boundedness of semigroups on L^p are all governed by structural properties of the operator, which can be effectively captured by suitable generalizations of p-ellipticity.

1.4 Trilinear embedding

The trilinear embedding theorem, introduced in [22], represents a significant advance in the analysis of elliptic operators with complex coefficients. It concerns triples of divergence-form operators together with triples of Hölder exponents and extends the bilinear embedding framework developed in [19, 20] to a multilinear setting, thereby pushing the Bellman-heat technique further and extending the scope of p-ellipticity.

A key feature of the trilinear context is that symmetry is lost compared to the bilinear case: knowing one exponent p does not determine the other two exponents, unlike in the bilinear situation where they are mutually conjugate. On the other hand, p-ellipticity is naturally suited for conjugation, being invariant under the conjugation of p. This asymmetry makes the trilinear case substantially different from, and technically more challenging than the bilinear one.

Beyond its intrinsic theoretical appeal, the trilinear embedding has several important applications in harmonic analysis and partial differential equations, including results on paraproducts, square functions associated with the corresponding operator semigroups, and Kato–Ponce type inequalities for elliptic operators with complex coefficients [22]. Moreover, it was shown in [22] that the trilinear embedding implies the bilinear one, although so far only under specific assumptions on the coefficients and on the underlying domain. This limitation arises because, for arbitrary open sets $\Omega \subset \mathbb{R}^d$, the heat-flow method in [22] relies on an adaption of the approximation technique developed in [19]. In the trilinear context, this approach requires stronger assumptions on the coefficients. By contrast, when $\Omega = \mathbb{R}^d$, no such approximation is needed and the heat-flow method can be applied as in [20], thus creating a discrepancy between the two settings.

A natural and appealing goal, therefore, is to bridge this gap by developing a unified framework in which the trilinear embedding can be established under the same general hypotheses on the coefficients both on $\mathbb{R}^d$ and on arbitrary open subsets. Achieving this would not only yield a conceptually simpler and more coherent theory, but would also establish the implication from the trilinear to the bilinear embedding in full generality.

Interestingly—depending on one's perspective, for better or worse—this discrepancy does not affect the aforementioned applications, since in those contexts the trilinear embedding is applied to triplets of matrices for which the two assumptions on the coefficients coincide. It would nevertheless be of considerable interest to identify new applications where an improved version of the trilinear embedding theorem plays a genuinely decisive role.

The final chapter of this thesis is devoted to this objective, presenting a new proof of the bilinear embedding designed to pave the way for such a unified trilinear result.

The thesis is structured into several chapters, which we outline as follows.

- In Chapter 2, we provide the necessary theoretical background required for the rest of the thesis. We begin by fixing the notation and conventions that will be used throughout. We then introduce sectorial operators and analytic semigroups, which form the foundation of the operator-theoretic framework employed in this work. Next, we present the notions of the H^∞-functional calculus and L^p-maximal regularity, emphasizing their connection. Finally, we discuss sesquilinear forms and their associated operators, which provide a convenient setting for the classes of operators considered in this thesis.

- Chapter 3 is devoted to presenting the proofs of the main results in the unperturbed setting, namely the dimension-free bilinear embedding for divergence-form operators with complex coefficients on arbitrary domains, as well as the L^p-contractivity of the generated semigroups. The goal is to keep the exposition of the thesis self-contained and to acquaint the reader with the core techniques employed, as these will later be nontrivially adapted in the treatment of more general operators.

- Chapter 4 turns to divergence-form operators with first-order perturbations. We introduce two new algebraic conditions on the coefficients, one strictly stronger than the other. Under the weaker condition, we establish L^p-contractivity of the associated semigroups. Under the stronger condition—which generalizes p-ellipticity—we extend the dimension-free bilinear embedding theorem of Carbonaro and Dragičević, and consequently show that the perturbed operators admit a bounded H^∞-functional calculus on L^p and satisfy L^p-maximal regularity.

- Chapter 5 applies the same analytical framework developed in Chapter 4 to Schrödinger-type operators with negative potentials, $-\mathrm{div}(A\nabla\cdot) + V$. We establish L^p-contractivity of the associated semigroup, prove a dimension-free bilinear embedding theorem, and derive both the boundedness of the H^∞-fucntional calculus and L^p-maximal regularity. The class of potentials considered consists of α-form-bounded functions, and the algebraic condition introduced for the bilinear embedding involved both the matrix A and the parameter α, generalizing the notion of p-ellipticiy as well.

- Finally, in Chapter 6, we present a new proof of the dimension-free bilinear embedding for unperturbed divergence-form operators. The key novelty lies in an alternative approximation technique, designed with a view toward future extensions. In particular, this approach aims to improve the existing trilinear embedding theorem by weakening the assumptions on the coefficient matrices in the context of arbitrary domains.

The results presented in this thesis are based on the following published and submitted works:

1. Andrea Poggio. Bilinear embedding for perturbed divergence-form operators with complex coefficients on irregular domains. *The Journal of Geometric Analysis* **35**, 58 (2025), 61 pp [78].
   This work corresponds to Chapter 4 of the present thesis.

2. Andrea Poggio. Bilinear embedding for divergence-form operators with negative potentials (submitted) [77].
   The main results of this paper are presented in Chapter 5.

In addition, Chapter 6 contains original and unpublished material, forming part of an ongoing project devoted to developing new techniques for the bilinear embedding with the long-term goal of improving the trilinear embedding theorem.

CHAPTER 2

Preliminaries

2.1 Notation

Throughout the thesis we adopt the following notational conventions.

Given two quantities X and Y, we adopt the convention whereby $X \lesssim Y$ means that there exists an absolute constant $C > 0$ such that $X \leqslant CY$. If both $X \lesssim Y$ and $Y \lesssim X$, then we write $X \sim Y$. If $\{\alpha_1, \ldots, \alpha_n\}$ is a set of parameters, then $C(\alpha_1, \ldots, \alpha_n)$ denotes a constant depending only on $\alpha_1, \ldots, \alpha_n$. When $X \leqslant C(\alpha_1, \ldots, \alpha_n)Y$, we will often write $X \lesssim_{\alpha_1,\ldots,\alpha_n} Y$.

If $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ and w is likewise, we write

$$\langle z, w \rangle_{\mathbb{C}^d} = \sum_{j=1}^{d} z_j \overline{w}_j$$

and $|z|^2 = \langle z, z \rangle_{\mathbb{C}^d}$. When the dimension is obvious, we sometimes omit the index $\mathbb{C}^d$ and only write $\langle z, w \rangle$. When both z and w belong to $\mathbb{R}^d$, we sometimes emphasize this by writing $\langle z, w \rangle_{\mathbb{R}^d}$.

Let $\mathbb{C}^{d\times d}$ be the space of all complex $d \times d$ matrices. For $M \in \mathbb{C}^{d\times d}$ denote its conjugate transpose by M^* and define its *symmetric part* M_{s} and *antisymmetric part* M_{a} by

$$M_{\mathrm{s}} := \frac{M + M^T}{2}, \qquad M_{\mathrm{a}} := \frac{M - M^T}{2}.$$

If f, g are complex functions on some sets U, V respectively, then $f \otimes g$ is the abbreviation for the function on $U \times V$ mapping $(x, y) \mapsto f(x)g(y)$. Similarly, for $m \in \mathbb{N}_+$ and $u, v \in \mathbb{C}^m$, we denote by $u \otimes v \in \mathbb{C}^{m\times m}$ the rank-one matrix whose (i, j)-entry is $u_i v_j$.

We denote by $\mathbf{z}$ and $\mathbf{1}$ the complex functions on a subset $U \subseteq \mathbb{C}$ defined by $\mathbf{z}(x) = x$ and $\mathbf{1}(x) = 1$, respectively.

Unless specified otherwise, for every $r \in [1, \infty]$ we denote by r' its conjugate exponent, i.e., $1/r + 1/r' = 1$. When working with a fixed exponent $p \in (1, \infty)$, we set q to be its conjugate exponent, so as to simplify the notation in the definition and subsequent use of the associated Bellman function introduced in (11).

Given a Banach space $\mathcal{X}$, we denote its norm by $\|\cdot\|_{\mathcal{X}}$, or simply $\|\cdot\|$ when no ambiguity arises. Its topological dual is denoted by $\mathcal{X}^*$. The *dual pairing* $\langle v, u \rangle$ represents the action of an element $v \in \mathcal{X}^*$ on an element $u \in \mathcal{X}$, that is, $\langle v, u \rangle = v(u)$. If $\mathcal{H}$ is a Hilbert space, we denote its inner product by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, or simply $\langle \cdot, \cdot \rangle$.

Function spaces

Let (Ω, μ) be a measure space. For $1 \leq p \leq \infty$, the *Lebesgue space* $L^p(\Omega)$ consists of (equivalence classes of) measurable functions $f : \Omega \to \mathbb{C}$ such that $\|f\|_p < \infty$, where

$$\|f\|_p := \begin{cases} (\int_\Omega |f(x)|^p \, \mathrm{d}\mu(x))^{1/p}, & \text{if } p < \infty, \\ \operatorname{ess\,sup}_{x\in\Omega} |f(x)|, & \text{if } p = \infty. \end{cases}$$

When μ is the Lebesgue measure on a subset $\Omega \subseteq \mathbb{R}^d$, we will write $\mathrm{d}x$ instead of $\mathrm{d}\mu(x)$ or simply omit the measure when no ambiguity arises.

For a Banach space $\mathcal{X}$ and $1 \leq p \leq \infty$, the *Bochner-Lebesgue space* $L^p(\Omega; \mathcal{X})$ consists of (equivalence classes of) strongly measurable functions $f : \Omega \to X$ such that $\|f\|_{L^p(\Omega;\mathcal{X})} < \infty$, where

$$\|f\|_{L^p(\Omega;\mathcal{X})} := \begin{cases} (\int_\Omega \|f(x)\|_{\mathcal{X}}^p \, \mathrm{d}\mu(x))^{1/p}, & \text{if } p < \infty, \\ \operatorname{ess\,sup}_{x\in\Omega} \|f(x)\|_{\mathcal{X}}, & \text{if } p = \infty. \end{cases}$$

Let $\Omega \subseteq \mathbb{R}^d$ be an open set. For $m \in \mathbb{N}$ we denote by

- $C^m(\Omega)$ the space of m-times continuously differentiable functions on Ω,
- $C^m(\overline{\Omega})$ the space of functions whose derivatives up to order m extend continuously to $\overline{\Omega}$.
- $C^\infty(\Omega)$ the space of infinitely differentiable functions on Ω,
- $C_c^m(\Omega)$ the space of functions in $C^m(\Omega)$ with compact support in Ω,
- $C_c^\infty(\Omega)$ the space of smooth functions with compact support in Ω.

If $m = 0$, the superscript will be omitted.

For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, the *Sobolev space* $W^{m,p}(\Omega)$ consists of functions $f \in L^p(\Omega)$ whose weak derivatives up to order m belong to $L^p(\Omega)$. The norm is

$$\|f\|_{W^{m,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|_p^p\right)^{1/p}, & \text{if } p < \infty, \\ \max_{|\alpha|\leq m} \|D^\alpha f\|_\infty, & \text{if } p = \infty. \end{cases}$$

When no ambiguity arises, we simply write $\|\cdot\|_{W^{m,p}}$ or $\|\cdot\|_{m,p}$. The subspace $W_0^{m,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Unless otherwise specified, we omit the domain from the notation for all these function spaces; for example, we write L^p or $W^{m,p}$ instead of $L^p(\Omega)$ or $W^{m,p}(\Omega)$ whenever the underlying set is clear from the context.

Operator theory

If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, we denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the Banach space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, and write $\mathcal{L}(\mathcal{X}) := \mathcal{L}(\mathcal{X}, \mathcal{X})$. When T acts between Lebesgue spaces, we sometimes write $\|T\|_{p-r}$ for the operator norm from L^p to L^r.

The identity operator on a Banach space $\mathcal{X}$ is denoted by $I_{\mathcal{X}}$, or simply I when no confusion arises.

For a (possibly unbounded) operator $\mathscr{L}$ on a Banach space $\mathcal{X}$, we denote by $\varrho(\mathscr{L})$ its *resolvent set*, by $\sigma(\mathscr{L})$ its *spectrum*, by $\mathrm{N}(\mathscr{L})$ its *null space* (or kernel), and by $\mathrm{R}(\mathscr{L})$ its *range*. The resolvent operator at $\zeta \in \varrho(\mathscr{L})$ is written as $(\zeta - \mathscr{L})^{-1}$.

2.2 Sectorial operators and semigroups

The purpose of this section is to introduce the abstract framework in which the main results of the thesis are developed. In the thesis we consider divergence-form operators whose coefficients guarantee the *sectoriality* of these operators.

Since the core results of the thesis are bilinear embedding theorems—bilinear estimates involving the *semigroups* generated by (the negative of) such operators—we recall here the main aspects of both sectorial operators and semigroups. After presenting their definitions and fundamental properties, we conclude by highlighting the connection between the two concepts.

Only the material required in later chapters is included, and most proofs are omitted. For comprehensive treatments, we refer to [43, 47] concerning sectorial operators and to [43, 73] for semigroup theory. See also [39] for a concise overview of sectorial operators on Hilbert spaces.

2.2.1 Sectorial operators

In this subsection, we introduce sectorial operators on Banach spaces. These are closed operators whose spectrum is contained in a sector of the complex plane and for which the norm of the resolvent can be controlled in a precise way. All divergence-form operators considered in this thesis will eventually be shown to belong to this class.

For $\omega \in (0, \pi]$ we denote by $\mathbf{S}_\omega$ the open sector symmetric about the positive real axis with vertex 0 and opening angle 2ω. Also set $\mathbf{S}_0 = (0, \infty)$. Explicitly,

$$\mathbf{S}_\omega := \begin{cases} \{z \in \mathbb{C} : |\arg z| < \omega\}, & \text{if } \omega \in (0, \pi], \\ (0, \infty), & \text{if } \omega = 0. \end{cases}$$

Definition 2.1. Let $\omega \in [0, \pi)$. An operator $\mathscr{L}$ on a Banach space $\mathcal{X}$ is called *sectorial* of angle ω if

(i) $\sigma(\mathscr{L}) \subset \overline{\mathbf{S}}_\omega$,

(ii) for each $\varphi \in (\omega, \pi)$ there exists C_φ such that

$$\|(\zeta - \mathscr{L})^{-1}\| \leqslant C_\varphi |\zeta|^{-1}, \quad \forall \zeta \in \mathbb{C} \setminus \mathbf{S}_\varphi. \tag{2.1}$$

Given a sectorial operator $\mathscr{L}$, we call

$$\omega(\mathscr{L}) := \inf\{\omega \in [0, \pi) : \mathscr{L} \text{ is sectorial of angle } \omega\}$$

the *sectoriality angle* of $\mathscr{L}$.

A family of operators $(\mathscr{L}_n)_n$ is called *uniformly sectorial* of angle $\omega \in [0, \pi)$ if each $\mathscr{L}_n$ is sectorial of angle ω and the constant C_φ in (2.1) can be chosen independently of n for all $\varphi \in (\omega, \pi)$.

Sectorial operators are automatically closed, since they have a non-empty resolvent set.

Another notion of sectoriality is often used for operators on Hilbert spaces. We introduce it below and then clarify the connection with the previous definition.

Definition 2.2. Let $\omega \in [0, \pi/2)$. An operator $\mathscr{L}$ on a complex Hilbert space $\mathcal{H}$ is called *sectorial in the sense of Kato* (or *Kato-sectorial*) of angle ω if

$$|\mathrm{Im}\, \langle \mathscr{L}u, u \rangle| \leqslant (\tan \omega)\, \mathrm{Re}\, \langle \mathscr{L}u, u \rangle, \tag{2.2}$$

for all $u \in \mathrm{D}(\mathscr{L})$.

A family of operators $(\mathscr{L}_n)_n$ is called *uniformly Kato-sectorial* of angle $\omega \in [0, \pi/2)$ if each $\mathscr{L}_n$ is Kato-sectorial of angle ω.

The *numerical range* of an operator $\mathscr{L}$ on $\mathcal{H}$ is the set

$$\mathcal{N}(\mathscr{L}) := \{\langle \mathscr{L}u, u\rangle : u \in \mathrm{D}(\mathscr{L}), \|u\| = 1\}.$$

Clearly, $\mathscr{L}$ satisfies (2.2) if and only if its numerical range $\mathcal{N}(\mathscr{L})$ is contained in the sector $\overline{\mathbf{S}}_\omega$.

Moreover, if $\mathscr{L}$ is Kato-sectorial of angle ω then

$$\|(\zeta - \mathscr{L})^{-1}\| \leqslant \frac{1}{\operatorname{dist}(\zeta, \overline{\mathbf{S}}_\omega)}, \tag{2.3}$$

for all $\zeta \in \rho(\mathscr{L})$ such that $\operatorname{dist}(\zeta, \overline{\mathbf{S}}_\omega) > 0$. Indeed, let $u \in \mathrm{D}(\mathscr{L})$ be such that $\|u\| = 1$. By the Cauchy-Schwarz inequality,

$$\begin{aligned}\|(\zeta - \mathscr{L})u\| &\geqslant |\,\langle (\zeta - \mathscr{L})u, u\rangle\,| \\ &= |\zeta - \langle \mathscr{L}u, u\rangle\,| \\ &\geqslant \operatorname{dist}(\zeta, \overline{\mathbf{S}}_\omega).\end{aligned}$$

Therefore, since $\zeta \in \rho(\mathscr{L})$ and $\operatorname{dist}(\zeta, \overline{\mathbf{S}}_\omega) > 0$, we obtain (2.3).

These observation highlights the connection between the two notions of sectoriality, formalized in the following lemma, whose proof is extracted from the argument in [72, Theorem 1.53].

Lemma 2.3. *Let $\mathscr{L}$ be a densely defined operator on a complex Hilbert space $\mathcal{H}$. Assume that $\mathscr{L}$ is Kato-sectorial of angle ω and that there exists $\zeta_0 \in \rho(\mathscr{L})$ with $\operatorname{dist}(\zeta_0, \overline{\mathbf{S}}_\omega) > 0$. Then $\mathscr{L}$ is sectorial of angle ω. In particular, it satisfies*

$$\|(\zeta - \mathscr{L})^{-1}\| \leqslant \frac{1}{\operatorname{dist}(\zeta, \overline{\mathbf{S}}_\omega)}, \qquad \forall \zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\omega,$$

and consequently,

$$\|(\zeta - \mathscr{L})^{-1}\| \leqslant \frac{1}{\sin(\varphi - \omega)}|\zeta|^{-1}, \qquad \forall \zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\varphi,\ \forall \varphi \in (\omega, \pi).$$

Proof. Let ζ_0 be as in the statement, and let $\zeta \in \mathbb{C}$. We write

$$\begin{aligned}\zeta - \mathscr{L} &= \zeta_0 - \mathscr{L} + \zeta - \zeta_0 \\ &= (\zeta_0 - \mathscr{L})[I + (\zeta - \zeta_0)(\zeta_0 - \mathscr{L})^{-1}].\end{aligned}$$

By (2.3), we have

$$\|(\zeta - \zeta_0)(\zeta_0 - \mathscr{L})^{-1}\| \leq \frac{|\zeta - \zeta_0|}{\operatorname{dist}(\zeta_0, \overline{\mathbf{S}}_\omega)}.$$

Hence, $\zeta - \mathscr{L}$ is invertible whenever $|\zeta - \zeta_0| < \operatorname{dist}(\zeta_0, \overline{\mathbf{S}}_\omega)$. By repeating this argument along a polygonal path contained in $\mathbb{C} \setminus \overline{\mathbf{S}}_\omega$, we deduce that $\zeta \in \rho(\mathscr{L})$ for all ζ such that $\operatorname{dist}(\zeta, \overline{\mathbf{S}}_\omega) > 0$, namely, $\sigma(\mathscr{L}) \subset \overline{\mathbf{S}}_\omega$. In particular, estimate (2.3) holds for every $\zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\omega$, that is,

$$\|(\zeta - \mathscr{L})^{-1}\| \leqslant \frac{1}{\operatorname{dist}(\zeta, \overline{\mathbf{S}}_\omega)}, \qquad \forall \zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\omega. \tag{2.4}$$

It remains to derive the explicit resolvent bound in terms of $|\zeta|$. Fix $\varphi \in (\omega, \pi)$. For every $\zeta \in \mathbb{C} \setminus \mathbf{S}_\varphi$, a simple geometric argument yields

$$\sin(\varphi - \omega) \leqslant \frac{\operatorname{dist}(\zeta, \overline{\mathbf{S}}_\omega)}{|\zeta|}.$$

Combining this with (2.4) gives

$$\|(\zeta - \mathscr{L})^{-1}\| \leqslant \frac{1}{\sin(\varphi - \omega)}|\zeta|^{-1}, \qquad \forall \zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\varphi.$$

Therefore, $\mathscr{L}$ is sectorial of angle ω. □

We now summarize some fundamental structural properties of sectorial operators that will be useful in the sequel. A remarkable feature of these operators is that knowledge of the location of their spectrum and the decay of the resolvent already provides substantial information about their functional-analytic properties. In particular, when convenient and assuming that the Banach space $\mathcal{X}$ is reflexive, these results allow us to treat the operators under consideration as injective, which simplifies the analysis of their functional calculus.

Proposition 2.4. *Let $\mathscr{L}$ be a sectorial operator on a Banach space $\mathcal{X}$.*

(i) *We have $\mathrm{N}(\mathscr{L}) \cap \overline{\mathrm{R}(\mathscr{L})} = \{0\}$. If $\overline{\mathrm{R}(\mathscr{L})} = \mathcal{X}$, then $\mathscr{L}$ is injective.*

(ii) *If the Banach space $\mathcal{X}$ is reflexive, one has $\overline{\mathrm{D}(\mathscr{L})} = \mathcal{X}$ and*

$$\mathcal{X} = \mathrm{N}(\mathscr{L}) \oplus \overline{\mathrm{R}(\mathscr{L})}.$$

Proof. See the proof of [47, Proposition 2.1.1]. □

2.2.2 Semigroups

We now recall the basic definitions and fundamental properties of operator semigroups on Banach spaces, which play a central role in the analysis of evolution equations. We focus on one-parameter strongly continuous semigroups, paying particular attention to the class of analytic semigroups, which are closely linked to sectorial operators. Our aim is to introduce the terminology and key facts that will be used throughout the thesis, without providing full proofs; for these, we refer to standard sources such as [43, 73]. For a rapid course in semigroup see [72, Section 1.3.2].

We begin with the definition of a semigroup.

Definition 2.5. A family $(T_t)_{t\geqslant 0}$ of bounded linear operators on a Banach space $\mathcal{X}$ is called a *one-parameter semigroup*, or only *semigroup*, if it satisfies:

- $T_0 = I$;
- $T_{s+t} = T_s T_t$ for all $s, t \geq 0$.

If, in addition,

- $\lim_{t\to 0} T_t u = u$ for every $u \in \mathcal{X}$,

we call $(T_t)_{t\geqslant 0}$ a *strongly continuous* semigroup.

For every strongly continuous semigroup $(T_t)_{t\geqslant 0}$, there exist constants $w \in \mathbb{R}$ and $M \geqslant 1$ such that

$$\|T_t\|_{\mathcal{L}(\mathcal{X})} \leqslant M e^{wt}, \tag{2.5}$$

for all $t \geqslant 0$ [43, Proposition I.5.5].

Definition 2.6. A strongly continuous semigroup $(T_t)_{t\geqslant 0}$ is called *bounded* if we can take $w = 0$ in (2.5), and *contractive* if $w = 0$ and $M = 1$.

The infinitesimal behavior of a strongly continuous semigroup is encoded in its generator.

Definition 2.7. The *generator* $\mathscr{L} : \mathrm{D}(\mathscr{L}) \subseteq \mathcal{X} \to \mathcal{X}$ of a strongly continuous semigroup $(T_t)_{t\geqslant 0}$ on a Banach space $\mathcal{X}$ is the operator

$$\mathrm{D}(\mathscr{L}) = \left\{ x \in \mathcal{X} \,:\, \lim_{t\to 0^+} \frac{T_t u - u}{t} \text{ exists} \right\},$$
$$\mathscr{L} u = \lim_{t\to 0^+} \frac{T_t u - u}{t}.$$

The generator $\mathscr{L}$ is a closed and densely defined operator and determines the semigroup uniquely [43, Theorem II.1.4]. Moreover, for all $f \in \mathrm{D}(\mathscr{L})$ we have $T_t f \in \mathrm{D}(\mathscr{L})$ and

$$\frac{d}{dt} T_t f = -\mathscr{L} T_t f = -T_t \mathscr{L} f,$$

for every $t \geqslant 0$ [43, Lemma 1.3(ii)].

The theory of strongly continuous semigroups was developed in order to study existence and uniqueness of solutions to the the Cauchy problem

$$(CP) \begin{cases} \dfrac{d}{dt} u(t) + \mathscr{L} u(t) = 0, & t \geqslant 0, \\ u(0) = f, \end{cases}$$

where $u : [0, \infty) \to \mathcal{X}$ is the unknown function taking values in $\mathrm{D}(\mathscr{L})$ for $t > 0$, and $f \in \mathcal{X}$ is the prescribed initial datum. If $\mathscr{L}$ is the generator of a strongly continuous semigroup $(T_t)_{t \geqslant 0}$, then for every $f \in \mathrm{D}(\mathscr{L})$ the problem *(CP)* admits a unique solution, given by $u(t) = T_t f$.

Among all semigroups, *analytic semigroups* play a distinguished role, as they arise from sectorial operators and are essential in the study of evolution equations and functional calculus.

Definition 2.8. Let $\vartheta \in (0, \pi/2]$. A family of bounded linear operator $(T_z)_{z \in \mathbf{S}_\vartheta \cup \{0\}}$ is called an *analytic* (or *holomorphic*) *semigroup* of angle ϑ if

- $T_0 = I$ and $T_{z_1 + z_2} = T_{z_1} T_{z_2}$ for all $z_1, z_2 \in \mathbf{S}_\vartheta$;
- The map $z \mapsto T_z$ is analytic in $\mathbf{S}_\vartheta$;
- $\lim_{z \to 0, z \in \overline{\mathbf{S}}_\varphi} T_z u = u$ for all $u \in \mathcal{X}$ and $\varphi \in (0, \vartheta)$.

If, in addition,

- $\|T_z\|_{\mathcal{L}(\mathcal{X})}$ is bounded in $\mathbf{S}_\varphi$ for every $\varphi \in (0, \vartheta)$,

we call $(T_z)_{z \in \mathbf{S}_\vartheta \cup \{0\}}$ a *bounded* analytic semigroup.

Analytic semigroups are intimately connected with sectorial operators. The following classical theorem characterizes the generators of bounded holomorphic semigroups and ensures existence and uniqueness of the solution to *(CP)* for *every* initial value $f \in \mathcal{X}$.

Theorem 2.9. *For an operator $\mathscr{L}$ on a Banach space $\mathcal{X}$, the following statements are equivalent*

(i) *$\mathscr{L}$ is densely defined and sectorial of angle ω.*

(ii) *$-\mathscr{L}$ generates a bounded analytic semigroup $(T_z)_{z \in \mathbf{S}_{\pi/2 - \omega}}$.*

(iii) *There exists $\varphi \in (0, \pi/2)$ such that the operators $-e^{\pm i\varphi} \mathscr{L}$ generate a bounded strongly continuous semigroup.*

(iv) *$-\mathscr{L}$ generates a bounded strongly continuous semigroup $(T_t)_{t>0}$ on $\mathcal{X}$ such that $\mathrm{R}(T_t) \subset \mathrm{D}(\mathscr{L})$ for all $t > 0$, and*

$$\sup_{t>0} \|t \mathscr{L} T_t\| < \infty.$$

Proof. See, for instance, [43, Theorem II.4.6]. □

2.3 H^∞-functional calculus and maximal regularity

As anticipated in the introduction, the main application of the bilinear embedding theorems developed in this thesis is the boundedness of an H^∞-functional calculus for divergence-form operators and, as a consequence, their parabolic maximal regularity.

This section introduces the notion of an H^∞-functional calculus, outlines its construction, and presents sufficient conditions for its boundedness that will be employed in later chapters. We then recall the definition of maximal regularity and discuss its close connection with the H^∞-calculus.

As in the previous section, we confine ourselves to the material required for the forthcoming analysis, omitting detailed proofs and referring the reader to standard references for the complete theory, in particular [26, 39, 47].

The concept of a *functional calculus* provides a systematic way to define and analyze functions of a linear operator. A familiar example is the Laplacian on $\mathbb{R}^d$, for which the functional calculus can be formulated through the Fourier transform: for every measurable function f on $[0,\infty)$, one defines $f(-\Delta)$ as the operator acting via multiplication by $f(4\pi^2|\xi|^2)$ in the Fourier domain.

In this section, we are going to develop this idea in more generality, focusing on sectorial operators. The reproducing structure

$$f(\lambda) = \frac{1}{2\pi i}\int_\gamma f(\zeta)\,(\zeta-\lambda)^{-1}\,\mathrm{d}\zeta$$

of the Cauchy integral formula for holomorphic functions provides the inspiration for a rigorous framework to define functions of operators via contour integrals, as introduced by Dunford and Riesz (see [37, Section VII.11] for a historical account). Replacing formally λ by an operator $\mathscr{L}$, one interprets the right-hand side in terms of resolvents in order to define

$$f(\mathscr{L}) = \frac{1}{2\pi i}\int_\gamma f(\zeta)\,(\zeta-\mathscr{L})^{-1}\,\mathrm{d}\zeta.$$

This formal expression motivates the general definition of a *holomorphic functional calculus*.

The class of sectorial operator is particularly well-suited for developing a functional calculus because the resolvent estimates guaranteed by sectoriality ensure the convergence of the above integral for a large class of holomorphic functions. Specifically, one can define $f(\mathscr{L})$ for bounded holomorphic functions on a sector containing the spectrum of $\mathscr{L}$, provided that f satisfies suitable decay conditions at infinity and near zero. The construction begins with such core class of functions and is then extended by first defining $(1+\mathbf{z})^{-1}(\mathscr{L}) = (1+\mathscr{L})^{-1}$ and $\mathbf{1}(\mathscr{L}) = I$, and subsequently by regularization for a larger class of functions.

The resulting theory, known as the *H^∞-functional calculus*, plays a central role in modern analysis of elliptic and parabolic operators.

2.3.1 Space of holomorphic functions

We introduce three classes of holomorphic functions, on which the functional calculus will be defined progressively.

For $\varphi \in (0,\pi]$ let $H^\infty(S_\varphi)$ be the Banach algebra of *bounded holomorphic functions* on $\mathbf{S}_\varphi$, endowed with the norm

$$\|f\|_\infty = \|f\|_{L^\infty(\mathbf{S}_\varphi)} = \sup\{|f(z)| : z \in \mathbf{S}_\varphi\}.$$

We denote by $H_0^\infty(\mathbf{S}_\varphi)$ the so-called *Dunford-Riesz class* on $\mathbf{S}_\varphi$, which consists of all functions $g \in H^\infty(\mathbf{S}_\varphi)$ satisfying

$$|g(z)| \leqslant C\min\{|z|^s, |z|^{-s}\},$$

for some $C, s > 0$ and all $z \in \mathbf{S}_\varphi$. Clearly, $H_0^\infty(\mathbf{S}_\varphi)$ is an algebra ideal in the algebra $H^\infty(\mathbf{S}_\varphi)$.

It is obvious that neither the rational function $(1+z)^{-1}$ nor the constant function $\mathbf{1}$ is contained in $H_0^\infty(\mathbf{S}_\varphi)$. We therefore define

$$\mathcal{E}(\mathbf{S}_\varphi) := H_0^\infty(\mathbf{S}_\varphi) \oplus \langle((1+z)^{-1}\rangle \oplus \langle \mathbf{1}\rangle,$$

called the *extended Dunford-Riesz class*. This set is in fact a subalgebra of $H^\infty(\mathbf{S}_\varphi)$, due to the identity

$$\frac{1}{(1+z)^2} = \frac{1}{1+z} - \frac{z}{(1+z)^2}.$$

We remark that both algebras $H_0^\infty(\mathbf{S}_\varphi)$ and $\mathcal{E}(\mathbf{S}_\varphi)$ are invariant under inversion, i.e., with $f(z)$ also $f(z^{-1})$ belongs to the set. This is clear for $H_0^\infty(\mathbf{S}_\varphi)$. For the larger algebra $\mathcal{E}(\mathbf{S}_\varphi)$ it follows from

$$\frac{1}{1+z^{-1}} = \frac{z}{1+z} = \mathbf{1} - \frac{1}{1+z}. \tag{2.6}$$

If the precise sector $\mathbf{S}_\varphi$ is understood from the context or unimportant, we simply write H_0^∞ instead of $H_0^\infty(\mathbf{S}_\varphi)$ in the sequel. The same applies to the other function spaces.

2.3.2 The primary functional calculus

We now define the functional calculus first for functions in the space H_0^∞, and subsequently for functions in $\mathcal{E}$.

For every $\varphi \in (0, \pi)$ we denote by $\gamma_\varphi := \partial \mathbf{S}_\varphi$ the boundary of the sector $\mathbf{S}_\varphi$, oriented in the positive sense, i.e.,

$$\gamma_\varphi := -\mathbb{R}_+ e^{i\varphi} \oplus \mathbb{R}_+ e^{-i\varphi}.$$

Let $\mathscr{L}$ be a sectorial operator of angle ω on a Banach space $\mathcal{X}$. For $\varphi \in (\omega, \pi)$ and $f \in H_0^\infty(\mathbf{S}_\varphi)$ we define

$$f(\mathscr{L}) := \frac{1}{2\pi i} \int_{\gamma_{\omega'}} f(\zeta)(\zeta - \mathscr{L})^{-1} \, \mathrm{d}\zeta, \tag{2.7}$$

where $\omega' \in (\omega, \varphi)$ is arbitrary. The integral in absolutely convergent in $\mathscr{L}(\mathcal{X})$ whenever $f \in H_0^\infty(\mathbf{S}_\varphi)$, thanks to the resolvent estimates for $\mathscr{L}$, see Definition 2.1. A standard argument using Cauchy's integral theorem shows that this definition is independent of ω'.

We extend the definition of $f(\mathscr{L})$ from $f \in H_0^\infty(\mathbf{S}_\varphi)$ to all $f \in \mathcal{E}(\mathbf{S}_\varphi)$. This can be done by defining

$$g(\mathscr{L}) := f(\mathscr{L}) + c(1+\mathscr{L})^{-1} + d, \tag{2.8}$$

whenever $g = f + c(1+z)^{-1} + d$ with $f \in H_0^\infty(\mathbf{S}_\varphi)$ and $c, d \in \mathbb{C}$.

Theorem 2.10. *Let $\mathscr{L}$ be a sectorial operator of angle ω on $\mathcal{X}$, and let $\varphi \in (\omega, \pi)$. The mapping*

$$\begin{aligned} \Phi_{\mathscr{L}} := \mathcal{E}(\mathbf{S}_\varphi) &\to \mathcal{L}(\mathcal{X}) \\ g &\mapsto g(\mathscr{L}) \end{aligned}$$

defined by (2.8) is a homomorphism of algebras. Moreover, $(\mathbf{z}(1+\mathbf{z})^{-1})(\mathscr{L}) = \mathscr{L}(1+\mathscr{L})^{-1}$.

Proof. See the proof of [47, Theorem 2.3.3]. □

In particular, we have

$$(\mathbf{z}(1+\mathbf{z})^{-2})(\mathscr{L}) = (\mathbf{z}(1+\mathbf{z})^{-1})(\mathscr{L})((1+\mathbf{z})^{-1})(\mathscr{L}) = \mathscr{L}(1+\mathscr{L})^{-2}. \tag{2.9}$$

Semigroups via functional calculus

Let $\mathscr{L}$ be a sectorial operator of angle $\omega \in [0, \pi/2)$ and assume that it is densely defined (which is always the case when $\mathcal{X}$ is reflexive, see Proposition 2.4(ii)). By Theorem 2.9, $-\mathscr{L}$ generates a bounded holomorphic semigroup $(T_z)_{\mathbf{S}_\vartheta}$, with $\vartheta := \pi/2 - \omega$. For $z \in \mathbf{S}_\vartheta$, the function $\mathbf{S}_\varphi \ni \zeta \mapsto e^{-z\zeta}$ belongs to $\mathcal{E}(\mathbf{S}_\varphi)$ for each $\varphi \in (\omega, \pi/2 - |\arg z|)$. This allows us to define the map

$$\mathbf{S}_\vartheta \ni z \longmapsto e^{-z\mathscr{L}} \in \mathcal{L}(\mathcal{X}).$$

using the primary functional calculus (Theorem 2.10). It turns out that the family $(e^{-z\mathscr{L}})_{z\in\mathbf{S}_\vartheta}$ is a bounded holomorphic semigroup whose generator is $-\mathscr{L}$; see [47, Proposition 3.4.1] or [43, Proposition 4.3 & Proposition 4.4]. Since the generator uniquely determines the semigroup, we have the identity

$$T_z = e^{-z\mathscr{L}}, \qquad z \in \mathbf{S}_\vartheta. \tag{2.10}$$

Moreover, the semigroup can be expressed via a Cauchy integral formula [47, Lemma 2.3.2]. Fix $z \in \mathbf{S}_\vartheta$ and $\delta > 0$. Choose $\varphi \in (0, \pi/2)$ such that $|\arg z| < \pi/2 - \varphi < \vartheta$, and denote by γ the positively oriented boundary of $\mathbf{S}_\varphi \cup \{\zeta \in \mathbb{C} : |\zeta| < \delta\}$. Then

$$T_z = e^{-z\mathscr{L}} = \int_\gamma e^{-z\zeta} (\zeta - \mathscr{L})^{-1} \, \mathrm{d}\zeta. \tag{2.11}$$

2.3.3 The extended functional calculus

There are many important functions f, for which we intuitively know what $f(\mathscr{L})$ should be, but which are not contained in $\mathcal{E}$. The simplest example is $\mathbf{z}(z) = z$, which should lead the possibly unbounded operator $\mathscr{L}$ itself. Enlarging the class of functions for the functional calculus to suitable unbounded functions, will, in general, make us leave the comfort zone of bounded operators and we have to include closed unbounded operators. We accomplish this extension via a regularization procedure. First, denote by $\mathrm{M}(\mathbf{S}_\varphi)$ the space of meromorphic functions on $\mathbf{S}_\varphi$.

Definition 2.11. Let $\mathscr{L}$ be a sectorial operator on a Banach space $\mathcal{X}$ of angle ω and let $\varphi \in (\omega, \pi)$.

- The set of *regularizers* on $\mathbf{S}_\varphi$ is

$$\mathrm{Reg}_{\mathscr{L}}(\mathbf{S}_\varphi) := \{e \in \mathcal{E}(\mathbf{S}_\varphi) \,:\, e(\mathscr{L}) \text{ is injective}\}.$$

- The set of *regularizable meromorphic functions* of $\mathbf{S}_\varphi$ is

$$\mathrm{M}_{\mathscr{L}}(\mathbf{S}_\varphi) := \{f \in \mathrm{M}(\mathbf{S}_\varphi) \,:\, ef \in \mathcal{E}(\mathbf{S}_\varphi) \text{ for some } e \in \mathrm{Reg}_{\mathscr{L}}(\mathbf{S}_\varphi)\}.$$

  If $f \in \mathrm{M}_{\mathscr{L}}(\mathbf{S}_\varphi)$ with regularizer $e \in \mathrm{Reg}_{\mathscr{L}}(\mathbf{S}_\varphi)$, we say that e regularizes f.

The set $\mathrm{Reg}_{\mathscr{L}}(\mathbf{S}_\varphi)$ contains at least $\mathbf{1}$ and the non-trivial regularizer $(1 + \mathbf{z})^{-1}$. If $\mathscr{L}$ is injective, then also $\mathbf{z}(1 + \mathbf{z})^{-2}$ is a regularizer thanks to (2.9). Moreover, $\mathbf{z}(1 + \mathbf{z})^{-2}$ regularizes every $f \in H^\infty(\mathbf{S}_\varphi)$, since $\mathbf{z}(1 + \mathbf{z})^{-2} \in H^\infty_0(\mathbf{S}_\varphi)$ and $H^\infty_0(\mathbf{S}_\varphi)$ is an algebra ideal in the algebra $H^\infty(\mathbf{S}_\varphi)$.

Regularizers allow us to extend the functional calculus in the following way. For $f \in \mathrm{M}_{\mathscr{L}}(\mathbf{S}_\varphi)$ pick some regularizer e. Then the primary functional calculus provide us with bounded operators $(ef)(\mathscr{L})$ and $e(\mathscr{L})$ and we know that $e(\mathscr{L})$ is injective. Thus, $e(\mathscr{L})^{-1}$ is a closed operator that heuristically corresponds to '$e^{-1}(\mathscr{L})$'. It is then rather natural to try the definition

$$\begin{aligned} \mathrm{D}(f(\mathscr{L})) :&= \{x \in \mathcal{X} \,:\, (ef)(\mathscr{L})x \in \mathrm{R}(e(\mathscr{L}))\}, \\ f(\mathscr{L}) :&= e(\mathscr{L})^{-1}(ef)(\mathscr{L}), \end{aligned} \tag{2.12}$$

in order to extend the functional calculus.

This idea gives rise to something well-defined, since the definition (2.12) is independent of the particular choice of the regularizer e and consistent with the primary functional calculus [47, Lemma 1.2.1].

Definition 2.12. Let $\varphi \in (\omega(\mathscr{L}), \pi)$. For $f \in \mathrm{M}_{\mathscr{L}}(\mathbf{S}_\varphi)$ define $f(\mathscr{L})$ as in (2.12). The emerging *functional calculus* $\mathrm{M}_{\mathscr{L}}(\mathbf{S}_\varphi) \ni f \mapsto f(\mathscr{L}) \in \{\text{closed operators in } \mathcal{X}\}$ will (with a slight abuse f notation) again denoted by $\Phi_{\mathscr{L}}$.

With this definition at hand, we can introduce the operator $\mathbf{z}(\mathscr{L})$. Indeed, the regularizer $e(z) = (1+z)^{-1}$ regularizes $\mathbf{z}$ thanks to the decomposition (2.6). We now verify that $\mathbf{z}(\mathscr{L}) = \mathscr{L}$. In fact, we have $(e\mathbf{z})(\mathscr{L}) = -(I + \mathscr{L})^{-1} + I$ and

$$\mathbf{z}(\mathscr{L}) = e(\mathscr{L})^{-1}(e\mathbf{z})(\mathscr{L}) = (I + \mathscr{L})(-(I + \mathscr{L})^{-1} + I) = \mathscr{L}.$$

For the extended functional calculus to be meaningful, it should retain, at least partially, the algebraic structure of the elementary one. Since closed operators do not form an algebra, we cannot expect a full homomorphism property, but the following theorem provides a satisfactory substitute.

Theorem 2.13. *Let $f, g \in \mathrm{M}_{\mathscr{L}}(\mathbf{S}_\varphi)$ and $\lambda, \mu \in \mathbb{C}$. Then*

$$\begin{aligned} \lambda f(\mathscr{L}) + \mu g(\mathscr{L}) &\subseteq (\lambda f + \mu g)(\mathscr{L}); \\ f(\mathscr{L})g(\mathscr{L}) &\subseteq (fg)(\mathscr{L}). \end{aligned} \tag{2.13}$$

Moreover, $D(f(\mathscr{L})g(\mathscr{L})) = D((fg)(\mathscr{L})) \cap D(g(\mathscr{L}))$. Finally, if $g(\mathscr{L}) \in \mathcal{L}(\mathcal{X})$, then the inclusions (2.13) become equalities.

Proof. See the proof of [47, Theorem 1.3.2]. □

Let $\mathscr{L}$ be a sectorial operator of angle $\omega \in [0, \pi/2)$. The functions $\mathbf{z}e^{-\mathbf{z}}$, $e^{-\mathbf{z}}$ and $\mathbf{z}$ belong respectively to $H^\infty_0(\mathbf{S}_{\pi/2})$, $\mathcal{E}(\mathbf{S}_{\pi/2})$ and $\mathrm{M}_{\mathscr{L}}(\mathbf{S}_{\pi/2})$. Hence, by Theorem 2.10 and Theorem 2.13, we obtain

$$\mathscr{L}e^{-\mathscr{L}} = \mathbf{z}(\mathscr{L})e^{-\mathbf{z}}(\mathscr{L}) = (\mathbf{z}e^{-\mathbf{z}})(\mathscr{L}) \in \mathcal{L}(\mathcal{X}).$$

Similarly, for every $z \in \mathbf{S}_{\pi/2-\omega}$ we have

$$z\mathscr{L}e^{-z\mathscr{L}} = (z\mathbf{z}e^{-z\mathbf{z}})(\mathscr{L}). \tag{2.14}$$

2.3.4 The H^∞-functional calculus

In many situations it can be decisive to know whether, for some given sectorial operator $\mathscr{L}$ and some holomorphic function f, the operator $f(\mathscr{L})$ is bounded. In particular, given $\varphi \in (\omega(\mathscr{L}), \pi)$, it is natural to ask if $f(\mathscr{L}) \in \mathcal{L}(\mathcal{X})$ for all $f \in H^\infty(\mathbf{S}_\varphi)$.
In order to define $f(\mathscr{L})$ in the first place, we are led to assuming that $\mathscr{L}$ is injective, because only then we can use the universal regularizer $e = \mathbf{z}(1 + \mathbf{z})^{-2}$ for f. If $\mathcal{X}$ is reflexive, however, this assumption is not a substantial restriction as long as we are not limited to one specific Banach space. More precisely, we will see that every sectorial operator becomes injective and sectorial on the closed subspace $\overline{\mathrm{R}(\mathscr{L})} \subseteq \mathcal{X}$.

In general it is not true that $f(\mathscr{L})$ is bounded for every $f \in H^\infty(\mathbf{S}_\varphi)$. The counterexamples are quite pathological, and in [47] is presented a general method which relies on Schauder bases for constructing such disproving examples. This observation motivates the following definition.

Definition 2.14. Let $\mathscr{L}$ be an injective sectorial operator on $\mathcal{X}$. We say that $\mathscr{L}$ has a *bounded* H^∞-functional calculus of angle $\varphi \in (\omega(\mathscr{L}), \pi)$ if there is a constant C such that for all $f \in H^\infty(\mathbf{S}_\varphi)$ we have $f(\mathscr{L}) \in \mathcal{L}(\mathcal{X})$ with norm bound

$$\|f(\mathscr{L})\|_{\mathcal{L}(\mathcal{X})} \leqslant C\|f\|_\infty.$$

The infimum of such angles φ is called the H^∞-angle of $\mathscr{L}$ and will be denoted by $\omega_{H^\infty}(\mathscr{L})$.

If $\mathscr{L}$ has a bounded H^∞-calculus of angle φ, then

$$\begin{aligned} \Phi_{\mathscr{L}} \colon H^\infty(\mathbf{S}_\varphi) &\longrightarrow \mathcal{L}(\mathcal{X}), \\ f &\longmapsto f(\mathscr{L}); \end{aligned}$$

is a bounded algebra homomorphism by Theorem 2.13.

A particular family of holomorphic functions plays a central role in this framework: imaginary power. Let $\varphi \in (0, \pi)$. For every $s \in \mathbb{R}$, the function $\mathbf{z}^{is}$ is holomorphic and bounded on each sector $\mathbf{S}_\varphi$, since

$$|z^{is}| = |e^{is \log z}| = e^{-s \arg z} \leqslant e^{\varphi|s|}, \qquad z \in \mathbf{S}_\varphi.$$

This motivates the following definition.

Definition 2.15. The *imaginary powers* of an injective sectorial operator $\mathscr{L}$ on a Banach space $\mathcal{X}$ is the family of operators $(\mathscr{L}^{is})_{s\in\mathbb{R}}$ defined via the extended functional calculus.

We say that $\mathscr{L}$ has *bounded imaginary powers (BIP)* if $\overline{\mathrm{D}(\mathscr{L}) \cap \mathrm{R}(\mathscr{L})} = \mathcal{X}$ and $\mathscr{L}^{is} \in \mathcal{L}(\mathcal{X})$ for all $s \in \mathbb{R}$.

If $\mathscr{L}$ has bounded imaginary powers, by [47, Corollary 3.5.7], we know that

$$\vartheta(\mathscr{L}) := \inf\{\vartheta \geqslant 0 : \exists C \text{ s.t. } \|\mathscr{L}^{is}\| \leqslant Ce^{\vartheta|s|} \ \forall s \in \mathbb{R}\} < \infty.$$

Clearly, every injective sectorial operator $\mathscr{L}$ with dense domain and dense range has BIP whenever it admits a bounded H^∞-functional calculus. In particular, $\vartheta(\mathscr{L}) \leqslant \omega_{H^\infty}(\mathscr{L})$. This is the case, for instance, when $\mathcal{X}$ is reflexive, by Proposition 2.4(ii).

In the special case of Hilbert spaces, the converse is also true: the BIP property is sufficient to guarantee the boundedness of the H^∞-functional calculus [64, 90]. This equivalence, however, fails in general for Banach spaces, as shown in [26].

Finally, the BIP property plays a crucial role in determining whether an operator admits *maximal parabolic regularity*, as will be discussed in Section 2.3.6.

H^∞-calculus for non-injective operators on reflexive spaces

So far, we have discussed the H^∞-functional calculus for injective sectorial operators. In order to extend this concept to general sectorial operators, we make use of the so-called *injective part* of a sectorial operator, which allows us to reduce the problem to the injective setting whenever the underlying Banach space is reflexive.

Definition 2.16. Let $\mathscr{L}$ be a sectorial operator on a Banach space $\mathcal{X}$. The *injective part* of $\mathscr{L}$, denoted by $\mathscr{L}_{||}$, is defined as the maximal restriction of $\mathscr{L}$ to the closure of its range $\overline{\mathrm{R}(\mathscr{L})}$. More precisely,

$$\begin{aligned} \mathrm{D}(\mathscr{L}_{||}) &:= \mathrm{D}(\mathscr{L}) \cap \overline{\mathrm{R}(\mathscr{L})}, \\ \mathscr{L}_{||}u &:= \mathscr{L}u, \qquad u \in \mathrm{D}(\mathscr{L}_{||}). \end{aligned}$$

Since $\sigma(\mathscr{L}_{||}) \subseteq \sigma(\mathscr{L})$ and $(\zeta - \mathscr{L}_{||})^{-1} = (\zeta - \mathscr{L})^{-1}|_{\overline{\mathrm{R}(\mathscr{L})}}$ for all $\zeta \in \rho(\mathscr{L})$, the operator $\mathscr{L}_{||}$ is sectorial of angle $\omega(\mathscr{L})$ on $\overline{\mathrm{R}(\mathscr{L})}$. Moreover, if $\mathcal{X}$ is reflexive, then $\mathscr{L}_{||}$ is injective by Proposition 2.4(ii).

The previous considerations allow us to define $f(\mathscr{L})$ as a closed operator in $\overline{\mathrm{R}(\mathscr{L})}$ via

$$f(\mathscr{L}) := f(\mathscr{L}_{||}).$$

Thanks to the following result, this definition is compatible with $f(\mathscr{L})$ whenever $f \in \mathrm{M}_{\mathscr{L}}(\mathbf{S}_\varphi)$.

Proposition 2.17. *Let $\mathscr{L}$ be a sectorial operator on a reflexive space $\mathcal{X}$, $\varphi \in (\omega(\mathscr{L}), \pi)$ and $f \in \mathrm{M}_{\mathscr{L}}(\mathbf{S}_\varphi)$. Then $\overline{\mathrm{R}(\mathscr{L})}$ is invariant under the action of $f(\mathscr{L})$, and one has $f(\mathscr{L}_{||}) = f(\mathscr{L})|_{\overline{\mathrm{R}(\mathscr{L})}}$.*

Proof. See [47, Theorem 2.3.3d) & Corollary 2.3.9]. □

Definition 2.18. Let $\mathscr{L}$ be a sectorial operator on a reflexive space $\mathcal{X}$. We say that $\mathscr{L}$ has a *bounded* H^∞-functional calculus of angle $\varphi \in (\omega(\mathscr{L}), \pi)$ if $\mathscr{L}_{||}$ has a bounded H^∞-functional calculus of angle φ.

The infimum of such angles φ—which coincides with $\omega_{H^\infty}(\mathscr{L}_{||})$—is called the H^∞-angle of $\mathscr{L}$ and will be denoted by $\omega_{H^\infty}(\mathscr{L})$.

2.3.5 A sufficient condition for a bounded H^∞-functional calculus

Characterizing the boundedness of an H^∞-functional calculus has been a natural goal in the theory of sectorial operators. In Hilbert spaces, several equivalent conditions are known, such as boundedness of the imaginary powers or quadratic estimates [27, 90]. The situation is considerably more delicate in Banach spaces. For instance, as already mentioned, bounded imaginary powers are only a necessary condition [26].

In this section, we recall a sufficient criterion that will be useful in later chapters, where we prove that various classes of divergence-form operators admit a bounded H^∞-calculus. The material presented here is taken from [26], where the notion of a *dual pair* plays a central role.

Definition 2.19. We say that the Banach spaces $\langle \mathcal{X}, \mathcal{Y} \rangle$ form a *dual pair* if there is a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{X} \times \mathcal{Y}$ such that

$$\begin{aligned} |\langle u, v \rangle| &\leqslant C_0 \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}, \qquad \forall u \in \mathcal{X}, \forall v \in \mathcal{Y}, \\ \|u\|_{\mathcal{X}} &\leqslant C_1 \sup\{|\langle u, v \rangle| : v \in \mathcal{Y}, \|v\|_{\mathcal{Y}} = 1\}, \qquad \forall u \in \mathcal{X}, \\ \|v\|_{\mathcal{Y}} &\leqslant C_2 \sup\{|\langle u, v \rangle| : u \in \mathcal{X}, \|u\|_{\mathcal{X}} = 1\}, \qquad \forall v \in \mathcal{Y}. \end{aligned}$$

The Hahn-Banach Theorem implies that $\langle \mathcal{X}, \mathcal{X}^* \rangle$ always forms a dual pair of Banach spaces.

Let $\langle \mathcal{X}, \mathcal{Y} \rangle$ be a dual pair. A necessary condition for an injective sectorial operator $\mathscr{L}$ on $\mathcal{X}$, with dense domain and dense range, to admit a bounded H^∞-functional calculus of angle φ is the validity of the following dual estimate for all $\psi \in H^\infty_0(\mathbf{S}_\varphi)$ [26, Theorem 4.2]:

$$\int_0^\infty |\langle \psi(t\mathscr{L})u, v \rangle| \, \frac{\mathrm{d}t}{t} \leqslant C \|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}} \quad \forall u \in \mathcal{X}, \forall v \in \mathcal{Y}. \tag{2.15}$$

However, in order for such an inequality to be strong enough to imply the boundedness of the H^∞-calculus, additional conditions on the function ψ are required. This motivates the following definition.

Definition 2.20. Suppose that $\nu \in (0,\pi)$. For all $\mu \in (0,\nu)$ we define the space $\Psi_\nu(\mathbf{S}_\mu)$ by

$$\Psi_\nu(\mathbf{S}_\mu) := \left\{ \psi \in H_0^\infty(\mathbf{S}_\mu) \,:\, e^{-\nu|\cdot|} \left| \widehat{(\psi \circ \exp)} \right|^{-1} \in L^\infty(\mathbb{R}) \right\},$$

and we set

$$\Psi_\nu(\mathbf{S}_\nu-) := \bigcap_{\mu\in(0,\nu)} \Psi_\nu(\mathbf{S}_\mu).$$

Here, $\widehat{(\psi \circ \exp)}$ denotes the Fourier transform on $\mathbb{R}$ of the function $t \mapsto \psi(e^t)$, namely, a variant of the Mellin transform of ψ.

A classical example of a function belonging to this space is $\psi(z) = ze^{-z}$, defined on $\mathbf{S}_{\pi/2}$. Indeed, if $\mu < \pi/2$, then $\psi \in H_0^\infty(\mathbf{S}_\mu)$. Moreover,

$$\widehat{(\psi \circ \exp)}(\lambda) = \int_0^\infty \tau^{-i\lambda} e^{-\tau} \,\mathrm{d}\tau = \Gamma(1 - i\lambda), \qquad \lambda \in \mathbb{R}.$$

Therefore, from the identity

$$|\Gamma(1 - i\lambda)|^2 = \frac{\pi\lambda}{\sinh \pi\lambda}, \qquad \lambda \in \mathbb{R},$$

we obtain $|\Gamma(1 - i\lambda)| \geqslant Ce^{-\pi|\lambda|/2}$ for all $\lambda \in \mathbb{R}$, which shows that $\psi \in \Psi_{\pi/2}(\mathbf{S}_{\pi/2}-)$.

The next theorem was initially stated in the special case $\vartheta = 0$ [26, Theorem 4.4]. As a consequence, whenever $\psi \in \Psi_\nu(\mathbf{S}_\nu-)$, the dual estimate (2.15) was shown to be also a sufficient condition for $\mathscr{L}$ to admit a bounded H^∞-calculus of any angle $\varphi > \nu$ [26, Corollary 4.5]. However, applying that result directly has a limitation: one cannot guarantee that the H^∞-angle of $\mathscr{L}$ lies in $(0,\nu)$. This becomes an issue, for instance, when taking $\psi(z) = ze^{-z}$, for which $\nu = \pi/2$. In that case, no matter how small ω is, one cannot go below $\pi/2$. The introduction of the extra parameter ϑ in the next theorem resolves this limitation.

Theorem 2.21. *Let $\langle \mathcal{X}, \mathcal{Y} \rangle$ be a dual pair. Suppose that $\mathscr{L}$ is an injective sectorial operator of angle ω on a Banach space $\mathcal{X}$, with dense domain and dense range. Let $\mu, \nu \in (0,\pi)$ and $\vartheta \in [0,\pi)$ such that*

$$\omega < \mu < \nu, \qquad \vartheta < \mu - \omega.$$

If there exist $\psi \in \Psi_\nu(\mathbf{S}_\mu)$ and $C > 0$ such that

$$\int_0^\infty \left| \left\langle \psi(te^{\varepsilon i\vartheta}\mathscr{L})u, v \right\rangle \right| \frac{\mathrm{d}t}{t} \leqslant C\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}, \qquad \forall x \in \mathcal{X},\, \forall v \in \mathcal{Y},\, \forall \varepsilon \in \{\pm 1\},$$

then $\mathscr{L}$ has a bounded H^∞-functional calculus of angle $\varphi > 2\nu - \mu - \vartheta$.

Proof. See the proof of [26, Theorem 4.6]. □

Applying Theorem 2.21 with $\psi(z) = ze^{-z}$ yields the following corollary [26, Example 4.8].

Corollary 2.22. *Let $\langle \mathcal{X}, \mathcal{Y} \rangle$ be a dual pair. Suppose that $\mathscr{L}$ is an injective sectorial operator of angle $\omega < \pi/2$ on a Banach space $\mathcal{X}$, with dense domain and dense range.*

Let $\vartheta \in (0, \pi/2 - \omega)$. If there exists $C > 0$ such that

$$\int_0^\infty \left| \left\langle \mathscr{L}e^{-te^{\varepsilon i\vartheta}\mathscr{L}}u, v \right\rangle \right| \mathrm{d}t \leqslant C\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}}, \qquad \forall x \in \mathcal{X},\, \forall v \in \mathcal{Y},\, \forall \varepsilon \in \{\pm 1\}, \tag{2.16}$$

then $\mathscr{L}$ has a bounded H^∞-functional calculus of angle $\varphi > \pi/2 - \vartheta$.

Proof. Let $\vartheta \in (0, \pi/2 - \omega)$ be such that (2.16) holds, and let $\varphi > \pi/2 - \vartheta$. Then, using (2.14) we obtain

$$\begin{aligned}\int_0^\infty \left|\left\langle \psi(te^{\varepsilon i\vartheta}\mathscr{L})u, v\right\rangle\right| \frac{\mathrm{d}t}{t} &= \int_0^\infty \left|\left\langle te^{\varepsilon i\vartheta}\mathscr{L}e^{-te^{\varepsilon i\vartheta}\mathscr{L}}u, v\right\rangle\right| \frac{\mathrm{d}t}{t},\\ &= \int_0^\infty \left|\left\langle \mathscr{L}e^{-te^{\varepsilon i\vartheta}\mathscr{L}}u, v\right\rangle\right| \mathrm{d}t,\\ &\leqslant C\|u\|_{\mathcal{X}}\|v\|_{\mathcal{Y}},\end{aligned}$$

for all $u \in \mathcal{X}$, $v \in \mathcal{Y}$ and $\varepsilon \in \{\pm 1\}$. By choosing $\mu \in (\vartheta + \omega, \pi/2)$ sufficiently close to $\pi/2$ so that $\varphi > \pi - \mu - \vartheta$, the conclusion follows directly from Theorem 2.21. □

In the previous section we have seen that it is possible to define an H^∞-calculus for non-injective sectorial operators whenever the underlying space X is reflexive. This is done by considering the injective part of such operators, thereby reducing the problem to injective operators acting on subspaces of $\mathcal{X}$. In order to apply Corollary 2.22 in the context of non-injective operators, it is thus convenient to introduce the following notions.

Definition 2.23. Suppose that $\langle \mathcal{X}, \mathcal{Y}\rangle$ form a dual pair.

- If $\mathcal{X}_0$ and $\mathcal{Y}_0$ are closed subspaces of $\mathcal{X}$ and $\mathcal{Y}$ respectively, and if $\langle \mathcal{X}_0, \mathcal{Y}_0\rangle$ forms a dual pair, with the same bilinear form of $\langle \mathcal{X}, \mathcal{Y}\rangle$, then we call $\langle \mathcal{X}_0, \mathcal{Y}_0\rangle$ a *dual subpair* of the dual pair $\langle \mathcal{X}, \mathcal{Y}\rangle$.
- If $\mathscr{L}$ and $\mathscr{T}$ are operators in $\mathcal{X}$ and $\mathcal{Y}$ respectively, then $\mathscr{L}$ and $\mathscr{T}$ are said to be *dual operators*, or $\mathscr{T}$ is said to be dual to $\mathscr{L}$, if both $\mathscr{L}$ and $\mathscr{T}$ are closed and

$$\langle \mathscr{L}u, v\rangle = \langle u, \mathscr{T}v\rangle, \qquad \forall u \in \mathrm{D}(\mathscr{L}),\ \forall v \in \mathrm{D}(\mathscr{T}).$$

Remark 2.24.
- If $\mathscr{L}$ is closed and densely defined, then $\mathscr{L}$ and its adjoint $\mathscr{L}^*$ are dual operators.
- Let $\mathscr{L}$ and $\mathscr{T}$ be dual operators in the dual pair $\langle \mathcal{X}, \mathcal{Y}\rangle$. Suppose that both are sectorial of angle ω and that $\mathcal{X}$ is reflexive. Then $\mathcal{Y}$ is also reflexive, and by Proposition 2.4(ii) we have
$$\mathcal{X} = \mathrm{N}(\mathscr{L}) \oplus \overline{\mathrm{R}(\mathscr{L})}, \quad \mathcal{Y} = \mathrm{N}(\mathscr{T}) \oplus \overline{\mathrm{R}(\mathscr{T})}.$$
Moreover, the annihilator of $\overline{\mathrm{R}(\mathscr{L})}$ in $\mathcal{Y}$ is $\mathrm{N}(\mathscr{T})$, and the annihilator of $\overline{\mathrm{R}(\mathscr{T})}$ in $\mathcal{X}$ is $\mathrm{N}(\mathscr{L})$. Hence, $\overline{\mathrm{R}(\mathscr{T})}$ is isomorphic to the dual space of $\overline{\mathrm{R}(\mathscr{L})}$, and conversely, that is, $\left\langle \overline{\mathrm{R}(\mathscr{L})}, \overline{\mathrm{R}(\mathscr{T})}\right\rangle$ is a dual subpair of $\langle \mathcal{X}, \mathcal{Y}\rangle$. Thus, $\mathscr{L}_{||}$ and $\mathscr{T}_{||}$—which are injective sectorial operator of angle ω on $\overline{\mathrm{R}(\mathscr{L})}$ and $\overline{\mathrm{R}(\mathscr{T})}$, respectively, with dense domain and dense range—are dual operators in the dual pair $\left\langle \overline{\mathrm{R}(\mathscr{L})}, \overline{\mathrm{R}(\mathscr{T})}\right\rangle$. Therefore, at least in reflexive spaces, there is no essential loss of generality if we assume that a dual pair of operators are injective and have dense range, provided that one is willing to work with dual subpairs.

2.3.6 Maximal parabolic regularity

The boundedness of the H^∞-calculus is closely related to the problem of *maximal regularity* for inhomogeneous Cauchy problems. In an abstract setting, this reduces to determining when the sum of two generators of strongly continuous semigroups on a Banach space is closed [32, 79]. The celebrated Dore–Venni Theorem [32] provides a practical criterion for this, requiring that the operators involved have bounded imaginary powers. In particular, a sectorial operator satisfies this condition whenever it admits a bounded H^∞-calculus.

In this section we recall the Dore–Venni Theorem. Before stating it, we introduce the notions needed to formulate it precisely. We begin by recalling the definition of maximal regularity.

Definition 2.25. Let $-\mathscr{L}$ be the generator of a strongly continuous semigroup $(T_t)_{t>0}$ on a complex Banach space $\mathcal{X}$. Let $\tau \in (0,\infty)$ and $r \in (1,\infty)$.
We say that $\mathscr{L}$ has *maximal L^r-regularity in* $(0,\tau)$ if for every $v \in L^r(0,\tau;\mathcal{X})$ the unique *mild solution*

$$u(t) := \int_0^t T_{t-s} v(s)\,\mathrm{d}s, \quad (0 < t < \tau) \tag{2.17}$$

to the Cauchy problem

$$\begin{cases} u'(t) + \mathscr{L}u(t) = v(t), \\ u(0) = 0, \end{cases}$$

belongs to $W^{1,r}(0,\tau;\mathcal{X}) \cap L^r(0,\tau;\mathrm{D}(\mathscr{L}))$.

Here, $u \in W^{1,r}(0,\tau;\mathcal{X})$ means that there exists $w \in L^r(0,\tau;\mathcal{X})$ such that

$$u(t) = \int_0^t w(s)\,\mathrm{d}s,$$

while $\mathrm{D}(\mathscr{L})$ is endowed with the graph norm $\|\cdot\|_{\mathrm{D}(\mathscr{L})} = \|\cdot\|_{\mathcal{X}} + \|\mathscr{L}\cdot\|_{\mathcal{X}}$.

The terminology stems from the fact that both summands on the left-hand side of the equation

$$u'(t) + \mathscr{L}u(t) = v(t)$$

should have the maximal "amount of regularity" that one can reasonably expect, given that the right-hand side v is in L^r.

This property does not depend on $\tau > 0$ and $r > 1$ [32]. This motivates the following notion.

Definition 2.26. We say that $\mathscr{L}$ has *(parabolic) maximal regularity* if for some, equivalently all, $r > 1$ and $\tau > 0$ the operator $\mathscr{L}$ has maximal L^r-regularity in $(0,\tau)$.

Remark 2.27. Suppose that $\mathcal{X}$ is reflexive and $\mathscr{L}$ is sectorial. As a consequence of Proposition 2.4(ii), we can always factor off the nullspace of $\mathscr{L}$ and study the maximal regularity for $v \in L^r(0,\tau;\mathcal{X})$ such that $v(s) \in \overline{\mathrm{R}(\mathscr{L})}$, for a.e. $s \in (0,\tau)$. Indeed, if $v(s) \in \mathrm{N}(\mathscr{L})$ for a.e. $s \in (0,\tau)$, then the mild solution

$$u(t) = \int_0^t v(s)\,\mathrm{d}s, \qquad t \in (0,\tau),$$

clearly belongs to $W^{1,r}(0,\tau;\mathcal{X})$. Moreover, since $v(s)$ lies in the null space of $\mathscr{L}$, for all $\zeta \in \varrho(\mathscr{L})$ we have

$$(\zeta - \mathscr{L})^{-1} v(s) = \frac{v(s)}{\zeta}.$$

Hence, by the functional calculus representation (2.11), together with the residue theorem and Cauchy's theorem, it follows that

$$T_h v(s) = v(s), \qquad h > 0.$$

Consequently,

$$u(t) = \int_0^t v(s)\,\mathrm{d}s \in \mathrm{N}(\mathscr{L}), \qquad t \in (0,\tau),$$

which, in particular, gives $u \in L^r(0,\tau;\mathrm{D}(\mathscr{L}))$.

Therefore, the component in $\mathrm{N}(\mathscr{L})$ of the prescribed data does not affect maximal regularity.

A central result due to Dore and Venni shows that maximal regularity can be deduced from spectral properties of $\mathscr{L}$, namely from the boundedness of its imaginary powers. This connection is valid under a structural assumption on the underlying Banach space, known as the *UMD property*, which we recall below.

Definition 2.28. A Banach space $\mathcal{X}$ is said to be *UMD* if for every σ-finite measure space (Ω, μ) and all $p \in (1, \infty)$ there exists a constant $C > 0$ such that for all finite martingale $(f_k)_{k=0}^n$ in $L^p(S; \mathcal{X})$ and scalars $|\varepsilon_k| = 1$, we have

$$\left\| \sum_{k=1}^{n} \varepsilon_k (f_k - f_{k-1}) \right\|_{L^p(\Omega;\mathcal{X})} \leqslant C \left\| \sum_{k=1}^{n} (f_k - f_{k-1}) \right\|_{L^p(\Omega;\mathcal{X})}.$$

Every UMD Banach space is reflexive, and standard examples include all reflexive Lebesgue spaces (see, e.g., [51]).

Theorem 2.29 (Dore-Venni)**.** *Let $\mathscr{L}$ be an injective sectorial operator on a UMD Banach space $\mathcal{X}$. Suppose that $\mathscr{L}$ has BIP with $\vartheta(\mathscr{L}) < \pi/2$. Then $\mathscr{L}$ has maximal regularity.*

Proof. See the proof in [32]. □

2.4 Sesquilinear forms

In this thesis, we focus on divergence-form operators with bounded measurable coefficients defined on arbitrary domains in Euclidean space. The framework of *sesquilinear forms* offers a natural and effective tool for constructing such operators and associating them with analytic semigroups on L^2. This approach has the advantage of avoiding the reliance on advanced results from partial differential equation theory or on Sobolev embedding properties, which typically require boundary smoothness. Moreover, a distinctive feature of this method is that many key properties of the generated semigroups can be expressed in terms of concrete, verifiable inequalities involving their associated forms.

This section is devoted to introducing the notion of sesquilinear form and of the associated operator, together with the main definitions and fundamental results related to them, without entering into the full details of the proofs. The material presented here is primarily based on [72]; see also [53].

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. Let $\mathfrak{a}$ be a sesquilinear form defined on a linear subspace $\mathrm{D}(\mathfrak{a})$ of $\mathcal{H}$, called the domain of $\mathfrak{a}$. That is,

$$\mathfrak{a} : \mathrm{D}(\mathfrak{a}) \times \mathrm{D}(\mathfrak{a}) \to \mathbb{C}$$

is a map which satisfies

$$\mathfrak{a}(\alpha u + v, w) = \alpha \mathfrak{a}(u, w) + \mathfrak{a}(u, w),$$
$$\mathfrak{a}(u, \alpha v + w) = \overline{\alpha} \mathfrak{a}(u, v) + \mathfrak{a}(u, w),$$

for every $\alpha \in \mathbb{C}$ and $u, v, w \in \mathrm{D}(\mathfrak{a})$.

Definition 2.30. Let $\mathfrak{a} : \mathrm{D}(\mathfrak{a}) \times \mathrm{D}(\mathfrak{a}) \to \mathbb{C}$ be a sesquilinear form. We say that

(i) $\mathfrak{a}$ is *densely defined* if

$$\mathrm{D}(\mathfrak{a}) \text{ is dense in } \mathcal{H}. \tag{2.18}$$

(ii) $\mathfrak{a}$ is *accretive* if

$$\operatorname{Re} \mathfrak{a}(u, u) \geqslant 0, \qquad \forall u \in \mathrm{D}(\mathfrak{a}). \tag{2.19}$$

(iii) $\mathfrak{a}$ is *continuous* if there exists a non-negative constant M such that

$$|\mathfrak{a}(u,v)| \leqslant M\|u\|_{\mathfrak{a}}\|v\|_{\mathfrak{a}}, \qquad \forall u,v \in \mathrm{D}(\mathfrak{a}) \tag{2.20}$$

where $\|u\|_{\mathfrak{a}} := \sqrt{\mathrm{Re}\,\mathfrak{a}(u,u) + \|u\|^2}$.

(iv) $\mathfrak{a}$ is *closed* if

$$(\mathrm{D}(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}}) \text{ is a complete space.} \tag{2.21}$$

If $\mathfrak{a}$ satisfies (2.18)-(2.21), then $\|\cdot\|_{\mathfrak{a}}$ is a norm on $\mathrm{D}(\mathfrak{a})$. It is called the norm associated with the form $\mathfrak{a}$.

Definition 2.31. Let $\mathfrak{a}$ be a sesquilinear form on $\mathcal{H}$. The *adjoint form* of $\mathfrak{a}$ is the sesquilinear form $\mathfrak{a}^*$ defined by:

$$\mathfrak{a}^*(u,v) = \overline{\mathfrak{a}(v,u)}, \quad \mathrm{D}(\mathfrak{a}^*) = \mathrm{D}(\mathfrak{a}).$$

We say that $\mathfrak{a}$ is a symmetric form if $\mathfrak{a}^* = \mathfrak{a}$, that is,

$$\mathfrak{a}(u,v) = \overline{\mathfrak{a}(v,u)},$$

for all $u, v \in \mathrm{D}(\mathfrak{a})$.

Definition 2.32. For any form $\mathfrak{a}$, the two forms $\mathfrak{h}$ and $\mathfrak{k}$, defined by

$$\mathfrak{h} = \frac{1}{2}(\mathfrak{a} + \mathfrak{a}^*), \quad \mathfrak{k} = \frac{1}{2i}(\mathfrak{a} - \mathfrak{a}^*), \quad \mathrm{D}(\mathfrak{h}) = \mathrm{D}(\mathfrak{k}) = \mathrm{D}(\mathfrak{a}).$$

will be called the *real* and *imaginary parts* of $\mathfrak{a}$, respectively, and be denoted by $\mathfrak{h} = \mathrm{Re}\,\mathfrak{a}$, $\mathfrak{k} = \mathrm{Im}\,\mathfrak{a}$.

This notation is justified by

$$\mathfrak{h}(u,u) = \mathrm{Re}\,\mathfrak{a}(u,u), \qquad \mathfrak{k}(u,u) = \mathrm{Im}\,\mathfrak{a}(u,u), \qquad u \in \mathrm{D}(\mathfrak{a}),$$

although $\mathfrak{h}(u,v)$ and $\mathfrak{k}(u,v)$ are not real-valued and have nothing to do with $\mathrm{Re}\,[\mathfrak{a}(u,v)]$ and $\mathrm{Im}\,[\mathfrak{a}(u,v)]$. Both are symmetric and

$$\mathfrak{a} = \mathfrak{h} + i\mathfrak{k}.$$

A stronger assumption than continuity is sectoriality, which we introduce in the following definition.

Definition 2.33. A sesquilinear form $\mathfrak{a}$, acting on a complex Hilbert space $\mathcal{H}$, is called *sectorial* of angle $\omega \in [0, \pi/2)$ if

$$|\mathrm{Im}\,\mathfrak{a}(u,u)| \leqslant (\tan\omega)\mathrm{Re}\,\mathfrak{a}(u,u) \text{ for all } u \in \mathrm{D}(\mathfrak{a}). \tag{2.22}$$

A family of sesquilinear forms $(\mathfrak{a}_n)_n$ is called *uniformly sectorial* of angle $\omega \in [0, \pi/2)$ if each $\mathfrak{a}_n$ is sectorial of angle ω.

The *numerical range* of $\mathfrak{a}$ is the set

$$\mathcal{N}(\mathfrak{a}) := \{\mathfrak{a}(u,u) : u \in \mathrm{D}(\mathfrak{a}), \|u\| = 1\}.$$

Clearly, $\mathfrak{a}$ satisfies (2.22) if and only if the numerical range $\mathcal{N}(\mathfrak{a})$ is contained in the closed sector $\overline{\mathbf{S}}_\omega$.

Proposition 2.34. *Every sectorial form acting on a complex Hilbert space $\mathcal{H}$ is continuous.*

Proof. See the proof of [72, Proposition 1.8]. □

2.4.1 Associated operator

Let $\mathfrak{a}$ be a densely defined, accretive, continuous, and closed sesquilinear form on $\mathcal{H}$. One can define in terms of $\mathfrak{a}$ an unbounded operator $\mathscr{L}$, defined on a linear subspace $\mathrm{D}(\mathscr{L})$ of $\mathcal{H}$ as follows:

$u \in \mathrm{D}(\mathfrak{a})$ is in the domain $\mathrm{D}(\mathscr{L})$ of $\mathscr{L}$, if and only if there exists $v \in \mathcal{H}$ such that the equality $\mathfrak{a}(u,w) = \langle v, w\rangle$ holds for all $w \in \mathrm{D}(\mathfrak{a})$. We then set $\mathscr{L}u := v$.

We rewrite this as

$$\begin{aligned}\mathrm{D}(\mathscr{L}) &= \{u \in \mathcal{H} : \exists v \in \mathcal{H} \text{ s.t. } \mathfrak{a}(u,w) = \langle v, w\rangle \,\forall w \in \mathrm{D}(\mathfrak{a})\},\\ \mathscr{L}u &= v.\end{aligned}$$

Observe also that $\mathrm{D}(\mathscr{L})$ is the set of vectors $u \in \mathrm{D}(\mathfrak{a})$ for which the mapping $w \to \mathfrak{a}(u,w)$ is continuous on $\mathrm{D}(\mathfrak{a})$ with respect to the norm of $\mathcal{H}$.

Definition 2.35. The linear operator $\mathscr{L}$, defined above, is called the *operator associated* with the form $\mathfrak{a}$.

There are several important properties of operators which are associated with the sesquilinear forms. We start with the following result.

Proposition 2.36. *Denote by $\mathscr{L}$ the operator associated with a densely defined, accretive, continuous, and closed sesquilinear form $\mathfrak{a}$. Then $\mathscr{L}$ is densely defined and for every $s > 0$, the operator $s + \mathscr{L}$ is invertible and its inverse $(s+\mathscr{L})^{-1}$ is a bounded operator on $\mathscr{L}$. In addition,*

$$\|s(s+\mathscr{L})^{-1} f\| \leqslant \|f\|, \qquad \forall s > 0,\, \forall f \in \mathcal{H}.$$

Proof. See the proof of [72, Proposition 1.22]. □

It is clear that the operator associated with a sectorial form is a Kato–sectorial operator. The converse also holds; see [72, Proposition 1.27]. In particular, assuming that the form is sectorial allows us to obtain stronger information on the associated operator and its resolvent than that provided by Proposition 2.36. We formulate this in the following theorem.

Theorem 2.37. *Let $\mathfrak{a}$ be a densely defined, accretive, continuous, and closed sesquilinear form acting on a complex Hilbert space $\mathcal{H}$. Suppose that $\mathfrak{a}$ is sectorial of angle ω. Denote by $\mathscr{L}$ the operator associated with $\mathfrak{a}$.*

(i) *The operator $\mathscr{L}$ is sectorial of angle ω and satisfies the resolvent estimates*

$$\|(\zeta - \mathscr{L})^{-1}\| \leqslant \frac{1}{\operatorname{dist}(\zeta, \overline{\mathbf{S}}_\omega)}, \qquad \forall \zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\omega, \tag{2.23}$$

$$\|(\zeta - \mathscr{L})^{-1}\| \leqslant \frac{1}{\sin(\varphi - \omega)} |\zeta|^{-1}, \qquad \forall \zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\varphi,\, \forall \varphi \in (\omega, \pi). \tag{2.24}$$

In particular, if a sequence $(\mathfrak{a}_i)_i$ of densely defined, accretive, continuous, and closed sesquilinear forms is uniformly sectorial of angle ω, then the corresponding sequence $(\mathscr{L}_i)_i$ of associated operators is uniformly sectorial of angle ω.

(ii) *The operator $-\mathscr{L}$ generates a strongly continuous semigroup $(T_t)_{t>0}$ on $\mathcal{H}$ which is analytic and contractive on the sector $\mathbf{S}_{\pi/2-\omega}$.*

Proof. From the sectoriality of $\mathfrak{a}$, it follows that $\mathscr{L}$ is Kato-sectorial of angle ω. By combining Proposition 2.3 with Proposition 2.36, we deduce that $\mathscr{L}$ is sectorial of angle ω and that estimates (2.23) and (2.24) hold true. The uniform version follows from the previous one and (2.24).

By Theorem 2.9, the sectoriality of $\mathscr{L}$ implies that $-\mathscr{L}$ generates a bounded holomorphic semigroup on $\mathbf{S}_{\pi/2-\omega}$. It remains to show that T_z is a contraction operator for every $z \in \mathbf{S}_{\pi/2-\omega}$. Fix ϑ such that $|\vartheta| \in [0, \pi/2 - \omega)$ and consider the semigroup $(T_{te^{i\vartheta}})_{t>0}$, whose generator is $-e^{i\vartheta}\mathscr{L}$. For every $u \in \mathcal{H}$ and $t > 0$ we have

$$\begin{aligned}
\frac{d}{dt}\|T_{te^{i\vartheta}}u\|^2 &= 2\,\mathrm{Re}\left\langle \frac{d}{dt}T_{te^{i\vartheta}}u, T_{te^{i\vartheta}}u\right\rangle \\
&= -2\,\mathrm{Re}\left\langle e^{i\vartheta}\mathscr{L}T_{te^{i\vartheta}}u, T_{te^{i\vartheta}}u\right\rangle \\
&= -2\left[\mathrm{Re}\,\langle \mathscr{L}T_{te^{i\vartheta}}u, T_{te^{i\vartheta}}u\rangle\cos\vartheta - \mathrm{Im}\,\langle \mathscr{L}T_{te^{i\vartheta}}u, T_{te^{i\vartheta}}u\rangle\sin\vartheta\right] \\
&\leqslant -2\left(\frac{\cos\vartheta}{\tan\omega} - |\sin\vartheta|\right)|\mathrm{Im}\,\langle \mathscr{L}T_{te^{i\vartheta}}u, T_{te^{i\vartheta}}u\rangle|\,.
\end{aligned}$$

Since $|\vartheta| \in [0, \pi/2 - \omega)$, it follows that $\cos\vartheta/\tan\omega - |\sin\vartheta| > 0$. This shows that $\|T_{te^{i\vartheta}}u\|^2$ is non-increasing as a function of t. Hence,

$$\|T_{te^{i\vartheta}}u\| \leqslant \|u\|, \quad \forall t > 0,$$

which completes the proof. □

Note that if the sesquilinear form $\mathfrak{a}$ satisfies (2.18)-(2.21), then the adjoint form $\mathfrak{a}^*$ satisfies the same conditions. One then associates an operator with $\mathfrak{a}^*$. It turns out that this operator is the adjoint of $\mathscr{L}$.

Proposition 2.38. *The operator associated with $\mathfrak{a}^*$ is $\mathscr{L}^*$. In particular, if $\mathfrak{a}$ is symmetric then $\mathscr{L}$ is self-adjoint.*

Proof. See the proof of [72, Proposition 1.24]. □

2.4.2 L^p-dissipativity and L^p-contractivity

One of the main objectives of this thesis is to determine under which conditions divergence-form operators generate semigroups that are contractive on L^p. This property will be established through the study of the associated sesquilinear form, using a characterization due to Nittka, stated below.

Let (Ω, μ) be a measure space, and let $\mathfrak{b}$ be a sesquilinear form defined on a domain $\mathrm{D}(\mathfrak{b}) \subset L^2 = L^2(\Omega)$. For $1 < p < \infty$, define

$$\mathrm{D}_p(\mathfrak{b}) := u \in \mathrm{D}(\mathfrak{b}) : |u|^{p-2}u \in \mathrm{D}(\mathfrak{b}). \tag{2.25}$$

We say that $\mathfrak{b}$ is *L^p-dissipative* if

$$\mathrm{Re}\,\mathfrak{b}(u, |u|^{p-2}u) \geqslant 0, \quad \forall u \in \mathrm{D}_p(\mathfrak{b}).$$

The notion of L^p-dissipativity for sesquilinear forms was first introduced by Cialdea and Maz'ya in [24] for forms defined on $C_c^1(\Omega)$, and later extended by Carbonaro and Dragičevi'c in [20, Definition 7.1].

The following theorem, due to Nittka [69], plays a crucial role in establishing the L^p-contractivity of semigroups. We reproduce it in the form presented in [21, Theorem 2.2].

Theorem 2.39 (Nittka)**.** *Let (Ω, μ) be a measure space. Suppose that the sesquilinear form $\mathfrak{a}$ on $L^2 = L^2(\Omega, \mu)$ is densely defined, accretive, continuous and closed. Let $\mathscr{L}$ be the operator associated with $\mathfrak{a}$.*

Take $p \in (1, \infty)$ and define $B^p := \{u \in L^2 \cap L^p : \|u\|_p \leqslant 1\}$. Let $\mathbf{P}_{B^p}$ be the orthogonal projection $L^2 \to B^p$. Then the following assertions are equivalent:

- $\|\exp(-t\mathscr{L})f\|_p \leqslant \|f\|_p$ *for all* $f \in L^2 \cap L^p$ *and all* $t \geqslant 0$*;*
- $\mathrm{D}(\mathfrak{a})$ *is invariant under* $\mathbf{P}_{B^p}$ *and* $\mathfrak{a}$ *is* L^p*-dissipative.*

Proof. See the proof of [69, Theorem 4.1]. □

2.4.3 A monotone convergence theorem

In many applications, it is often convenient to approximate a given operator by a family of "better" operators—typically simpler or more regular ones—for which the desired property can be more easily established. To make such an approximation useful, one must then be able to recover the corresponding property for the original operator by means of a suitable convergence argument. These approximations are usually carried out at the level of the coefficients (for instance by regularization or truncation), and thus in terms of the associated sesquilinear forms.

The aim of this section is to extend the argument used in [21, Section 3.4] to a more general setting. This level of generality will allow us to apply the same reasoning to the two classes of divergence-form operators considered in Chapters 4 and 5, thereby covering both cases within a unified framework and avoiding unnecessary repetition.

Accordingly, we remain in an abstract setting and show first how the convergence of a sequence of sectorial sesquilinear forms, whose real and imaginary parts form monotone sequences, implies the convergence of the resolvents of the corresponding associated operators, as done in [21, Lemma 3.8]. The idea is to combine classical monotone convergence theorems for sequences of symmetric forms, summarized below, with a vector-valued version of Vitali's theorem [3, Theorem A.5].

Let $\mathfrak{b}_1$ and $\mathfrak{b}_2$ be two symmetric and accretive forms on $\mathcal{H}$. The notation $\mathfrak{b}_1 \geqslant \mathfrak{b}_2$ means that $\mathrm{D}(\mathfrak{b}_1) \subseteq \mathrm{D}(\mathfrak{b}_2)$ and $\mathfrak{b}_1(u, u) \geqslant \mathfrak{b}_2(u, u)$ for all $u \in \mathrm{D}(\mathfrak{b}_1)$.

Definition 2.40. A sequence of $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ of symmetric and accretive forms is *nonincreasing* (*nondecreasing*) if $\mathfrak{b}_n \geqslant \mathfrak{b}_{n+1}$ ($\mathfrak{b}_n \leqslant \mathfrak{b}_{n+1}$) for all $n \in \mathbb{N}$.

We now combine [53, Theorem 3.11] and [53, Theorem 3.13a] into a single statement. The original theorems are formulated under more general assumptions, but we present them here in this simplified form, which is sufficient for the applications we will consider later.

Theorem 2.41. *Let $\mathfrak{b}$ be a densely defined, accretive, closed and symmetric forms on $\mathcal{H}$. Let $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ be a sequence of densely defined, accretive, closed and symmetric forms on $\mathcal{H}$ such that*

$$\mathrm{D}(\mathfrak{b}) \subseteq \mathrm{D}(\mathfrak{b}_n) = \mathrm{D}(\mathfrak{b}_0), \qquad \forall\, n \in \mathbb{N},$$
$$\lim_{n\to\infty} \mathfrak{b}_n(u, u) = \mathfrak{b}(u, u), \qquad \forall u \in \mathrm{D}(\mathfrak{b}).$$

Suppose that one of the following conditions holds:

(i) $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ *is nondecreasing;*

(ii) $(\mathfrak{b}_n)_{n\in\mathbb{N}}$ *is nonincreasing and* $\mathrm{D}(\mathfrak{b}) = \mathrm{D}(\mathfrak{b}_n)$ *for all* $n \in \mathbb{N}$.

Let $\mathscr{L}$ and $\mathscr{L}_n$ be the associated operators with $\mathfrak{b}$ and $\mathfrak{b}_n$, respectively. Then, for all $f \in \mathcal{H}$ and $s > 0$,

$$(s + \mathscr{L}_n)^{-1} f \to (s + \mathscr{L})^{-1} f \quad \text{in } \mathcal{H}, \quad \text{as } n \to \infty.$$

Proof. See [53, Theorem3.11, p. 459] or [81, Theorem 3.2] for the case of nonincreasing sequences and [53, Theorem 3.13a, p. 461] or [81, Theorem 3.1] for nondecreasing sequences. □

The following corollary is modeled after [21, Lemma 3.8], whose proof is based on idea of Ouhabaz [71]. In [21] the result is stated for divergence-form operators with nonnegative potentials.

Corollary 2.42. *Let $\mathfrak{a}$ be a densely defined, closed, and sectorial sesquilinear form on $\mathcal{H}$ of angle ω. Let $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ be a sequence of densely defined, closed, and uniformly sectorial forms on $\mathcal{H}$ of the same angle ω, such that*

$$\begin{aligned} \mathrm{D}(\mathfrak{a}) \subseteq \mathrm{D}(\mathfrak{a}_n) = \mathrm{D}(\mathfrak{a}_0), &\quad \forall n \in \mathbb{N}, \\ \mathfrak{a}_n(u,u) \to \mathfrak{a}(u,u), &\quad \forall u \in \mathrm{D}(\mathfrak{a}). \end{aligned}$$

Suppose that one of the following conditions holds:

(i) *both $(\operatorname{Re}\mathfrak{a}_n)_{n\in\mathbb{N}}$ and $(\operatorname{Im}\mathfrak{a}_n)_{n\in\mathbb{N}}$ are nondecreasing;*

(ii) *both $(\operatorname{Re}\mathfrak{a}_n)_{n\in\mathbb{N}}$ and $(\operatorname{Im}\mathfrak{a}_n)_{n\in\mathbb{N}}$ are nonincreasing, and $\mathrm{D}(\mathfrak{a}_n) = \mathrm{D}(\mathfrak{a})$ for all $n \in \mathbb{N}$;*

(iii) *$(\operatorname{Re}\mathfrak{a}_n)_{n\in\mathbb{N}}$ is nondecreasing, and $(\operatorname{Im}\mathfrak{a}_n)_{n\in\mathbb{N}}$ is nonincreasing;*

(iv) *$(\operatorname{Re}\mathfrak{a}_n)_{n\in\mathbb{N}}$ is nonincreasing, $(\operatorname{Im}\mathfrak{a}_n)_{n\in\mathbb{N}}$ is nondecreasing, and $\mathrm{D}(\mathfrak{a}_n) = \mathrm{D}(\mathfrak{a})$ for all $n \in \mathbb{N}$.*

Let $\mathscr{L}$ and $\mathscr{L}_n$ be the operators associated with $\mathfrak{a}$ and $\mathfrak{a}_n$, respectively. Then, for every $f \in \mathcal{H}$ and every $s > 0$,

$$(s + \mathscr{L}_n)^{-1} f \longrightarrow (s + \mathscr{L})^{-1} f \quad \text{in } \mathcal{H}, \qquad \text{as } n \to \infty.$$

Proof. The sectoriality of $\mathfrak{a}$ and the uniform sectoriality of the sequence $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ implies that $\mathscr{L}$ and each operator $\mathscr{L}_n$ are sectorial of angle ω by Theorem 2.37(i). For every $z \in \mathbb{C}$ and $n \in \mathbb{N}$, define the sesquilinear forms $\mathfrak{a}_z$ and $\mathfrak{a}_{n,z}$ by

$$\begin{aligned} \mathfrak{a}_z &:= \operatorname{Re}\mathfrak{a} + z \operatorname{Im}\mathfrak{a}, & \mathrm{D}(\mathfrak{a}_z) &= \mathrm{D}(\mathfrak{a}), \\ \mathfrak{a}_{n,z} &:= \operatorname{Re}\mathfrak{a}_n + z \operatorname{Im}\mathfrak{a}_n, & \mathrm{D}(\mathfrak{a}_{n,z}) &= \mathrm{D}(\mathfrak{a}_n). \end{aligned}$$

It is straightforward to verify that these forms are densely defined, closed and sectorial for all $z \in \mathcal{O} := \{z \in \mathbb{C} : |\operatorname{Re} z| < \delta\}$ and $n \in \mathbb{N}$, where $\delta = \cot\omega$. Moreover, they are also symmetric if $z \in (-\delta, \delta)$. Note in particular that $\mathfrak{a} = \mathfrak{a}_i$.

Let $\mathscr{L}_z$ and $\mathscr{L}_{n,z}$ denote the operators associated with $\mathfrak{a}_z$ and $\mathfrak{a}_{n,z}$, respectively. From Proposition 2.36 we know that the operator $s + \mathscr{L}_z$ is invertible and satisfies the estimate $\|(s + \mathscr{L}_z)^{-1} f\|_2 \leqslant \|f\|/s$, for $z \in \mathcal{O}$ and $s > 0$. A theorem due to Kato [53, p. 395], recently reproven in [84], shows that $z \mapsto (s + \mathscr{L}_z)^{-1}$ is holomorphic as a map from $\mathcal{O}$ to the space of bounded linear operators on $\mathcal{H}$. The same holds for the map $z \to (s + \mathscr{L}_{n,z})^{-1}$.

On the other hand, the sequence $(\mathfrak{a}_{n,z})_{n\in\mathbb{N}}$ of densely defined, closed and sectorial symmetric forms is

- nondecreasing for all $z \in [0, \delta)$, under the assumption (i);
- nonincreasing for all $z \in [0, \delta)$ with $\mathrm{D}(\mathfrak{a}_z) = \mathrm{D}(\mathfrak{a}_{n,z})$, under the assumption (ii);

- nondecreasing for all $z \in (-\delta, 0]$, under the assumption (iii);
- nonincreasing for all $z \in (-\delta, 0]$ with $\mathrm{D}(\mathfrak{a}_z) = \mathrm{D}(\mathfrak{a}_{n,z})$, under the assumption (iv).

Therefore, by applying Theorem 2.41, we deduce the existence of a subset $\mathcal{U} \subseteq \mathcal{O}$ with accumulation points in $\mathcal{O}$ such that for all $s > 0$, $z \in \mathcal{U}$ and $f \in \mathcal{H}$ we have

$$(s + \mathscr{L}_{n,z})^{-1} f \to (s + \mathscr{L}_z)^{-1} f \quad \text{in } \mathcal{H}, \quad \text{as } m \to \infty. \tag{2.26}$$

The holomorphy of the maps $z \to (s+\mathscr{L}_z)^{-1}$ and $z \to (s+\mathscr{L}_{n,z})^{-1}$, together with the convergence in (2.26) and a vector-valued version of Vitali's theorem [3, Theorem A.5] imply that $(s + \mathscr{L}_{n,z})^{-1} f \to (s + \mathscr{L}_z)^{-1} f$ for all $z \in \mathcal{O}$. In particular, taking $z = i$, we obtain the desired conclusion. □

An application: sequence of potentials

We now leave the abstract setting and consider a more concrete, yet still fairly general, situation: sequences of sesquilinear forms on $L^2(\Omega)$, where Ω is an arbitrary open subset of $\mathbb{R}^d$, consisting of a fixed part—independent of the sequence parameter—plus a sequence of functions.

The motivation for this analysis comes from the main goal of this thesis: to establish a bilinear embedding for divergence-form operators, possibly with potentials. As we shall see in the next chapters, it will be convenient to approximate the potentials by a sequence of "well-behaved" potentials, prove the bilinear embedding in that setting, and then handle the general case by following the argument developed by Carbonaro and Dragičević in [21, Section 3.4]. In their work, nonnegative potentials were truncated, the bilinear embedding was proved for bounded potentials, and the general case with unbounded potentials was deduced from this approximation argument.

Here, we extend the argument of [21, Section 3.4] to a more general setting—general enough to cover the cases of interest in Chapters 4 and 5.

Let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary open. Let $\mathfrak{b}$ be a densely defined sesquilinear form on $L^2(\Omega)$ such that $W_0^{1,2}(\Omega) \subseteq \mathrm{D}(\mathfrak{b}) \subseteq W^{1,2}(\Omega)$. Assume the following condition holds:

(B1) For every sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathrm{D}(\mathfrak{b})$ and $u \in \mathrm{D}(\mathfrak{b})$ such that

$$\begin{aligned} \nabla u_n &\rightharpoonup \nabla u \qquad \text{in } L^2(\Omega, \mathbb{C}^d), \\ u_n &\to u \qquad \text{in } L^2(\Omega), \end{aligned} \qquad \text{as } n \to \infty.$$

we have

$$\begin{aligned} \mathfrak{b}(u_n, u) &\to \mathfrak{b}(u, u), \\ \mathfrak{b}(u, u_n) &\to \mathfrak{b}(u, u), \end{aligned} \qquad \text{as } n \to \infty.$$

Here the symbol $\rightharpoonup$ denotes weak convergence (that is, convergence in the weak topology).

Let $\varrho \in \mathbb{C}$, $V \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R})$ and $(V_n)_{n\in\mathbb{N}}$ be a sequence of functions in $L^1_{\mathrm{loc}}(\Omega, \mathbb{R})$. Define the sesquilinear forms $\mathfrak{a}$ and $\mathfrak{a}_n$ by

$$\begin{aligned} \mathfrak{a}(u, v) &= \mathfrak{b}(u, v) + \int_\Omega \varrho V u \overline{v}, \\ \mathfrak{a}_n(u, v) &= \mathfrak{b}(u, v) + \int_\Omega \varrho V_n u \overline{v}, \end{aligned}$$

with domains

$$\begin{aligned} \mathrm{D}(\mathfrak{a}) &= \{u \in \mathrm{D}(\mathfrak{b}) : \int_\Omega |V||u|^2 < \infty\}, \\ \mathrm{D}(\mathfrak{a}_n) &= \{u \in \mathrm{D}(\mathfrak{b}) : \int_\Omega |V_n||u|^2 < \infty\}. \end{aligned}$$

The forms $\mathfrak{a}$ and $\mathfrak{a}_n$ are densely defined, since their domains contain $C_c^\infty(\Omega)$. For notational convenience, we set $\mathfrak{a}_\infty := \mathfrak{a}$ and $V_\infty := V$.

Assume the following conditions hold:

(V1) $|V_n| \leqslant |V|$ for all $n \in \mathbb{N}$;

(V2) either $V_n \nearrow V$ a.e. on Ω and $\operatorname{Re} \varrho \geqslant 0$, or $V_n \searrow V$ a.e. on Ω and $\mathrm{D}(\mathfrak{a}) = \mathrm{D}(\mathfrak{a}_n)$ for all $n \in \mathbb{N}$;

(A1) the form $\mathfrak{a}_n$ is closed for every $n \in \mathbb{N} \cup \{\infty\}$ and the sequence $(\mathfrak{a}_n)_{n\in\mathbb{N}\cup\{\infty\}}$ is uniformly sectorial of angle ω;

(A2) there exists a constant $C > 0$ such that

$$\operatorname{Re} \mathfrak{a}_n(u,u) \geqslant C \left(\|\nabla u\|_2^2 + \||V_n|^{1/2} u\|_2^2 \right),$$

for all $u \in \mathrm{D}(\mathfrak{a}_n)$ and all $n \in \mathbb{N} \cup \{\infty\}$.

For every $n \in \mathbb{N}\cup\{\infty\}$, we denote by $\mathscr{L}_n$ the associated operator with $\mathfrak{a}_n$. Since $(\mathfrak{a}_n)_{n\in\mathbb{N}\cup\{\infty\}}$ is uniformly sectorial of angle ω, it follows from Theorem 2.37 that

$$\|(\zeta - \mathscr{L}_n)^{-1}\| \leqslant \frac{1}{\operatorname{dist}(\zeta, \overline{\mathbf{S}}_\omega)}, \qquad \zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\omega,\ n \in \mathbb{N} \cup \{\infty\}, \tag{2.27}$$

and that $-\mathscr{L}_n$ generates an analytic and contractive semigroup $(T_{n,z})_{z\in\mathbf{S}_\vartheta}$ on $L^2(\Omega)$, with $\vartheta := \omega - \pi/2$.

We now state the theorem that will allow us, in the following chapters, to deduce the bilinear embedding for classes of divergence-form operators with possibly unbounded potentials. It extends [21, Theorem 3.6].

Theorem 2.43. *Let $\varrho \in \mathbb{C}$, $(V_n)_{n\in\mathbb{N}\cup\{\infty\}}$ and $(\mathfrak{a}_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be as above. For all $f \in L^2(\Omega)$ and all $z \in \mathbf{S}_\vartheta$ we have*

$$\begin{aligned} \nabla T_{n,z} f &\to \nabla T_z f \quad \text{in} \quad L^2(\Omega, \mathbb{C}^d), \\ |V_n|^{1/2} T_{n,z} f &\to |V|^{1/2} T_z f \quad \text{in} \quad L^2(\Omega), \end{aligned}$$

as $n \to \infty$.

The proof will be given later. Before proceeding, we need the following auxiliary result, which generalizes [21, Proposition 3.9].

Proposition 2.44. *Let $\varrho \in \mathbb{C}$, $(V_n)_{n\in\mathbb{N}\cup\{\infty\}}$ and $(\mathfrak{a}_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be as above. For all $f \in L^2(\Omega)$ and all $\zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\omega$, we have*

$$\begin{aligned} (\zeta - \mathscr{L}_n)^{-1} f &\to (\zeta - \mathscr{L})^{-1} f \quad \text{in} \quad L^2(\Omega), \\ \nabla(\zeta - \mathscr{L}_n)^{-1} f &\to \nabla(\zeta - \mathscr{L})^{-1} f \quad \text{in} \quad L^2(\Omega, \mathbb{C}^d), \\ |V_n|^{1/2}(\zeta - \mathscr{L}_n)^{-1} f &\to |V|^{1/2}(\zeta - \mathscr{L})^{-1} f \quad \text{in} \quad L^2(\Omega), \end{aligned} \tag{2.28}$$

as $n \to \infty$.

Proof. Recall the notation $V_\infty = V$. Fix $f \in L^2(\Omega)$. For $n \in \mathbb{N} \cup \{\infty\}$ and $\zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\omega$ set

$$u_n = u_n(\zeta) := (\mathscr{L}_n - \zeta)^{-1} f \in \mathrm{D}(\mathscr{L}_n) \subseteq \mathrm{D}(\mathfrak{a}) \subseteq W^{1,2}(\Omega).$$

By (A2), for every $n \in \mathbb{N} \cup \{\infty\}$ and $\zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\omega$ we have

$$\begin{aligned} C\left(\|\nabla u_n\|_2^2 + \||V_n|^{1/2} u_n\|_2^2\right) &\leqslant \operatorname{Re} \mathfrak{a}_n(u_n, u_n) \\ &= \operatorname{Re} \int_\Omega (\mathscr{L}_n u_n)\overline{u}_n \\ &= \operatorname{Re} \int_\Omega f\overline{u}_n + (\operatorname{Re}\zeta) \int_\Omega |u_n|^2 \\ &\leqslant \|f\|_2 \|u_n\|_2 + |\operatorname{Re}\zeta| \cdot \|u_n\|_2^2. \end{aligned}$$

Therefore, the uniform sectoriality estimate (2.27) gives, for all $n \in \mathbb{N} \cup \{\infty\}$ and $\zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\omega$,

$$\|u_n(\zeta)\|_2 + \|\nabla u_n(\zeta)\|_2 + \||V_n|^{1/2} u_n(\zeta)\|_2 \leqslant C_\omega(\zeta)\|f\|_2, \tag{2.29}$$

where $C_\omega(\zeta) > 0$ is continuous in ζ.

Now temporarily fix $s > 0$ and set

$$\begin{aligned} u_n &= u_n(-s), \\ u &= u_\infty(-s). \end{aligned}$$

Condition (V2) ensures that the sequences $(\operatorname{Re}\mathfrak{a}_n)_{n\in\mathbb{N}}$ and $(\operatorname{Im}\mathfrak{a}_n)_{n\in\mathbb{N}}$ satisfy one of the assumptions (i)–(iv) of Corollary 2.42. Indeed, if $V_n \nearrow V$ a.e. on Ω and $\operatorname{Re}\varrho \geqslant 0$, then we are in case (i) or (iii), depending on the sign of $\operatorname{Im}\varrho$. Conversely, if $V_n \searrow V$ a.e. on Ω and $\mathrm{D}(\mathfrak{a}) = \mathrm{D}(\mathfrak{a}_n)$ for all $n \in \mathbb{N}$, then one of the cases (i)–(iv) applies, depending on the signs of $\operatorname{Re}\varrho$ and $\operatorname{Im}\varrho$. In combination with (A1), this enables us to apply Corollary 2.42, which reads

$$\lim_{n\to\infty} u_n = u \quad \text{in } L^2(\Omega), \quad \forall s > 0. \tag{2.30}$$

By (2.29), for all $n \in \mathbb{N}$ the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,2}(\Omega)$, hence it admits a weakly convergent subsequence. That is, there exist a subsequence of indices $(n_j)_{j\in\mathbb{N}}$ and function $\omega \in W^{1,2}(\Omega)$ such that

$$u_{n_k} \rightharpoonup \omega \quad \text{in } W^{1,2}(\Omega),$$

as $k \to \infty$. This implies that $\omega = u$. Thus $u_{n_k} \rightharpoonup u$ in $W^{1,2}(\Omega)$.

Again by (2.29), the sequences $\left((V_n)_\pm^{1/2} u_n\right)_{n\in\mathbb{N}}$ are bounded in $L^2(\Omega)$. From (2.30) and a standard theorem, we derive a subsequence $(u_{n_l})_{l\in\mathbb{N}}$ such that $u_{n_l} \to u$ almost everywhere on Ω. Recall that $V_n \to V$ almost everywhere on Ω by (V2). Hence, $(V_{n_l})_\pm^{1/2} u_{n_l} \to (V)_\pm^{1/2} u$ almost everywhere on Ω. Now a well-known theorem [50, Theorem 13.44] gives $(V_{n_l})_\pm^{1/2} u_{n_l} \rightharpoonup (V)_\pm^{1/2} u$ in $L^2(\Omega)$.

So far, we proved that there exists a subsequence of indices $(n_k)_{k\in\mathbb{N}}$ such that in L^2 we have

$$u_n \to u, \quad \nabla u_{n_k} \rightharpoonup \nabla u, \quad (V_{n_k})_\pm^{1/2} u_{n_k} \rightharpoonup (V)_\pm^{1/2} u. \tag{2.31}$$

We now show that the weak convergences in (2.31) are actually in the normed topology of $L^2(\Omega)$.

By (A2),

$$\begin{aligned}
J_{n_k} &:= s\|u_{n_k}-u\|_2^2 + C\left(\|\nabla u_{n_k}-\nabla u\|_2^2 + \left\||V_n|^{1/2}u_{n_k}-|V_n|^{1/2}u\right\|_2^2\right)\\
&\leqslant s\|u_{n_k}\|^2 + s\|u\|^2 - 2s\mathrm{Re}\int_\Omega u_{n_k}\overline{u} + \mathrm{Re}\,\mathfrak{a}_n(u_{n_k}-u,u_{n_k}-u)\\
&= s\|u_{n_k}\|^2 + s\|u\|^2 - 2s\mathrm{Re}\int_\Omega u_{n_k}\overline{u}\\
&\quad + \mathrm{Re}\,\mathfrak{b}(u_{n_k}-u,u_{n_k}-u)\\
&\quad + \mathrm{Re}\,\varrho\int_\Omega V_{n_k}|u_{n_k}-u|^2\\
&= s\|u_{n_k}\|^2 + s\|u\|^2 - 2s\mathrm{Re}\int_\Omega u_{n_k}\overline{u}\\
&\quad + \mathrm{Re}\,\mathfrak{b}(u_{n_k},u_{n_k}) + \mathrm{Re}\,\mathfrak{b}(u,u) - \mathrm{Re}\,\mathfrak{b}(u_{n_k},u) - \mathrm{Re}\,\mathfrak{b}(u,u_{n_k})\\
&\quad + \mathrm{Re}\,\varrho\left[\int_\Omega V_{n_k}|u_{n_k}|^2 + \int_\Omega V_{n_k}|u|^2 - 2\mathrm{Re}\int_\Omega V_{n_k}u_{n_k}\overline{u}\right]\\
&= I_{n_k}^0 + I_{n_k}^1 + I_{n_k}^2 + I_{n_k}^3 + I_{n_k}^{4,+} + I_{n_k}^{4,-},
\end{aligned}$$

where

$$\begin{aligned}
I_{n_k}^0 &= s\|u\|_2^2 + \mathrm{Re}\,\mathfrak{a}_{n_k}(u,u),\\
I_{n_k}^1 &= s\|u_{n_k}\|_2^2 + \mathrm{Re}\,\mathfrak{a}_{n_k}(u_{n_k},u_{n_k}),\\
I_{n_k}^2 &= -2s\mathrm{Re}\int_\Omega u_{n_k}\overline{u} - 2\mathrm{Re}\,\varrho\cdot\mathrm{Re}\left[\int_\Omega (V_{n_k})_+^{1/2}u_{n_k}V_+^{1/2}\overline{u} - \int_\Omega (V_{n_k})_-^{1/2}u_{n_k}V_-^{1/2}\overline{u}\right],\\
I_{n_k}^3 &= -\mathrm{Re}\left(\mathfrak{b}(u_{n_k},u)+\mathfrak{b}(u,u_{n_k})\right),\\
I_{n_k}^{4,+} &= -2\mathrm{Re}\,\varrho\cdot\mathrm{Re}\int_\Omega (V_{n_k})_+^{1/2}u_{n_k}\cdot\left((V_{n_k})_+^{1/2}-V_+^{1/2}\right)\overline{u},\\
I_{n_k}^{4,-} &= 2\mathrm{Re}\,\varrho\cdot\mathrm{Re}\int_\Omega (V_{n_k})_-^{1/2}u_{n_k}\cdot\left((V_{n_k})_-^{1/2}-V_-^{1/2}\right)\overline{u}.
\end{aligned}$$

Notice that (V1) and the fact that $u\in\mathrm{D}(\mathscr{L})$ yield

$$|V_{n_k}||u|^2 \leqslant |V||u|^2 \in L^1(\Omega), \tag{2.32}$$

$$\left|V_\pm^{1/2}-(V_{n_k})_\pm^{1/2}\right|^2|u|^2 \leqslant 4V_\pm|u|^2\in L^1(\Omega). \tag{2.33}$$

Therefore, sending $k\to\infty$, we obtain

$$I_{n_k}^0 \to s\|u\|_2^2 + \mathrm{Re}\,\mathfrak{a}(u,u) = \mathrm{Re}\int_\Omega\left((s+\mathscr{L})u\right)\overline{u} = \mathrm{Re}\int_\Omega f\overline{u},$$

by (2.32) and $u\in\mathrm{D}(\mathscr{L})$;

$$I_{n_k}^1 = \mathrm{Re}\int_\Omega\left((s+\mathscr{L}_{n_k})u_{n_k}\right)\overline{u}_{n_k} = \mathrm{Re}\int_\Omega f\overline{u}_{n_k} \to \mathrm{Re}\int_\Omega f\overline{u},$$

because $u_{n_k}\in\mathrm{D}(\mathscr{L}_{n_k})$ and $u_{n_k}\to u$ in $L^2(\Omega)$;

$$I_{n_k}^2 \to -2s\|u\|_2^2 - 2\mathrm{Re}\,\varrho\left[\|V_+^{1/2}u\|_2^2 - \|V_-^{1/2}u\|_2^2\right],$$

by (2.31), since $u\in\mathrm{D}(\mathfrak{a})$ implies $V_+^{1/2}\overline{u}$, $V_-^{1/2}\overline{u}\in L^2(\Omega)$;

$$I^3_{n_k,} \to -2\mathrm{Re}\,\mathfrak{b}(u,u)$$

by (2.31) and (B1); and finally

$$|I^{4,\pm}_{n_k}| \leqslant 2|\mathrm{Re}\,\varrho| \left(\int_\Omega \left|V_\pm^{1/2} - (V_{n_k})_\pm^{1/2}\right|^2 |u|^2\right)^{1/2} \|(V_{n_k})_\pm^{1/2} u_{n_k}\|_2 \to 0,$$

by the Cauchy-Schwarz inequality, (2.29) and (2.33).

Therefore, using that $u \in \mathrm{D}(\mathscr{L})$, we obtain, as $k \to \infty$,

$$\begin{aligned} I^0_{n_k} + I^1_{n_k} &\to 2\mathrm{Re} \int_\Omega f\overline{u}, \\ I^2_{n_k} + I^3_{n_k} &\to -2\mathrm{Re} \int_\Omega ((s+\mathscr{L}^U)u)\overline{u} = -2\mathrm{Re} \int_\Omega f\overline{u}, \\ I^{4,+}_{n_k} + I^{4,-}_{n_k} &\to 0. \end{aligned}$$

It follows that $J_{n_k} \to 0$ as $k \to \infty$, so

$$\nabla u_{n_k} \to \nabla u \quad \text{in } L^2(\Omega), \tag{2.34}$$

and $\left\||V_n|^{1/2} u_{n_k} - |V_{n_k}|^{1/2} u\right\|_2 \to 0$, as $k \to \infty$. From the latter convergence, (2.33) and the following estimate

$$\||V_{n_k}|^{1/2} u_{n_k} - |V|^{1/2} u\|_2^2 \lesssim \left\||V_{n_k}|^{1/2} u_{n_k} - |V_{n_k}|^{1/2} u\right\|_2^2 + \left\|\left(|V_{n_k}|^{1/2} - |V|^{1/2}\right) u\right\|_2^2,$$

we obtain

$$|V_{n_k}|^{1/2} u_{n_k} \to |V|^{1/2} u \qquad \text{in } L^2(\Omega), \tag{2.35}$$

as $k \to \infty$.

By repeating verbatim the argument following (2.31), we may prove that every *subsequence* of $(u_n)_{n\in\mathbb{N}}$ has its own subsequences for which (2.34) and (2.35) hold. Therefore, by a standard convergence argument involving subsequences, (2.28) holds for all $\zeta = -s$, $s > 0$.

It remains to prove (2.28) for all $\zeta \in (\mathbb{C} \setminus \overline{\mathbf{S}}_\omega) \setminus (-\infty, 0)$. Fix $f \in L^2(\Omega)$ and for each $n \in N \cup \{\infty\}$ consider the function

$$G_n : \mathbb{C} \setminus \overline{\mathbf{S}}_\omega \to L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega; \mathbb{C}^d) =: \mathcal{H}$$

given by

$$G_n(\zeta) = (u_n(\zeta), |V_n|^{1/2} u_n(\zeta), \nabla u_n(\zeta)).$$

By (2.29), the family $\{G_n : n \in \mathbb{N}\}$ is locally uniformly bounded in $\mathbb{C} \setminus \overline{\mathbf{S}}_\omega$. Hence [3, Proposition A.3] implies that the functions G_n are holomorphic, because the complex-valued function $\langle G_n(\cdot), \mathbf{g}\rangle_{\mathcal{H}}$ is holomorphic for all $\mathbf{g}$ from the norming subspace $L^2(\Omega) \times C_c^\infty(\Omega) \times C_c^\infty(\Omega; \mathbb{C}^d)$ of $\mathcal{H}$. Moreover, since we have already proven that $G_n(-s) \to G_\infty(-s)$ in $\mathcal{H}$ for all $s > 0$, it follows from Vitali's theorem [3, Theorem A.5] that the convergence holds true for all $\zeta \in \mathbb{C} \setminus \overline{\mathbf{S}}_\omega$. □

Proof of Theorem 2.43. We use the standard representation of the analytic semigroup $(T_{n,z})_{z\in\mathbf{S}_\vartheta}$, $n \in \mathbb{N} \cup \{\infty\}$, by means of a Cauchy integral (see (2.11)). Fix $z \in \mathbf{S}_\vartheta$ and $\delta > 0$. Choose

$\varphi \in (0, \pi/2)$ such that $|\arg z| < \varphi^* < \vartheta$, and denote by γ the positively oriented boundary of $\mathbf{S}_\varphi \cup \{\zeta \in \mathbb{C} : |\zeta| < \delta\}$. Then

$$\|\nabla T_{n,z} f - \nabla T_z f\|_2 \lesssim \int_\gamma e^{-\operatorname{Re}(z\zeta)} \|\nabla(\zeta - \mathscr{L}_n)^{-1} f - \nabla(\zeta - \mathscr{L})^{-1} f\|_2 \, |\mathrm{d}\zeta|,$$
$$\||V_n|^{1/2} T_{n,z} f - |V|^{1/2} T_z f\|_2 \lesssim \int_\gamma e^{-\operatorname{Re}(z\zeta)} \||V_n|^{1/2}(\zeta - \mathscr{L}_n)^{-1} f - |V|^{1/2}(\zeta - \mathscr{L})^{-1} f\|_2 \, |\mathrm{d}\zeta|.$$

By Proposition 2.44, the integrands converge to zero, as $n \to \infty$. An examination of the constant $C_\omega(\zeta)$ from (2.29) shows that for ζ along the curve γ we have $C_\omega(\zeta) \lesssim 1$ uniformly in $n \in \mathbb{N} \cup \{\infty\}$ and $\zeta \in \gamma$. This means that we can apply the dominated convergence theorem in the two integrals above and complete the proof. □

CHAPTER 3

Second-order divergence-form operators

The purpose of this chapter is to summarize the main notions, ideas, and results from [19–21], so as to make the thesis as self-contained as possible and to familiarize the reader with the tools and methods that will be employed in the subsequent chapters in technically more involved settings.

More specifically, given a uniformly elliptic matrix A defined on an arbitrary subset Ω of $\mathbb{R}^d$, we consider the second-order divergence-form operator

$$\mathscr{L} = -\mathrm{div}(A\nabla\cdot),$$

and the associated semigroup

$$T_t = e^{-t\mathscr{L}}, \qquad t > 0.$$

The main result of [20] is a dimension-free bilinear embedding for a pair of such operators on $\mathbb{R}^d$, which extends the result of Dragičević and Volberg [35]—originally proved for *real* and *equal* coefficients—to the more general setting of *complex* and possibly *different* matrices. The key new idea introduced in [20] is the notion of *p-ellipticity*, a structural condition on the coefficient matrices that naturally interpolates between complex and real ellipticity, and provides a sufficient—and sharp—criterion for the dimension-free bilinear embedding to hold.

In this chapter, particular attention will be devoted to highlighting the technique developed in [19] to extend the bilinear embedding of [20] to arbitrary open sets. These techniques were introduced to overcome an integration by parts arising in the so-called *heat-flow method*. This approach will serve as a guiding framework when addressing analogous questions for divergence-form operators with first-order perturbations, as well as for Schrödinger-type operators with negative potentials. Schrödinger-type operators with non-negative potentials are also treated in [21], but for the sake of clarity we have decided to limit our discussion in this chapter to the case without potentials.

The concept of p-ellipticity is deeply connected with the L^p-contractivity and analyticity of the semigroup $(T_t)_{t>0}$ generated by $\mathscr{L}$. Indeed, thanks to the ellipticity of A, this semigroup is always contractive and analytic on $L^2(\Omega)$; see Section 3.1. A classical question in harmonic analysis is whether these properties persist on $L^p(\Omega)$ for $p \in (1,\infty)\setminus\{2\}$. In other words, under which structural conditions on A does $(T_t)_{t>0}$ extend to a contractive and analytic semigroup on $L^p(\Omega)$?

This question is interesting in its own right, but also plays a key role in the study of the Cauchy problem

$$\begin{cases} u'(t) + \mathscr{L}u(t) = 0, \\ u(0) = f, \end{cases}$$

when the initial datum f lies in $L^p(\Omega)$. In particular, analyticity of the semigroup on $L^p(\Omega)$ ensures the existence and uniqueness of the solution u for all initial data $f \in L^p(\Omega)$. The results of Carbonaro and Dragičević in [19–21] provide a comprehensive answer, revealing the central role of p-ellipticity in this theory.

In Section 3.1, we rigorously introduce divergence-form operators using the theory of sesquilinear forms, specifying the boundary conditions under which these operators are defined. Sections 3.2 and 3.3 are devoted to recalling the notion of p-ellipticity, the concept of generalized convexity, and the connection between them—both introduced in [20]. In Section 3.4, we establish the L^p-contractivity and analyticity of the semigroup associated with these operators. Finally, in Section 3.5 we prove the dimension-free bilinear embedding.

3.1 Divergence-form operators

Let $\Omega \subseteq \mathbb{R}^d$ be a nonempty open set. Denote by $\mathcal{A}(\Omega)$ the class of all complex uniformly strictly elliptic $d \times d$ matrix-valued functions on Ω with L^∞ coefficients (in short, elliptic matrices). That is to say, $\mathcal{A}(\Omega)$ is the class of all measurable $A : \Omega \to \mathbb{C}^{d\times d}$ for which there exist λ, $\Lambda > 0$ such that for almost all $x \in \Omega$ we have

$$\operatorname{Re} \langle A(x)\xi, \xi\rangle \geqslant \lambda|\xi|^2\,, \qquad \forall \xi \in \mathbb{C}^d; \tag{3.1}$$

$$|\langle A(x)\xi, \sigma\rangle| \leqslant \Lambda\,|\xi|\,|\sigma|\,, \qquad \forall \xi, \sigma \in \mathbb{C}^d. \tag{3.2}$$

We denote by $\lambda(A)$ and $\Lambda(A)$ the optimal λ and Λ, respectively.

Suppose that $A \in \mathcal{A}(\Omega)$. Fix a closed subspace $\mathscr{V}$ of the Sobolev space $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$. Denote by $\mathscr{L} = \mathscr{L}_2^{A,\mathscr{V}}$ the unbounded operator on $L^2(\Omega)$ associated with the densely defined, accretive continuous sesquilinear form

$$\mathfrak{a}_{A,\mathscr{V}}(u, v) \;= \int_\Omega \langle A\nabla u, \nabla v\rangle_{\mathbb{C}^d}\,, \qquad \mathrm{D}(\mathfrak{a}_{A,\mathscr{V}}) = \mathscr{V}. \tag{3.3}$$

Namely,

$$\mathrm{D}(\mathscr{L}) := \Big\{u \in \mathrm{D}(\mathfrak{a}) : \exists w \in L^2(\Omega) :\ \mathfrak{a}(u, v) = \langle w, v\rangle_{L^2(\Omega)}\ \ \forall v \in \mathrm{D}(\mathfrak{a})\Big\}$$

and

$$\langle \mathscr{L}u, v\rangle_{L^2(\Omega)} = \int_\Omega \langle A\nabla u, \nabla v\rangle\,, \quad \forall u \in \mathrm{D}(\mathscr{L}), \quad \forall v \in \mathrm{D}(\mathfrak{a})\,. \tag{3.4}$$

Ellipticity of A implies that the form $\mathfrak{a}_{A,\mathscr{V}}$ is Kato-sectorial of angle $\omega = \omega(\lambda, \lambda)$. Therefore, from Theorem 2.37 it follows that $\mathscr{L}$ is sectorial of angle ω and $-\mathscr{L}$ generates of a strongly continuous semigroup on $L^2(\Omega)$

$$T_t = T_t^{A,\mathscr{V}}, \quad t > 0,$$

which is analytic and contractive in the cone $\mathbf{S}_{\pi/2-\omega}$.

Formally, $\mathscr{L}$ is given by the expression

$$\mathscr{L}u = -\operatorname{div}(A\nabla u).$$

3.1.1 Boundary conditions

Here we describe certain classes of closed subspace $\mathscr{V}$ of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$ that satisfy additional conditions which will be imposed throughout this thesis. The exposition follows [21].

We say that the space $\mathscr{V} \subseteq W^{1,2}(\Omega)$ is *invariant* under:

- the function $p : \mathbb{C} \to \mathbb{C}$, if $u \in \mathscr{V}$ implies $p(u) := p \circ u \in \mathscr{V}$;

- the family $\mathscr{P}$ of functions $\mathbb{C} \to \mathbb{C}$, if its invariant under all $p \in \mathscr{P}$.

Define functions $P, T : \mathbb{C} \to \mathbb{C}$ by

$$P(\zeta) = \begin{cases} \zeta; & |\zeta| \leqslant 1, \\ \zeta/|\zeta|; & |\zeta| \geqslant 1, \end{cases}$$
$$T(\zeta) = (\operatorname{Re}\zeta)_+.$$

Thus $P(\zeta) = \min\{1, |\zeta|\}\operatorname{sign}\zeta$, where sign is defined as [72, (2.2)]:

$$\operatorname{sign}\zeta := \begin{cases} \zeta/|\zeta|; & \zeta \neq 0, \\ 0; & \zeta = 0. \end{cases}$$

Let $\mathscr{V}$ be a closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$. We consider the invariance conditions

$$\mathscr{V} \text{ is invariant under the function } P, \tag{3.5}$$
$$\mathscr{V} \text{ is invariant under the function } T, \tag{3.6}$$

and assume either only (3.5) or both (3.5) and (3.6), depending on the context. In general, (3.5) does not imply (3.6); see [72, Example 4.3.2].

It is well know (see [72, Propositions 4.4&4.11], [23, Section 2.1] and [21, Appendix A]) that (3.5) and (3.6) are satisfied in these notable cases:

(a) $\mathscr{V} = W_0^{1,2}(\Omega)$,

(b) $\mathscr{V} = W^{1,2}(\Omega)$,

(c) $\mathscr{V} = \widetilde{W_D}^{1,2}(\Omega)$, the closure in $W^{1,2}(\Omega)$ of $\{u \in W^{1,2}(\Omega) : \operatorname{dist}(\operatorname{supp} u, D) > 0\}$, where D is a (possibly empty) closed subset of $\partial\Omega$,

(d) $\mathscr{V} = W_D^{1,2}(\Omega)$, the closure in $W^{1,2}(\Omega)$ of $\{u_{|\Omega} : u \in C_c^\infty(\mathbb{R}^d \setminus D)\}$, where D is a (possibly empty) proper closed subset of $\partial\Omega$.

When $\mathscr{V}$ falls into any of the special cases (a)-(d), we say that $\mathscr{L}$ is subject to (a) *Dirichlet*, (b) *Neumann*, (c) *mixed*, or (d) *good mixed boundary conditions*.

When working with a pair of spaces $\mathscr{V}$ and $\mathscr{W}$, we will sometimes need to select them appropriately from the spaces listed above. Certain combinations will not be considered. The following assumption is introduced because [19, Lemma 19] may fail if $(\mathscr{V}, \mathscr{W})$ does not satisfy it. In particular, if one space is of type (d), the other must belong either to the same class or to that described in (a). See Remark 3.28. We will formulate and prove the bilinear embedding theorem under the following additional requirement on $\mathscr{V}$ and $\mathscr{W}$ described in (a)-(d). The general case, where $\mathscr{V}$ and $\mathscr{W}$ are arbitrary combinations of the types listed in (a)-(d), cannot be treated using the heat-flow method of [19].

Assumption BE. We say that the pair $(\mathscr{V}, \mathscr{W})$ satisfies the Assumption BE if either of the following holds:

- $\mathscr{V}$ and $\mathscr{W}$ fall into any of the special cases (a)-(c), or
- $\mathscr{V}$ and $\mathscr{W}$ are of the type described in (a) or (d).

The combinations allowed by Assumption BE are summarized in Table 3.1.

Remark 3.1. This assumption is imposed only in the bilinear embedding theorem and must be added to the statements of the corresponding theorems in [19, 21, 78]. We emphasize that all results in [19, 78] derived from the bilinear embedding—such as the boundedness of the H^∞-functional calculus and L^p-maximal regularity—remain unaffected by the error, since they follow from applying the bilinear embedding to an operator and its adjoint, which are subject to the same boundary conditions. The fact that this assumption is necessary for the statement of [19, Lemma 19] does not imply that it is required for the bilinear embedding itself; nevertheless, a new approach must be pursued.

The space described in (c) is new in the context of bilinear embedding on arbitrary domains. It is introduced to allow combinations of mixed and Neumann boundary conditions, which would otherwise be prohibited by Assumption BE.

	$W_0^{1,2}(\Omega)$	$W_D^{1,2}(\Omega)$	$\widetilde{W_D}^{1,2}(\Omega)$	$W^{1,2}(\Omega)$
$W_0^{1,2}(\Omega)$	✓	✓	✓	✓
$W_D^{1,2}(\Omega)$	✓	✓	×	×
$\widetilde{W_D}^{1,2}(\Omega)$	✓	×	✓	✓
$W^{1,2}(\Omega)$	✓	×	✓	✓

Table 3.1: Assumption BE: allowed and forbidden pairs of Sobolev spaces; the symbol ✓ indicates that the pair is allowed, while × that it is not.

Invariance properties of the form domain

We present now some invariance properties of such closed subspaces $\mathscr{V}$ of $W^{1,2}(\Omega)$. Moreover, by means of Proposition 3.5below, we lay the groundwork to explain the necessity of Assumption BE for the validity of [19, Lemma19], which will be addressed later in Remark 3.28.

Denote by $\mathcal{L}$ the set of all Lipischitz functions $\Phi : \mathbb{C} \to \mathbb{C}$ with $\Phi(0) = 0$. If the Lipschitz constant of Φ is 1, Φ is said to be a *normal contraction*. We denote by $\mathscr{N}$ the set of all normal contractions.

Proposition 3.2. *Let $\mathscr{V}$ be a closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$. Then $\mathscr{V}$ satisfies (3.5) and (3.6) if and only if it is invariant under the (whole) class $\mathcal{L}$.*

Proof. In view of [21, Proposition A.1.] and the fact that $\mathscr{N} \subseteq \mathcal{L}$, it suffices to prove that $\mathscr{V}$ is invariant under $\mathcal{L}$ if it is invariant under $\mathscr{N}$.

Let $\Phi \in \mathcal{L}$ and $u \in \mathscr{V}$. Then $\Phi/\mathrm{Lip}(\Phi) \in \mathcal{N}$. Hence, the invariance of $\mathscr{V}$ under $\mathscr{N}$ gives

$$\Phi(u) = \mathrm{Lip}(\Phi) \cdot \frac{\Phi}{\mathrm{Lip}(\Phi)}(u) \in \mathscr{V}. \qquad \square$$

We aim to establish an analogous invariance result in the multivariable setting for the spaces described in (a)–(d). We proceed much as in [41, Lemma 4]. See also [19, Lemma 19].

Lemma 3.3. *Let $\mathscr{V}$ be a closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$. Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathscr{V}$ and $u \in L^2(\Omega)$ such that*

- *$(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathscr{V}$,*
- *$u_n \to u$ in $L^2(\Omega)$ as $n \to \infty$.*

Then $u \in \mathscr{V}$.

Proof. Since $(\Phi_n(u))_{n\in\mathbb{N}}$ is bounded in $\mathscr{V}$, it admits a subsequence with weak limit $u_\infty \in \mathscr{V}$. Then $u = u_\infty \in \mathscr{V}$, as u is the strong limit of $(u_n)_{n\in\mathbb{N}}$ in $L^2(\Omega)$. □

Proposition 3.4. *Suppose that $\mathscr{V}$ falls into any of the special cases (a)-(c) and $\mathscr{W} = W^{1,2}(\Omega)$. Let $u \in \mathscr{V}$, $v \in \mathscr{W}$ and $\Phi : \mathbb{C}^2 \to \mathbb{C}$ be a Lipschitz function such that $\Phi(0,\eta) = 0$ for all $\eta \in \mathbb{C}$. Then $\Phi \circ (u,v) \in \mathscr{V}$.*

Furthermore, if $\mathscr{V}$ is of the type described in (d) the same conclusion holds provided that $\mathscr{W}$ is of the type described either in (a) or in (d).

Proof. First case: $\mathscr{V}$ falls into any of the special cases (a)-(c).

If $\mathscr{V} = W^{1,2}(\Omega)$, the assertion follows from [4, Corollary 2.7]. Moreover, we have

$$\|\Phi \circ (u,v)\|_{1,2} \lesssim \mathrm{Lip}(\Phi)\left(\|u\|_{1,2} + \|v\|_{1,2}\right). \tag{3.7}$$

Suppose that $\mathscr{V} = \widetilde{W_D}^{1,2}(\Omega)$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,2}(\Omega)$ with $\mathrm{dist}(\mathrm{supp}\, u_n, D) > 0$ converging to u in $W^{1,2}(\Omega)$. We set $w_n := \Phi \circ (u_n, v)$. From the previous case $w_n \in W^{1,2}(\Omega)$ and thanks to $\Phi(0,\cdot) = 0$ we have $\mathrm{dist}(\mathrm{supp}\, w_n, D) > 0$. Thus $w_n \in \widetilde{W_D}^{1,2}(\Omega)$. Estimate (3.7) shows that $(w_n)_n$ is bounded in $\widetilde{W_D}^{1,2}(\Omega)$. On the other hand, we have $\|w_n - \Phi\circ(u,v)\|_2 \leqslant \mathrm{Lip}(\Phi)\|u_n - u\|_2$, so that $w_n \to \Phi\circ(u,v)$ strongly in $L^2(\Omega)$ as $n \to \infty$. Thus, Lemma 3.3 gives $\Phi \circ (u,v) \in \widetilde{W_D}^{1,2}(\Omega)$ as required.

When $\mathscr{V} = W_0^{1,2}(\Omega)$ the proof is similar to the previous one.

Second case:

$\mathscr{V}$ is of the type described in (d) and $\mathscr{W}$ of the type described either in (a) or in (d).

Suppose that $\mathscr{V} = W_D^{1,2}(\Omega)$ and $\mathscr{W} = W_{D'}^{1,2}(\Omega)$ with D, D' being (possibly empty) closed subsets of $\partial\Omega$. Let $(u_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d \setminus D)$ and $(v_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d \setminus D')$ such that $u_{n|\Omega}$ and $v_{n|\Omega}$ converge to u and v in $W^{1,2}(\Omega)$ as $n \to \infty$, respectively. We set $w_n := \Phi \circ (u_n, v_n)$. Then w_n is Lipschitz continuous on $\mathbb{R}^d$ and has compact support vanishing in a neighborhood of D since $\Phi(0,\cdot) = 0$. We conclude $w_n \in W_D^{1,2}(\Omega)$ because the required approximations in $C_c^\infty(\mathbb{R}^d \setminus D)$ can explicitly be constructed by convolution with smooth, compactly supported kernels. At this point, we repeat the same final argument of the first case where we made use of (3.7) and Lemma 3.3. □

The next proposition illustrates the necessity of imposing additional assumptions on $\mathscr{W}$ in the case when $\mathscr{V} = W_D^{1,2}(\Omega)$. In fact, in general $\Phi \circ (u,v) \notin W_D^{1,2}(\Omega)$ whenever $u \in W_D^{1,2}(\Omega)$ and $v \in W^{1,2}(\Omega)$.

Before presenting the detailed counterexample, let us illustrate heuristically what the problem is. We try to proceed as in the proof of Proposition 3.4. Let $(u_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^d \setminus D)$ be such that $u_n|_\Omega \to u$ in $W^{1,2}(\Omega)$ as $n \to \infty$. We would like to have $\Phi \circ (u_n, v) \in W_D^{1,2}(\Omega)$. The problem is that, in general, v cannot be extended to a sufficiently regular function on $\mathbb{R}^d$, and therefore neither can $\Phi \circ (u_n, v)$. More precisely, the difficulty stems from the fact that $W_\emptyset^{1,2}(\Omega) \neq W^{1,2}(\Omega)$ for arbitrary open sets Ω.

Proposition 3.5. *Let $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < |x| < 1, 0 < y < 1\}$ and $\Phi : \mathbb{C}^2 \to \mathbb{C}$ be a Lipschitz function such that $\Phi(0,\cdot) = 0$. Suppose that there exist $\zeta_0, \eta_0 \in \mathbb{C}$ such that*

$$\Phi(\zeta_0, \eta_0) \neq \Phi(\zeta_0, 0). \tag{3.8}$$

Then there exist $D \subseteq \partial\Omega$ (possibly empty) closed, $u \in W_D^{1,2}(\Omega)$ and $v \in W^{1,2}(\Omega)$ such that $\Phi \circ (u,v) \notin W_D^{1,2}(\Omega)$.

Proof. Take $D = \emptyset$ and define

$$u := \zeta_0, \qquad v := \begin{cases} \eta_0, & \text{if } x > 0, \\ 0, & \text{if } x < 0. \end{cases}$$

By construction $u \in W^{1,2}_{\emptyset}(\Omega)$, $v \in W^{1,2}(\Omega)$ and

$$\Phi \circ (u, v) = \begin{cases} \Phi(\zeta_0, \eta_0), & \text{if } x > 0, \\ \Phi(\zeta_0, 0), & \text{if } x < 0. \end{cases}$$

Therefore, from (3.8) we deduce that for sufficiently small $\widetilde{\varepsilon} > 0$ no function $\phi \in C^\infty_c(\mathbb{R}^2)$ can satisfy $\|\Phi \circ (u, v) - \phi_{|\Omega}\|_{1,2} < \widetilde{\varepsilon}$. Hence, $\Phi(u, v) \notin W^{1,2}_{\emptyset}(\Omega)$. □

Remark 3.6.

- The function v defined in the proof of Proposition 3.5 not only belongs to $W^{1,2}(\Omega)$, but in fact lies in $\widetilde{W_{D'}}^{1,2}(\Omega)$, where $D' = \{(-1, y) : 0 \leqslant y \leqslant 1\}$. Therefore, we also provided a counterexample of $u \in W^{1,2}_D(\Omega)$ and $v \in \widetilde{W_{D'}}^{1,2}(\Omega)$ such that $\Phi \circ (u, v) \notin W^{1,2}_D(\Omega)$.
- In order to obtain a counterexample with $D \neq \emptyset$, we may keep D' and v as before and take $D = D'$ and $u = (\zeta_0 \psi \otimes \mathbf{1})_{|\Omega}$, where ψ is a smooth function on $[-1, 1]$ such that $\psi = 0$ on $[-1, -2/3]$, $\psi = 1$ on $[-1/3, 1]$ and $0 \leqslant \psi \leqslant 1$ otherwise. Set $\Omega' = \{(x, y) \in \Omega : x > -1/3\}$. Then, by the same previous argument, we infer that for sufficiently small $\widetilde{\varepsilon} > 0$ no function $\phi \in C^\infty_c(\mathbb{R}^2)$ can satisfy $\|\Phi \circ (u, v) - \phi_{|\Omega'}\|_{W^{1,2}(\Omega')} < \widetilde{\varepsilon}$. Hence, no function $\phi \in C^\infty_c(\mathbb{R}^2 \setminus D)$ can satisfy $\|\Phi \circ (u, v) - \phi_{|\Omega}\|_{W^{1,2}(\Omega)} < \widetilde{\varepsilon}$.

3.2 p-ellipticity

The notion of *p-ellipticity* plays a central role throughout this thesis, as it provides the structural condition on the coefficients underlying all the main results. In particular, we will extend and apply this concept in two different settings in the following chapters.

The p-ellipticity condition was introduced independently by Carbonaro and Dragičević in the context of dimension-free bilinear embeddings [20], and by Dindoš and Pipher in the study of boundary value problems [30]. It may be viewed as an algebraic strengthening of the standard ellipticity condition, coinciding with it when $p = 2$, and offering a natural interpolation between real and complex elliptic matrices.

A detailed account of the genesis of p-ellipticity can be found in [22, Section 1.2]. Its conception by Carbonaro and Dragičević originated in [20] after several years of gradually refining the *heat-flow method* associated with a specific *Bellman function* $\mathcal{Q}$, first introduced by Nazarov and Treil [67] and further developed through a sequence of works on bilinear embeddings and spectral multipliers [16–18, 34–36]. In [36], a detailed study of $\mathcal{Q}$ revealed delicate convexity properties that proved to be crucial for establishing the bilinear embedding theorem. These discoveries suggested that $\mathcal{Q}$ might possess deeper structural convexity features, a hypothesis that was later confirmed and systematically analyzed in subsequent works [16–18], culminating in [20]. In that context, the focus gradually shifted from the Bellman function itself to the convexity of its elementary components—namely, the power functions $|\zeta|^p$—considered with respect to the matrix A. This led to the introduction of a generalized notion of convexity adapted to matrix coefficients, which coincides with the classical one when A is the identity (see Section 3.3 for the definition). The p-ellipticity condition emerged precisely from this framework: it can be interpreted as the uniform convexity of $|\zeta|^p$ relative to A, later reformulated in the algebraic form recalled below (3.9).

As already mentioned, before the general theory was established, convexity relative to A had been explored in several particular configurations: when A is either the identity [34,36,67], real accretive [35], of the form $e^{i\phi}I_d$ [17], or of the form $e^{i\phi}B$ with B real, constant and with a symmetric part which is positive definite [18]. The introduction of these complex rotations in [17] was motivated by the novel idea of studying bilinear embeddings along complex times, rather than restricting the analysis to real times as in all previous work. This insight opened the way to further developments of the theory in the direction of divergence-form elliptic operators with complex coefficients [20].

We now recall the precise definition introduced by Carbonaro and Dragičević in [20]. Given $A \in \mathcal{A}(\Omega)$ and $p \in (1,\infty)$, we say that A is *p-elliptic* if $\Delta_p(A) > 0$, where

$$.\Delta_p(A) := \operatorname*{ess\,inf}_{x\in\Omega} \min_{|\xi|=1} \operatorname{Re} \left\langle A(x)\xi, \xi + |1-2/p|\overline{\xi} \right\rangle_{\mathbb{C}^d}.$$

Equivalently, A is p-elliptic if here exists $C = C(A,p) > 0$ such that for a.e. $x \in \Omega$,

$$\operatorname{Re} \left\langle A(x)\xi, \xi + |1-2/p|\overline{\xi} \right\rangle_{\mathbb{C}^d} \geqslant C|\xi|^2, \quad \forall \xi \in \mathbb{C}^d. \tag{3.9}$$

From the definition, it follows that Δ_p is invariant under conjugation of p, meaning that $\Delta_p(A) = \Delta_{p'}(A)$, as $|1-2/p| = |1-2/p'|$. The case $p = 2$ recovers the standard ellipticity condition, as $\Delta_2(A) = \lambda(A)$, thereby illustrating the generalizing nature of p-ellipticity. On the other hand, for any fixed $p \neq 2$, the class of complex p-elliptic matrices is strictly smaller than the class of all elliptic matrices. Therefore, the p-ellipticity condition not only generalize classical ellipticity, but also strengthens it. More precisely, denoting by $\mathcal{A}_p(\Omega)$ the class of all p-elliptic matrix functions on Ω, the family $\{\mathcal{A}_p : p \in [2,\infty)\}$ forms a decreasing chain of matrix classes such that

$$\{\text{elliptic matrices on } \Omega\} = \mathcal{A}_2(\Omega),$$
$$\{\text{real elliptic matrices on } \Omega\} = \bigcap_{p\in[2,\infty)} \mathcal{A}_p(\Omega).$$

Thus, the notion of p-ellipticity bridges the gap between real and complex elliptic matrices. In addition, it provides regularity estimates for local solutions of $\mathscr{L}u = 0$ with complex coefficients [30], serving as a substitute for the De Giorgi-Nash-Moser theory that is only available in the real case.

A condition similar to (3.9), namely $\Delta_p(A) \geqslant 0$, was formulated in a different manner by Cialdea and Maz'ya in [24, (2.25)]. See [20, Remark 5.14].

In the next lemma, we list several important properties of p-ellipticity. Some of them have already been mentioned, while others—such as invariance under adjointness, stability under small complex rotations, and continuity with respect to the parameter p—are highlighted here for completeness. To begin with, for every $p \in [1,+\infty)$ we define the $\mathbb{R}$-linear map $\mathcal{I}_p : \mathbb{C}^d \to \mathbb{C}^d$ by

$$\mathcal{I}_p(\xi) = \xi + (1-2/p)\overline{\xi}, \quad \xi \in \mathbb{C}^d. \tag{3.10}$$

Equivalently,

$$\mathcal{I}_p(\alpha + i\beta) = 2\left(\frac{\alpha}{p'} + i\frac{\beta}{p}\right), \qquad \alpha,\beta \in \mathbb{R}^d.$$

Lemma 3.7. [20] *Let $A \in \mathcal{A}(\Omega)$, $p \in (1,\infty)$ and denote by $q = p/(p-1)$ its conjugate exponent. Then*

(i) $\Delta_p(A) = \Delta_q(A)$.

(ii) $\Delta_p(A) = \operatorname*{ess\,inf}_{x\in\Omega} \min_{|\xi|=1} \langle A(x)\xi, \mathcal{I}_p\xi\rangle$.

(iii) $\Delta_p(A) \geqslant 0$ *if and only if* $\Delta_q(A^*) \geqslant 0$. *The same holds for strict inequalities.*

(iv) *The function* $p \mapsto \Delta_p(A)$ *is Lipschitz continuous on* $[1,\infty)$, *nondecreasing on* $[1,2]$ *and nonincresing on* $[2,\infty)$.

(v) *The function* $\varphi \mapsto \Delta_p(e^{i\varphi}A)$ *is Lipschitz continuous in the interval* $(-\pi/2,\pi/2)$.

(vi) $\Delta_1(A) \leqslant 0$, *with equality precisely for real* A.

(vii) $\operatorname{Im} A(x) = 0$ *for a.e.* $x \in \Omega$ *if and only if* $\Delta_p(A) \geqslant 0$ *for all* $p \geqslant 1$.

Proof. Item (i) has been already observed and it follows by definition (3.2).

When $p \geqslant 2$, item (ii) follows by definitions of $\Delta_p(A)$ and $\mathcal{I}_p$. If $p < 2$, the conclusion follows by (i), the previous case applied with $q \geqslant 2$ and the connection $\mathcal{I}_q(i\xi) = i\mathcal{I}_p(\xi)$, as they imply, in order,

$$\Delta_p(A) = \Delta_q(A) = \operatorname*{ess\,inf}_{x\in\Omega} \min_{|\xi|=1} \langle A(x)\xi, \mathcal{I}_q\xi\rangle = \operatorname*{ess\,inf}_{x\in\Omega} \min_{|\xi|=1} \langle A(x)\xi, \mathcal{I}_p\xi\rangle\,.$$

Regarding item (iii), since $(A^*)^* = A$ it suffices to prove that $\Delta_p(A^*) \geqslant 0$ provided that $\Delta_p(A) \geqslant 0$. Set $p^* = \max\{p,q\}$. Take $\xi \in \mathbb{C}^d$ with $|\xi| = 1$, write $X = \mathcal{I}_p\xi$ and observe that $\mathcal{I}_q\mathcal{I}_p = (4/pq)I_{\mathbb{C}^d}$. This implies $\xi = (pq/4)\mathcal{I}_q X$ and thus for almost every $x \in \Omega$ we have

$$\operatorname{Re}\,\langle A(x)^*\xi, \mathcal{I}_p\xi\rangle_{\mathbb{C}^d} = \frac{pq}{4}|X|^2 \operatorname{Re}\left\langle \mathcal{I}_q\frac{X}{|X|}, A(x)\frac{X}{|X|}\right\rangle_{\mathbb{C}^d} \geqslant \frac{pq}{4}|\mathcal{I}_p\xi|^2\Delta_q(A),$$

where in the last inequality we used (ii). We know from item (i) that $\Delta_q(A) = \Delta_p(A)$. Therefore, since $\min\{|\mathcal{I}_p\xi| : |\xi| = 1\} = 2/p^*$ and $\Delta_p(A) \geqslant 0$, we conclude that $\Delta_p(A^*) \geqslant 0$ by minimizing over ξ and x and by using (ii).

Item (iv) follows from the formula

$$\Delta_p(A) = \operatorname*{ess\,inf}_{x\in\Omega} \min_{|\xi|=1} \left(\operatorname{Re}\,\langle A(x)\xi,\xi\rangle - |1-2/p|\cdot\left|\left\langle A(x)\xi, \overline{\xi}\right\rangle\right|\right),$$

which emerges from [20, (5.9) & (5.12)].

It quickly follows from the definition (3.2) of $\Delta_p(A)$ that for $A, B \in L^\infty(\Omega;\mathbb{C}^{d\times d})$ we have

$$|\Delta_p(A) - \Delta_p(B)| \leqslant 2\frac{\|A-B\|_\infty}{\min\{p,q\}}. \tag{3.11}$$

In particular, $\varphi \mapsto \Delta_p(e^{i\varphi}A)$ is Lipschitz continuous on $(-\pi/2,\pi/2)$. This proves item (v).

Prove now item (vi). It suffices to consider the case where A is a constant matrix, since the general (non-constant) case then follows from the identity $\Delta_p(A) = \operatorname{ess\,inf}_{x\in\Omega}\Delta_p(A(x))$. For all $\xi = \alpha + i\beta \in \mathbb{C}^d$ we have

$$\operatorname{Re}\,\langle A\xi, \mathcal{I}_1\xi\rangle = 2\operatorname{Re}\,\langle A\xi, i\beta\rangle = 2\,\langle \operatorname{Re} A\beta, \beta\rangle + 2\,\langle \operatorname{Im} A\alpha, \beta\rangle\,.$$

Therefore, if A is real we get $\Delta_1(A) = 0$ by (ii) and by taking real ξ as minimizers. Otherwise, for every $\rho \in \mathbb{R}$ choose $\alpha = \alpha(\rho) \in \mathbb{R}^d$ such that $\rho^2|\operatorname{Im} A\,\alpha|^2 + |\alpha|^2 = 1$ and define $\beta = \beta(\rho) := -\rho \operatorname{Im} A\,\alpha \in \mathbb{R}^d$. In this way $\xi(\rho) := \alpha(\rho) + i\beta(\rho)$ satisfies $|\xi(\rho)| = 1$ for all $\rho \in \mathbb{R}$. Therefore, since $\operatorname{Im} A \neq 0$ we obtain

$$\operatorname{Re}\,\langle A\xi(\rho), \mathcal{I}_1\xi(\rho)\rangle = 2\rho\left(\rho\,\langle \operatorname{Re} A \operatorname{Im} A\,\alpha(\rho), \operatorname{Im} A\,\alpha(\rho)\rangle - |\operatorname{Im} A\,\alpha(\rho)|^2\right),$$

which is negative for small $\rho > 0$. Hence, (ii) yields

$$\Delta_1(A) \leqslant \inf_{\rho\in\mathbb{R}} \operatorname{Re}\,\langle A\xi(\rho), \mathcal{I}_1\xi(\rho)\rangle < 0.$$

Finally, prove item (vii). Suppose that $\operatorname{Im} A(x) = 0$ for a.e. $x \in \Omega$. Then $\Delta_1(A) = 0$ by (vi) and for all $p \in (1, \infty)$ we have

$$\operatorname{Re} \langle A(x)\xi, \mathcal{I}_p\xi\rangle = \frac{2}{q}\langle A(x)\operatorname{Re}\xi, \operatorname{Re}\xi\rangle + \frac{2}{p}\langle A(x)\operatorname{Im}\xi, \operatorname{Im}\xi\rangle \geqslant \frac{2\lambda(A)}{\max\{p,q\}}|\xi|^2,$$

for almost all $x \in \Omega$ and all $\xi \in \mathbb{C}^d$. Hence, $\Delta_p(A) \geqslant 0$ for all $p \geqslant 1$ by (ii). The other implication follows from (vi). □

Remark 3.8. Given $A \in \mathcal{A}_p(\Omega)$, it follows from Lemma 3.7 (iv) and (v) that there exist

- $\varepsilon > 0$ such that $A \in \mathcal{A}_s(\Omega)$ for all $s \in [p - \varepsilon, p + \varepsilon]$;
- $\vartheta \in (0, \pi/2)$ such that $e^{i\varphi}A \in \mathcal{A}_p(\Omega)$ for all $\varphi \in [-\vartheta, \vartheta]$;

respectively.

Remark 3.9. Let us recall an equivalent formulation of p-ellipticity established in [20]. Let $A \in \mathcal{A}(\Omega)$. We define the constant $\mu(A)$ by

$$\mu(A) := \operatorname*{ess\,inf}_{x\in\Omega} \inf_{\xi\in\mathbb{C}^d,\, \langle A(x)\xi,\bar{\xi}\rangle\neq 0} \frac{\operatorname{Re}\langle A(x)\xi, \xi\rangle}{|\langle A(x)\xi, \bar{\xi}\rangle|}. \tag{3.12}$$

In view of the basic assumptions (3.1) and (3.2), the quantity $\mu(A)$ admits the trivial lower bound $\mu(A) \geq \lambda(A)/\Lambda(A)$. We also have the sharp universal upper bound $\mu(A) \leqslant 1$, which follows from considering $\xi \in \mathbb{R}^d$ in (3.12). Hence,

$$\frac{\lambda(A)}{\Lambda(A)} \leqslant \mu(A) \leqslant 1. \tag{3.13}$$

It was proved in [20, Proposition 5.15] that A is p-elliptic if and only if

$$\mu(A) > |1 - 2/p|. \tag{3.14}$$

More precisely, given $p \in [1, \infty]$, one has

$$\Delta_p(A) = 0 \qquad \text{if and only if} \qquad |1 - 2/p| = \mu(A), \tag{3.15}$$

and, whenever these equalities are not satisfied,

$$\frac{\lambda(A)}{\mu(A)} \leqslant \frac{\Delta_p(A)}{\mu(A) - |1 - 2/p|} \leqslant \Lambda(A). \tag{3.16}$$

In particular, by (3.13), condition (3.14) is automatically satisfied whenever $|1-2/p| < \lambda(A)/\Lambda(A)$. Furthermore, (3.15) together with Lemma 3.7 (vi) implies that $\mu(A) = 1$ when A is real-valued.

The formulation (3.14) has the advantage of decoupling p and A.

3.3 Generalized convexity

In [20], Carbonaro and Dragičević introduced the concept of convexity with respect to matrices. As we will see in Sections 3.4 and 3.5, this notion plays a key role both in the study of the L^p-dissipativity of the form $\mathfrak{a}$, which—via Nittka's Theorem—allows one to establish the L^p-contractivity of the associated semigroup, and in the application of the heat-flow method used to prove the bilinear embedding.

In the context of L^p-contractivity, the function whose generalized convexity is analyzed is the power function of exponent p. In contrast, the bilinear embedding relies on the convexity properties of a specially designed Bellman function, which will be introduced in Section 3.5.1.

We will see that the power function of exponent p is convex with respect to a matrix A if and only if $\Delta_p(A) \geqslant 0$ (Lemma 3.14), and later that the Bellman function is strictly convex with respect to a pair of matrices, provided that both are p-elliptic (Theorem 3.22).

3.3.1 Real form of complex operators

We explicitly identify $\mathbb{C}^d$ with $\mathbb{R}^{2d}$ as follows. For each $d \in \mathbb{N}_+$ consider the operator $\mathcal{V}_d : \mathbb{C}^d \to \mathbb{R}^d \times \mathbb{R}^d$, defined by

$$\mathcal{V}_d(\xi_1 + i\xi_2) = (\xi_1, \xi_2), \quad \xi_1, \xi_2 \in \mathbb{R}^d.$$

One has

$$\text{Re } \langle \xi, \sigma \rangle_{\mathbb{C}^d} = \langle \mathcal{V}_d(\xi), \mathcal{V}_d(\sigma) \rangle_{\mathbb{R}^{2d}}, \quad \xi, \sigma \in \mathbb{C}^d. \tag{3.17}$$

If $A \in \mathbb{C}^{d\times d}$ we shall frequently use its real form:

$$\mathcal{M}(A) = \mathcal{V}_d A \mathcal{V}_d^{-1} = \begin{bmatrix} \text{Re } A & -\text{Im } A \\ \text{Im } A & \text{Re } A \end{bmatrix}.$$

For $\xi, \sigma \in \mathbb{C}^d$ we have the following extension of (3.17):

$$\text{Re } \langle A\xi, \sigma \rangle_{\mathbb{C}^d} = \langle \mathcal{M}(A)\mathcal{V}_d(\xi), \mathcal{V}_d(\sigma) \rangle_{\mathbb{R}^{2d}}. \tag{3.18}$$

Let $k, d \in \mathbb{N}_+$. We define another identification operator

$$\mathcal{W}_{k,d} : \underbrace{\mathbb{C}^d \times \cdots \times \mathbb{C}^d}_{k-\text{times}} \longrightarrow \underbrace{\mathbb{R}^{2d} \times \cdots \times \mathbb{R}^{2d}}_{k-\text{times}}$$

by the rule

$$\mathcal{W}_{k,d}(\xi^1, \ldots, \xi^k) = \left(\mathcal{V}_d(\xi^1), \ldots, \mathcal{V}_d(\xi^k)\right), \quad \xi^j \in \mathbb{C}^d, \ j = 1, \ldots, k.$$

3.3.2 Gradient and Hessian forms

Let $\Phi : \mathbb{C}^k \to \mathbb{R}$ be of class C^2. We associate the function Φ on $\mathbb{C}^k$ with the the following function on $\mathbb{R}^{2k}$

$$\Phi_{\mathcal{W}} := \Phi \circ \mathcal{W}_{k,1}^{-1}. \tag{3.19}$$

We denote, respectively, by $D^2\Phi(\omega)$ and by $D\Phi(\omega)$ the Hessian matrix and the gradient of the function $\Phi_{\mathcal{W}} : \mathbb{R}^{2k} \to \mathbb{R}$ calculated at the point $\mathcal{W}_{k,1}(\omega) \in \mathbb{R}^{2k}$.

We will write

$$|D\Phi| = \left(\sum_{j=1}^{2k} |\partial_{x_j} \Phi_{\mathcal{W}}|^2 \right)^{1/2}$$

and likewise

$$|D^2\Phi| = \left(\sum_{j,l=1}^{2k} |\partial^2_{x_j x_l} \Phi_{\mathcal{W}}|^2 \right)^{1/2}.$$

Sometimes we may also write D_j for ∂_{x_j} and similarly D^2_{jl} for $\partial^2_{x_j x_l}$.

For $w = w_1 + iw_2 \in \mathbb{C}$, introduce the complex derivative

$$\partial_w = \frac{\partial_{w_1} - i\partial_{w_2}}{2}.$$

Then, for $\omega = (\omega^1, \ldots, \omega^k) \in \mathbb{C}^k$, we define

$$\partial_{\omega^j} \Phi(\omega) = \left(\partial_{\omega^j} \Phi_{\mathcal{W}}\right)\left(\mathcal{W}_{k,1}(\omega)\right), \qquad j = 1, \ldots, k.$$

3.3.3 Convexity with respect to complex matrices

Given a matrix $D = [d_{ij}]_{i,j} \in \mathbb{R}^{M,N}$ and $d \in \mathbb{N}$, we let $D \otimes I_{\mathbb{R}^d} \in \mathbb{R}^{Md,Nd}$ be the Kronecker product of D with the identity on $\mathbb{R}^d$, that is,

$$D \otimes I_{\mathbb{R}^d} := [d_{ij} \cdot I_{\mathbb{R}^d}]_{i,j}.$$

For example,

$$\begin{bmatrix} a & b \end{bmatrix} \otimes \mathbb{R}^3 = \begin{bmatrix} a & & & b & & \\ & a & & & b & \\ & & a & & & b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes I_{\mathbb{R}^2} = \begin{bmatrix} a & & b & \\ & a & & b \\ c & & d & \\ & c & & d \end{bmatrix}.$$

Following [20] we only treat the one-dimensional and the two-dimensional case separately, in order to make the text more readable. One could provide a single definition for the k-dimensional case, as Carbonaro and Dragičević did in [19].

One-dimensional case

Let $\phi : \mathbb{C} \to \mathbb{R}$ be of class C^2 and $A \in \mathbb{C}^{d \times d}$.

Definition 3.10. [20] We define the *generalized Hessian* of ϕ at the point $\zeta \in \mathbb{C}$ with respect to A as

$$H_\phi^A(\zeta) = \mathcal{M}(A^*) \cdot \left((D^2\phi)(\zeta) \otimes I_{\mathbb{R}^d} \right).$$

In accordance with [19, 20], we introduce a special notation for denoting the quadratic form associated with the generalized Hessians. We define

$$H_\phi^A[\zeta; X] = \left\langle H_\phi^A(\zeta)\mathcal{V}_d(X), \mathcal{V}_d(X) \right\rangle_{\mathbb{R}^{2d}}, \qquad X \in \mathbb{C}^d.$$

Definition 3.11. [20] We say that ϕ is *A-convex* in $\mathbb{C}$ if $H_\phi^A[\zeta; X]$ is nonnegative for all $\zeta \in \mathbb{C}$, $X \in \mathbb{C}^d$.

Two-dimensional case

Let $\Phi : \mathbb{C}^2 \to \mathbb{R}$ be of class C^2 and $A, B \in \mathbb{C}^{d \times d}$.

Definition 3.12. [20] We define the *generalized Hessian* of Φ at the point $\omega \in \mathbb{C}^2$ with respect to A, B as

$$H_\Phi^{(A,B)}(\omega) = (\mathcal{M}(A^*) \oplus \mathcal{M}(B^*)) \cdot \left((D^2\Phi)(\omega) \otimes I_{\mathbb{R}^d} \right),$$

where $\mathcal{M}(A^*) \oplus \mathcal{M}(B^*)$ is the $4d \times 4d$ block diagonal real matrix with the $2d \times 2d$ blocks $\mathcal{M}(A^*), \mathcal{M}(B^*)$ along the main diagonal.

As in the one-dimensional case, we denote the quadratic form associate with the generalized Hessian as

$$H_\Phi^{(A,B)}[\omega; \Xi] = \left\langle H_\Phi^{(A,B)}(\omega)\mathcal{W}_{2,d}(\Xi), \mathcal{W}_{2,d}(\Xi) \right\rangle_{\mathbb{R}^{4d}}, \qquad \Xi \in \mathbb{C}^{2d}.$$

Definition 3.13. [20] We say that Φ is *(A, B)-convex* in $\mathbb{C}^2$ if $H_\Phi^{(A,B)}[\omega; \Xi]$ is nonnegative for all $\omega \in \mathbb{C}^2$, $\Xi \in \mathbb{C}^{2d}$.

We maintain the same notation when instead of matrices we consider matrix-valued *functions* $A, B \in L^\infty(\Omega, \mathbb{C}^{d\times d})$; in this case however we require that all the conditions are satisfied for a.e. $x \in \Omega$.

3.3.4 Power functions and p-ellipticity

We now show the connection between p-ellipticity and the generalized convexity of power functions. This relation emerges from the computation of the Hessian of these functions, leading to identity (3.23), where the operator $\mathcal{I}_p$ appears—the same operator that features in one of the equivalent formulations of the p-ellipticity constant Δ_p.

Let $r > 0$. Define the power function $F_r : \mathbb{C} \to \mathbb{R}_+$ by

$$F_r(\zeta) = |\zeta|^r, \quad \zeta \in \mathbb{C}. \tag{3.20}$$

Let $\mathbf{1}$ denote the constant function of value 1 on $\mathbb{C}$, that is, $\mathbf{1} = F_0$. We extend Δ_r and $\mathcal{I}_r$ to $r > 0$ by the same rules (3.2) anda (3.10).

A rapid calculation [17, 19–21] shows that

$$(DF_r)(\zeta) = r\,|\zeta|^{r-2}\,\mathcal{V}_1(\zeta), \qquad \forall\,\zeta \in \mathbb{C}, \tag{3.21}$$

$$(D^2F_r)(\zeta) = r\,|\zeta|^{r-2}\left(I_{\mathbb{R}^2} + (r-2)\frac{\mathcal{V}_1(\zeta)}{|\zeta|} \otimes \frac{\mathcal{V}_1(\zeta)}{|\zeta|}\right), \qquad \forall\,\zeta \in \mathbb{C}\setminus\{0\}. \tag{3.22}$$

As shown in [19, p.13], for all $\zeta \in \mathbb{C}$, $X \in \mathbb{C}^d$ we have

$$\left[\left(\frac{\mathcal{V}_1(\zeta)}{|\zeta|} \otimes \frac{\mathcal{V}_1(\zeta)}{|\zeta|}\right) \otimes I_{\mathbb{R}^d}\right] \mathcal{V}_d(X) = \mathcal{V}_d\left(\mathrm{sign}\zeta \cdot \mathrm{Re}\,(\mathrm{sign}\overline{\zeta} \cdot X)\right).$$

Therefore, from (3.22) we infer

$$\left(\mathcal{V}_d^{-1} \circ \left[(D^2F_r)(\zeta) \otimes I_{\mathbb{R}^d}\right] \circ \mathcal{V}_d\right)(X) = \frac{r^2}{2}|\zeta|^{r-2}\mathrm{sign}\zeta \cdot \mathcal{I}_r(\mathrm{sign}\overline{\zeta} \cdot X), \tag{3.23}$$

for all $\zeta \in \mathbb{C}\setminus\{0\}$, $X \in \mathbb{C}^d$. Hence, by applying the definition of the generalized Hessian with respect to A and (3.18), we deduce the following.

Lemma 3.14. [20, Lemma 5.6] *Let $A \in \mathcal{A}(\Omega)$, $p \in (1,\infty)$. Then for a.e. $x \in \Omega$ and every $\zeta \in \mathbb{C}\setminus\{0\}$ and $X \in \mathbb{C}^d$ we have*

$$H_{F_p}^{A(x)}[\zeta; X] = \frac{p^2}{2}|\zeta|^{p-2}\mathrm{Re}\left\langle A(x)\left(\mathrm{sign}\overline{\zeta} \cdot X\right), \mathcal{I}_p\left(\mathrm{sign}\overline{\zeta} \cdot X\right)\right\rangle. \tag{3.24}$$

In particular, for all $\zeta \in \mathbb{C}\setminus\{0\}$ we have

$$\operatorname*{ess\,inf}_{x\in\Omega} \min_{|X|=1} H_{F_p}^{A(x)}[\zeta; X] = \frac{p^2}{2}|\zeta|^{p-2}\Delta_p(A),$$

and

$$H_{F_p}^{A(x)}[\zeta; X] \geqslant \frac{p^2}{2}\Delta_p(A)|\zeta|^{p-2}|X|^2,$$

for almost every $x \in \Omega$.

3.4 L^p-contractivity and analyticity of $(T_t^{A,\mathscr{V}})_{t>0}$

Let $A \in \mathcal{A}(\Omega)$ and let $\mathscr{V}$satisfy (3.5). In this section, we establish the L^p-contractivity and analyticity of the semigroup $(T_t^{A,\mathscr{V}})_{t>0}$. Regarding L^p-contractivity, the strategy is to apply Theorem 2.39. To this end, recall that two conditions must be verified:

- the domain of the form (3.3) associated with the semigroup must be invariant under the projection $\mathbf{P}_{B^p}$ onto $B^p = \{u \in L^2 \cap L^p : \|u\|_p \leqslant 1\}$;
- the form (3.3) must be L^p-dissipative.

We will show that the invariance property holds for every $p \in [1, \infty]$, while the L^p-dissipativity relies crucially on the p-ellipticity of the matrix A. In fact, the L^p-dissipativity of the form (9) is closely related to the generalized Hessian $H_{F_p}^A$, as we show in the next two lemmas.

Lemma 3.15. [21, Lemma B.6] *Let $u \in W^{1,2}(\Omega)$ and $p \in (1, \infty)$. The function $|u|^{p-2}u$ belongs to $W^{1,2}(\Omega)$ if and only if $|u|^{p-2}u \in L^2(\Omega)$ and $|u|^{p-2}\nabla u \in L^2(\Omega; \mathbb{C}^d)$. In this case,*

$$\nabla(|u|^{p-2}u) = \frac{p}{2}|u|^{p-2}\mathrm{sign}u \cdot \mathcal{I}_p(\mathrm{sign}\overline{u} \cdot \nabla u)\mathbb{1}_{\{u\neq 0\}}.$$

Consequently,

$$|\nabla(|u|^{p-2}u)| \sim |u|^{p-2}|\nabla u|\mathbb{1}_{\{u\neq 0\}}.$$

Lemma 3.16. *Let $B \in \mathbb{C}^{d\times d}$. Suppose that $p > 1$ and $u \in W^{1,2}(\Omega)$ are such that $|u|^{p-2}u \in W^{1,2}(\Omega)$. Then,*

$$H_{F_p}^B[u; \nabla u]\mathbb{1}_{\{u\neq 0\}} = p\,\mathrm{Re}\left\langle B\nabla u, \nabla\left(|u|^{p-2}\,u\right)\right\rangle \mathbb{1}_{\{u\neq 0\}}.$$

Proof. The claim follows by combining Lemma 3.14 and Lemma 3.15. □

We are now ready to prove the L^p-contractivity of the semigroup $(T_t^{A,\mathscr{V}})_{t>0}$. This result has been established progressively in increasingly general settings: initially for operators subject to Dirichlet boundary conditions [20, $(a) \Rightarrow (b)$ of Theorem 1.3], then extended to operators with Neumann and good mixed boundary conditions [41, Theorem 2] and [19, Proposition 1], and finally in [21, Theorem 1.2], under the basic assumption that the space $\mathscr{V}$ satisfies the condition (3.5). We reproduce the result in the form it appears in [21, Theorem 1.2], in the case $V = 0$.

Theorem 3.17. *Choose $p > 1$, $A \in \mathcal{A}(\Omega)$, $\phi \in \mathbb{R}$ such that $|\phi| < \pi/2 - \omega$ and $\Delta_p(e^{i\phi}A) \geqslant 0$. Then, for every $\mathscr{V}$ satisfying (3.5),*

$$\left(e^{-te^{i\phi}\mathscr{L}}\right)_{t>0}$$

extends to a strongly continuous semigroup of contractions on $L^p(\Omega)$.

Proof. We will apply Nittka's invariance criterion (Theorem 2.39). Under our assumptions on ϕ, the sesquilinear form $\mathfrak{b} := e^{i\phi}\mathfrak{a}$ is densely defined, closed and sectorial. In particular, it is accretive and, by Proposition 2.34, continuous. Hence, $\mathfrak{b}$ fits within the framework of Nittka's criterion. The operator associated with $\mathfrak{b}$ is $e^{i\phi}\mathscr{L}^A$.

In order to apply Theorem 2.39 we must check the following:

(i) $\mathrm{D}(\mathfrak{b}) = \mathrm{D}(\mathfrak{a})$ is invariant under $\mathbf{P}_{B^p}$;

(ii) $\mathfrak{b}$ is L^p-dissipative.

Let us start with (i). Let $-\Delta$ denote the operator associated with the form $\mathfrak{a}_{I,\mathscr{V}}$. By the basic assumption (3.5) and [72, Theorem 4.9], the semigroup $(e^{t\Delta})_{t>0}$ is contractive on $L^\infty(\Omega)$, and thus, by interpolation with the L^2-estimates and by duality, on $L^p(\Omega)$ for all $p \in [1,\infty]$. Hence Nittka's Theorem 2.39 gives that $\mathrm{D}(\mathfrak{a}_{I,\mathscr{V}})$ is invariant under $\mathbf{P}_{B^p}$. Now use that $\mathrm{D}(\mathfrak{a}_{I,\mathscr{V}}) = \mathrm{D}(\mathfrak{a}) = \mathrm{D}(\mathfrak{b})$.

Prove (ii). Recall the definition (2.25) of $\mathrm{D}_p(\mathfrak{b})$. Let $u \in \mathrm{D}_p(\mathfrak{b})$. By Lemma 3.16, applied with $B = e^{i\phi}A$, we get

$$\operatorname{Re}\mathfrak{b}(u,|u|^{p-2}u) = p^{-1}\int_\Omega H_{F_p}^{e^{i\phi}A}[u;\nabla u].$$

Hence the statement (ii) follows from Lemma 3.14. □

We reproduce now [21, Corollary 1.3] in the case when $V = 0$. See also [19, Lemma 17].

Corollary 3.18. *Choose $p > 1$, $A \in \mathcal{A}_p(\Omega)$. Let $\mathscr{V}$ satisfy (3.5). Then there exists $\vartheta = \vartheta(p,A) > 0$ such that if $|1-2/r| \leqslant |1-2/p|$, then $\{T_z : z \in \mathbf{S}_\vartheta\}$ is analytic and contractive in $L^r(\Omega)$.*

Proof. By Lemma 3.7 (i),(iv) and (v), there exists $\vartheta > 0$ such that $\Delta_r(e^{i\phi}A) > 0$ for all $\phi \in [-\vartheta,\vartheta]$ and all $r > 1$ satisfying $|1-2/r| \leqslant |1-2/p|$. The contractivity part now follows from Theorem 3.17 and the relation

$$T_{te^{i\phi}} = \exp\left(-te^{i\phi}\mathscr{L}\right), \tag{3.25}$$

whereupon analyticity is a consequence of Theorem 2.9. □

3.5 A bilinear embedding for (unperturbed) divergence-form operators

We prove now the main result of this chapter: the dimension-free bilinear embedding. This result has been established progressively in increasingly general settings. Initially, in [35], the matrices were supposed to be equal and real, and Ω was the whole space $\mathbb{R}^d$. In this case, the bilinear embedding holds for all $p \in (1,\infty)$. It was later extended in [20] to all complex p-elliptic matrices (possibly different), while still assuming that $\Omega = \mathbb{R}^d$. This is a generalization of the previous case due to Lemma 3.7(vii). Finally, the assumption that Ω coincides with the whole space $\mathbb{R}^d$ was removed in [19]. In all three cases, a heat-flow method associated with a Bellman function was employed.

However, when working with general open sets, new difficulties arise that prevent the direct application of the heat-flow method as done in [20, 35]. In fact, in [20], reducing to smooth matrices A, B with bounded derivatives allowed the authors to justify each step of the heat-flow method, since in that setting they could exploit either the boundedness on L^∞ of the semigroups $(T_t^A)_{t>0}$ and $(T_t^B)_{t>0}$ [20, Sections 4.1 & 6] or their analyticity on $L^r(\mathbb{R}^d)$ with $\mathrm{D}(\mathscr{L}_r^A) = \mathrm{D}(\mathscr{L}_r^B) = W^{2,r}(\mathbb{R}^d)$, for all $1 < r < \infty$ [19, Section 6.2]. As explained in [19, Section 1.4], for divergence-form operators $\mathscr{L}^A$ on $\Omega \subset \mathbb{R}^d$ with, say, Neumann boundary conditions $\mathscr{V} = W^{1,2}(\Omega)$ the situation is different. On one hand, extrapolation of T_t^A in $L^\infty(\Omega)$ is not expected, even for complex constant A [19, Appendix A.5], and it is not clear if there exists an operator core of bounded functions for $\mathscr{L}_p^A$, $1 < p < \infty$. One the other hand, even the domain of the Neumann Laplacian $\mathscr{L}^I$ in L^p is unknown and, in general, it is not included in $W^{2,p}(\Omega)$ [25, 45] or even in $W^{1,p}(\Omega)$ [52]. For general open sets, working with smooth matrices therefore provides no advantage.

Consequently, in [19], it was necessary to modify the Bellman-heat-flow argument used in [20]. We shall present the new method of [19] in this section.

Before enunciating the theorem, we introduce further notation. For $A, B \in \mathcal{A}_p(\Omega)$ we denote

$$\begin{aligned} \lambda(A,B) &= \min\{\lambda(A), \lambda(B)\}, \\ \Lambda(A,B) &= \max\{\Lambda(A), \Lambda(B)\}, \\ \Delta_p(A,B) &= \min\{\Delta_p(A), \Delta_p(B)\}. \end{aligned} \tag{3.26}$$

Theorem 3.19. ([20, Theorem 1.1],[19, Theorem 2]) *Suppose that $\Omega \subseteq \mathbb{R}^d$ is an open set. Let $\mathscr{V}$ and $\mathscr{W}$ be two closed subspaces of $W^{1,2}(\Omega)$ satisfying Assumption BE. Let $p > 1$, $q = p/(p-1)$ and $A, B \in \mathcal{A}(\Omega)$. Suppose that A, B are p-elliptic. Then there exists $C > 0$, depending only on $p, \Delta_p(A,B), \lambda(A,B)$ and $\Lambda(A,B)$, such that*

$$\int_0^\infty \int_\Omega \left|\nabla T_t^{A,\mathscr{V}} f(x)\right| \left|\nabla T_t^{B,\mathscr{W}} g(x)\right| \,\mathrm{d}x \,\mathrm{d}t \leqslant C \|f\|_p \|g\|_q, \tag{3.27}$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

We will provide the proof in Section 3.5.4. Theorem 3.19 was further extended in [21] to divergence-form operators with nonnegative potentials, by only assuming the p-ellipticity of the matrices.

Remark 3.20. Recently, by adapting the heat-flow method from [19], Böhnlein, Egert and Rehberg proved in [12, Theorem 1.4] a bilinear embedding for (unperturbed) elliptic operators in divergence form, subject to dynamical boundary conditions, provided that the coefficient matrices are p-elliptic.

Remark 3.21. In general, when $\Delta_p(A,B) < 0$ the *dimension-free* bilinear embedding (3.27) fails. For all $p \in (1,\infty)$ denote

$$\phi_p := \arccos|1 - 2/p|.$$

It was showed in [20, Proposition 1.2] that for all $\phi \in (\phi_p, \pi/2)$ the quantity

$$\sup_{f,g \in C_c^\infty \setminus \{0\}} \frac{1}{\|f\|_p \|g\|_q} \int_0^\infty \int_{\mathbb{R}^d} |\nabla T_t^{e^{i\phi} I_d} f(x)| |\nabla T_t^{e^{-i\phi} I_d} g(x)| \,\mathrm{d}x \,\mathrm{d}t$$

is unbounded as d ranges over $\mathbb{N}_+$. This result follows from determining the L^p-contractivity domain of the classical heat semigroup with complex time.

The angle ϕ_p above is known to be the optimal angle of holomorphy of symmetric contraction semigroup on L^p [17, 48, 56, 61]. Its complementary angle, $\pi/2 - \phi_p$, was proven in [17] to be the optimal angle in holomorphic functional calculus for generators of symmetric contraction semigroups.

3.5.1 The Bellman function of Nazarov and Treil

We want to study the monotonicity of the flow

$$\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^{A,\mathscr{V}} f, T_t^{B,\mathscr{W}} g)$$

associated with a particular explicit *Bellman function* $\mathcal{Q}$ invented by Nazarov and Treil [67]. Here we use a simplified variant introduced in [35] which comprises only two variables:

$$\mathcal{Q}(\zeta,\eta) = |\zeta|^p + |\eta|^q + \delta \begin{cases} |\zeta|^2 |\eta|^{2-q}, & |\zeta|^p \leqslant |\eta|^q; \\ (2/p)\,|\zeta|^p + (2/q - 1)\,|\eta|^q, & |\zeta|^p \geqslant |\eta|^q\,, \end{cases} \tag{3.28}$$

where $p > 2$, $q = p/(p-1)$, $\zeta, \eta \in \mathbb{C}$ and $\delta > 0$ is a positive parameter that will be fixed later. It was noted in [20, p. 3195] that $\mathcal{Q} \in C^1(\mathbb{C}^2) \cap C^2(\mathbb{C}^2 \setminus \Upsilon)$, where

$$\Upsilon = \{\eta = 0\} \cup \{|\zeta|^p = |\eta|^q\},$$

and that for $(\zeta, \eta) \in \mathbb{C} \times \mathbb{C}$ we have

$$\begin{aligned} 0 \leqslant \mathcal{Q}(\zeta,\eta) &\lesssim_{p,\delta} (|\zeta|^p + |\eta|^q), \\ |(\partial_\zeta \mathcal{Q})(\zeta,\eta)| &\lesssim_{p,\delta} \max\{|\zeta|^{p-1}, |\eta|\}, \\ |(\partial_\eta \mathcal{Q})(\zeta,\eta)| &\lesssim_{p,\delta} |\eta|^{q-1}. \end{aligned} \tag{3.29}$$

One of the earliest examples of what is now known as the Bellman function technique is the construction of the original Nazarov–Treil function, introduced shortly beforehand in harmonic analysis by Nazarov et al. [65]. The term "Bellman function" originates from stochastic optimal control theory (see [66] for further details). In the same work [66], the authors also elucidate the connection between the Nazarov–Treil–Volberg framework and Burkholder's earlier contributions to martingale inequalities [13–15]. For a comprehensive account of recent developments in martingale inequalities, we refer the reader to [70]. An overview of the origins of Bellman functions and of the general method can be found in [67, 86, 88]. Notable recent applications of Bellman heat-flow techniques are presented in [11, 12, 16–22, 28, 31, 33–36, 55, 63, 75, 89].

In [20] the authors established the (A, B)-convexity of the Bellman function $\mathcal{Q}$ under the assumption that the matrices A and B are p-elliptic. We present the result as stated in [21, Theorem 3.1] which also includes a lower bound for the first-order derivatives of $\mathcal{Q}$.

Theorem 3.22. [21, Theorem 3.1] *Choose $p \geqslant 2$ and $A, B \in \mathcal{A}_p(\Omega)$. Then there exists a continuous function $\tau : \mathbb{C}^2 \to [0, +\infty)$ such that $\tau^{-1} = 1/\tau$ is locally integrable on $\mathbb{C}^2 \setminus \{(0,0)\}$, and $\delta \in (0,1)$ such that $\mathcal{Q} = \mathcal{Q}_{p,\delta}$ as in (11) admits the following properties:*

(i) *for any $\omega = (\zeta, \eta) \in \mathbb{C}^2 \setminus \Upsilon$, $X, Y \in \mathbb{C}^d$, and a.e. $x \in \Omega$, we have*

$$H_{\mathcal{Q}}^{(A(x),B(x))}[\omega, (X,Y)] \gtrsim \tau|X|^2 + \tau^{-1}|Y|^2;$$

(ii) *for any $\omega = (\zeta, \eta) \in \mathbb{C}^2$, we have*

$$(\partial_\zeta \mathcal{Q})(\zeta,\eta) \cdot \zeta \gtrsim \tau|\zeta|^2 \quad \text{and} \quad (\partial_\eta \mathcal{Q})(\zeta,\eta) \cdot \eta \gtrsim \tau^{-1}|\eta|^2.$$

The implied constants depending on p, A, B, but not on the dimension d.

We may take $\tau(\zeta,\eta) = \max\{|\zeta|^{p-2}, |\eta|^{2-q}\}$.

In the region $\{|\zeta|^p > |\eta|^q\}$, the Bellman function $\mathcal{Q}$ is a linear combination of power functions. Thus, the strictly (A, B)-convexity of $\mathcal{Q}$ in such region follows directly from Lemma 3.14 together with the inequality between the arithmetic and geometric mean. In the complementary region, $\{|\zeta^p < |\eta|^q\}$, the argument is more delicate due to the presence of the term $|\zeta|^2|\eta|^{2-q}$. Nevertheless, by choosing δ sufficiently small and exploiting the B-convexity of F_q along with the A-convexity of F_2, one can establish the desired estimate in this region as well. For the detailed proof, see [20, Theorem 5.2] or [21, Theorem 3.1].

Remark 3.23. A careful examination of the proof of [21, Theorem 3.1] shows that the conclusion of the previous theorem holds for all parameters smaller than the specific δ given above, provided that τ is chosen accordingly, depending on δ. More precisely, there exists $\delta_0 \in (0,1)$ such that for all $\delta \in (0, \delta_0)$ there exist $C = C(\delta) > 0$ and $\tau = \tau_\delta : \mathbb{C}^2 \setminus \Upsilon \to (0, +\infty)$ such that, a.e. $x \in \Omega$,

$$H_{\mathcal{Q}}^{(A(x),B(x))}[\omega, (X,Y)] \geqslant 2\delta C(\delta) \left(\tau|X|^2 + \tau^{-1}|Y|^2\right),$$

for all $\omega \in \mathbb{C}^2 \setminus \Upsilon$ and $X, Y \in \mathbb{C}^d$.

Here τ may be choosen as

$$\tau(\zeta,\eta) = \rho(\zeta,\eta) := \begin{cases} (p-1)|\zeta|^{p-2}, & |\zeta|^p \geqslant |\eta|^q > 0, \\ D|\eta|^{2-q}, & |\zeta|^p > |\eta|^q, \end{cases} \tag{3.30}$$

where $D = D(\delta)$ is a positive constant depending on $\delta, p, \lambda(A), B$ such that

$$C(\delta)D^{-1}(\delta) \to +\infty, \quad \text{as } \delta \searrow 0. \tag{3.31}$$

Regularization of $\mathcal{Q}$

Denote by $*$ the convolution in $\mathbb{R}^4$ and let $(\varphi_\nu)_{\nu>0}$ be a nonnegative, smooth, and compactly supported approximation of the identity on $\mathbb{R}^4$. Explicitly, $\varphi_\nu(y) = \nu^{-4}\varphi(y/\nu)$, where φ is smooth, nonnegative, radial, of integral 1, and supported in the closed unit ball in $\mathbb{R}^4$. If $\Phi : \mathbb{C}^2 \to \mathbb{R}$, define $\Phi * \varphi_\nu = (\Phi_{\mathcal{W}} * \varphi_\nu) \circ \mathcal{W} : \mathbb{C}^2 \to \mathbb{R}$. Explicitly, for $\omega \in \mathbb{C}^2$,

$$\begin{aligned} (\Phi * \varphi_\nu)(\omega) &= \int_{\mathbb{R}^4} \Phi_{\mathcal{W}}(\mathcal{W}(\omega) - z)\varphi_\nu(z)\,\mathrm{d}z \\ &= \int_{\mathbb{R}^4} \Phi(\omega - \mathcal{W}^{-1}(z))\varphi_\nu(z)\,\mathrm{d}z. \end{aligned} \tag{3.32}$$

We shall need estimates of the first-order partial derivatives and the generalized Hessian of $\mathcal{Q} * \varphi_\nu$. As a consequence of (3.29) we have

$$\begin{aligned} &0 \leqslant (\mathcal{Q} * \varphi_\nu)(\zeta,\eta) \lesssim_{p,\delta} (|\zeta| + \nu)^p + (|\eta| + \nu)^q, \\ &|\partial_\zeta(\mathcal{Q} * \varphi_\nu)(\zeta,\eta)| \lesssim_{p,\delta} \max\{(|\zeta| + \nu)^{p-1}, |\eta| + \nu\}, \\ &|\partial_\eta(\mathcal{Q} * \varphi_\nu)(\zeta,\eta)| \lesssim_{p,\delta} (|\eta| + \nu)^{q-1}, \end{aligned} \tag{3.33}$$

for all $\zeta, \eta \in \mathbb{C}$ and $\nu \in (0,1)$. See also [20, Corollary 5.4]. In addition, the following estimates hold uniformly in ν.

Lemma 3.24. *There exists $C = C(p,\delta) > 0$ such that*

(i) $|(\mathcal{Q} * \varphi_\nu)(\omega)| \leqslant C(|\omega|^p + |\omega|^q + 1)$;

(ii) $|D(\mathcal{Q} * \varphi_\nu)(\omega)| \leqslant C(|\omega|^{p-1} + |\omega|^{q-1})$;

(iii) $|D^2(\mathcal{Q} * \varphi_\nu)(\omega)| \leqslant \nu^{q-2}C(|\zeta|^{p-2} + |\eta|^{2-q} + 1)$

for all $\nu \in (0,1]$ and $\omega = (\zeta,\eta) \in \mathbb{C} \times \mathbb{C}$.

Proof. See the proof of [19, Lemma 14]. □

Regarding the (A,B)-convexity of the regularized Bellman function, we have the following result.

Corollary 3.25. [20, Corollary 5.5] *Let $p \geqslant 2$. Suppose that $A, B \in \mathcal{A}_p(\Omega)$. Then there exists $\delta \in (0,1)$ such that for $\mathcal{Q} = \mathcal{Q}_{p,\delta}$ a above and $\nu > 0$ we have, for almost every $x \in \Omega$,*

$$H^{(A(x),B(x))}_{\mathcal{Q}*\varphi_\nu}[\omega;(X,Y)] \gtrsim |X||Y|,$$

for any $\omega = (\zeta,\eta) \in \mathbb{C}^2$ and $X, Y \in \mathbb{C}^d$.

Proof. Since $\mathcal{Q}_{\mathcal{W}} \in C^1(\mathbb{R}^4)$ and its second-order partial derivatives exist on $\mathbb{R}^4 \setminus \mathcal{W}(\Upsilon)$ and are locally integrable in $\mathbb{R}^4$, by the ACL characterization of Sobolev spaces (see, for example, [80, Théorème V, p. 57] or [57, Theorem 11.45]) we have

$$H^{(A,B)}_{\mathcal{Q}*\varphi_\nu}[\omega;(X,Y)] = \int_{\mathbb{R}^4} H^{(A,B)}_{\mathcal{Q}}[\omega - \mathcal{W}^{-1}_{2,1}(z);(X,Y)]\varphi_\nu(z)\,\mathrm{d}z. \tag{3.34}$$

By combining Theorem 3.22(i) with the inequality between the arithmetic and geometric mean, we deduce

$$H^{(A,B)}_{\mathcal{Q}}[\omega;(X,Y)] \gtrsim |X||Y|, \tag{3.35}$$

for all $\omega \in \mathbb{C}^2 \setminus \Upsilon$ and $X, Y \in \mathbb{C}^d$. We conclude by merging (3.34) and (3.35). □

3.5.2 The heat-flow method

The main reason for introducing the notion of convexity with respect to complex matrices (generalized convexity) is its link with the monotonicity of certain functionals associated with semigroups [17–21]. In what follows we explain this link at a formal level. In the applications, the justification of the formal passages is part of the problem and it will not be addressed here.

Let $\Omega \subseteq \mathbb{R}^d$, $A, B \in \mathcal{A}(\Omega)$ and $\mathscr{V}, \mathscr{W}$ of the type described in Section 3.1.1. Let $\Phi : \mathbb{C}^2 \to \mathbb{R}_+$ be of class C^1. Given $f, g \in L^2(\Omega)$, define the function

$$\mathcal{E}(t) = \int_\Omega \Phi\left(T^{A,\mathscr{V}}_t f, T^{B,\mathscr{W}}_t g\right), \quad t > 0.$$

a) Suppose that we can differentiate and interchange derivative and integral. Then a calculation (see [20]) shows that

$$\begin{aligned} -\mathcal{E}'(t) = 2\mathrm{Re} \int_\Omega \Big(&\partial_\zeta \Phi\left(T^{A,\mathscr{V}}_t f, T^{B,\mathscr{W}}_t g\right) \mathscr{L}^A T^{A,\mathscr{V}}_t f \\ &+ \partial_\eta \Phi\left(T^{A,\mathscr{V}}_t f, T^{B,\mathscr{W}}_t g\right) \mathscr{L}^B T^{B,\mathscr{W}}_t g \Big). \end{aligned} \tag{3.36}$$

b) Suppose that $\Phi \in C^2$ and that $\partial_\zeta \Phi(T^A_t f, T^B_t g)$ and $\partial_\eta \Phi(T^A_t f, T^B_t g)$ belong to the form domains $\mathscr{V}$ and $\mathscr{W}$, respectively. Then we can integrate by parts in the sense of (3.4) on the right-hand side of (3.36) and, by applying the chain rule (see [20]), we get

$$-\mathcal{E}'(t) = \int_\Omega H^{(A,B)}_\Phi \left[\left(T^{A,\mathscr{V}}_t f, T^{B,\mathscr{W}}_t g\right); \left(\nabla T^{A,\mathscr{V}}_t f, \nabla T^{B,\mathscr{W}}_t g\right)\right]. \tag{3.37}$$

It follows that if Φ is (A,B)-convex on $\mathbb{C}^2$ then the function $\mathcal{E}$ is nonincreasing on $(0,+\infty)$. When Φ is *strictly* (A,B)-convex and satisfies a suitable size estimate, this formal method can be used for proving bilinear inequalities in the spirit of [17–21].

Since the Bellman function $\mathcal{Q}$ satisfies a suitable estimate, namely the first estimate in (3.29), and is strictly (A,B)-convex, one may consider applying the method with $\Phi = \mathcal{Q}$. Indeed, from the first estimate in (3.29) we immediately obtain the following *upper bound*:

$$-\int_0^\infty \mathcal{E}'(t)\,\mathrm{d}t \leqslant \mathcal{E}(0) \lesssim \|f\|_p^p + \|g\|_q^q.$$

On the other hand, if (3.37) held with $\Phi = \mathcal{Q}$, the strict (A,B)-convexity of $\mathcal{Q}$, namely (3.35), would yield the following *lower bound*:

$$-\mathcal{E}'(t) \gtrsim \int_\Omega |\nabla T^{A,\mathscr{V}}_t f||\nabla T^{B,\mathscr{W}}_t g|.$$

Combining these two estimates would lead to

$$\int_0^\infty \int_\Omega \left|\nabla T_t^{A,\mathscr{V}} f(x)\right| \left|\nabla T_t^{B,\mathscr{W}} g(x)\right| \,\mathrm{d}x\,\mathrm{d}t \lesssim \left(\|f\|_p^p + \|g\|_q^q\right). \tag{3.38}$$

Once (3.38) is proved for all $f, g \in (L^p \cap L^q)(\Omega)$, (3.27) follows by replacing f and g in (3.38) with sf and $s^{-1}g$ and minimizing the right-hand side with respect to $s > 0$.

However, we cannot apply the heat-flow method directly to the function $\mathcal{Q}$. First, in order to obtain (3.37), the chain rule must be applied to

$$\nabla[\partial_\zeta \mathcal{Q}(T_t^{A,\mathscr{V}} f, T_t^{B,\mathscr{W}} g)] \quad \text{and} \quad \nabla[\partial_\eta \mathcal{Q}(T_t^{A,\mathscr{V}} f, T_t^{B,\mathscr{W}} g)],$$

which produces the Hessian matrix of $\mathcal{Q}$. By standard results on the chain rule for Sobolev functions, this step demands that $\mathcal{Q}$ be of class C^2. For this reason, $\mathcal{Q}$ is first replaced by its smooth regularization $\mathcal{Q} * \varphi_\nu$. However, a further obstruction appears when attempting to apply the method to $\Phi = \mathcal{Q} * \varphi_\nu$. In order to integrate by parts in (3.36), we need

$$\partial_\zeta \mathcal{Q}(T_t^{A,\mathscr{V}} f, T_t^{B,\mathscr{W}} g) \in \mathscr{V} \quad \text{and} \quad \partial_\eta \mathcal{Q}(T_t^{A,\mathscr{V}} f, T_t^{B,\mathscr{W}} g) \in \mathscr{W}.$$

This inclusion cannot be guaranteed by Proposition 3.4, since $\partial_\zeta(\mathcal{Q} * \varphi_\nu)$ and $\partial_\zeta(\mathcal{Q} * \varphi_\nu)$ are not Lipschitz continuous.

The idea is therefore to approximate $\mathcal{Q}$ by a suitably constructed sequence of functions. In the next section, we briefly recall the definition of this approximating sequence and state the main properties that will allow us to apply the heat-flow method. For a detailed account of the construction and motivation behind this sequence, we refer the reader to [19]. See also [22, Sections 7.0-7.5]. Roughly speaking, the construction in [19] proceeds by multiplying $\mathcal{Q} * \varphi_\nu$ with compactly supported smooth functions that are identically equal to 1 on gradually larger sets, and then adding a carefully designed correction term that restores (A, B)–convexity. This yields a family of (A, B)-convex approximants, to which the heat-flow method can be applied, and Fatou's lemma can be subsequently invoked in the limiting argument. In Chapter 6 we will provide a new approximating sequence. As in [19], we multiply $\mathcal{Q} * \varphi_\nu$ by compactly supported smooth cutoffs that are equal to 1 on larger and larger sets; however, in our construction these sets are tailored to the specific structure of $\mathcal{Q}$, and no additional correction term is added. This finer geometric adaptation allows us to work directly with the resulting sequence and to pass to the limit by means of the Lebesgue dominated convergence theorem, rather than relying only on Fatou's lemma.

3.5.3 The approximation sequence $(\mathcal{R}_{n,\nu})_{n\in\mathbb{N}_+}$ of [19]

Let $p > 2$ and $A, B \in \mathcal{A}_p(\Omega)$. By Lemma 3.7(iv) there exists $\varepsilon > 0$ such that $\Delta_{p+\varepsilon}(A, B) > 0$. For this particular $\varepsilon > 0$ and all $n \in \mathbb{N}_+$ define f_n by

$$f_n(t) := \begin{cases} n^{-\varepsilon} t^{p+\varepsilon}, & 0 \leqslant t \leqslant n, \\ \frac{p+\varepsilon}{2} n^{p-2} t^2 + (1 - \frac{p+\varepsilon}{2}) n^p, & t \geqslant n. \end{cases}$$

For every $k \in \mathbb{N}_+$ define $\mathcal{F}_n : \mathbb{C}^k \to \mathbb{R}_+$ by

$$\mathcal{F}_n(\omega) := f_n(|\omega|), \quad \omega \in \mathbb{C}^k.$$

Let $K_{p+\varepsilon}$ be the constant defined in [19, (23)]. Its role is to ensure that the following function is (A, B)-convex on $\mathbb{C}^2$ [19, Proposition 11]:

$$\mathcal{P}_n(\omega) := \mathcal{F}_n(\omega) + K_{p+\varepsilon}\left(\mathcal{F}_n(\zeta) + \mathcal{F}_n(\eta)\right), \quad \omega = (\zeta, \eta) \in \mathbb{C} \times \mathbb{C}. \tag{3.39}$$

Let $q = p/(p-1)$ and $\mathcal{Q} = \mathcal{Q}_\delta$ denote the Nazarov-Treil Bellman function introduced in (11) with $\delta > 0$ chosen so that Theorem 3.22 holds true. Fix a radial function $\varphi \in C_c^\infty(\mathbb{R}^4)$ such that $0 \leqslant \varphi \leqslant 1$, $\operatorname{supp}\varphi \subset B_{\mathbb{R}^4}(0,1)$ and $\int \varphi = 1$. Also, fix a radial function $\psi \in C_c^\infty(\mathbb{C}^2)$ such that $\psi \geqslant 0$, $\psi = 1$ on $\{|\omega| \leqslant 3\}$ and $\psi = 0$ on $\{|\omega| > 4\}$. For $\nu \in (0,1]$ and $n \in \mathbb{N}_+$ define $\varphi_\nu(\omega) = \nu^{-4}\varphi(\omega/\nu)$ and $\psi_n(\omega) = \psi(\omega/n)$.

Recall the notations (3.19) and (3.32). For every $n \in \mathbb{N}_+$ and all $\nu \in (0,1]$, define

$$\begin{aligned} \mathcal{Q}_{n,\nu} &:= \psi_n \cdot (\mathcal{Q} * \varphi_\nu); \\ \mathcal{R}_{n,\nu} &:= \mathcal{Q}_{n,\nu} + C_1 \nu^{q-2}(\mathcal{P}_n * \varphi_\nu), \end{aligned} \tag{3.40}$$

where $C_1 = C_1(p, A, B, \psi) > 0$ is a constant not depending on ν which was fixed in [19, Theorem 16] to achieve the (A,B)-convexity of $\mathcal{R}_{n,\nu}$ on all $\mathbb{C}^2$ for all $n \in \mathbb{N}_+$ and $\nu \in (0,1)$.

We collect the main properties of $\mathcal{R}_{n,\nu}$ in the following theorem, proved in [19].

Theorem 3.26. [19, Theorem 16] *Let $\nu \in (0,1]$. There exists $C_1 = C_1(p, A, B, \psi) > 0$, not depending on ν, such that $\mathcal{R}_{n,\nu}$ is (A,B)-convex in $\mathbb{C}^2$ for all $n \in \mathbb{N}_+$. Moreover, the following statements hold.*

(i) $D^2\mathcal{R}_{n,\nu} \in L^\infty(\mathbb{C}^2; \mathbb{R}^{4\times 4})$.

(ii) *We have*

$$\begin{aligned} D\mathcal{R}_{n,\nu} &\to D(\mathcal{Q} * \varphi_\nu), \\ D^2\mathcal{R}_{n,\nu} &\to D^2(\mathcal{Q} * \varphi_\nu) \end{aligned}$$

pointwise in $\mathbb{C}^2$ as $n \to \infty$.

(iii) *For any $n \in \mathbb{N}_+$ there exists $C(n, \nu, C_1) > 0$ such that*

$$|(D\mathcal{R}_{n,\nu})(\omega)| \leqslant C(n,\nu)|\omega|, \quad \forall \omega \in \mathbb{C}^2.$$

(iv) *There exists $C = C(\nu) > 0$ that does not depend on n such that*

$$|(D\mathcal{R}_{n,\nu})(\omega)| \leqslant C(\nu)\left(|\omega|^{p-1} + |\omega|^{q-1}\right),$$

for all $\omega \in \mathbb{C}^2$, $n \in \mathbb{N}_+$ and $\nu \in (0,1]$.

(v) *For any $n \in \mathbb{N}_+$ and $\nu > 0$ we have*

$$(\partial_\zeta \mathcal{R}_{n,\nu})(0,\eta) = 0, \quad (\partial_\eta \mathcal{R}_{n,\nu})(\zeta, 0) = 0,$$

for all $\zeta, \eta \in \mathbb{C}$.

Moreover, it is easy to see that there exists $C = C(\nu) > 0$ that does not depend on n such that

$$|D^2\mathcal{R}_{n,\nu}(\omega)| \leqslant C(\nu)(|\omega|^{p-2} + 1), \tag{3.41}$$

for all $\omega \in \mathbb{C}^2$, $n \in \mathbb{N}_+$ and $\nu \in (0,1)$.

The sequence $(\mathcal{R}_{n,\nu})_{n\in\mathbb{N}_+}$ has also been exploited to prove the bilinear embedding in [12, Theorem 1.4].

This approximating sequence was constructed, among other reasons, to allow integration by parts in (3.36). This can be justified once we know that

$$\partial_\zeta \mathcal{R}_{n,\nu}(T_t^{A,\mathscr{V}} f, T_t^{B,\mathscr{W}} g) \in \mathscr{V} \quad \text{and} \quad \partial_\eta \mathcal{R}_{n,\nu}(T_t^{A,\mathscr{V}} f, T_t^{B,\mathscr{W}} g) \in \mathscr{W}.$$

This necessary property was established in [19, Lemma 19]. However, as already mentioned earlier in Section 3.1.1, the statement of that lemma is incorrect, since it does not hold for arbitrary choices of spaces $\mathscr{V}$ and $\mathscr{W}$ of the type described in (a)–(d). To ensure its validity, we need the Assumption BE.

Lemma 3.27. *Suppose that $(\mathscr{V},\mathscr{W})$ satisfies Assumption BE. Let $u\in\mathscr{V}$ and $v\in\mathscr{W}$. Then*

$$\partial_\zeta\mathcal{R}_{n,\nu}(u,v)\in\mathscr{V}\quad\text{and}\quad\partial_\eta\mathcal{R}_{n,\nu}(u,v)\in\mathscr{W},$$

for all $n\in\mathbb{N}_+$ and $\nu>0$.

Proof. From Theorem 3.26(i), the functions $\partial_\zeta\mathcal{R}_{n,\nu}$ and $\partial_\eta\mathcal{R}_{n,\nu}$ are in particular Lipschitz continuous on $\mathbb{C}^2$ for all $n\in\mathbb{N}$ and $\nu\in(0,1)$. Hence, by also suing Theorem 3.26(v), we can apply Proposition 3.4 with $\Phi=\partial_\zeta\mathcal{R}_{n,\nu}$ and $\Phi=\partial_\eta\mathcal{R}_{n,\nu}$, and conclude. □

Remark 3.28. Proposition 3.5 shows the necessity of Assumption BE for the validity of Lemma 3.27. In fact, a trivial computation shows that

$$\begin{aligned}\partial_\zeta\mathcal{Q}(1,2^{1/q})&=\frac{p}{2}+\delta\cdot2^{(2-q)/2}>\frac{p}{2}+\delta=\partial_\zeta\mathcal{Q}(1,0),\\ \partial_\zeta\mathcal{P}_n(1,2^{1/q})&=\frac{p+\varepsilon}{2}n^{-\varepsilon}\left[K_{p+\varepsilon}+(1+2^{2/q})^{(p+\varepsilon-2)/2}\right]\\&>\frac{p+\varepsilon}{2}n^{-\varepsilon}\left[K_{p+\varepsilon}+1\right]=\partial_\zeta\mathcal{P}_n(1,0),\end{aligned}$$

for all $n\geqslant(1+2^{2/q})^{1/2}$. Therefore, since both $\mathcal{Q}$ and $\mathcal{P}_n$ are continuous on $\mathbb{C}^2$, for all $n\geqslant(1+2^{2/q})^{1/2}$ there exists $\nu_0(n)$ such that for any $\nu\in(0,\nu_0(n))$ we have

$$\begin{aligned}\partial_\zeta\mathcal{R}_{n,\nu}(1,2^{1/q})&=\partial_\zeta(\mathcal{Q}*\varphi_\nu)(1,2^{1/q})+C_1\nu^{q-2}\partial_\zeta(P_n*\varphi_\nu)(1,2^{1/q})\\&>\partial_\zeta(\mathcal{Q}*\varphi_\nu)(1,0)+C_1\nu^{q-2}\partial_\zeta(P_n*\varphi_\nu)(1,0)=\partial_\zeta\mathcal{R}_{n,\nu}(1,0).\end{aligned}$$

Hence, Proposition 3.5 implies that there exist $D,D'\subseteq\partial\Omega$ closed, $u\in W^{1,2}_D(\Omega)$ and $v\in\widetilde{W_{D'}}^{1,2}(\Omega)$ such that $\partial_\zeta\mathcal{R}_{n,\nu}(u,v)\notin W^{1,2}_D(\Omega)$ for all $n\geqslant(1+2^{2/q})^{1/2}$ and $\nu\in(0,\nu_0(n))$.

3.5.4 Proof of the bilinear embedding

By Lemma 3.7(i), we have $\Delta_p(A,B)=\Delta_q(A,B)$. So, thanks to the symmetry of Assumption BE, it suffices to prove (3.38) when $p\geqslant2$. Fix $p\geqslant2$, $A,B\in\mathcal{A}(\Omega)$ such that $\Delta_p(A,B)>0$ and $\mathscr{V}$ and $\mathscr{W}$ satisfying Assumption BE.

We now start the heat-flow method of Section 3.5.2, but for simplicity we omit the superscript $\mathscr{V}$ and $\mathscr{W}$.

Proposition 3.29. *Let $p\geqslant2$. Let A,B, $\mathscr{V},\mathscr{W}$ as above. Assume that*

$$\int_\Omega|\nabla u||\nabla v|\lesssim2\,\mathrm{Re}\int_\Omega\Big((\partial_\zeta\mathcal{Q})(u,v)\mathscr{L}^Au+(\partial_\eta\mathcal{Q})(u,v)\mathscr{L}^Bv\Big)\tag{3.42}$$

for $u\in\mathrm{D}(\mathscr{L}^A)$, $v\in\mathrm{D}(\mathscr{L}^B)$ such that $u,v,\mathscr{L}^Au,\mathscr{L}^Bv\in(L^p\cap L^q)(\Omega)$. Then the bilinear estimate (3.38) holds for all $f,g\in(L^p\cap L^q)(\Omega)$.

Proof. Fix $f,g\in(L^p\cap L^q)(\Omega)$. Define

$$\mathcal{E}(t)=\int_\Omega\mathcal{Q}\left(T^A_tf,T^B_tg\right),\quad t>0.$$

The estimates (3.29), the analyticity and the contractivity of $(T^{\mathscr{A}}_t)_{t>0}$ and $(T^{\mathscr{B}}_t)_{t>0}$ both in L^p and in L^q (see Theorem 4.22 and Corollary 4.24) and [19, Proposition C.1] imply that $\mathcal{E}$ is well defined, continuous on $[0,\infty)$, differentiable on $(0,\infty)$ with a continuous derivative and

$$-\mathcal{E}'(t)=2\,\mathrm{Re}\int_\Omega\left(\partial_\zeta\mathcal{Q}\left(T^A_tf,T^B_tg\right)\mathscr{L}^AT^A_tf+\partial_\eta\mathcal{Q}\left(T^A_tf,T^B_tg\right)\mathscr{L}^BT^B_tg\right).\tag{3.43}$$

The first estimate in (3.29) yields

$$-\int_0^\infty \mathcal{E}'(t)\,\mathrm{d}t \leqslant \mathcal{E}(0) \lesssim \|f\|_p^p + \|g\|_q^q. \tag{3.44}$$

Analyticity of the semigroups gives $T_t^A f \in \mathrm{D}(\mathscr{L}_p^A) \cap \mathrm{D}(\mathscr{L}_q^A)$ and $T_t^B g \in \mathrm{D}(\mathscr{L}_p^B) \cap \mathrm{D}(\mathscr{L}_q^B)$. By the consistency of the semigroups and Hölder's inequality, we have

$$\mathrm{D}(\mathscr{L}_p^A) \cap \mathrm{D}(\mathscr{L}_q^A) \subset \mathrm{D}(\mathscr{L}_2^A) \subset \mathscr{V}$$

and

$$\mathrm{D}(\mathscr{L}_p^B) \cap \mathrm{D}(\mathscr{L}_q^B) \subset \mathrm{D}(\mathscr{L}_2^B) \subset \mathscr{W}.$$

Therefore, (3.42) applied with the couple $(u,v) = (T_t^A f, T_t^B g)$, combined with (3.43) and (3.44), gives the desired estimate (3.38). □

Proof of (3.42)

By using Theorem 3.26 (ii) and (iv), Lemma 3.24(ii), the fact that $\mathcal{Q} \in C^1(\mathbb{R}^4)$ and Lebesgue's dominated convergence theorem twice, we deduce that

$$\begin{aligned} &2\mathrm{Re}\int_\Omega \Big((\partial_\zeta \mathcal{Q})(u,v)\mathscr{L}^A u + (\partial_\eta \mathcal{Q})(u,v)\mathscr{L}^B v\Big) \\ &= \lim_{\nu\to 0_+}\lim_{n\to+\infty} 2\mathrm{Re}\int_\Omega \Big((\partial_\zeta \mathcal{R}_{n,\nu})(u,v)\mathscr{L}^A u + (\partial_\eta \mathcal{R}_{n,\nu})(u,v)\mathscr{L}^B v\Big). \end{aligned} \tag{3.45}$$

By Lemma 3.27 we have $(\partial_\zeta \mathcal{R}_{n,\nu})(u,v) \in \mathscr{V}$ and $(\partial_\eta \mathcal{R}_{n,\nu})(u,v) \in \mathscr{W}$. Hence we can integrate by parts the integral on the right-hand side of (3.45) and, by means of the chain-rule for the composition of smooth functions with vector-valued Sobolev functions, deduce that

$$\begin{aligned} &2\mathrm{Re}\int_\Omega \Big((\partial_\zeta \mathcal{R}_{n,\nu})(u,v)\mathscr{L}^A u + (\partial_\eta \mathcal{R}_{n,\nu})(u,v)\mathscr{L}^B v\Big) \\ &= \int_\Omega H_{\mathcal{R}_{n,\nu}}^{(A,B)}\left[(u,v)\,;(\nabla u,\nabla v)\right]. \end{aligned} \tag{3.46}$$

By Theorem 3.26, the function $\mathcal{R}_{n,\nu}$ is (A,B)-convex in $\mathbb{C}^2$, so the integrand on the right-hand side of (3.46) is nonnegative for all $n \in \mathbb{N}_+$. Hence, by Fatou's lemma, Theorem 3.26 (ii) and Corollary 3.25,

$$\begin{aligned} \lim_{n\to+\infty} 2\mathrm{Re}\int_\Omega &\Big((\partial_\zeta \mathcal{R}_{n,\nu})(u,v)\mathscr{L}^A u + (\partial_\eta \mathcal{R}_{n,\nu})(u,v)\mathscr{L}^B v\Big) \\ &\geqslant \int_\Omega H_{\mathcal{Q}*\varphi_\nu}^{(A,B)}\left[(u,v)\,;(\nabla u,\nabla v)\right] \\ &\geqslant C_0\int_\Omega |\nabla u||\nabla v|, \end{aligned}$$

for all $\nu \in (0,1)$, where $C_0 = C_0(p, \Delta_p(A,B), \lambda(A,B), \Lambda(A,B)) > 0$ does not depend on ν. The desired inequality (3.42) now follows from (3.45).

CHAPTER 4

First-oder perturbations

In the previous chapter, we saw how the condition of p-ellipticity plays a fundamental role in the L^p-theory of (unperturbed) divergence-form operators. It was shown to be deeply connected to the contractivity and analyticity of the semigroup generated by such operators on L^p, as well as to be the correct structural assumption for extending the dimension-free bilinear embedding from the real to the complex coefficient setting. In [21], Schrödinger-type operators with non-negative potentials were studied, namely operators of the form

$$\mathscr{L}u = -\mathrm{div}(A\nabla u) + Vu.$$

Remarkably, under the sole assumption of p-ellipticity—without imposing any additional condition on the potentials—analogous L^p-results were obtained, in particular a dimension-free bilinear embedding.

However, to the best of our knowledge, no such results are currently available for more general operators of the form

$$\mathscr{L}u = -\mathrm{div}(A\nabla u) + \left\langle \nabla u, \bar{b} \right\rangle - \mathrm{div}(cu) + Vu,$$

where b, c are bounded vector-valued functions on Ω. The main objective of this chapter is to extend the bilinear embedding and associated semigroup theory to this broader class. To this end, we introduce a new structural condition on the coefficients, which reduces to p-ellipticity when $b = c = 0$.

Importantly, this new bilinear embedding is not derived from the known one via classical perturbative techniques. Instead, we rely on the heat-flow method developed in the previous chapter. The primary challenge lies in adapting this technique to accommodate the presence of first-order terms. In particular, these terms obstruct a direct use of the approximation procedure described in Sections 3.5.3 and 3.5.4 of the previous chapter. To overcome this difficulty, we establish the bilinear embedding in successive steps, gradually removing regularity assumptions on the coefficients. The strategy unfolds as follows:

- We first prove the bilinear embedding for smooth coefficients—in the same spirit as [20]—where the first-order terms and the potentials are compactly supported.
- Next, we allow the coefficients to be not regular, but still assume the first-order terms and potentials are compactly supported and the potentials are bounded.
- We then extend the result to bounded potentials without compact support.
- Finally, we treat the case of unbounded potentials, using arguments inspired by [21].

In addition, by applying Nittka's theorem—as done in the unperturbed setting—we extend the L^p-contractivity and analyticity results from the previous chapter to the perturbed case. We compare these results with those of Cialdea and Mazýa, who required stronger regularity assumptions on both the coefficients and the domain.

Finally, following the idea of [19], as a byproduct of the bilinear embedding, we also establish that such operators admit a bounded H^∞-functional calculus and L^p maximal regularity. This is made possible by the fact that the new condition, which generalizes p-ellipticity, preserves key invariance properties.

The chapter is organized as follows. In Section 4.1 we introduce the divergence-form operators with first-order perturbations. Section 4.2 presents the new structural condition generalizing p-ellipticity and develops its fundamental properties. In Section 4.3, we review and extend core concepts, including the generalized convexity framework introduced in Section 3.3 of the previous chapter. In Section 4.4, we establish a new generalized convexity for power functions of one complex variable and for the Bellman function $\mathcal{Q}$. Section 4.5 is devoted to proving L^p-contractivity and analyticity results for semigroups. The proof of the bilinear embedding is given in Sections 4.6-4.8, and finally, in Section 4.9, we prove the boundedness of the H^∞-functional calculus on L^p and the validity of L^p-maximal regularity.

4.1 Perturbed divergence-form operators

Let $\Omega \subseteq \mathbb{R}^d$ be a nonempty open set. Suppose that $A \in \mathcal{A}(\Omega)$. Let $b, c \in L^\infty(\Omega, \mathbb{C}^d)$ and $V \in L^1_{\mathrm{loc}}(\Omega)$ be nonnegative. Fix a closed subspace $\mathscr{V}$ of the Sobolev space $W^{1,2}(\Omega)$ containing $W^{1,2}_0(\Omega)$ of the type described in Section 3.1.1. Consider the sesquilinear form $\mathfrak{a} = \mathfrak{a}_{A,b,c,V,\mathscr{V}}$ defined by

$$\begin{aligned} \mathrm{D}(\mathfrak{a}) &= \{u \in \mathscr{V} : \textstyle\int_\Omega V|u|^2 < \infty\}, \\ \mathfrak{a}(u,v) &= \displaystyle\int_\Omega \langle A\nabla u, \nabla v\rangle_{\mathbb{C}^d} + \left\langle \nabla u, \overline{b}\right\rangle_{\mathbb{C}^d} \overline{v} + u \langle c, \nabla v\rangle_{\mathbb{C}^d} + Vu\overline{v}. \end{aligned} \tag{4.1}$$

Clearly, $\mathfrak{a}$ is densely defined in $L^2(\Omega)$.

Lemma 4.1. *Suppose that for almost all $x \in \Omega$*

$$\mathrm{Re}\,\langle A(x)\xi, \xi\rangle + \mathrm{Re}\left\langle \overline{b}(x) + c(x), \xi\right\rangle + V(x) \geqslant 0, \quad \forall \xi \in \mathbb{C}^d.$$

Then $\mathfrak{a}$ is accretive.

Proof. By using the abbreviation $\mathfrak{a}(u) = \mathfrak{a}(u,u)$ and the *ad hoc* notation $\xi = \nabla u/u$, the accretivity of $\mathfrak{a}$ follows from the identity

$$\mathrm{Re}\,\mathfrak{a}(u) = \int_\Omega |u|^2 \Big[\mathrm{Re}\,\langle A\xi, \xi\rangle + \mathrm{Re}\left\langle \overline{b} + c, \xi\right\rangle + V\Big]. \tag{4.2}$$

□

Lemma 4.2. *Suppose that there exists $\mu \in (0,1]$ such that for almost all $x \in \Omega$*

$$\mathrm{Re}\,\langle A(x)\xi, \xi\rangle + \mathrm{Re}\left\langle \overline{b}(x) + c(x), \xi\right\rangle + V(x) \geqslant \mu(|\xi|^2 + V(x)), \quad \forall \xi \in \mathbb{C}^d. \tag{4.3}$$

Then $\mathfrak{a}$ is accretive, closed and

$$\mathrm{Re}\,\mathfrak{a}(u,u) \geqslant \mu\left(\|\nabla u\|_2^2 + \|V^{1/2}u\|_2^2\right), \tag{4.4}$$

for all $u \in \mathrm{D}(\mathfrak{a})$.

Additionally, if we assume that there exists $M > 0$ *such that for almost all* $x \in \Omega$

$$|\bar{b}(x) - c(x)| \leqslant M\sqrt{V}, \tag{4.5}$$

then $\mathfrak{a}$ *is also sectorial of angle* ω. *The angle* ω *depends only on* μ, M *and* $\Lambda(A)$.

Proof. The accretivity of $\mathfrak{a}$ is a consequence of Lemma 4.1, while (4.4) follows by (4.2) and (4.3).

In order to prove that $\mathfrak{a}$ is closed we follow the proof of [72, Proposition 4.30]. Let $(u_n)_n$ be a Cauchy sequence in $(\mathrm{D}(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$. Recall that $\|\cdot\|_{\mathfrak{a}}^2 = \operatorname{Re}\mathfrak{a}(\cdot) + \|\cdot\|_2^2$. Then, by (4.4),

$$\mu\|\nabla u_n - \nabla u_m\|_2^2 + \mu\|V^{1/2}u_n - V^{1/2}u_m\|_2 + \|u_n - u_m\|_2^2 \leqslant \|u_n - u_m\|_{\mathfrak{a}}^2 < \varepsilon.$$

Therefore, $(u_n)_n$ is a Cauchy sequence in $(\mathscr{V}, \|\cdot\|_{W^{1,2}(\Omega)})$. On the other hand, $\mathscr{V}$ is closed in $W^{1,2}(\Omega)$, hence $(u_n)_n$ converges in $\mathscr{V}$. Let $u \in \mathscr{V}$ be its limit. In particular, $u_n \to u$ in $L^2(\Omega)$, so we can take a subsequence $(u_{n_k})_k$ such that $u_{n_k}(x) \to u(x)$, for almost all $x \in \Omega$. Therefore, by Fatou's Lemma and by (4.2) and (4.3),

$$\int_\Omega V|u|^2 = \int_\Omega \lim_k V|u_{n_k}|^2 \leqslant \liminf_k \int_\Omega V|u_{n_k}|^2 \lesssim \liminf_k \|u_{n_k}\|_{\mathfrak{a}}^2 < \infty.$$

Therefore, $u \in \mathrm{D}(\mathfrak{a})$. By taking a subsequence $(u_{n_k})_k$ such that $\partial_j u_{n_k}(x)$ converges to $\partial_j u(x)$ and $u_{n_k}(x)$ converges to $u(x)$ for a.e. $x \in \Omega$ and all $j \in \{1, \dots, d\}$ and by applying Fatou's lemma, (4.2) and (4.3), we obtain

$$\begin{aligned}\operatorname{Re}\mathfrak{a}(u_n - u) \leqslant \liminf_{k\to\infty} \int_\Omega \operatorname{Re}\Big[&\langle A\nabla(u_n - u_{n_k}), \nabla(u_n - u_{n_k})\rangle + \left\langle \nabla(u_n - u_{n_k}), \bar{b}\right\rangle (\overline{u_n - u_{n_k}}) \\ &+ (u_n - u_{n_k})\langle c, \nabla(u_n - u_{n_k})\rangle + V|u_n - u_{n_k}|^2\Big] \\ &\leqslant \liminf_{k\to\infty} \|u_n - u_{n_k}\|_{\mathfrak{a}}^2.\end{aligned}$$

From this it follows that $\operatorname{Re}\mathfrak{a}(u_n - u) \to 0$ as $n \to \infty$. Thus, the form $\mathfrak{a}$ is closed.

Suppose now that (4.5) also holds. With the same notation as in the proof of Lemma 4.1, we find

$$\operatorname{Im}\mathfrak{a}(u) = \int_\Omega |u|^2\Big[\operatorname{Im}\,\langle A\xi, \xi\rangle - \operatorname{Im}\left\langle \bar{b} - c, \xi\right\rangle\Big].$$

Therefore, boundedness of A and (4.5) yield

$$|\operatorname{Im}\mathfrak{a}(u)| \lesssim_{\Lambda, M} \|\nabla u\|_2^2 + \|V^{1/2}u\|_2^2,$$

which, combined with (4.4), implies that $\mathfrak{a}$ is sectorial of some angle $\omega = \omega(\mu, M, \Lambda) \in (0, \pi/2)$. □

Denote by $\mathcal{B}_{\mu,M}(\Omega)$ the class of all $\mathscr{A} = (A, b, c, V) \in \mathcal{A}(\Omega) \times (L^\infty(\Omega, \mathbb{C}^d))^2 \times L^1_{\mathrm{loc}}(\Omega, \mathbb{R}_+)$ for which (4.3) and (4.5) hold for some fixed $\mu, M > 0$. Moreover, we set

$$\mathcal{B}(\Omega) = \bigcup_{\mu,M>0} \mathcal{B}_{\mu,M}(\Omega).$$

For $\mathscr{A} = (A, b, c, V)$, we denote by $\mu(\mathscr{A})$ and $M(\mathscr{A})$ the optimal μ and M, respectively.

For every $\mathscr{A} \in \mathcal{B}(\Omega)$ we denote by $\mathscr{L} = \mathscr{L}_2^{\mathscr{A},\mathscr{V}}$ the unbounded operator on $L^2(\Omega)$ associated with the sesquilinear form $\mathfrak{a}$. That is,

$$\mathrm{D}(\mathscr{L}) := \Big\{u \in \mathrm{D}(\mathfrak{a}) : \exists w \in L^2(\Omega) :\ \mathfrak{a}(u, v) = \langle w, v\rangle_{L^2(\Omega)}\ \ \forall v \in \mathrm{D}(\mathfrak{a})\Big\}$$

and

$$\langle \mathscr{L}u, v\rangle_{L^2(\Omega)} = \mathfrak{a}(u,v), \quad \forall u \in \mathrm{D}(\mathscr{L}), \quad \forall v \in \mathrm{D}(\mathfrak{a})\,. \tag{4.6}$$

Formally, $\mathscr{L}$ is given by the expression

$$\mathscr{L}u = -\mathrm{div}\,(A\nabla u) + \left\langle \nabla u, \overline{b}\right\rangle - \mathrm{div}\,(uc) + Vu.$$

Corollary 4.3. *Suppose that $\mathscr{A} = (A, b, c, V) \in \mathcal{B}_{\mu,M}(\Omega)$ for some $\mu \in (0,1]$ and $M > 0$. Then $\mathscr{L}$ is sectorial of angle $\omega = \omega(\mu, M, \Lambda(A))$ and $-\mathscr{L}$ is the generator of a strongly continuous semigroup on $L^2(\Omega)$*

$$T_t = T_t^{\mathscr{A},\mathscr{V}}, \quad t > 0,$$

which is analytic and contractive in the cone $\mathbf{S}_{\pi/2-\omega}$.

Proof. This follows by combining Lemma 4.2 with Theorem 2.37. □

4.1.1 Standard assumptions

Unless otherwise specified, henceforth we assume that

- $\Omega \subseteq \mathbb{R}^d$ is a nonempty open set;
- $A, B \in \mathcal{A}(\Omega)$;
- $b, c, \beta, \gamma \in L^\infty(\Omega, \mathbb{C}^d)$;
- $V, W \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}_+)$.

These assumptions shall be called the *standard assumptions*.

4.2 The classes $\mathcal{S}_p$ and $\mathcal{B}_p$

Let $p > 1$ and $\mathscr{A} = (A, b, c, V)$ satisfy the standard assumptions of Section 4.1.1. Denote by $q = p/(p-1)$ its conjugate exponent. For every $p \in (1, +\infty)$, we consider the operator

$$\mathcal{J}_p(\xi) = \xi + (p-2)\mathrm{Re}\,\xi. \tag{4.7}$$

Observe that:

(a) $\mathcal{J}_p$ is $\mathbb{R}$-linear;

(b) $\mathrm{Re}\,\langle \mathcal{J}_p\xi, \sigma\rangle = \mathrm{Re}\,\langle \xi, \mathcal{J}_p\sigma\rangle$ for all $\xi, \sigma \in \mathbb{C}^d$;

(c) $\mathcal{J}_p = (p/2)\,\mathcal{I}_p$, where $\mathcal{I}_p$ is the operator defined in (3.10), introduced by Carbonaro and Dragičević in [20, (1.4)];

(d) $\mathcal{J}_p\mathcal{J}_q = I_{\mathbb{C}^d}$ [20, p. 3201];

(e) $\mathcal{J}_p(i\xi) = i(p-1)\mathcal{J}_q\xi$ for all $\xi \in \mathbb{C}^d$;

(f) $\|\mathcal{J}_p\| = \max\{p-1, 1\} = \max\{p, 2\} - 1$.

We define the function $\Gamma_p^{\mathscr{A}} = \Gamma_p^{A,b,c,V} : \Omega \times \mathbb{C}^d \to \mathbb{R}$ by

$$\Gamma_p^{\mathscr{A}}(x,\xi) = \operatorname{Re} \langle A(x)\xi, \mathcal{J}_p\xi\rangle_{\mathbb{C}^d} + \operatorname{Re} \left\langle \bar{b}(x) + (\mathcal{J}_p c)(x), \xi \right\rangle_{\mathbb{C}^d} + V(x). \tag{4.8}$$

By using dilations $\xi \rightsquigarrow t\xi$, $t \to \infty$, we see that if for almost all $x \in \Omega$

$$\Gamma_p^{\mathscr{A}}(x,\xi) \geqslant 0, \quad \forall \xi \in \mathbb{C}^d, \tag{4.9}$$

then $\Delta_p(A) \geqslant 0$. Moreover, when $b = c = 0$ the two conditions coincide. A condition slightly weaker than (4.9) was formulated in a different manner by Cialdea and Maz'ya in [24, (2.25)] (see Section 4.5.1).

We denote by $\mathcal{S}_p(\Omega)$ the class of all $\mathscr{A} = (A, b, c, V) \in \mathcal{A}_p(\Omega) \times (L^\infty(\Omega, \mathbb{C}^d))^2 \times L^1_{\mathrm{loc}}(\Omega, \mathbb{R}_+)$ for which

- there exists $\mu > 0$ such that for a.e. $x \in \Omega$

$$\Gamma_p^{\mathscr{A}}(x,\xi) \geqslant \mu(|\xi|^2 + V(x)), \quad \forall \xi \in \mathbb{C}^d, \tag{4.10}$$

- the condition (4.5) holds.

We denote by $\mu_p(\mathscr{A})$, or just μ_p, the largest admissible μ in (4.10). We list a few observations on $\mathcal{S}_p$ and μ_p:

- By applying the dilation argument again, we get

$$\mu_p(\mathscr{A}) \leqslant \frac{p}{2}\Delta_p(A). \tag{4.11}$$

  Moreover, if $V \neq 0$ we also have

$$\mu_p(\mathscr{A}) \leqslant \min\left\{\frac{p}{2}\Delta_p(A), 1\right\}. \tag{4.12}$$

- If $\bar{b} + \mathcal{J}_p c \neq 0$, then $V \neq 0$ and $\mu_p(\mathscr{A}) < 1$. In fact, assuming by contradiction that $V = 0$, for almost all $x \in \Omega$ we can find $\xi \in \mathbb{C}^d$ such that $\operatorname{Re} \left\langle \bar{b}(x) + \mathcal{J}_p c(x), \xi \right\rangle_{\mathbb{C}^d} \neq 0$ and, as a consequence, $t \in \mathbb{R}$, sufficiently close to 0, such that $\Gamma_p^{A,b,c,0}(x, t\xi) < \mu t^2 |\xi|^2$. By using the same contradiction argument, we also infer that $\mu_p(\mathscr{A}) < 1$.

- $(A, b, c, V) \in \mathcal{S}_2(\Omega)$ if and only if (4.3) and (4.5) hold, that is, $\mathcal{S}_2(\Omega) = \mathcal{B}(\Omega)$.

- when $b = c = 0$, this new condition coincides with p-ellipticity by Lemma 3.7(ii), in the sense that $(A, 0, 0, V) \in \mathcal{S}_p(\Omega)$ if and only if $A \in \mathcal{A}_p(\Omega)$.

Finally, we define the class

$$\mathcal{B}_p(\Omega) = \mathcal{S}_p(\Omega) \cap \mathcal{S}_q(\Omega),$$

which also coincides with $\mathcal{A}_p(\Omega)$ when $b = c = 0$, in the sense that $(A, 0, 0, V) \in \mathcal{B}_p(\Omega)$ if and only if $A \in \mathcal{A}_p(\Omega)$, by means of the invariance of p-ellipticity under conjugation; see Lemma 3.7(i). It also coincides with $\mathcal{B}(\Omega)$ when $p = 2$.

Our motivation for introducing $\mathcal{B}_p$ on top of $\mathcal{S}_p$ was to obtain a class which, other than generalizing p-ellipticity, retains the following properties that the classes $\mathcal{A}_p$ possess, as stated in in Lemma 3.7(i), (iii), (iv):

(i) invariance under conjugation, in the sense that $\mathcal{A}_p(\Omega) = \mathcal{A}_q(\Omega)$;

(ii) a decrease with respect to p, in the sense that $\{\mathcal{A}_p(\Omega) : p \in [2,\infty)\}$ is a decreasing chain of matrix classes;

(iii) invariance under adjointness, in the sense that $A \in \mathcal{A}_p(\Omega) \iff A^* \in \mathcal{A}_q(\Omega)$.

We shall see that $\mathcal{S}_p$ satisfies the analogue of (iii) (see Proposition 4.5(i)), but not (i) and (ii) (see Proposition 4.6). On the other hand, $\mathcal{B}_p$ is invariant under conjugation by definition and it turns out that it is also decreasing with respect to p; see Proposition 4.5(ii). Therefore, the class $\mathcal{B}_p$ seems a more adequate generalization of p-ellipticity. Moreover, we shall see that the strict positivity of Γ_p is preserved for exponents close enough to p and for small complex rotations of the coefficients (compare with Lemma 3.7(iv), (v)). Finally, we shall see that the first order terms b, c are appropriately controlled by the potential V.

See Remark 4.10 for explanations about the origin of the definition (4.8) and conditions (4.9) and (4.10).

We shall prove now elementary properties of $\mathcal{S}_p(\Omega)$ and $\mathcal{B}_p(\Omega)$ which we mentioned before and which we shall use in the rest of the paper.

Proposition 4.4. *Let $1 < p < \infty$. Suppose that $\mathscr{A} = (A, b, c, V) \in \mathcal{S}_p(\Omega)$. Then the following assertions hold.*

(i) *For every $s \in (1,\infty)$ there exists $\widetilde{C} > 0$ such that for a.e. $x \in \Omega$.*

$$\left|\overline{b}(x) + (\mathcal{J}_s c)(x)\right| \leqslant \widetilde{C}\sqrt{V(x)}.$$

The constant $\widetilde{C}$ can be chosen so as to (continuously) depend only on $p, s, \Lambda(A)$, $M(\mathscr{A})$ and $\mu_p(\mathscr{A})$.

(ii) *There exists $C > 0$ such that $|b(x)|$, $|c(x)| \leq C\sqrt{V(x)}$ for a.e. $x \in \Omega$. The constant C can be chosen so as to (continuously) depend only on $p, \Lambda(A), M(\mathscr{A})$ and $\mu_p(\mathscr{A})$.*

(iii) *There exists $\varepsilon > 0$ such that $(A, b, c, V) \in \mathcal{S}_s(\Omega)$ for all $s \in [p-\varepsilon, p+\varepsilon]$.*

(iv) *There exists $\vartheta \in (0, \pi/2)$ such that $(e^{i\varphi}A, e^{i\varphi}b, e^{i\varphi}c, (\cos\varphi)V) \in \mathcal{S}_p(\Omega)$ for all $\varphi \in [-\vartheta, \vartheta]$.*

Proof. We will be using properties (a)-(f) of the operator $\mathcal{J}_p$, recalled on page 70.

(i) First consider the case $s = p$. Set $\xi_0 = \overline{b} + \mathcal{J}_p c \in L^\infty(\Omega, \mathbb{C}^d)$ and let $\varepsilon > 0$. By ellipticity of A and by choosing $\xi = -\varepsilon\,\xi_0$ in (4.10), we obtain

$$\varepsilon\,[1 - \varepsilon(\|\mathcal{J}_p\|\Lambda - \mu_p)]\,|\xi_0|^2 \leqslant (1-\mu_p)V. \tag{4.13}$$

Observe that $\|\mathcal{J}_p\|\Lambda - \mu_p \geqslant 0$, since $\mu_p \leqslant (p/2)\Delta_p(A) \leqslant \|\mathcal{J}_p\|\Lambda$, by (4.11) and (c). If $\|\mathcal{J}_p\|\Lambda - \mu_p = 0$, then we would have $\xi_0 = 0$, by sending ε to $+\infty$ in (4.13). Thus, the statement would clearly hold. Otherwise, by maximizing the left-hand side in (4.13) with respect to $\varepsilon > 0$, we get

$$|\xi_0|^2 \leqslant 4(1-\mu_p)(\|\mathcal{J}_p\|\Lambda - \mu_p)V.$$

We notice that this inequality also shows that $V \neq 0$ and $\mu_p < 1$ provided that $\overline{b} + \mathcal{J}_p c \neq 0$, as we have already observed on page 70.

If $s \neq p$, the assertion follows by the previous case, (ii) and the identity

$$\overline{b} + \mathcal{J}_s c = \overline{b} + \mathcal{J}_p c + (s-p)\operatorname{Re} c. \tag{4.14}$$

(ii) We shall prove that $|\mathrm{Re}\, c|$, $|\mathrm{Re}\, b|$, $|\mathrm{Im}\, b|$, $|\mathrm{Im}\, c| \lesssim \sqrt{V}$ in this order. The first estimate follows by applying (i) for $s = p$, (4.5) and the identity

$$p\,\mathrm{Re}\, c = \mathrm{Re}\,(\overline{b} - c) - \mathrm{Re}\,\left(\overline{b} + \mathcal{J}_p c\right).$$

Clearly, by (4.5) we have

$$|\mathrm{Re}\, b - \mathrm{Re}\, c| \lesssim \sqrt{V}, \tag{4.15}$$
$$|\mathrm{Im}\, b + \mathrm{Im}\, c| \lesssim \sqrt{V}. \tag{4.16}$$

Combining (4.15) with the estimate of $|\mathrm{Re}\, c|$, we get $|\mathrm{Re}\, b| \lesssim \sqrt{V}$. On the other hand, by (i) and (4.14) we have

$$|\mathrm{Im}\,\overline{b} + \mathrm{Im}\, c| = |\mathrm{Im}\,\left(\overline{b} + \mathcal{J}_p c\right)| \lesssim \sqrt{V}. \tag{4.17}$$

Combining (4.16) and (4.17), we obtain $|\mathrm{Im}\, b|, |\mathrm{Im}\, c| \lesssim \sqrt{V}$. All the constants which appear in the inequalities (continuously) depend only on $p, \Lambda, M, \mu_p(\mathscr{A})$.

(iii) By (4.14), $\mathcal{J}_{p+\delta}\xi = \mathcal{J}_p\xi + \delta\,\mathrm{Re}\,\xi$, hence (ii) implies

$$\begin{aligned}
\Gamma^{\mathscr{A}}_{p+\delta}(\cdot,\xi) &= \Gamma^{\mathscr{A}}_{p}(\cdot,\xi) + \delta\,\mathrm{Re}\,[\,\langle A\xi, \mathrm{Re}\,\xi\rangle + \langle \xi, \mathrm{Re}\, c\rangle\,] \\
&\geqslant \mu_p(|\xi|^2 + V) - |\delta|(\Lambda|\xi|^2 + |\xi||\mathrm{Re}\, c|) \\
&\geqslant (\mu_p - |\delta|\Lambda)|\xi|^2 - |\delta|C|\xi|\sqrt{V} + \mu_p V \\
&\geqslant (\mu_p - |\delta|\Lambda - |\delta|(C/2))|\xi|^2 + [\mu_p - |\delta|(C/2)]V \\
&\gtrsim |\xi|^2 + V,
\end{aligned}$$

if $|\delta|$ is small enough.

(iv) Denote by $q = p/(p-1)$ the conjugate exponent of p. From the properties (b), (e) of $\mathcal{J}_p$, we infer

$$\begin{aligned}
&\mathrm{Re}\,\left\langle e^{i\vartheta}A\xi, \mathcal{J}_p\xi\right\rangle + \mathrm{Re}\,\left\langle e^{-i\vartheta}\overline{b} + \mathcal{J}_p(e^{i\vartheta}c), \xi\right\rangle + (\cos\vartheta)V \\
&\qquad = \cos\vartheta\,\mathrm{Re}\,[\,\langle A\xi, \mathcal{J}_p\xi\rangle + \langle b + \mathcal{J}_p c, \xi\rangle + V] \\
&\qquad\quad - \sin\vartheta\,\mathrm{Im}\,[\,\langle A\xi, \mathcal{J}_p\xi\rangle - \left\langle \overline{b} - (p-1)\mathcal{J}_q c, \xi\right\rangle\,].
\end{aligned}$$

By (ii) we get $\left|\overline{b} - (p-1)\mathcal{J}_q c\right| \leqslant C\sqrt{V}$. Therefore, we may proceed as

$$\geqslant (\mu_p\cos\vartheta - |\sin\vartheta|\,\|\mathcal{J}_p\|\Lambda)|\xi|^2 - C\,|\sin\vartheta|\,\sqrt{V}|\xi| + \mu_p(\cos\vartheta)V.$$

We conclude choosing ϑ small enough. □

In Lemma 3.7(iv) we showed that $\mathcal{A}_p$ is invariant under taking adjoints. The following proposition shows that this invariance still holds for $\mathcal{S}_p$. Moreover, in Lemma 3.7(iii) it was proved that $\{\mathcal{A}_p(\Omega) : p \in [2,\infty)\}$ is a decreasing chain of matrix classes. In Proposition 4.6 we shall show that $\{\mathcal{S}_p(\Omega) : p \in [2,\infty)\}$ is not. However, an interpolation property holds for $\mathcal{S}_p$ which implies that $\{\mathcal{B}_p(\Omega) : p \in [2,\infty)\}$ is a decreasing chain.

Proposition 4.5. *Let Ω, A, b, c, V satisfy the standard assumptions of Section 4.1.1. Let $1 < r < p < \infty$. Denote by q the conjugate exponent of p. Then the following assertions hold.*

(i) $(A, b, c, V) \in \mathcal{S}_p(\Omega)$ *if and only if* $(A^*, \overline{c}, \overline{b}, V) \in \mathcal{S}_q(\Omega)$.

(ii) *We have*

$$\mathcal{S}_r(\Omega) \cap \mathcal{S}_p(\Omega) \subset \mathcal{S}_s(\Omega)$$

for all $s \in [r, p]$.

In particular, $\{\mathcal{B}_p(\Omega) : p \in [2, \infty)\}$ *is a decreasing chain.*

Proof. (i) By (d) and (b), we obtain for all $\xi \in \mathbb{C}^d$

$$\operatorname{Re} \langle A^*\xi, \mathcal{J}_q\xi \rangle = \operatorname{Re} \langle A\left(\mathcal{J}_q\xi\right), \mathcal{J}_p\left(\mathcal{J}_q\xi\right) \rangle, \tag{4.18}$$

$$\operatorname{Re} \left\langle c + \mathcal{J}_q\overline{b}, \xi \right\rangle = \operatorname{Re} \left\langle \mathcal{J}_q\left(\overline{b} + \mathcal{J}_p c\right), \xi \right\rangle = \operatorname{Re} \left\langle \overline{b} + \mathcal{J}_p c, \mathcal{J}_q\xi \right\rangle. \tag{4.19}$$

Since $|\mathcal{J}_s\xi| \sim_s |\xi|$ for all $\xi \in \mathbb{C}^d$ and all $s \in (1, \infty)$, we conclude combining (4.10), (4.18) and (4.19).

(ii) Let $t \in (0, 1)$ such that $tp + (1-t)r = s$. By assumptions we have for all $\xi \in \mathbb{C}^d$

$$\begin{aligned} t\left(\operatorname{Re} \langle A\xi, \mathcal{J}_p\xi \rangle + \operatorname{Re} \left\langle \overline{b} + \mathcal{J}_p c, \xi \right\rangle + V\right) &\gtrsim t\left(|\xi|^2 + V\right), \\ (1-t)\left(\operatorname{Re} \langle A\xi, \mathcal{J}_r\xi \rangle + \operatorname{Re} \left\langle \overline{b} + \mathcal{J}_r c, \xi \right\rangle + V\right) &\gtrsim (1-t)\left(|\xi|^2 + V\right). \end{aligned}$$

Using the identity

$$t\mathcal{J}_p + (1-t)\mathcal{J}_r = \mathcal{J}_s$$

and summing the terms above we conclude. □

We conclude this subsection underlining that $\{\mathcal{S}_p(\Omega) : p \in [2, \infty)\}$ is not a decreasing chain. Moreover, $\mathcal{S}_p(\Omega)$ is not invariant under conjugation of p. This further justifies our introduction of another class $(\mathcal{B}_p)$.

Proposition 4.6. *For all* $V \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}_+) \setminus \{0\}$, $p, r \in (1, \infty)$, $p \neq r$, $B \in \mathcal{A}_r(\Omega)$, *there exist* $b = b(p, r, B, V)$, $c = c(p, r, B, V) \in L^\infty(\Omega, \mathbb{C}^d)$ *such that*

- $(A, b, c, V) \in \mathcal{S}_p(\Omega)$ *for all* $A \in \mathcal{A}_p(\Omega)$,
- $(B, b, c, V) \notin \mathcal{S}_r(\Omega)$.

In particular, for all $V \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}_+) \setminus \{0\}$, $p, r \in (1, \infty)$, $p \neq r$, $A \in \mathcal{A}_p(\Omega) \cap \mathcal{A}_r(\Omega)$, *there exist* $b, c \in L^\infty(\Omega, \mathbb{C}^d)$ *such that* $(A, b, c, V) \in \mathcal{S}_p(\Omega) \setminus \mathcal{S}_r(\Omega)$.

Proof. Fix $V \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}_+) \setminus \{0\}$, $p, r \in (1, \infty)$, with $p \neq r$ and $B \in \mathcal{A}_r(\Omega)$. Let $\varrho > 0$ and $v = v_\varrho \in L^\infty(\Omega, \mathbb{R}^d) \setminus \{0\}$ such that

$$(r-p)^2|v|^2 = \varrho V.$$

We define

$$b = b_\varrho = -(p-1)v + i\frac{p}{2}v \qquad \text{and} \qquad c = c_\varrho = v + i\frac{p}{2}v.$$

Then

$$\overline{b} + \mathcal{J}_p c = 0, \quad |\overline{b} - c|^2 = \frac{2p^2}{(r-p)^2}\varrho V, \qquad \overline{b} + \mathcal{J}_r c = (r-p)v.$$

Therefore $(A, b_\varrho, c_\varrho, V) \in \mathcal{S}_p(\Omega)$ for all $A \in \mathcal{A}_p(\Omega)$ and $\varrho > 0$. Moreover, setting $\sigma := v/|v|$, we have for all $t \geqslant 0$ and $x \in \Omega$ such that $V(x) \neq 0$

$$\begin{aligned} \Gamma_r^{B,b,c,V}(x, -t\sigma) &= t^2 \operatorname{Re} \langle B(x)\sigma, \mathcal{J}_r\sigma \rangle - t(r-p)|v(x)| + V(x) \\ &= t^2 \operatorname{Re} \langle B(x)\sigma, \mathcal{J}_r\sigma \rangle - t\sqrt{\varrho}\sqrt{V(x)} + V(x) \underset{\varrho \to +\infty}{\longrightarrow} -\infty. \end{aligned}$$

□

Remark 4.7. Let $p \in (1,\infty)$. The condition $(A,b,c,V) \in \mathcal{S}_p(\Omega)$ is defined by a global inequality on $\mathbb{C}^d$ involving the simultaneous contribution of second-order, first-order, and potential terms. Its direct verification is generally not immediate, as one must account for the interplay between these coefficients. To overcome this difficulty, it is useful to provide simpler, albeit more restrictive, sufficient conditions by estimating the contribution of each coefficient individually.

Suppose that $\overline{b} + \mathcal{J}_p c = 0$ almost everywhere on $\Omega \setminus \operatorname{supp} V$ and

$$\left\| V^{-1/2} \left(\overline{b} + \mathcal{J}_p c \right) \right\|^2_{L^\infty(\operatorname{supp} V)} < 2p\Delta_p(A). \tag{4.20}$$

We show that (A,b,c,V) satisfies (4.10) under these assumptions. In particular, if (4.5) is fulfilled, then $(A,b,c,V) \in \mathcal{S}_p(\Omega)$.

From (4.20) there exists a positive constant $\mu_p < \min\{(p/2)\Delta_p(A), 1\}$ such that

$$V^{-1} \left| \overline{b} + \mathcal{J}_p c \right|^2 \leqslant \left\| V^{-1/2} \left(\overline{b} + \mathcal{J}_p c \right) \right\|^2_{L^\infty(\operatorname{supp} V)} \leqslant 4 \left(\frac{p}{2} \Delta_p(A) - \mu_p \right) (1 - \mu_p),$$

almost everywhere on $\operatorname{supp} V$. Since $\overline{b} + \mathcal{J}_p c$ vanishes on $\Omega \setminus \operatorname{supp} V$, we deduce that

$$\left(\frac{p}{2} \Delta_p(A) - \mu_p \right) |\xi|^2 - |\overline{b} + \mathcal{J}_p c||\xi| + (1 - \mu_p) V \geqslant 0$$

for all $\xi \in \mathbb{C}^d$ and almost everywhere on Ω. Hence, (A,b,c,V) satisfies (4.10).

Assume now that the potential is uniformly positive, namely $V_{\min} := \operatorname{ess\,inf} V > 0$. Under this assumption, condition (4.5) is automatically satisfied. Then one can similarly show that $(A,b,c,V) \in \mathcal{S}_p(\Omega)$ provided that

$$\left(\|b\|_\infty + \|\mathcal{J}_p\| \|c\|_\infty \right)^2 < 2p\Delta_p(A) V_{\min}. \tag{4.21}$$

This inequality determines a non-empty interval around $p = 2$ whenever

$$\|b\|_\infty + \|c\|_\infty < 2\sqrt{V_{\min} \lambda(A)}. \tag{4.22}$$

In order to determine an explicit range of exponents for which $(A,b,c,V) \in \mathcal{S}_p(\Omega)$, one needs to decouples A and p in the right-hand side of (4.21).

Set $p^* = \max\{p, p'\}$. If A is real-valued, the estimate $\Delta_p(A) \geqslant 2\lambda(A)/p^*$ implies that (4.21) holds, and hence $(A,b,c,V) \in \mathcal{S}_p(\Omega)$, whenever

$$\left(\|b\|_\infty + \|\mathcal{J}_p\| \|c\|_\infty \right)^2 < 2p \left(1 - |1 - 2/p| \right) \lambda(A) V_{\min}. \tag{4.23}$$

It is straightforward to verify that (4.22) is necessary for (4.23), by examining the behaviour of both sides as functions of p. Analyzing separately the cases $p \geq 2$ and $p \leq 2$, and using the explicit expression for $\|\mathcal{J}_p\|$ (see (f)), one obtains a concrete range of admissible exponents. Indeed, p satisfies (4.23) if and only if (4.22) holds and

$$\underline{p} := 1 + \frac{(\|b\|_\infty + \|c\|_\infty)^2}{4V_{\min}\lambda(A)} < p < 1 + \frac{2\sqrt{V_{\min}\lambda(A)} - \|b\|_\infty}{\|c\|_\infty} =: \overline{p},$$

where we set $\overline{p} = \infty$ if $c = 0$. Condition (4.22) guarantees that $\underline{p} < 2 < \overline{p}$. In particular, it follows that $(A,b,c,V) \in \mathcal{B}_p(\Omega)$ for all p satisfying

$$\max\{\overline{p}', \underline{p}\} < p < \min\{\overline{p}, \underline{p}'\}.$$

For general complex-valued p-elliptic matrices, the separation of p and A can be achieved through the constant $\mu(A)$ defined in (3.12). In fact, from (3.16) we have

$$\Delta_p(A) \geqslant \lambda(A)\,\frac{\mu(A) - |1 - 2/p|}{\mu(A)}. \tag{4.24}$$

Therefore, (4.21) is satisfied, and hence $(A, b, c, V) \in \mathcal{S}_p(\Omega)$, provided that

$$(\|b\|_\infty + \|\mathcal{J}_p\|\|c\|_\infty)^2 < 2p\,\frac{\mu(A) - |1 - 2/p|}{\mu(A)}\lambda(A)V_{\min}. \tag{4.25}$$

Again, (4.22) is necessary for (4.25). Moreover, (4.25) coincides with (4.23) when A is real-valued, since $\mu(A) = 1$ in this case; see Remark 3.9.

In the particular case $A = e^{i\phi}I_d$, one has $\Delta_p(e^{i\phi}I_d) = \cos\phi - |1 - 2/p|$ [20] and $\mu(e^{i\phi}I_d) = \lambda(e^{i\phi}I_d) = \cos\phi$. Hence, (4.21) and (4.25) coincide and reduce to

$$(\|b\|_\infty + \|\mathcal{J}_p\|\|c\|_\infty)^2 < 2p\,(\cos\phi - |1 - 2/p|)\,V_{\min}.$$

4.3 Heat-flow monotonicity and generalized convexity

4.3.1 Convexity with respect to complex matrices, complex vectors and real scalars

In Section 3.3, we introduced the notion of generalized convexity of a function with respect to a matrix (or a collection of matrices). Here we extend this notion to 4-tuples (A, b, c, V). So the novelty is the presence of b, c and V. Our aim is to demonstrate the value of this extended notion by proving that it implies properties of $\mathscr{L}$ as shown earlier for $b = c = 0$.

Due to the presence of the first- and zero-order terms, following [20] we only treat the one-dimensional and the two-dimensional case separately, in order to make the text more readable. One could provide a single definition for the k-dimensional case, as Carbonaro and Dragičević did in [19].

Let $d \in \mathbb{N}_+$. Take

$$\begin{aligned} A, B &\in \mathbb{C}^{d\times d}, \\ b, \beta, c, \gamma &\in \mathbb{C}^d, \\ V, W &\in \mathbb{R}, \\ \omega = (\zeta, \eta) &\in \mathbb{C}\times\mathbb{C}, \\ \Xi = (X, Y) &\in \mathbb{C}^d \times \mathbb{C}^d. \end{aligned}$$

Denote

$$\begin{aligned} \mathscr{A} &= (A, b, c, V), \\ \mathscr{B} &= (B, \beta, \gamma, W). \end{aligned}$$

Recall the notations introduced in Sections 3.3.1, 3.3.2 and 3.3.3.

One-dimensional case

Let $\phi : \mathbb{C} \to \mathbb{R}$ be of class C^2. We define

$$\begin{aligned} H_\phi^{(b,c)}[\zeta; X] &= \left\langle \left[D^2\phi(\zeta)\otimes I_{\mathbb{R}^d}\right]\mathcal{V}_d(X), \mathcal{V}_d(\zeta c)\right\rangle_{\mathbb{R}^{2d}} + \left\langle D\phi(\zeta), \mathcal{V}_1\left(\left\langle X, \bar{b}\right\rangle\right)\right\rangle_{\mathbb{R}^2}, \\ G_\phi^V[\zeta] &= \langle D\phi(\zeta), \mathcal{V}_1(V\zeta)\rangle_{\mathbb{R}^2}. \end{aligned} \tag{4.26}$$

Finally, we define

$$\mathbf{H}_\phi^{\mathscr{A}}[\zeta; X] = H_\phi^A[\zeta; X] + H_\phi^{(b,c)}[\zeta; X] + G_\phi^V[\zeta]. \tag{4.27}$$

Definition 4.8. We say that ϕ is $\mathscr{A}$-*convex* in $\mathbb{C}$ if $\mathbf{H}_\phi^{\mathscr{A}}[\zeta; X]$ is nonnegative for all $\zeta \in \mathbb{C}$, $X \in \mathbb{C}^d$.

In accordance with Definition 3.11 we say that ϕ is A-*convex* if it is $(A, 0, 0, 0)$-convex.

Two-dimensional case

Let $\Phi : \mathbb{C}^2 \to \mathbb{R}$ be of class C^2. We define

$$\begin{aligned} H_\Phi^{(b,\beta,c,\gamma)}[\omega; \Xi] &= \left\langle \left[D^2\Phi(\omega) \otimes I_{\mathbb{R}^d} \right] \mathcal{W}_{2,d}(\Xi), \mathcal{W}_{2,d}(\zeta c, \eta\gamma) \right\rangle_{\mathbb{R}^{4d}} \\ &\quad + \left\langle D\Phi(\omega), \mathcal{W}_{2,1}\left(\left\langle X, \overline{b} \right\rangle, \left\langle Y, \overline{\beta} \right\rangle \right) \right\rangle_{\mathbb{R}^4}, \\ G_\Phi^{(V,W)}[\omega] &= \langle D\Phi(\omega), \mathcal{W}_{2,1}(V\zeta, W\eta) \rangle_{\mathbb{R}^4}. \end{aligned} \tag{4.28}$$

Finally, we define

$$\mathbf{H}_\Phi^{(\mathscr{A},\mathscr{B})}[\omega; \Xi] = H_\Phi^{(A,B)}[\omega; \Xi] + H_\Phi^{(b,\beta,c,\gamma)}[\omega; \Xi] + G_\Phi^{(V,W)}[\omega]. \tag{4.29}$$

Definition 4.9. We say that Φ is $(\mathscr{A}, \mathscr{B})$-*convex* in $\mathbb{C}^2$ if $\mathbf{H}_\Phi^{(\mathscr{A},\mathscr{B})}[\omega; \Xi]$ is nonnegative for all $\omega \in \mathbb{C}^2$, $\Xi \in \mathbb{C}^{2d}$.

In accordance with Definition 3.13 we say that Φ is (A, B)-*convex* if it is $(A, 0, 0, 0, B, 0, 0, 0)$-convex.

We maintain the same notation when instead of matrices we consider matrix-valued *functions* $A, B \in L^\infty(\Omega, \mathbb{C}^{d\times d})$, vector-valued functions $b, \beta, c, \gamma \in L^\infty(\Omega, \mathbb{C}^d)$ and scalar functions $V, W \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}_+)$; in this case however we require that all the conditions are satisfied for a.e. $x \in \Omega$.

Given $f, g \in L^2(\Omega)$, define the functions

$$\mathcal{F}(t) = \mathcal{F}(\phi, t) = \int_\Omega \phi\left(T_t^{\mathscr{A}} f\right), \qquad \mathcal{E}(t) = \mathcal{E}(\Phi, t) = \int_\Omega \Phi\left(T_t^{\mathscr{A}} f, T_t^{\mathscr{B}} g\right), \quad t > 0. \tag{4.30}$$

Definitions (4.26), (4.27), (4.28) and (4.29) are motivated by the fact that, formally,

$$\begin{aligned} -\mathcal{F}'(t) &= \int_\Omega \mathbf{H}_\phi^{\mathscr{A}}\left[T_t^{\mathscr{A}} f; \nabla T_t^{\mathscr{A}} f\right], \\ -\mathcal{E}'(t) &= \int_\Omega \mathbf{H}_\Phi^{(\mathscr{A},\mathscr{B})}\left[\left(T_t^{\mathscr{A}} f, T_t^{\mathscr{B}} g\right); \left(\nabla T_t^{\mathscr{A}} f, \nabla T_t^{\mathscr{B}} g\right)\right]. \end{aligned} \tag{4.31}$$

It follows that if Φ is $(\mathscr{A}, \mathscr{B})$-convex on $\mathbb{C}^2$ then the function $\mathcal{E}$ is nonincreasing on $(0, +\infty)$. When Φ is *strictly* $(\mathscr{A}, \mathscr{B})$-convex and satisfies a suitable size estimate, this formal method can be used for proving bilinear inequalities in the spirit of [17–20].

Remark 4.10. Let us briefly explain the origin of the definition (4.8) and of conditions (4.9) and (4.10). Let $p \in (1, \infty)$. As discussed in Section 3.2 and Section 3.3.4, p-ellipticity was introduced as the condition on the matrix A ensuring the A-convexity of the power function F_p, defined in (3.20). The weaker condition $\Delta_p(A) \geqslant 0$ is in fact equivalent to the A-convexity of F_p, thanks to the identity (3.24).

When dealing with divergence-form operators with lower-order terms, the first step is therefore to find a condition on $\mathscr{A}$ equivalent to the $\mathscr{A}$-convexity of F_p. To this end, one looks for an expression of $\mathbf{H}_{F_p}^{\mathscr{A}}$ analogous to (3.24). It turns out that

$$\mathbf{H}_{F_p}^{\mathscr{A}(x)}[\zeta; X] = p|\zeta|^p \, \Gamma_p^{\mathscr{A}}\left(x, \frac{X}{\zeta}\right),$$

for almost every $x \in \Omega$ and all $\zeta \in \mathbb{C} \setminus \{0\}$, $X \in \mathbb{C}^d$ (Proposition 4.12). This explains the origin of (4.8) and (4.9).

The formal identity (4.31) shows that L^p-contractivity of $(T_t^{\mathscr{A}})_{t>0}$ is closely related to $\mathscr{A}$-convexity of the power function F_p, and hence to condition (4.9). Indeed, $(T_t^{\mathscr{A}})_{t>0}$ is L^p-contractive if and only if the flow $\mathcal{F} = \mathcal{F}(F_p, \cdot)$ in (4.30) is nonincreasing for all $f \in (L^2 \cap L^p)(\Omega)$, which would follow from

$$-\mathcal{F}'(t) \geqslant 0, \qquad \forall f \in (L^2 \cap L^p)(\Omega), \ \forall t > 0.$$

This provides the heuristic behind the proof of the L^p-contractivity of $(T_t^{\mathscr{A}})_{t>0}$. However, the information available on the semigroup $T_t^{\mathscr{A}}$ is not sufficient: *a priori*, $\mathcal{F}$ may not be well defined or regular. Even if it were, the identity (4.31) itself is only formal, since it relies on an integration by parts in the sense of (4.6). In particular, one would need

$$T_t^{\mathscr{A}} f \, |T_t^{\mathscr{A}} f|^{p-2} \sim \partial_\zeta F_p(T_t^{\mathscr{A}} f) \in \mathrm{D}(\mathfrak{a}_{\mathscr{A}}),$$

which is not available in general.

These issues are circumvented by Nittka's Theorem (Theorem 2.39), which allows one to transfer the question of L^p-contractivity of the semigroup to the L^p-dissipativity of the associated sesquilinear form. In this approach, the semigroup does not appear, so no issues of well-definedness or regularity of the flow arise. Moreover, no domain problems occur during the integration by parts, since one works directly under the assumption $u|u|^{p-2} \in \mathrm{D}(\mathfrak{a}_{\mathscr{A}})$.

Finally, condition (4.10) stems from the fact that, in order to prove the bilinear embedding, the generalized convexity of power functions with respect to $\mathscr{A}$ is not sufficient; in the same spirit as [19–21], one also needs an explicit lower bound for their generalized Hessian with respect to $\mathscr{A}$.

4.4 Generalized convexity of power functions and Bellman function

Recall the definition (11) of the Bellman function $\mathcal{Q}$. Owing to the tensor structure of $\mathcal{Q}$, the generalized convexity of $\mathcal{Q}$ is related to that of its elementary building blocks: the power functions (see [20]).

In Lemma 3.14 we showed that F_p is A-convex if and only if $\Delta_p(A) \geqslant 0$, while in Theorem 3.22 that $\mathcal{Q}$ is strictly (A, B)-convex provided that $A, B \in \mathcal{A}_p(\Omega)$.

In the next subsections we shall prove that F_p is $\mathscr{A}$-convex if and only if $\Gamma_p^{\mathscr{A}} \geqslant 0$, and $\mathcal{Q}$ is strictly $(\mathscr{A}, \mathscr{B})$-convex provided that $\mathscr{A} \in (\mathcal{S}_p \cap \mathcal{S}_2)(\Omega)$ and $\mathscr{B} \in \mathcal{S}_q(\Omega)$, generalizing the previous results.

4.4.1 Generalized convexity of power functions

Let $r > 0$. Before enunciating the next lemma, extend $\mathcal{J}_r$, Γ_r and $\mathcal{S}_r$ to $r > 0$ by the same rules (4.7), (4.8) and (4.10). Recall the definitions (4.26) and (4.27). From (3.21) and (3.22) it follows

that for all $\zeta \in \mathbb{C} \setminus \{0\}$ and all $X \in \mathbb{C}^d$

$$\mathbf{H}^{\mathscr{A}}_{F_r}[\zeta; X] = |\zeta|^{r-2} \left(H^{A}_{F_r}[\zeta/|\zeta|; X] + |\zeta| H^{(b,c)}_{F_r}[\zeta/|\zeta|; X] + |\zeta|^2 G^{V}_{F_r}[\zeta/|\zeta|] \right). \tag{4.32}$$

The next result is modeled after Lemma 3.14. We define

$$\sigma = e^{-i \arg(\zeta)} X \in \mathbb{C}^d.$$

By (3.17), (3.21), (3.22) and by adequately modifying the proof of Lemma 3.14 we get the following result.

Lemma 4.11. *Let $r > 0$, $b, c \in \mathbb{C}^d$, $V \in \mathbb{R}$, $\zeta \in \mathbb{C}$ and $X \in \mathbb{C}^d$. Then, for $|\zeta| = 1$,*

$$\begin{aligned} r^{-1} H^{(b,c)}_{F_r}[\zeta; X] &= \operatorname{Re} \left\langle \overline{b} + \mathcal{J}_r c, \sigma \right\rangle, \\ r^{-1} G^{V}_{F_r}[\zeta] &= V |\zeta|^2. \end{aligned}$$

Proposition 4.12. *Let Ω and $\mathscr{A} = (A, b, c, V)$ satisfy the standard assumptions of Section 4.1.1. Let $r > 0$, $\zeta \in \mathbb{C} \setminus \{0\}$ and $X \in \mathbb{C}^d$. Suppose that for a.e. $x \in \Omega$*

$$\Gamma^{\mathscr{A}}_r(x, \xi) \geqslant 0, \quad \forall \xi \in \mathbb{C}^d.$$

Then

$$\mathbf{H}^{\mathscr{A}(x)}_{F_r}[\zeta; X] \geqslant 0.$$

Moreover, if $\mathscr{A} \in \mathcal{S}_r(\Omega)$, then for a.e. $x \in \Omega$ we have

$$\mathbf{H}^{\mathscr{A}(x)}_{F_r}[\zeta; X] \geqslant r \mu_r(A) |\zeta|^{r-2} (|X|^2 + V(x)|\zeta|^2).$$

Proof. By (4.32), Lemma 3.14 and Lemma 4.11 we get

$$\begin{aligned} \mathbf{H}^{\mathscr{A}}_{F_r}[\zeta; X] &= r|\zeta|^{r-2} \left(\operatorname{Re} \langle A\sigma, \mathcal{J}_r \sigma \rangle + |\zeta| \operatorname{Re} \left\langle \overline{b} + \mathcal{J}_r c, \sigma \right\rangle + |\zeta|^2 V \right) \\ &= r|\zeta|^r \Gamma^{\mathscr{A}}_r(x, \sigma/|\zeta|). \end{aligned}$$

Therefore, we conclude. □

4.4.2 Generalized convexity of the Bellman function of Nazarov and Treil

Let $p \geqslant 2$ and denote by $q = p/(p-1)$ its conjugate exponent. Now we shall prove the $(\mathscr{A}, \mathscr{B})$-convexity of the Bellman function provided that $\mathscr{A} \in (\mathcal{S}_p \cap \mathcal{S}_2)(\Omega)$ and $\mathscr{B} \in \mathcal{S}_q(\Omega)$. This lemma is the analogue of [20, Lemma 5.11].

Lemma 4.13. *Let $1 < q < 2$ and $b, c, \beta, \gamma \in \mathbb{C}^d$. Take $\omega = (\zeta, \eta) \in \mathbb{C} \times \mathbb{C}$ such that $|\zeta| < |\eta|^{q-1}$ and $X, Y \in \mathbb{C}^d$. Then*

$$\begin{aligned} H^{(b,\beta,c,\gamma)}_{F_2 \otimes F_{2-q}}[\omega; (X, Y)] = {} & |\eta|^{2-q} H^{(b,c)}_{F_2}[\zeta, X] + |\zeta|^2 H^{(\beta,\gamma)}_{F_{2-q}}[\eta, Y] \\ & + 2(2-q)|\zeta|^2 |\eta|^{1-q} \operatorname{Re} \langle c, \operatorname{Re} \sigma_2 \rangle \\ & + 2(2-q)|\zeta||\eta|^{2-q} \operatorname{Re} \langle \gamma, \operatorname{Re} \sigma_1 \rangle. \end{aligned}$$

Proof. This follows from combining the definition of $H^{(b,\beta,c,\gamma)}_{F_2 \otimes F_{2-q}}[\omega; (X, Y)]$ (see (4.28)) and the identity

$$\partial^2_{\zeta_j \eta_k} (F_2 \otimes F_{2-q})(\zeta, \eta) = 2(2-q) \zeta_j \eta_k |\eta|^{-q} \quad \text{for } j, k = 1, 2.$$ □

Lemma 3.14, Lemma 4.13 and [20, Lemma 5.11] immediately give the following estimate.

Corollary 4.14. *Let $\Omega, \mathscr{A} = (A, b, c, V), \mathscr{B} = (B, \beta, \gamma, W)$ satisfy the standard assumptions of Section 4.1.1. Let $1 < q < 2$. Take $\omega = (\zeta, \eta) \in \mathbb{C} \times \mathbb{C}$ such that $|\zeta| < |\eta|^{q-1}$ and $X, Y \in \mathbb{C}^d$. Then, for almost every $x \in \Omega$, we have*

$$\begin{aligned}
&\mathbf{H}_{F_2 \otimes F_{2-q}}^{(\mathscr{A}(x), \mathscr{B}(x))}[\omega; (X, Y)] \\
&\geqslant |\eta|^{2-q} \mathbf{H}_{F_2}^{\mathscr{A}(x)}[\zeta; X] + \frac{(2-q)^2}{2} \Delta_{2-q}(B) |\eta|^{q-2} |Y|^2 \\
&\quad - (2-q) \left| \overline{\beta}(x) + (\mathcal{J}_{2-q}\gamma)(x) \right| |\zeta|^2 |\eta|^{1-q} |Y| + (2-q) W(x) |\zeta|^2 |\eta|^{2-q} \\
&\quad - 4(2-q)\Lambda |X||Y| - 2(2-q) |\mathrm{Re}\, c(x)| |\zeta|^2 |\eta|^{1-q} |Y| - 2(2-q) |\mathrm{Re}\, \gamma(x)| |\eta| |X|.
\end{aligned}$$

We are ready now to prove an analogous result of Theorem 3.22.

Theorem 4.15. *Let $\Omega, \mathscr{A} = (A, b, c, V), \mathscr{B} = (B, \beta, \gamma, W)$ satisfy the standard assumptions of Section 4.1.1. Let $p \geq 2$ and denote by $q = p/(p-1)$ its conjugate exponent. Assume that $\mathscr{A} \in (\mathcal{S}_p \cap \mathcal{S}_2)(\Omega)$ and $\mathscr{B} \in \mathcal{S}_q(\Omega)$. Then there exist $\delta \in (0, 1)$ and $C > 0$ such that for $\mathcal{Q} = \mathcal{Q}_{p,\delta}$ we have, for almost every $x \in \Omega$,*

$$\mathbf{H}_{\mathcal{Q}}^{(\mathscr{A}(x), \mathscr{B}(x))}[\omega; (X, Y)] \geqslant C \sqrt{|X|^2 + V(x)|\zeta|^2} \sqrt{|Y|^2 + W(x)|\eta|^2}$$

for any $\omega = (\zeta, \eta) \in (\mathbb{C} \times \mathbb{C}) \setminus \Upsilon$ and $X, Y \in \mathbb{C}^d$. The constant $C > 0$ can be chosen so as to (continuously) depend on $p, \lambda(A, B), \Lambda(A, B), M(\mathscr{A}, \mathscr{B}), \mu(\mathscr{A}), \mu_p(\mathscr{A})$ and $\mu_q(\mathscr{B})$, but not on the dimension d.

Proof. When $p = 2$ the Bellman function reads $\mathcal{Q}(\zeta, \eta) = (1+\delta)|\zeta|^2 + |\eta|^2$ for all $\zeta, \eta \in \mathbb{C}$, hence the theorem quickly follows from Proposition 4.12. Thus, from now on assume that $p > 2$.

If $\boxed{|\zeta|^p > |\eta|^q > 0}$, then by the second assertion of Proposition 4.12 we have, for almost every $x \in \Omega$,

$$\begin{aligned}
\mathbf{H}_{\mathcal{Q}}^{(\mathscr{A}(x), \mathscr{B}(x))}[\omega; (X, Y)] &= [1 + \delta(2/p)] \mathbf{H}_{F_p}^{\mathscr{A}(x)}[\zeta; X] + [1 + \delta(2/q - 1)] \mathbf{H}_{F_q}^{\mathscr{B}(x)}[\eta; Y] \\
&\geqslant p[1 + \delta(2/p)] \mu_p(\mathscr{A}) |\zeta|^{p-2} (|X|^2 + V(x)|\zeta|^2) \\
&\quad + q[1 + \delta(2/q - 1)] \mu_q(\mathscr{B}) |\eta|^{q-2} (|Y|^2 + W(x)|\eta|^2).
\end{aligned}$$

By assumptions we have $2 - q > 0$. So whenever $\delta > 0$, we may continue as

$$\begin{aligned}
&\geqslant \min\{\mu_p(\mathscr{A}), \mu_q(\mathscr{B})\} \left(p|\zeta|^{p-2} (|X|^2 + V(x)|\zeta|^2) + q|\eta|^{q-2} (|Y|^2 + W(x)|\eta|^2) \right) \\
&\geqslant \frac{p}{\sqrt{p-1}} \min\{\mu_p(\mathscr{A}), \mu_q(\mathscr{B})\} \sqrt{|X|^2 + V(x)|\zeta|^2} \sqrt{|Y|^2 + W(x)|\eta|^2}.
\end{aligned}$$

In the last step we have used the assumption $|\zeta|^p > |\eta|^q$ and the inequality between the arithmetic and geometric mean.

What remains is the case $\boxed{|\zeta|^p < |\eta|^q}$. Set $\Theta := \{p, \lambda, \Lambda, \mu, M, \mu_p(\mathscr{A}), \mu_q(\mathscr{B})\}$. By Proposition 4.4(i), (ii) there exists $C_0 = C_0(\Theta) > 0$ such that for a.e. $x \in \Omega$

$$|\mathrm{Re}\, c(x)| \leqslant C_0 \sqrt{V(x)}, \quad |\mathrm{Re}\, \gamma(x)| \leqslant C_0 \sqrt{W(x)}, \quad \left| \overline{\beta}(x) + (\mathcal{J}_{2-q}\gamma)(x) \right| \leqslant C_0 \sqrt{W(x)}.$$

Therefore, from Proposition 4.12 (applied first with $r = p$ and then with $r = q$ and $r = 2$) and Corollary 4.14 we get, for almost every $x \in \Omega$,

$$
\begin{aligned}
&\mathbf{H}_{\mathcal{Q}}{}^{(\mathscr{A}(x),\mathscr{B}(x))}[\omega;(X,Y)] \\
&\quad= \mathbf{H}_{F_p}^{\mathscr{A}(x)}[\zeta;X] + \mathbf{H}_{F_q}^{\mathscr{B}(x)}[\eta;Y] + \delta\mathbf{H}_{F_2\otimes F_{2-q}}^{(\mathscr{A}(x),\mathscr{B}(x))}[\omega;(X,Y)] \\
&\quad\geqslant \mathbf{H}_{F_q}^{\mathscr{B}(x)}[\eta;Y] + \delta\mathbf{H}_{F_2\otimes F_{2-q}}^{(\mathscr{A}(x),\mathscr{B}(x))}[\omega;(X,Y)] \\
&\quad\geqslant q\mu_q|\eta|^{q-2}\left(|Y|^2 + W(x)|\eta|^2\right) + 2\delta\mu_2|\eta|^{2-q}\left(|X|^2 + V(x)|\zeta|^2\right) + (2-q)\delta W|\zeta|^2|\eta|^{2-q} \qquad (4.33) \\
&\qquad+ \delta\frac{(2-q)^2}{2}\Delta_{2-q}(B)|\eta|^{q-2}|Y|^2 - 4\delta(2-q)\Lambda|X||Y| \qquad (4.34) \\
&\qquad- \delta(2-q)C_0|\zeta|^2|\eta|^{1-q}|Y|\left(\sqrt{W(x)} + 2\sqrt{V(x)}\right) - 2\delta(2-q)C_0\sqrt{W(x)}|\eta||X|. \qquad (4.35)
\end{aligned}
$$

The key terms are those in (4.33), particularly the one containing $|\eta|^{q-2}|Y|^2$. The strategy is to apply [21, Corollary 3.4] after combining the following terms:

- $|\eta|^{q-2}|Y|^2$ and $V|\zeta|^2|\eta|^{2-q}$ in (4.33), to compensate the term including $\sqrt{V}|\zeta|^2|\eta|^{1-q}|Y|$ in (4.35).
- $|\eta|^{q-2}|Y|^2$ and $W|\zeta|^2|\eta|^{2-q}$ in (4.33), to compensate the term including $\sqrt{W}|\zeta|^2|\eta|^{1-q}|Y|$ in (4.35).
- $|\eta|^{2-q}|X|^2$ and $W|\eta|^q$ in (4.33), to compensate the term including $\sqrt{W}|\eta||X|$ in (4.35).
- $|\eta|^{2-q}|X|^2$ and $|\eta|^{q-2}|Y|^2$ in (4.33), to compensate the terms in (4.34); recall that $\Delta_{2-q}(B) \leqslant 0$ since $q \in (1,2)$ [20].

Therefore, by splitting the term containing $|\eta|^{q-2}|Y|^2$ in (4.33) into a sum of three parts, the term including $|\eta|^{2-q}|X|^2$ in (4.33) into a sum of two parts and by using the fact that $|\eta|^{q-2} \geqslant |\zeta|^2|\eta|^{-q}$ in the current domain, we get for almost every $x \in \Omega$,

$$
\begin{aligned}
&\mathbf{H}_{\mathcal{Q}}{}^{(\mathscr{A}(x),\mathscr{B}(x))}[\omega;(X,Y)] \\
&\quad\geqslant \delta|\zeta|^2|\eta|^{-q}\left(\frac{q\mu_q}{3\delta}|Y|^2 - 2(2-q)C_0\sqrt{V(x)}|\eta||Y| + 2\mu_2 V(x)|\eta|^2\right) \\
&\qquad+ \frac{2-q}{2}\delta|\zeta|^2|\eta|^{-q}\left(\frac{2q\mu_q}{3(2-q)\delta}|Y|^2 - 2C_0\sqrt{W(x)}|\eta||Y| + 2W(x)|\eta|^2\right) \\
&\qquad+ \delta\left(2\mu_2|\eta|^{2-q}|X|^2 - 2(2-q)C_0\sqrt{W(x)}|\eta||X| + \frac{q\mu_q}{\delta}W(x)|\eta|^q\right) \\
&\qquad+ 2\delta\left(\frac{\mu_2}{2}|\eta|^{2-q}|X|^2 - 2(2-q)\Lambda|X||Y| + \Xi_\delta|\eta|^{q-2}|Y|^2\right),
\end{aligned} \qquad (4.36)
$$

where

$$
\Xi_\delta = \frac{q\mu_q}{6\delta} + \frac{(2-q)^2}{4}\Delta_{2-q}(B).
$$

Observing that $|\eta| = |\eta|^{(2-q)/2}|\eta|^{q/2}$, we intend to apply [21, Corollary 3.4] to the terms within the parentheses on the right-hand side of (4.36). Since $\mu_2, \mu_q > 0$, $q \in (1,2)$ and Ξ_δ grows to infinity as $\delta \searrow 0$, it suffices to choose a small enough $\delta > 0$ such that

$$
\frac{2q\mu_2\mu_q}{3\delta} > [(2-q)C_0]^2, \quad \frac{4q\mu_q}{3(2-q)\delta} > C_0^2, \quad \frac{2q\mu_2\mu_q}{\delta} > [(2-q)C_0]^2, \quad \frac{\mu_2\Xi_\delta}{2} > [(2-q)\Lambda]^2.
$$

By the definition of Ξ_δ and [20, (5.23)], we may choose δ (continuously) depending only on Θ. Now, [21, Corollary 3.4] ensures the existence of constants $C_1 = C_1(\delta, \Theta) > 0$ such that

$$\begin{aligned}
\frac{q\mu_q}{3\delta}|Y|^2 - 2(2-q)C_0\sqrt{V(x)}|\eta||Y| + 2\mu_2 V(x)|\eta|^2 &\geqslant C_1 V(x)|\eta|^2,\\
\frac{2q\mu_q}{3(2-q)\delta}|Y|^2 - 2C_0\sqrt{W(x)}|\eta||Y| + 2W(x)|\eta|^2 &\geqslant 0,\\
2\mu_2|\eta|^{2-q}|X|^2 - 2(2-q)C_0\sqrt{W(x)}|\eta||X| + \frac{q\mu_q}{\delta}W(x)|\eta|^q &\geqslant C_1 W(x)|\eta|^q,\\
\frac{\mu_2}{2}|\eta|^{2-q}|X|^2 - 2(2-q)\Lambda|X||Y| + \Xi_\delta|\eta|^{q-2}|Y|^2 &\geqslant C_1(|\eta|^{2-q}|X|^2 + |\eta|^{q-2}|Y|^2).
\end{aligned} \tag{4.37}$$

Therefore, by combining (4.36) and (4.37), we obtain for almost every $x \in \Omega$,

$$\begin{aligned}
\mathbf{H}_{\mathcal{Q}}^{(\mathscr{A}(x),\mathscr{B}(x))}[\omega;(X,Y)] &\geqslant \delta C_1|\eta|^{2-q}\left(|X|^2 + V(x)|\zeta|^2\right) + \delta C_1|\eta|^{q-2}\left(|Y|^2 + W(x)|\eta|^2\right)\\
&\geqslant \delta C_1\sqrt{|X|^2 + V(x)|\zeta|^2}\sqrt{|Y|^2 + W(x)|\eta|^2},
\end{aligned}$$

where in the last step we have used the inequality between the arithmetic and geometric mean. □

Remark 4.16. When $b = c = 0$, in Theorem 3.22 Carbonaro and Dragičević estimated $H_{\mathcal{Q}}^{(A,B)}$ and $G_{\mathcal{Q}}^{(V,W)}$ separately. However, this time, in order to use the nonnegativity of the function Γ_p, we cannot separate $H_{\mathcal{Q}}^{(A,B)}, H_{\mathcal{Q}}^{(b,c,\beta,\gamma)}$ and $G_{\mathcal{Q}}^{(V,W)}$ into individual pieces and estimate them one by one.

We would like to have an analogue of Corollary 3.25.

Lemma 4.17. *Let $A, B \in \mathbb{C}^{d\times d}$, $b, \beta, c, \gamma \in \mathbb{C}^d$ and $V, W \in \mathbb{R}_+$. Set $\mathscr{A} = (A, b, c, V)$ and $\mathscr{B} = (B, \beta, \gamma, W)$. Then, for all $\omega = (\zeta, \eta) \in \mathbb{C}^2$, $X, Y \in \mathbb{C}^d$, $\nu \in (0,1)$, we have*

$$\begin{aligned}
\mathbf{H}_{\mathcal{Q}*\varphi_\nu}^{(\mathscr{A},\mathscr{B})}[\omega;(X,Y)] = &\int_{\mathbb{R}^4} \mathbf{H}_{\mathcal{Q}}^{(\mathscr{A},\mathscr{B})}[\omega - \mathcal{W}_{2,1}^{-1}(z);(X,Y)]\varphi_\nu(z)\,\mathrm{d}z\\
&+ \mathcal{N}_\nu^{(c,\gamma)}[\omega;(X,Y)] + \mathcal{N}_\nu^{(V,W)}(\omega),
\end{aligned} \tag{4.38}$$

where

$$\begin{aligned}
\mathcal{N}_\nu^{(c,\gamma)} &= \mathcal{N}_\nu^{(c,\gamma)}[\omega;(X,Y)]\\
&= \int_{\mathbb{R}^4}\left\langle\left[D^2\mathcal{Q}(\omega - \mathcal{W}_{2,1}^{-1}(z)) \otimes I_{\mathbb{R}^d}\right]\mathcal{W}_{2,d}(X,Y), \mathcal{W}_{2,d}\left(\mathcal{V}_1^{-1}(z_1)c, \mathcal{V}_1^{-1}(z_2)\gamma\right)\right\rangle \times\\
&\qquad \times \varphi_\nu(z)\,\mathrm{d}z,\\
\mathcal{N}_\nu^{(V,W)} &= \mathcal{N}_\nu^{(V,W)}(\omega) = \int_{\mathbb{R}^4}\left\langle D\mathcal{Q}(\omega - \mathcal{W}_{2,1}^{-1}(z)), (Vz_1, Wz_2)\right\rangle\varphi_\nu(z)\,\mathrm{d}z.
\end{aligned}$$

Remark 4.18. We would like to use (4.38) in order to estimate $\mathbf{H}_{\mathcal{Q}*\varphi_\nu}^{(\mathscr{A},\mathscr{B})}$ from below. The integral $\int_{\mathbb{R}^4}\mathbf{H}_{\mathcal{Q}}$ in (4.38) will be estimated by means of Theorem 4.15. Recall that in the unperturbed case, Carbonaro and Dragičević [19, 20] proved (3.34), that is,

$$H_{\mathcal{Q}*\varphi_\nu}^{(A,B)}[\omega;(X,Y)] = \int_{\mathbb{R}^4} H_{\mathcal{Q}}^{(A,B)}[\omega - \mathcal{W}_{2,1}^{-1}(z);(X,Y)]\varphi_\nu(z)\,\mathrm{d}z.$$

This time, the presence of the terms $\mathcal{N}_\nu^{(c,\gamma)}$ and $\mathcal{N}_\nu^{(V,W)}$ in (4.38) is due to the fact that in the definition (4.28) the element $\omega = (\zeta, \eta)$ appears not only as a variable of the function $\mathcal{Q}$, but also in the components of $(\zeta c, \eta\gamma)$ and $(V\zeta, W\eta)$. For this reason we cannot proceed exactly as in [19]. The terms $\mathcal{N}_\nu^{(c,\gamma)}$ and $\mathcal{N}_\nu^{(V,W)}$ are remainders, in the sense that they are going to disappear as $\nu \to 0$. In order to prove that, we will need their upper estimates which will be established next; see Lemma 4.19.

Proof of Lemma 4.17. By definition (4.28) we have

$$\begin{aligned} H_{\mathcal{Q}}^{(b,\beta,c,\gamma)}[\omega;(X,Y)] &= \left\langle \left[D^2\mathcal{Q}(\omega)\otimes I_{\mathbb{R}^d}\right]\mathcal{W}_{2,d}(X,Y),\mathcal{W}_{2,d}(\zeta c,\eta\gamma)\right\rangle_{\mathbb{R}^{4d}} \\ &\quad + \left\langle D\mathcal{Q}(\omega),\mathcal{W}_{2,1}\left(\left\langle X,\overline{b}\right\rangle,\left\langle Y,\overline{\beta}\right\rangle\right)\right\rangle_{\mathbb{R}^4} \\ &=: H_{\mathcal{Q}}^{(c,\gamma)}[\omega;(X,Y)] + G_{\mathcal{Q}}^{(b,\beta)}[\omega;(X,Y)], \end{aligned}$$

for all $\omega=(\zeta,\eta)\in\mathbb{C}^2\setminus\Upsilon$ and all $X,Y\in\mathbb{C}^d$. Since $\mathcal{Q}_{\mathcal{W}}\in C^1(\mathbb{R}^4)$ and its second-order partial derivatives exist on $\mathbb{R}^4\setminus\mathcal{W}(\Upsilon)$ and are locally integrable in $\mathbb{R}^4$, by the ACL characterization of Sobolev spaces (see, for example, [80, Théorème V, p. 57] or [57, Theorem 11.45]) we have

$$\begin{aligned} \mathbf{H}_{\mathcal{Q}*\varphi_\nu}^{(\mathscr{A},\mathscr{B})}[\omega;(X,Y)] &= \int_{\mathbb{R}^4} H_{\mathcal{Q}}^{(A,B)}[\omega-\mathcal{W}_{2,1}^{-1}(z);(X,Y)]\varphi_\nu(z)\,\mathrm{d}z \\ &\quad + \int_{\mathbb{R}^4} G_{\mathcal{Q}}^{(b,\beta)}[\omega-\mathcal{W}_{2,1}^{-1}(z);(X,Y)]\varphi_\nu(z)\,\mathrm{d}z \\ &\quad + \left\langle\left[D^2(\mathcal{Q}*\varphi)(\omega)\otimes I_{\mathbb{R}^d}\right]\mathcal{W}_{2,d}(X,Y),\mathcal{W}_{2,d}(\zeta c,\eta\gamma)\right\rangle \\ &\quad + \left\langle D(\mathcal{Q}*\varphi)(\omega),\mathcal{W}_{2,1}(V\zeta,W\eta)\right\rangle, \end{aligned} \tag{4.39}$$

for all $\omega\in\mathbb{C}^2$ and all $X,Y\in\mathbb{C}^d$. Writing the third and the fourth terms of the right-hand side of (4.39) as

$$\begin{aligned} &\left\langle\left[D^2(\mathcal{Q}*\varphi)(\omega)\otimes I_{\mathbb{R}^d}\right]\mathcal{W}_{2,d}(X,Y),\mathcal{W}_{2,d}(\zeta c,\eta\gamma)\right\rangle \\ &\qquad = \int_{\mathbb{R}^4} H_{\mathcal{Q}}^{(c,\gamma)}[\omega-\mathcal{W}_{2,1}^{-1}(z);(X,Y)]\varphi_\nu(z)\,\mathrm{d}z + \mathcal{N}_\nu^{(c,\gamma)}[\omega;(X,Y)], \\ &\left\langle D(\mathcal{Q}*\varphi)(\omega),\mathcal{W}_{2,1}(V\zeta,W\eta)\right\rangle \\ &\qquad = \int_{\mathbb{R}^4} G_{\mathcal{Q}}^{(V,W)}[\omega-\mathcal{W}_{2,1}^{-1}(z)]\varphi_\nu(z)\,\mathrm{d}z + \mathcal{N}_\nu^{(V,W)}(\omega), \end{aligned}$$

we get (4.38). □

Lemma 4.19. *Let $\nu\in(0,1)$, $c,\gamma\in\mathbb{C}^d$ and $V,W\in\mathbb{R}_+$. Then*

(i) *for all $\omega=(\zeta,\eta)\in\mathbb{C}^2$ and all $X,Y\in\mathbb{C}^d$,*

$$\begin{aligned} |\mathcal{N}_\nu^{(c,\gamma)}[\omega;(X,Y)]| &\lesssim \nu^{q-1}\max\{|c|,|\gamma|\}\left(1+|\zeta|^{p-2}+|\eta|^{2-q}\right)|(X,Y)|, \\ |\mathcal{N}_\nu^{(V,W)}(\omega)| &\lesssim \nu\max\{V,W\}\left(1+|\zeta|^{p-1}+|\eta|^{q-1}+|\eta|\right); \end{aligned}$$

(ii) *$\mathcal{N}_\nu^{(c,\gamma)}[\omega;(X,Y)]$ and $\mathcal{N}_\nu^{(V,W)}(\omega)$ converge to 0 as $\nu\to0_+$ for all $\omega\in\mathbb{C}^2$, $X,Y\in\mathbb{C}^d$.*

Proof. By using Lemma 3.24(iii), the second and the third estimates of (3.33) and the fact that the support of both the integrand of $\mathcal{N}_\nu^{(c,\gamma)}$ and $\mathcal{N}_\nu^{(V,W)}$ is contained in $B_{\mathbb{R}^4}(0,\nu)$, we get item (i). Sending ν to 0, we prove item (ii). □

Corollary 4.20. *Let $\Omega,\mathscr{A}=(A,b,c,V),\mathscr{B}=(B,\beta,\gamma,W)$ satisfy the standard assumptions of Section 4.1.1. Let $p\geq2$ and denote by $q=p/(p-1)$ its conjugate exponent. Assume that $\mathscr{A}\in(\mathcal{S}_p\cap\mathcal{S}_2)(\Omega)$ and $\mathscr{B}\in\mathcal{S}_q(\Omega)$. Then for every $\omega=(\zeta,\eta)\in\mathbb{C}\times\mathbb{C}$, $X,Y\in\mathbb{C}^d$ and almost every $x\in\Omega$,*

$$\liminf_{\nu\to0}\mathbf{H}_{\mathcal{Q}*\varphi_\nu}^{(\mathscr{A}(x),\mathscr{B}(x))}[\omega;(X,Y)] \geq C\sqrt{|X|^2+V(x)|\zeta|^2}\sqrt{|Y|^2+W(x)|\eta|^2}.$$

The constant $C>0$ can be chosen so as to (continuously) depend on $p,\lambda(A,B)$, $\Lambda(A,B)$, $M(\mathscr{A},\mathscr{B})$, $\mu(\mathscr{A})$, $\mu_p(\mathscr{A})$ and $\mu_q(\mathscr{B})$, but not on the dimension d.

Proof. By combining Theorem 4.15 and Lemma 4.17, we infer that for almost every $x \in \Omega$,

$$\begin{aligned}
&\mathbf{H}_{\mathcal{Q}*\varphi_\nu}^{(\mathscr{A}(x),\mathscr{B}(x))}[\omega;(X,Y)] \\
&\quad\gtrsim \int_{\mathbb{R}^4} \sqrt{|X|^2+V(x)|\zeta-\mathcal{V}_1^{-1}(z_1)|^2}\sqrt{|Y|^2+W(x)|\eta-\mathcal{V}_1^{-1}(z_2)|^2}\varphi_\nu(z)\,\mathrm{d}z \\
&\qquad+\ \mathcal{N}_\nu^{(c(x),\gamma(x))}[\omega;(X,Y)]+\mathcal{N}_\nu^{(V(x),W(x))}(\omega),
\end{aligned} \tag{4.40}$$

for all $\omega=(\zeta,\eta)\in\mathbb{C}^2$ and $X,Y\in\mathbb{C}^d$.

Since the function $\mathbb{C}^2\ni(\zeta,\eta)\mapsto\sqrt{|X|^2+V(x)|\zeta|^2}\sqrt{|Y|^2+W(x)|\eta|^2}$ is continuous in $\mathbb{C}^2$ for all $X,Y\in\mathbb{C}^d$ and for all $x\in\Omega$, we get

$$\begin{aligned}
&\lim_{\nu\to 0}\int_{\mathbb{R}^4}\sqrt{|X|^2+V(x)|\zeta-\mathcal{V}_1^{-1}(z_1)|^2}\sqrt{|Y|^2+W(x)|\eta-\mathcal{V}_1^{-1}(z_2)|^2}\varphi_\nu(z)\,\mathrm{d}z \\
&\qquad=\ \sqrt{|X|^2+V(x)|\zeta|^2}\sqrt{|Y|^2+W(x)|\eta|^2}
\end{aligned} \tag{4.41}$$

for all $\zeta,\eta\in\mathbb{C}$, $X,Y\in\mathbb{C}^d$ and $x\in\Omega$. Combining Lemma 4.19(ii), (4.40) and (4.41), we conclude. □

4.5 L^p contractivity and analyticity of $(T_t^{\mathscr{A},\mathscr{V}})_{t>0}$

Let $\mathscr{A}=(A,b,c,V)\in\mathcal{B}(\Omega)$ and let $\mathscr{V}$ satisfy (3.5). As a first result, we aim to extend Theorem 3.17. The strategy remains essentially the same: applying Nittka's Theorem 2.39. In this context, the condition $\Delta_p(e^{i\phi}A)\geqslant 0$—crucial for establishing the L^p-dissipativity of the form (3.3)—will be replaced by the condition

$$\Gamma_p^{\mathscr{A}_\phi}\geqslant 0,$$

where

$$\mathscr{A}_\phi:=(e^{i\phi}A,\ e^{i\phi}b,\ e^{i\phi}c,\ (\cos\phi)V),\quad \phi\in\mathbb{R}.$$

Before proceeding with the proof of the L^p-contractivity of the semigroup $(T_t^{\mathscr{A},\mathscr{V}})_{t>0}$, we present an analogue of Lemma 3.16. In fact, the L^p-dissipativity of the form (4.1) is closely related to our $\mathbf{H}_{F_p}^{\mathscr{A}}$, as we show next.

Lemma 4.21. *Let $B\in\mathbb{C}^{d\times d}$, $\beta,\gamma\in\mathbb{C}^d$ and $W\in\mathbb{R}_+$. Suppose that $p>1$ and $f\in W^{1,2}(\Omega)$ are such that $|f|^{p-2}f\in W^{1,2}(\Omega)$. Then, setting $\mathscr{B}=(B,\beta,\gamma,W)$,*

$$\mathbf{H}_{F_p}^{\mathscr{B}}[f;\nabla f]=p\,\mathrm{Re}\left(\left\langle B\nabla f,\nabla\left(|f|^{p-2}f\right)\right\rangle+\left\langle\nabla f,\overline{\beta}\right\rangle|f|^{p-2}\overline{f}+f\left\langle\gamma,\nabla\left(|f|^{p-2}f\right)\right\rangle\right)+W|f|^p.$$

Proof. The claim follows by combining Lemma 3.14, Lemma 3.15 and Lemma 4.11. □

We prove now the L^p-contractivity of the semigroup $(T_t^{\mathscr{A},\mathscr{V}})_{t>0}$ by adapting the argument used in the proof of Theorem 3.17 to the present, perturbed setting. On one hand, to establish the invariance of the domain of the form under the projection $\mathbf{P}_{B^p}$ we replace [72, Theorem 4.9] with [72, Theorem 4.31*2)*], as it has been done in [21, Theorem 1.2], in the context of Schrödinger-type operators with unbounded potentials. On the other hand, the condition $\Gamma_p^{\mathscr{A}_\phi}\geqslant 0$ ensures the L^p-dissipativity of the form.

Theorem 4.22. *Choose $p>1$, $\mathscr{A}\in\mathcal{B}(\Omega)$, $\phi\in\mathbb{R}$ such that $|\phi|<\pi/2-\omega$ and $\Gamma_p^{\mathscr{A}_\phi}(x,\xi)\geqslant 0$ for almost all $x\in\Omega$ and all $\xi\in\mathbb{C}^d$. Then, for every $\mathscr{V}$ satisfying (3.5),*

$$\left(e^{-te^{i\phi}\mathscr{L}}\right)_{t>0}$$

extends to a strongly continuous semigroup of contractions on $L^p(\Omega)$.

Proof. We will use Nittka's invariance criterion (Theorem 2.39). Under our assumptions on ϕ, the sesquilinear form $\mathfrak{b} := e^{i\phi}\mathfrak{a}$ is densely defined, closed and sectorial. In particular, it is accretive and, by Proposition 2.34, continuous. Therefore, it falls into the framework of Nittka's criterion. The operator associated with $\mathfrak{b}$ is $e^{i\phi}\mathscr{L}^{A,b,c,V}$.

Let $-\Delta+V$ denote the operator associated with the form $\mathfrak{a}_{I,0,0,V,\mathscr{V}}$. By the basic assumption (3.5) and [72, Theorem 4.31*2)*], the semigroup $(e^{-t(-\Delta+V)})_{t>0}$ is contractive on $L^\infty(\Omega)$, and thus, by interpolation with the L^2-estimates and by duality, on $L^p(\Omega)$ for all $p \in [1,\infty]$. Hence Nittka's Theorem 2.39 gives that $\mathrm{D}(\mathfrak{a}_{I,0,0,V\mathscr{V}})$ is invariant under $\mathbf{P}_{B^p}$. Now use that $\mathrm{D}(\mathfrak{a}_{I,0,0,V,\mathscr{V}}) = \mathrm{D}(\mathfrak{a}) = \mathrm{D}(\mathfrak{b})$.

Recall the definition (2.25) of $\mathrm{D}_p(\mathfrak{b})$. Let $u \in \mathrm{D}_p(\mathfrak{b})$. By Lemma 4.21, applied with $\mathscr{B} = \mathscr{A}_\phi$, we get

$$\operatorname{Re}\mathfrak{b}(u,|u|^{p-2}u) = p^{-1}\int_\Omega \mathbf{H}_{F_p}^{\mathscr{A}_\phi}[u;\nabla u].$$

Hence the theorem follows from Proposition 4.12. □

Without assumptions on the lower order terms, we obtain the L^p-quasi-contractivity of the semigroup $(T_t)_{t>0}$ generated by $-\mathscr{L}$ (i.e., the L^p-contractivity of the semigroup generated by $\mathscr{L}+\omega$, for some $\omega \geqslant 0$) provided that A is p-elliptic. Compare the next result with [24, Theorem 6]; see also [20, Proposition 5.18].

Corollary 4.23. *Let $V \in L^1_{\mathrm{loc}}(\Omega,\mathbb{R})$ be bounded from below and choose $p > 1$ and $A \in \mathcal{A}(\Omega)$ such that $A \in \mathcal{A}_p(\Omega)$. Then, for every $b, c \in L^\infty(\Omega,\mathbb{C}^d)$ and $\mathscr{V}$ satisfying (3.5), the semigroup $(T_t^{\mathscr{A},\mathscr{V}})_{t>0}$, associated with $\mathscr{A} = (A,b,c,V)$, extends to a quasi-contraction semigroup on $L^p(\Omega)$.*

Proof. Let $b, c \in L^\infty(\Omega,\mathbb{C}^d)$ and let $\mathscr{V}$ satisfy (3.5). For every $\omega \in \mathbb{R}$ we denote $\mathcal{A}_\omega = (A,b,c,V+\omega)$. We want to find $\omega > |\operatorname{ess\,inf} V|$ such that $(T_t^{\mathscr{A}_\omega,\mathscr{V}})_{t>0}$ extends to a strongly continuous semigroup of contractions on $L^p(\Omega)$. To this end, in light of Theorem 4.22 applied with $\phi = 0$, it is sufficient to find $\omega, \mu > 0$ such that

$$\Gamma_2^{\mathscr{A}_\omega}(\xi) \geqslant \mu\left(|\xi|^2 + V + \omega\right),$$
$$\Gamma_p^{\mathscr{A}_\omega}(\xi) \geqslant 0,$$

for all $\xi \in \mathbb{C}^d$. The first inequality would follow if we had $\mu < \min\{\lambda, 1\}$, where $\lambda = \lambda(A)$ is such that

$$(\lambda-\mu)|\xi|^2 + \operatorname{ess\,inf}\min_{|\sigma|=1}\operatorname{Re}\left\langle \overline{b}+c,\sigma\right\rangle|\xi| + (1-\mu)(\operatorname{ess\,inf} V+\omega) \geqslant 0.$$

This holds for ω large enough. In the same way, we are able to find ω such that the second inequality holds. □

The next corollary, applied with r equal to the conjugate exponent of p, extends [21, Corollary 1.3], which in turn generalizes Corollary 3.18.

Corollary 4.24. *Choose $r \leqslant 2 \leqslant p$ such that $\mathscr{A} \in (\mathcal{S}_r \cap \mathcal{S}_p)(\Omega)$. Let $\mathscr{V}$ satisfy (3.5). Then there exists $\vartheta = \vartheta(p,r,\mathscr{A}) > 0$ such that $\{T_z : z \in \mathbf{S}_\vartheta\}$ is analytic and contractive in $L^s(\Omega)$ for all $s \in [r,p]$.*

Proof. By Proposition 4.5(ii) and Proposition 4.4(iv), there exists $\vartheta > 0$ such that $(e^{i\phi}A,\, e^{i\phi}b,\, e^{i\phi}c,\, (\cos\phi)V) \in \mathcal{S}_s(\Omega)$ for all $\phi \in [-\vartheta,\vartheta]$ and all $s \in [r,p]$. The contractivity part now follows from Theorem 4.22 and the relation

$$T_{te^{i\phi}} = \exp\left(-te^{i\phi}\mathscr{L}\right), \tag{4.42}$$

whereupon analyticity is a consequence of Theorem 2.9. □

4.5.1 Comparison between (4.9) and the Cialdea-Maz'ya condition [24, (2.25)]

Let $(C_c(\Omega,\mathbb{C}))^*$ denote the continuous dual of $C_c(\Omega,\mathbb{C})$ endowed with the uniform norm. In [24] Cialdea and Maz'ya studied the L^p-dissipativity of the sesquilinear form $\widetilde{\mathfrak{a}}$, defined by

$$\begin{aligned} \mathrm{D}(\widetilde{\mathfrak{a}}) &= C_c^1(\Omega), \\ \widetilde{\mathfrak{a}}(u,v) &= \int_\Omega \langle A\nabla u, \nabla v\rangle + \left\langle \nabla u, \overline{b}\right\rangle \overline{v} + u\,\langle c, \nabla v\rangle + Vu\overline{v}. \end{aligned} \tag{4.43}$$

Here A is a $d\times d$ matrix with entries $a_{hk}\in (C_c(\Omega,\mathbb{C}))^*$, $b=(b_1,\dots,b_d)$ and $c=(c_1,\dots,c_d)$ stand for vectors with $b_j, c_j \in (C_c(\Omega,\mathbb{C}))^*$ and V is a complex-valued scalar distribution in $(C_c^1(\Omega))^*$. By the Riesz-Markov theorem every element of $(C_c(\Omega,\mathbb{C}))^*$ may be interpreted as a regular complex Borel measure on Ω. Therefore, for instance, the meaning of the first and the last terms in (4.43) is

$$\int_\Omega \langle A\nabla u, \nabla v\rangle = \sum_{k,h=1}^d \int_\Omega \partial_k u\, \partial_h \overline{v}\, \mathrm{d}a_{hk},$$

$$\int_\Omega Vu\overline{v} = \langle V, u\overline{v}\rangle\,.$$

Notice that the sesquilinear form $\widetilde{\mathfrak{a}}$ in (4.43) differs from the one considered by Cialdea and Maz'ya by the sign of b and V.

They introduced the notion of L^p-dissipativity and, under the additional assumptions that $\Omega\subset\mathbb{R}^d$ is a bounded domain with sufficiently regular boundary [24, p. 1087], the entries of A and b belong to $C^1(\overline{\Omega})$, c is zero and $V\in C(\overline{\Omega})$, they proved in [24, Theorem 6] that the semigroup $T_t^{\mathscr{A},H_0^1}$ is L^p-quasicontractive if and only if

$$|p-2|\,|\langle \operatorname{Im} A(x)\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \operatorname{Re} A(x)\xi,\xi\rangle\,,$$

for any $x\in\Omega, \xi\in\mathbb{R}^d$. The above inequality is equivalent to saying that $\Delta_p(A_s)\geqslant 0$, where $A_s=(A+A^T)/2$ is the symmetric part of A; see [20, Proposition 5.18].

Without these further assumptions on Ω and on the coefficients A, b, c and V, in [24, Corollary 4] they showed that $\widetilde{\mathfrak{a}}$ is L^p-dissipative provided that there exist two real constants ϑ and ν such that

$$\begin{aligned} &\frac{4}{pp'}\,\langle \operatorname{Re} A\alpha,\alpha\rangle + \langle \operatorname{Re} A\beta,\beta\rangle + 2\left\langle \left(p^{-1}\operatorname{Im} A + (p')^{-1}\operatorname{Im} A^*\right)\alpha,\beta\right\rangle \\ &\quad + \langle \operatorname{Im}(-b+c),\beta\rangle + 2\left\langle \operatorname{Re}\left(\vartheta\frac{b}{p}+\nu\frac{c}{p'}\right),\alpha\right\rangle + \\ &\quad + \operatorname{Re}\left[V - \operatorname{div}\left((1-\vartheta)\frac{b}{p}+(1-\nu)\frac{c}{p'}\right)\right] \geqslant 0 \quad \forall \alpha,\beta\in\mathbb{R}^d. \end{aligned} \tag{4.44}$$

Condition (4.44) has to be understood in the sense of measures and distributions, that is, it means that, for every $\alpha,\beta\in\mathbb{R}^d$,

$$\begin{aligned} \int_\Omega \Bigg[&\left(\frac{4}{pp'}\,\langle \operatorname{Re} A\alpha,\alpha\rangle + \langle \operatorname{Re} A\beta,\beta\rangle + 2\left\langle \left(p^{-1}\operatorname{Im} A + (p')^{-1}\operatorname{Im} A^*\right)\alpha,\beta\right\rangle \right. \\ &\left. + \langle \operatorname{Im}(-b+c),\beta\rangle + 2\left\langle \operatorname{Re}\left(\vartheta\frac{b}{p}+\nu\frac{c}{p'}\right),\alpha\right\rangle + \operatorname{Re} V\right)\varphi\Bigg] \\ &- \left\langle \operatorname{div}\left((1-\vartheta)\frac{\operatorname{Re} b}{p}+(1-\nu)\frac{\operatorname{Re} c}{p'}\right),\varphi\right\rangle \geqslant 0, \end{aligned}$$

for any nonnegative $\varphi\in C_c^1(\Omega)$, where the last term of the left-hand side of the previous inequality stands for the action of the distribution $\operatorname{div}((1-\vartheta)(\operatorname{Re} b/p)+(1-\nu)(\operatorname{Re} c/p'))$ on φ.

The distributional divergence appears after two integrations by parts. In fact, let $p \geqslant 2$ and u be in $C_c^1(\Omega)$. Setting $v = |u|^{(p-2)/2}u$, which belongs to $C_c^1(\Omega)$, and noticing that $2\operatorname{Re}(\overline{v}\nabla v) = \nabla(|v|^2)$, they showed in [24, p. 1071 and p. 1078] that

$$\begin{aligned}\int_\Omega \operatorname{Re}\left(\left\langle \nabla u, \overline{b}\right\rangle |u|^{p-2}\overline{u}\right) &= \int_\Omega \frac{2}{p}\left\langle \operatorname{Re} b, \operatorname{Re}(\overline{v}\nabla v)\right\rangle - \left\langle \operatorname{Im} b, \operatorname{Im}(\overline{v}\nabla v)\right\rangle \\ &= \int_\Omega \left(\frac{2\vartheta}{p}\left\langle \operatorname{Re} b, \operatorname{Re}(\overline{v}\nabla v)\right\rangle - \left\langle \operatorname{Im} b, \operatorname{Im}(\overline{v}\nabla v)\right\rangle\right) \\ &\quad - \frac{1-\vartheta}{p}\left\langle \operatorname{div}(\operatorname{Re} b), |v|^2\right\rangle.\end{aligned}$$

Similarly, they got the divergence of $(1-\nu)(\operatorname{Re} c/p')$.

Our assumptions on the domain $\mathrm{D}(\mathfrak{a})$ of the form $\mathfrak{a}$, defined in (9), are weaker than those on the domain of $\widetilde{\mathfrak{a}}$, because the latter, which consists of all compactly supported functions on Ω of class C^1, is a proper subspace of $\mathrm{D}(\mathfrak{a})$. Moreover, since in generality that we consider the domain Ω may be completely irregular, $\mathrm{D}(\widetilde{\mathfrak{a}})$ might not even be dense in $\mathrm{D}(\mathfrak{a})$ with respect to the Sobolev norm $\|\cdot\|_{W^{1,2}(\Omega)}$, for instance if $\mathrm{D}(\mathfrak{a}) = W^{1,2}(\Omega)$. In particular, for an arbitrary $u \in \mathrm{D}_p(\mathfrak{a})$, we are not able to justify the distributional integration by parts

$$\int_\Omega \left\langle \operatorname{Re} b, \nabla(|v|^2)\right\rangle = -\left\langle \operatorname{div}(\operatorname{Re} b), |v|^2\right\rangle,$$

where $v = |u|^{(p-2)/2}u$ and $b \in L^\infty(\Omega, \mathbb{C}^d)$. Due to the weaker assumptions, our sufficient condition for the L^p-contractivity of the semigroup naturally turns out to imply (4.44).

Proposition 4.25. *Let $A : \Omega \to \mathbb{C}^{d\times d}$, b, $c : \Omega \to \mathbb{C}^d$, $V : \Omega \to \mathbb{R}_+$ and $p > 1$. Suppose that $\Gamma_p^{A,b,c,V}(x,\xi) \geqslant 0$ for almost all $x \in \Omega$ and all $\xi \in \mathbb{C}^d$. Then (4.44) holds with $\vartheta = \nu = 1$.*

Proof. Recalling (3.18), replacing α with $\alpha p/2$ and using the identity

$$\overline{b} + \mathcal{J}_p c = \operatorname{Re}(b + (p-1)c) + i\operatorname{Im}(-b + c),$$

the condition (4.44) with $\vartheta = \nu = 1$ can be expressed as: for a.e. $x \in \Omega$

$$\operatorname{Re}\left\langle A(x)(\alpha + i\beta), \mathcal{J}_p(\alpha + i\beta)\right\rangle + \operatorname{Re}\left\langle \overline{b} + \mathcal{J}_p c, \alpha + i\beta\right\rangle + V(x) \geqslant 0 \quad \forall \alpha, \beta \in \mathbb{R}^d. \qquad \square$$

4.6 Bilinear embedding for operators with regular coefficients

In case when $b = c = 0$, in [21, Theorem 1.4] Carbonaro and Dragičević proved that there exists $C > 0$ independent of the dimension d such that

$$\int_0^\infty \int_\Omega \sqrt{\left|\nabla T_t^{A,V,\mathscr{V}} f\right|^2 + V\left|T_t^{A,V,\mathscr{V}} f\right|^2}\sqrt{\left|\nabla T_t^{B,W,\mathscr{W}} g\right|^2 + W\left|T_t^{B,W,\mathscr{W}} g\right|^2} \leqslant C\|f\|_p\|g\|_q, \tag{4.45}$$

for all $A, B \in \mathcal{A}_p(\Omega)$, $V, W \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}_+)$ and all $f, g \in (L^p \cap L^q)(\Omega)$, where $\mathscr{V}$ and $\mathscr{W}$ are two subspaces of $W^{1,2}(\Omega)$ satisfying Assumption BE and $q = p/(p-1)$ is the conjugate exponent of p.

Given $b, c, \beta, \gamma \in L^\infty(\Omega, \mathbb{C}^d)$, we extend now the bilinear embedding in (4.45) to the semigroups $(T_t^{\mathscr{A},\mathscr{V}})_{t>0}$ and $(T_t^{\mathscr{B},\mathscr{W}})_{t>0}$, where $\mathscr{A} = (A, b, c, V), \mathscr{B} = (B, \beta, \gamma, W) \in \mathcal{B}(\Omega)$. In accordance with [19–21], we need a stronger condition than the one which implies the L^p contractivity of such semigroups. Recall (3.26). Consistent with [19–21], it is convenient to introduce further notation:

$$\begin{aligned}\mu(\mathscr{A}, \mathscr{B}) &= \min\{\mu(\mathscr{A}), \mu(\mathscr{B})\}, \\ M(\mathscr{A}, \mathscr{B}) &= \max\{M(\mathscr{A}), M(\mathscr{B})\}.\end{aligned}$$

In the next three sections we will prove the following result.

Theorem 4.26. *Let $\Omega, \mathscr{A} = (A, b, c, V), \mathscr{B} = (B, \beta, \gamma, W)$ satisfy the standard assumptions of Section 4.1.1 and $\mathscr{V}, \mathscr{W}$ satisfy Assumption BE. Suppose that $\mathscr{A}, \mathscr{B} \in \mathcal{B}(\Omega)$ and choose $p > 1$ such that $\mathscr{A}, \mathscr{B} \in \mathcal{B}_p(\Omega)$. Denote by $q = p/(p-1)$ its conjugate exponent. Then there exists $C > 0$ independent of the dimension d such that*

$$\int_0^\infty \int_\Omega \sqrt{\left|\nabla T_t^{\mathscr{A},\mathscr{V}} f\right|^2 + V\left|T_t^{\mathscr{A},\mathscr{V}} f\right|^2} \sqrt{\left|\nabla T_t^{\mathscr{B},\mathscr{W}} g\right|^2 + W\left|T_t^{\mathscr{B},W,\mathscr{W}} g\right|^2} \leqslant C\|f\|_p \|g\|_q, \tag{4.46}$$

for all $f, g \in (L^p \cap L^q)(\Omega)$. The constant $C > 0$ can be chosen so as to (continuously) depend only on $p, \lambda(A,B), \Lambda(A,B), \mu(\mathscr{A},\mathscr{B}), M(\mathscr{A},\mathscr{B}), \mu_p(\mathscr{A})$ and $\mu_q(\mathscr{B})$.

This result incorporates several earlier theorems as special cases, including:

- $b = c = 0$, $V = W$, $\Omega = \mathbb{R}^d$, A, B equal and real [35, Theorem 1]
- $b = c = 0$, $V = W = 0$, $\Omega = \mathbb{R}^d$ [20, Theorem 1.1] (Theorem 3.19)
- $b = c = 0$, $V = W = 0$ Theorem 3.19 [19, Theorem 2] (Theorem 3.19)
- $b = c = 0$ [21, Theorem 1.4].

We modify the heat-flow method of [19] and [21], explained in Sectionn 3.5. Observe that we cannot estimate $\mathbf{H}^{\mathscr{A},\mathscr{B}}$ splitting it into the three terms of the definition (4.29), as Carbonaro and Dragičević implicitly did in [21] when the first-order terms were zero. This is because our condition (4.10) involves the matrix (A), as well as the first- and zero-order terms (b, c and V, respectively) at the same time, without "decoupling". On the other hand, as we saw in Section 3.5.4, in the unperturbed setting we strongly exploited the (A,B)-convexity of the sequences appearing in the limit argument in the heat-flow method, so that we could use Fatou's lemma (twice). The crucial point there was that the (A,B)-convexity is invariant under convolution, while, as already observed in Remark 4.18, the $(\mathscr{A},\mathscr{B})$-convexity might be not. Therefore, we would like to justify the passage of the limits through the integrals by means of Lebesgue's dominated convergence theorem. However, under the general assumptions on the coefficients, it is not straightforward to uniformly dominate the integrands by integrable functions. Hence, we will initially prove the bilinear embedding under further assumptions on the coefficients and then we will deduce the arbitrary case by a limit argument.

Without loss of generality we assume $p \geqslant 2$. Let $\mathscr{A} = (A, b, c, V), \mathscr{B} = (B, \beta, \gamma, W)$ satisfy the standard assumptions of Section 4.1.1. Suppose that $\mathscr{A}, \mathscr{B} \in \mathcal{B}_p(\Omega)$. Fix two closed subspaces $\mathscr{V}$ and $\mathscr{W}$ of $W^{1,2}(\Omega)$ satisfying Assumption BE. In the spirit of [20], we will for the moment also

$$\text{assume that } A, B \in C^\infty(\Omega),\ b, c, \beta, \gamma, V, W \in C_c^\infty(\Omega). \tag{4.47}$$

The smoothness of the coefficients implies that $T_t^{\mathscr{A},\mathscr{V}} f, T_t^{\mathscr{B},\mathscr{W}} g \in C^\infty(\Omega)$ for all $f, g \in L^2(\Omega)$; see Lemma A.6. In particular, they and their respective gradients are locally bounded on Ω. This fact, combined with the assumption on the supports of the first- and zero-order terms, will allow us to uniformly dominate certain integrands by integrable functions and then apply Lebesgue's dominated convergence theorem. Once the proof with these further assumptions is over, we will apply the regularization argument from the Appendix to pass to the case of arbitrary coefficients.

Now, we will prove that

$$\int_0^\infty \int_\Omega \sqrt{\left|\nabla T_t^{\mathscr{A},\mathscr{V}} f\right|^2 + V\left|T_t^{\mathscr{A},\mathscr{V}} f\right|^2} \sqrt{\left|\nabla T_t^{\mathscr{B},\mathscr{W}} g\right|^2 + W\left|T_t^{\mathscr{B},\mathscr{W}} g\right|^2} \lesssim \left(\|f\|_p^p + \|g\|_q^q\right), \tag{4.48}$$

for all $f, g \in (L^p \cap L^q)(\Omega)$. Once (4.48) is proved, (4.46) follows by replacing f and g in (4.48) with sf and $s^{-1}g$ and minimizing the right-hand side with respect to $s > 0$.

We now start the heat-flow method. We simply denote

$$\begin{aligned} T_t^{\mathscr{A}} &= T_t^{\mathscr{A},\mathscr{V}}, & T_t^{\mathscr{B}} &= T_t^{\mathscr{B},\mathscr{W}}, & \forall t > 0, \\ \mathscr{L}^{\mathscr{A}} &= \mathscr{L}^{\mathscr{A},\mathscr{V}}, & \mathscr{L}^{\mathscr{B}} &= \mathscr{L}^{\mathscr{B},\mathscr{W}}. \end{aligned}$$

Observe that the boundedness of the potentials V, W implies that

$$\mathrm{D}(\mathfrak{a}_{\mathscr{A},\mathscr{V}}) = \mathscr{V}, \qquad \mathrm{D}(\mathfrak{a}_{\mathscr{B},\mathscr{W}}) = \mathscr{W}. \tag{4.49}$$

We also recall that for $r \in (1, \infty)$ the operator $-\mathscr{L}_r^{\mathscr{A}}$ stands for the generator of $(T_t^{\mathscr{A}})$ on $L^r(\Omega)$, whenever the semigroup extends on $L^r(\Omega)$.

Let $\mathcal{Q} = \mathcal{Q}_\delta$ as in (11). Fix $\delta > 0$ such that Theorem 4.15 holds. Let us start with a reduction in the spirit of [19, Section 6.1]; see also [22, Proposition 7.2]. Compare with Proposition 3.29.

Proposition 4.27. *Let $p \geqslant 2$. Let $\mathscr{A} = (A, b, c, V), \mathscr{B} = (B, \beta, \gamma, W)$, $\mathscr{V}, \mathscr{W}$ as above. Assume that*

$$\int_\Omega \sqrt{|\nabla u|^2 + V|u|^2}\sqrt{|\nabla v|^2 + W|v|^2} \lesssim 2\,\mathrm{Re}\int_\Omega \Big((\partial_\zeta \mathcal{Q})(u,v)\mathscr{L}^{\mathscr{A}}u + (\partial_\eta \mathcal{Q})(u,v)\mathscr{L}^{\mathscr{B}}v\Big) \tag{4.50}$$

for $u \in \mathrm{D}(\mathscr{L}^{\mathscr{A}})$, $v \in \mathrm{D}(\mathscr{L}^{\mathscr{B}})$ such that $u, v \in C^\infty(\Omega)$ and $u, v, \mathscr{L}^{\mathscr{A}}u, \mathscr{L}^{\mathscr{B}}v \in (L^p \cap L^q)(\Omega)$. Then the bilinear estimate (4.48) holds for all $f, g \in (L^p \cap L^q)(\Omega)$.

Proof. Fix $f, g \in (L^p \cap L^q)(\Omega)$. The proof consists in studying the monotonicity of the flow

$$\mathcal{E}(t) = \int_\Omega \mathcal{Q}\left(T_t^{\mathscr{A}} f, T_t^{\mathscr{B}} g\right), \quad t > 0,$$

exactly as in the proof of Proposition 3.29. In this case, we further restrict ourselves to functions $u, v \in C^\infty(\Omega)$, since the smoothness of $\mathscr{A}$ and $\mathscr{B}$, together with Lemma A.6, implies that $T_t^{\mathscr{A}} f$ and $T_t^{\mathscr{B}} g$ belong to $C^\infty(\Omega)$. □

As we saw in Chapter 3, when the first- and zero-order terms are zero, in order to establish an analogue of (4.50), that is (3.42), Carbonaro and Dragičević approximated the Bellman function $\mathcal{Q}$ by a sequence $(\mathcal{R}_{n,\nu})_{n\in\mathbb{N}}$, $\nu \in (0,1)$, of smooth (A,B)-convex functions proving that

$$2\,\mathrm{Re}\int_\Omega \Big((\partial_\zeta \mathcal{Q})(u,v)\mathscr{L}^{A}u + (\partial_\eta \mathcal{Q})(u,v)\mathscr{L}^{B}v\Big) = \lim_{\nu\to 0_+}\lim_{n\to+\infty}\int_\Omega H^{(A,B)}_{\mathcal{R}_{n,\nu}}[(u,v);(\nabla u, \nabla v)] \tag{4.51}$$

and

$$\lim_{\nu\to 0_+}\lim_{n\to+\infty}\int_\Omega H^{(A,B)}_{\mathcal{R}_{n,\nu}}[(u,v);(\nabla u, \nabla v)] \gtrsim \int_\Omega |\nabla u||\nabla v|; \tag{4.52}$$

see Section 3.5.4. While the identity (4.51) is easily adapted to our case since it follows from properties of the sequence $(\mathcal{R}_{n,\nu})_{n\in\mathbb{N}_+}$ which do not depend on the matrices A and B (Theorem 3.26 (ii) and (iv) and Lemma 3.27), the estimate (4.52) is not. In fact, it was proved by strongly exploiting the (A,B)-convexity of $\mathcal{R}_{n,\nu}$ for all $n \in \mathbb{N}_+$ and $\nu \in (0,1)$. The crucial point there was that the (A,B)-convexity is invariant under convolution, while, as already observed in Remark 4.18, the $(\mathscr{A},\mathscr{B})$-convexity might be not. Therefore, in this case we do not know if $\mathcal{R}_{n,\nu}$ are $(\mathscr{A},\mathscr{B})$-convex for all $n \in \mathbb{N}_+$ and $\nu \in (0,1)$.

4.6.1 Proof of (4.50)

Let u, v as in the assumptions of Proposition 4.27.

As done in Section 3.5.4, we deduce that

$$\begin{aligned} &2\,\mathrm{Re}\int_\Omega \Big((\partial_\zeta \mathcal{Q})(u,v)\mathscr{L}^{\mathscr{A}}u + (\partial_\eta \mathcal{Q})(u,v)\mathscr{L}^{\mathscr{B}}v\Big) \\ &\quad = \lim_{\nu\to 0_+}\lim_{n\to+\infty} 2\,\mathrm{Re}\int_\Omega \Big((\partial_\zeta \mathcal{R}_{n,\nu})(u,v)\mathscr{L}^{\mathscr{A}}u + (\partial_\eta \mathcal{R}_{n,\nu})(u,v)\mathscr{L}^{\mathscr{B}}v\Big). \end{aligned} \tag{4.53}$$

By definition of $\mathscr{L}^{\mathscr{A}}$ and $\mathscr{L}^{\mathscr{B}}$ (4.6), (4.49) and Lemma 3.27 we can integrate by parts the integral on the right-hand side of (4.53) and, by means of the chain-rule for the composition of smooth functions with vector-valued Sobolev functions, deduce that

$$\begin{aligned} &2\,\mathrm{Re}\int_\Omega \Big((\partial_\zeta \mathcal{R}_{n,\nu})(u,v)\mathscr{L}^{\mathscr{A}}u + (\partial_\eta \mathcal{R}_{n,\nu})(u,v)\mathscr{L}^{\mathscr{B}}v\Big) \\ &\quad = \int_\Omega \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{R}_{n,\nu}}[(u,v);(\nabla u,\nabla v)]. \end{aligned} \tag{4.54}$$

So far we have repeated the same steps of Section 3.5.4, adapting them to our case, establishing an analogue of (4.51). Now if we knew that

$$\lim_{n\to+\infty}\int_\Omega \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{R}_{n,\nu}}[(u,v);(\nabla u,\nabla v)] \geqslant \int_\Omega \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)] \tag{4.55}$$

and

$$\liminf_{\nu\to 0}\int_\Omega \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)] \gtrsim \int_\Omega \sqrt{|\nabla u|^2+V|u|^2}\sqrt{|\nabla v|^2+W|v|^2}, \tag{4.56}$$

we could prove (4.50) by combining these two estimates with (4.53) and (4.54).

Notice that, by Theorem 3.26 (ii) and Corollary 4.20, we have

$$\begin{aligned} &\lim_{n\to+\infty} \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{R}_{n,\nu}}[(u,v);(\nabla u,\nabla v)] = \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)], \\ &\liminf_{\nu\to 0} \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)] \gtrsim \sqrt{|\nabla u|^2+V|u|^2}\sqrt{|\nabla v|^2+W|v|^2}. \end{aligned}$$

Therefore, proving (4.55) and (4.56) reduces to justifying the passage of the limits through the integrals. While in Section 3.5.4, it was sufficient applying Fatou's lemma since both $\mathcal{R}_{n,\nu}$ and $\mathcal{Q}*\varphi_\nu$ are (A,B)-convex, now we will also need Lebesgue's dominated convergence theorem; see Section 3.5.4. We will strongly exploit that $u, v \in C^\infty(\Omega)$.

Proof of (4.55)

We should proceed in a slightly different way than Section 3.5.4, since we do not know that $\mathcal{R}_{n,\nu}$ are $(\mathscr{A},\mathscr{B})$-convex in $\mathbb{C}^2$ for all $n \in \mathbb{N}_+$ and $\nu \in (0,1)$; see Remark 4.18. However, by Theorem 3.26 $\mathcal{R}_{n,\nu}$ are (A,B)-convex for all $n \in \mathbb{N}_+$ and $\nu \in (0,1)$. Therefore, it is convenient to split the integrand of the right-hand side of (4.54) according to the definition (4.29), obtaining

$$\begin{aligned} &\int_\Omega \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{R}_{n,\nu}}[(u,v);(\nabla u,\nabla v)] \\ &= \int_\Omega H^{(A,B)}_{\mathcal{R}_{n,\nu}}[(u,v);(\nabla u,\nabla v)] + \int_\Omega \Big(H^{(b,\beta,c,\gamma)}_{\mathcal{R}_{n,\nu}}[(u,v);(\nabla u,\nabla v)] + G^{(V,W)}_{\mathcal{R}_{n,\nu}}[(u,v)]\Big) \\ &=: I_1^n + I_2^n. \end{aligned} \tag{4.57}$$

By using Theorem 3.26 (ii) and Fatou's lemma we deduce that

$$\liminf_{n\to+\infty} I_1^n \geqslant \int_\Omega H^{(A,B)}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)]. \tag{4.58}$$

Recall that $u, v \in C^\infty(\Omega)$, $b, \beta, c, \gamma, V, W$ are compactly supported and $p \geqslant 2$. Hence from Theorem 3.26 (iv) and (3.41) we get

$$\begin{aligned}
\left|H^{(b,\beta,c,\gamma)}_{\mathcal{R}_{n,\nu}}[(u,v);(\nabla u,\nabla v)]\right| &\lesssim \left(|D^2\mathcal{R}_{n,\nu}(u,v)|\,|(u,v)| + |D\mathcal{R}_{n,\nu}(u,v)|\right)|(\nabla u,\nabla v)|\mathbb{1}_K \\
&\lesssim C(\nu)\left(|(u,v)|^{p-1} + |(u,v)|^{q-1} + |(u,v)|\right)|(\nabla u,\nabla v)|\mathbb{1}_K \\
&\lesssim C(\nu)\left(|(u,v)|^{p-1} + |(u,v)|^{q-1}\right)|(\nabla u,\nabla v)|\mathbb{1}_K \in L^1(\Omega), \\
\left|G^{(V,W)}_{\mathcal{R}_{n,\nu}}[(u,v)]\right| &\lesssim C(\nu)\left(|(u,v)|^p + |(u,v)|^q\right)\mathbb{1}_K \in L^1(\Omega),
\end{aligned} \tag{4.59}$$

where K is compact in Ω containing the supports of $b, \beta, c, \gamma, V, W$. Therefore, by applying Theorem 3.26 (ii), (4.59) and Lebesgue's dominated convergence theorem we deduce that

$$\lim_{n\to+\infty} I_2^n = \int_\Omega \left(H^{(b,\beta,c,\gamma)}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)] + G^{(V,W)}_{\mathcal{Q}*\varphi_\nu}[(u,v)]\right). \tag{4.60}$$

Hence we obtain (4.55) by combining (4.57), (4.58) and (4.60).

Proof of (4.56)

By Corollary 4.20, for almost all $x \in \Omega$,

$$\liminf_{\nu\to 0} \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)] \gtrsim \sqrt{|\nabla u|^2 + V|u|^2}\sqrt{|\nabla v|^2 + W|v|^2}.$$

So, if we knew that

$$\liminf_{\nu\to 0} \int_\Omega \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{Q}*\varphi_\nu} \geqslant \int_\Omega \liminf_{\nu\to 0} \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{Q}*\varphi_\nu}, \tag{4.61}$$

we could get (4.56). But we cannot proceed as in Section 3.5.4 to justify (4.61): in fact, we cannot use Fatou's Lemma, not knowing that $\mathcal{Q} * \varphi_\nu$ are $(\mathscr{A},\mathscr{B})$-convex in $\mathbb{C}^2$. However, the identity (4.38) implies that

$$\begin{aligned}
\mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)] &= \int_{\mathbb{R}^4} \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{Q}}[(u,v) - \mathcal{W}^{-1}_{2,1}(z);(\nabla u,\nabla v)]\varphi_\nu(z)\,\mathrm{d}z \\
&\quad + \mathcal{N}^{(c,\gamma)}_\nu[(u,v);(\nabla u,\nabla v)] + \mathcal{N}^{(V,W)}_\nu(u,v) \\
&=: J_\nu + \mathcal{N}^{(c,\gamma)}_\nu + \mathcal{N}^{(V,W)}_\nu,
\end{aligned} \tag{4.62}$$

for all $\nu \in (0,1)$. The integral J_ν is nonnegative for all $\nu \in (0,1)$. Therefore, by using Fatou's lemma, Lemma 4.19(ii) and (4.62), we get

$$\liminf_{\nu\to 0} \int_\Omega J_\nu \geqslant \int_\Omega \liminf_{\nu\to 0} J_\nu = \int_\Omega \liminf_{\nu\to 0} \mathbf{H}^{(\mathscr{A},\mathscr{B})}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)]. \tag{4.63}$$

On the other hand, sending $\nu \to 1$ on the right-hand side of Lemma 4.19(i) implies that there exists a constant $C > 0$, that does not depend on ν, such that

$$\begin{aligned}
|\mathcal{N}^{(c,\gamma)}_\nu| &\leqslant C\left(1 + |u|^{p-2} + |v|^{2-q}\right)|(\nabla u,\nabla v)|\mathbb{1}_K \in L^1(\Omega), \\
|\mathcal{N}^{(V,W)}_\nu| &\leqslant C\left(1 + |u|^{p-1} + |v|^{q-1} + |v|\right)\mathbb{1}_K \in L^1(\Omega).
\end{aligned}$$

All the right-hand terms of these estimates are integrable functions, since $u, v \in C^\infty(\Omega)$. Therefore, by using Lemma 4.19(ii) and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{\nu\to 0} \int_\Omega \mathcal{N}_\nu^{(c,\gamma)}[(u,v);(\nabla u,\nabla v)] &= 0, \\ \lim_{\nu\to 0} \int_\Omega \mathcal{N}_\nu^{(V,W)}(u,v) &= 0. \end{aligned} \tag{4.64}$$

Then, by combining (4.62), (4.63) and (4.64), we get (4.61) and thus (4.56), as explained above.

4.7 Bilinear embedding for operators with bounded potentials

We now begin to gradually relax the regularity assumptions on the coefficients, retaining only the boundedness of the potentials, which will be further removed in the next section. The aim of the present section is first to eliminate the regularity requirement on the coefficients—while still assuming that the first- and zero-order terms are compactly supported—and subsequently to remove the compact support condition. At each stage, the strategy is to approximate the coefficients so that they satisfy the assumptions of the preceding step. For instance, in the case when the coefficients lack regularity and the first- and zero-order terms are compactly supported, we approximate them by means of convolution with smooth approximation of identity in order to obtain smooth coefficients satisfying (4.47). Once the problem is reduced to the setting of the previous step, we can apply the bilinear embedding to the approximating coefficients and then recover the bilinear embedding for the original coefficients by means of a limit argument.

In order to apply the bilinear embedding result from the previous step, it is necessary that the class $\mathcal{B}_p$ be stable under the approximations we employ. Moreover, for the limiting argument, it is essential to control the constant appearing in the bilinear embedding, and to ensure that an approximation of the coefficients corresponds to an approximation of the associated semigroup. For the last point, we rely on a regularization method that we learnt by Carbonaro and Dragičević in [20, Appendix].

4.7.1 Semigroup approximation

Let $\mathscr{V}$ be a closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$. For any $\phi \in (0,\pi/2)$ denote $\phi^* := \pi/2 - \phi$. Write $I = [0,1]$. Consider a one-parameter family $\{A_s : s \in I\}$ of $d \times d$ complex matrix functions on Ω, two one-parameter families $\{b_s : s \in I\}$, $\{c_s : s \in I\}$ of d-dimensional complex vector-valued functions on Ω and a one parameter family $\{V_s : s \in I\}$ of nonnegative functions on Ω. Suppose that there exist $\mu, M, C > 0$ such that

(H1) $\mathscr{A}_s := (A_s, b_s, c_s, V_s) \in \mathcal{B}_{\mu,M}(\Omega)$ for all $s \in I$;

(H2) $\max\{\|A_s\|_\infty, \|b_s\|_\infty, \|c_s\|_\infty, \|V_s\|_\infty\} \leqslant C$ for all $s \in I$.

(H3) For a.e. $x \in \Omega$

$$\begin{aligned} \lim_{s\to 0} \|A_s(x) - A_0(x)\|_{\mathcal{B}(\mathbb{C}^d)} &= 0, \\ \lim_{s\to 0} \|b_s(x) - b_0(x)\|_{\mathcal{B}(\mathbb{C}^d,\mathbb{C})} &= 0, \\ \lim_{s\to 0} \|c_s(x) - c_0(x)\|_{\mathcal{B}(\mathbb{C},\mathbb{C}^d)} &= 0, \\ \lim_{s\to 0} \|V_s(x) - V_0(x)\|_{\mathcal{B}(\mathbb{C})} &= 0. \end{aligned}$$

Denote by $\mathfrak{a}_s = \mathfrak{a}_{\mathscr{A}_s,\mathscr{V}}$ the sesquilinear form associated with the pair $(\mathscr{A}_s, \mathscr{V})$ and by $\mathscr{L}_s = \mathscr{L}^{\mathscr{A}_s}$ the operator on $L^2(\Omega)$ associated with $\mathfrak{a}_s$. From (H1) and Corollary 4.3 each $\mathscr{L}_s$ is sectorial

with sectoriality angle $\omega(\mathscr{L}_s) \leqslant \omega$, where $\omega = \omega(\mu, M, \Lambda)$ is a positive angle smaller than $\pi/2$. In particular,

$$\sup_{\zeta \in \mathbb{C} \setminus \mathbf{S}_\vartheta} \|\zeta(\zeta - \mathscr{L}_s)^{-1}\|_{L^2(\Omega)} < \infty, \quad \forall \vartheta \in (\omega, \pi/2). \tag{4.65}$$

Set $\mathcal{H}_0 = L^2(\Omega)$ and $\mathcal{H}_1 = L^2(\Omega, \mathbb{C}^d)$. For $\zeta \in \mathbb{C} \setminus \mathbf{S}_\omega$ define the operators $G_{\mathscr{L}_s}(\zeta) : \mathcal{H}_0 \to \mathcal{H}_1$ and $S_{\mathscr{L}_s}(\zeta) : C_c^\infty(\Omega, \mathbb{C}^d) \to \mathcal{H}_0$ by

$$G_{\mathscr{L}_s}(\zeta) := \nabla(\zeta - \mathscr{L}_s)^{-1}, \qquad S_{\mathscr{L}_s}(\zeta) := (\zeta - \mathscr{L}_s)^{-1}\mathrm{div}\,.$$

Lemma 4.28. *Assume (H1) holds. Then for every $\vartheta \in (\omega, \pi/2)$ there exists $C = C(\mu, M, \Lambda, \vartheta) > 0$ such that*

$$\begin{aligned} |\zeta|^{1/2}\|G_{\mathscr{L}_s}(\zeta)f\|_{\mathcal{H}_1} &\leqslant C\|f\|_{\mathcal{H}_0}, \\ |\zeta|^{1/2}\|S_{\mathscr{L}_s}(\zeta)F\|_{\mathcal{H}_0} &\leqslant C\|F\|_{\mathcal{H}_1}, \\ \|\nabla S_{\mathscr{L}_s}(\zeta)F\|_{\mathcal{H}_1} &\leqslant C\|F\|_{\mathcal{H}_1}, \end{aligned}$$

for all $s \in I$, $f \in L^2(\Omega)$, $F \in C_c^\infty(\Omega; \mathbb{C}^d)$ and $\zeta \in \mathbb{C} \setminus \mathbf{S}_\vartheta$. Moreover, the very same estimates hold with $\mathscr{L}_s$ replaced by $\mathscr{L}_s^*$.

Proof. Since the proof is basically the same as that of [20, Lemma A.1] for unperturbed divergence-form operators, let us prove only the first inequality. The condition (H1) implies that

$$\begin{aligned} \mu\|\nabla(\zeta - \mathscr{L}_s)^{-1}f\|_{\mathcal{H}_1}^2 &\leqslant \mathrm{Re}\,\mathfrak{a}_s\left((\zeta - \mathscr{L}_s)^{-1}f, (\zeta - \mathscr{L}_s)^{-1}f\right) \\ &= \mathrm{Re}\left\langle \mathscr{L}_s(\zeta - \mathscr{L}_s)^{-1}f, (\zeta - \mathscr{L}_s)^{-1}f\right\rangle_{\mathcal{H}_0} \\ &= \mathrm{Re}\left\langle \zeta(\zeta - \mathscr{L}_s)^{-1}f, (\zeta - \mathscr{L}_s)^{-1}f\right\rangle_{\mathcal{H}_0} - \mathrm{Re}\left\langle f, (\zeta - \mathscr{L}_s)^{-1}f\right\rangle_{\mathcal{H}_0}. \end{aligned}$$

The first inequality of the lemma now follows from (4.65). Since $\mathscr{L}_s^* = \mathscr{L}^{A_s^*, \overline{c_s}, \overline{b_s}, V_s}$ and $(A_s^*, \overline{c_s}, \overline{b_s}, V_s) \in \mathcal{B}_{\mu,M}(\Omega)$ by Proposition 4.5(i), the same estimate clearly holds with $\mathscr{L}_s$ replaced by $\mathscr{L}_s^*$. □

Remark 4.29. The preceding lemma implies that for $\zeta \in \mathbb{C} \setminus \mathbf{S}_\omega$ the operators $S_{\mathscr{L}_s}(\zeta)$ and $\nabla S_{\mathscr{L}_s}(\zeta)$ admit unique extensions to bounded operators $\mathcal{H}_1 \to \mathcal{H}_0$ and $\mathcal{H}_1 \to \mathcal{H}_1$, respectively. Moreover, $S_{\mathscr{L}_s}^*(\zeta) = G_{\mathscr{L}_s^*}(\overline{\zeta})$.

Lemma 4.30. *For every $s \in I$, $\zeta \in \mathbb{C} \setminus \mathbf{S}_\omega$ and $f \in L^2(\Omega)$ we have*

$$\begin{aligned} (\zeta - \mathscr{L}_0)^{-1}f - (\zeta - \mathscr{L}_s)^{-1}f = {} & S_{\mathscr{L}_s}(\zeta) \circ M_{A_0 - A_s} \circ G_{\mathscr{L}_0}(\zeta)f \\ & + (\zeta - \mathscr{L}_s)^{-1} \circ M_{b_0 - b_s} \circ G_{\mathscr{L}_0}(\zeta)f \\ & + S_{\mathscr{L}_s}(\zeta) \circ M_{c_0 - c_s} \circ (\zeta - \mathscr{L}_0)^{-1}f \\ & + (\zeta - \mathscr{L}_s)^{-1} \circ M_{V_0 - V_s} \circ (\zeta - \mathscr{L}_0)^{-1}f, \end{aligned}$$

where we denote

$$\begin{aligned} M_{A_0 - A_s} &: \mathcal{H}_1 \to \mathcal{H}_1 \text{ the operator of multiplication by } A_0 - A_s, \\ M_{b_0 - b_s} &: \mathcal{H}_1 \to \mathcal{H}_0 \text{ the operator } \left\langle \cdot, \overline{b_0 - b_s}\right\rangle, \\ M_{c_0 - c_s} &: \mathcal{H}_0 \to \mathcal{H}_1 \text{ the operator of multiplication by } c_0 - c_s, \\ M_{V_0 - V_s} &: \mathcal{H}_0 \to \mathcal{H}_0 \text{ the operator of multiplication by } V_0 - V_s. \end{aligned}$$

Proof. Let $\zeta \in \mathbb{C} \setminus \mathbf{S}_\omega$. The identity

$$(\zeta - \mathscr{L}_0)^{-1} - (\zeta - \mathscr{L}_s)^{-1} = (\zeta - \mathscr{L}_s)^{-1}\mathscr{L}_0(\zeta - \mathscr{L}_0)^{-1} - \mathscr{L}_s(\zeta - \mathscr{L}_s)^{-1}(\zeta - \mathscr{L}_0)^{-1}$$

implies that, for all $f \in L^2(\Omega)$ and $g \in \mathrm{D}(\mathscr{L}_s^*)$,

$$\begin{aligned}
&\left\langle (\zeta - \mathscr{L}_0)^{-1} f - (\zeta - \mathscr{L}_s)^{-1} f, g \right\rangle_{\mathcal{H}_0} \\
&\quad = \left\langle \mathscr{L}_0(\zeta - \mathscr{L}_0)^{-1} f, (\overline{\zeta} - \mathscr{L}_s^*)^{-1} g \right\rangle_{\mathcal{H}_0} - \overline{\left\langle \mathscr{L}_s^*(\overline{\zeta} - \mathscr{L}_s^*)^{-1} g, (\zeta - \mathscr{L}_0)^{-1} f \right\rangle}_{\mathcal{H}_0} \\
&\quad = \mathfrak{a}_0\left((\zeta - \mathscr{L}_0)^{-1} f, (\overline{\zeta} - \mathscr{L}_s^*)^{-1} g\right) - \mathfrak{a}_s\left((\zeta - \mathscr{L}_0)^{-1} f, (\overline{\zeta} - \mathscr{L}_s^*)^{-1} g\right) \\
&\quad = \left\langle M_{A_0 - A_s} G_{\mathscr{L}_0}(\zeta) f, G_{\mathscr{L}_s^*}(\overline{\zeta}) g \right\rangle_{\mathcal{H}_1} + \left\langle M_{b_0 - b_s} G_{\mathscr{L}_0}(\zeta) f, (\overline{\zeta} - \mathscr{L}_s^*)^{-1} g \right\rangle_{\mathcal{H}_0} \\
&\qquad + \left\langle M_{c_0 - c_s}(\zeta - \mathscr{L}_0)^{-1} f, G_{\mathscr{L}_s^*}(\overline{\zeta}) g \right\rangle_{\mathcal{H}_1} + \left\langle M_{V_0 - V_s}(\zeta - \mathscr{L}_0)^{-1} f, (\overline{\zeta} - \mathscr{L}_s^*)^{-1} g \right\rangle_{\mathcal{H}_0}.
\end{aligned}$$

We conclude by invoking Remark 4.29. □

Lemma 4.31. *Fix $f \in L^2(\Omega)$, $z \in \mathbf{S}_{\omega^*}$ and assume (H1), (H2) and (H3). Then we have*

$$V_s^{1/2} T_z^{\mathscr{A}_s} f \to V_0^{1/2} T_z^{\mathscr{A}_0} f \qquad \text{in } L^2(\Omega), \tag{4.66}$$

$$\nabla T_z^{\mathscr{A}_s} f \to \nabla T_z^{\mathscr{A}_0} f \qquad \text{in } L^2(\Omega; \mathbb{C}^d), \tag{4.67}$$

as $s \to 0$.

Proof. In order to prove (4.66), by (H2) and (H3) it suffices to prove that

$$T_z^{\mathscr{A}_s} f \to T_z^{\mathscr{A}_0} f \quad \text{in } L^2(\Omega), \tag{4.68}$$

as $s \to 0$. Let $\vartheta^* \in (0, \pi/2)$ be such that $|\arg z| < \vartheta^* < \omega^*$. Fix $\delta > 0$ and denote by γ the positively oriented boundary of $\mathbf{S}_\vartheta \cup \{\zeta \in \mathbb{C} : |\zeta| < \delta\}$. For $s \in I$ and $\zeta \in \gamma$ define

$$\begin{aligned}
U(s, \zeta) = {} & S_{\mathscr{L}_s}(\zeta) \circ M_{A_0 - A_s} \circ G_{\mathscr{L}_0}(\zeta) + (\zeta - \mathscr{L}_s)^{-1} \circ M_{b_0 - b_s} \circ G_{\mathscr{L}_0}(\zeta) \\
& + S_{\mathscr{L}_s}(\zeta) \circ M_{c_0 - c_s} \circ (\zeta - \mathscr{L}_0)^{-1} + (\zeta - \mathscr{L}_s)^{-1} \circ M_{V_0 - V_s} \circ (\zeta - \mathscr{L}_0)^{-1}.
\end{aligned}$$

Then by (2.11) and Lemma 4.30,

$$T_z^{\mathscr{A}_0} f - T_z^{\mathscr{A}_s} f = \frac{1}{2\pi i} \int_\gamma e^{-z\zeta} U(s, \zeta) f \, \mathrm{d}\zeta.$$

Therefore, by Minkowsky's integral inequality, (4.65) and the second estimate of Lemma 4.28,

$$\begin{aligned}
\|T_z^{\mathscr{A}_0} f - T_z^{\mathscr{A}_s} f\|_2 &\lesssim \int_\gamma |e^{-z\zeta}| \cdot \|U(s, \zeta) f\|_2 \, \mathrm{d}|\zeta| \\
&\lesssim \int_\gamma e^{-\mathrm{Re}\,(z\zeta)} |\zeta|^{-1/2} \|M_{A_0 - A_s} G_{\mathscr{L}_0}(\zeta) f\|_2 \, \mathrm{d}|\zeta| \\
&\quad + \int_\gamma e^{-\mathrm{Re}\,(z\zeta)} |\zeta|^{-1} \|M_{b_0 - b_s} G_{\mathscr{L}_0}(\zeta) f\|_2 \, \mathrm{d}|\zeta| \\
&\quad + \int_\gamma e^{-\mathrm{Re}\,(z\zeta)} |\zeta|^{-1/2} \|M_{c_0 - c_s}(\zeta - \mathscr{L}_0)^{-1} f\|_2 \, \mathrm{d}|\zeta| \\
&\quad + \int_\gamma e^{-\mathrm{Re}\,(z\zeta)} |\zeta|^{-1} \|M_{V_0 - V_s}(\zeta - \mathscr{L}_0)^{-1} f\|_2 \, \mathrm{d}|\zeta|.
\end{aligned} \tag{4.69}$$

By (H2), (H3), Lemma 4.28 and Lebesgue's dominated convergence theorem we obtain

$$\begin{aligned}\lim_{s\to 0}\|M_{A_0-A_s}G_{\mathscr{L}_0}(\zeta)f\|_2 &= 0,\\ \lim_{s\to 0}\|M_{b_0-b_s}G_{\mathscr{L}_0}(\zeta)f\|_2 &= 0,\\ \lim_{s\to 0}\|M_{c_0-c_s}(\zeta-\mathscr{L}_0)^{-1}f\|_2 &= 0,\\ \lim_{s\to 0}\|M_{V_0-V_s}(\zeta-\mathscr{L}_0)^{-1}f\|_2 &= 0,\end{aligned}$$

for all $\zeta\in\gamma$. Moreover, by Lemma 4.28 again and by (4.65), we have

$$\begin{aligned}\|M_{A_0-A_s}G_{\mathscr{L}_0}(\zeta)f\|_2 &\lesssim |\zeta|^{-1/2}\|f\|_2,\\ \|M_{b_0-b_s}G_{\mathscr{L}_0}(\zeta)f\|_2 &\lesssim |\zeta|^{-1/2}\|f\|_2,\\ \|M_{c_0-c_s}(\zeta-\mathscr{L}_0)^{-1}f\|_2 &\lesssim |\zeta|^{-1}\|f\|_2,\\ \|M_{V_0-V_s}(\zeta-\mathscr{L}_0)^{-1}f\|_2 &\lesssim |\zeta|^{-1}\|f\|_2,\end{aligned}$$

for all $\zeta\in\gamma$. Now (4.68) follows from Lebesgue's dominated convergence theorem.

Similarly one can prove (4.67), using in this case the first and the third estimate of Lemma 4.28 to get the analogue of (4.69). □

4.7.2 Regularization of all coefficients

Let $k\in C_c^\infty(\mathbb{R}^d)$ be a radial function such that $0\leqslant k\leqslant 1$, $\operatorname{supp} k\subset B_{\mathbb{R}^d}(0,1)$ and $\int k=1$. For $\varepsilon>0$ define $k_\varepsilon(x):=\varepsilon^{-d}k(x/\varepsilon)$. Let $A\in\mathcal{A}(\Omega)$, $b,c\in L^\infty(\Omega,\mathbb{C}^d)$ and $V\in L^\infty(\Omega,\mathbb{R}_+)$. Suppose that b,c,V are compactly supported in Ω and that

$$\operatorname{supp} b,\ \operatorname{supp} c\subseteq\operatorname{supp} V. \tag{4.70}$$

If $\mathscr{A}=(A,b,c,V)\in\mathcal{B}(\Omega)$, we define for all $\alpha>0$

$$\begin{aligned}A_{\varepsilon,\alpha}&:=(A_\alpha * k_\varepsilon)|_\Omega,\\ b_\varepsilon&:=(b*k_\varepsilon)|_\Omega,\\ c_\varepsilon&:=(c*k_\varepsilon)|_\Omega,\\ V_\varepsilon&:=(V*k_\varepsilon)|_\Omega,\\ \mathscr{A}_{\varepsilon,\alpha}&:=(A_{\varepsilon,\alpha},b_\varepsilon,c_\varepsilon,V_\varepsilon),\end{aligned}$$

where

$$A_\alpha(x):=\begin{cases}A(x), & \text{if } x\in\Omega;\\ \alpha I_{\mathbb{R}^d}, & \text{if } x\in\mathbb{R}^d\setminus\Omega.\end{cases}$$

Lemma 4.32. *For every $\mathscr{A}\in\mathcal{B}_{\mu,M}(\Omega)$ as above and $p\in(1,\infty)$ the following assertions hold true.*

(i) *For a.e. $x\in\Omega$ and all $\alpha>0$,*

$$\begin{aligned}\lim_{\varepsilon\to 0}\|A_{\varepsilon,\alpha}(x)-A(x)\|_{\mathcal{B}(\mathbb{C}^d)}&=0,\\ \lim_{\varepsilon\to 0}\|b_\varepsilon(x)-b(x)\|_{\mathcal{B}(\mathbb{C}^d,\mathbb{C})}&=0,\\ \lim_{\varepsilon\to 0}\|c_\varepsilon(x)-c(x)\|_{\mathcal{B}(\mathbb{C},\mathbb{C}^d)}&=0,\\ \lim_{\varepsilon\to 0}\|V_\varepsilon(x)-V(x)\|_{\mathcal{B}(\mathbb{C})}&=0.\end{aligned}$$

(ii) *There exists* $\varepsilon_0 \in (0,1]$ *such that* $\mathscr{A}_{\varepsilon,\alpha} \in \mathcal{B}_{\mu,M}(\Omega)$ *for all* $\varepsilon \in (0,\varepsilon_0)$ *and all* $\alpha \geqslant \mu_2(\mathscr{A})$.

(iii) *Suppose that* $\mathscr{A} \in \mathcal{S}_p(\Omega)$. *Then there exist* $\varepsilon_0 \in (0,1]$ *and* $\alpha_p \geqslant 1$ *such that*

$$\begin{aligned} \mathscr{A}_{\varepsilon,\alpha} &\in \mathcal{S}_p(\Omega), && \forall \alpha > 0, \forall \varepsilon \in (0,\varepsilon_0), \\ \mu_p(\mathscr{A}) &= \lim_{\varepsilon\to 0} \mu_p(\mathscr{A}_{\varepsilon,\alpha}), && \forall \alpha \geqslant \alpha_p. \end{aligned}$$

In particular, if $\mathscr{A} \in \mathcal{B}_p(\Omega)$, *then*

$$\mathscr{A}_{\varepsilon,\alpha} \in \mathcal{B}_p(\Omega), \quad \forall \alpha > 0, \forall \varepsilon \in (0,\varepsilon_0).$$

(iv) *For every* $\alpha > 0$ *there exists* $C = C(\alpha) > 0$ *such that*

$$\max\{\|A_{\varepsilon,\alpha}\|_\infty, \|b_\varepsilon\|_\infty, \|c_\varepsilon\|_\infty, \|V_\varepsilon\|_\infty\} \leqslant C,$$

for all $\varepsilon > 0$.

Proof. Let us prove only the first limit in (i). The others are proved in the same way. As in the proof of [20, Lemma A.5(i)], it is enough to show that for a.e. $x \in \Omega$ and all $\alpha > 0$ we have $A_{\varepsilon,\alpha}(x)\xi \to A(x)\xi$ for all $\xi \in \mathbb{C}^d$. This is true because each $(A_\alpha)_{ij} \in L^\infty(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$ so that $((A_\alpha)_{ij} * k_\varepsilon)(x)$ tends to $(A_\alpha)_{ij}(x) = A_{ij}(x)$ for a.e. $x \in \Omega$, as $\varepsilon \to 0$.

Let us prove now items (ii) and (iii). Let $r \in (1,\infty)$ and $\mathscr{A} \in \mathcal{S}_r(\Omega)$. Then, by the Cauchy-Schwarz inequality and the fact that $\int k = 1$, we get for all $x \in \Omega$ and all $\varepsilon > 0$,

$$\begin{aligned} |\overline{b_\varepsilon}(x) - c_\varepsilon(x)| &\leqslant \int_{\mathbb{R}^d} \left|\bar{b}(x-y) - c(x-y)\right| k_\varepsilon^{1/2}(y) \cdot k_\varepsilon^{1/2}(y)\,\mathrm{d}y \\ &\leqslant \left(\int_{\mathbb{R}^d} |\bar{b}(x-y) - c(x-y)|^2 k_\varepsilon(y)\,\mathrm{d}y\right)^{1/2} \\ &\leqslant M\left(\int_{\mathbb{R}^d} V(x-y)k_\varepsilon(y)\,\mathrm{d}y\right)^{1/2} \\ &= M\sqrt{V_\varepsilon(x)}. \end{aligned}$$

It remains to estimate $\Gamma_r^{\mathscr{A}_\varepsilon}$ from below. Let $\varepsilon_0 \in (0,1]$ such that

$$\mathrm{supp}(V) + \overline{B(0,\varepsilon)} \subseteq \Omega, \quad \forall \varepsilon \in (0,\varepsilon_0). \tag{4.71}$$

Let $x \in \Omega$. Then, by combining (4.70), (4.71) and the fact that $\mathrm{supp}\, k_\varepsilon \subset B_{\mathbb{R}^d}(0,\varepsilon)$, for all $\varepsilon \in (0,\varepsilon_0)$ and $\xi \in \mathbb{C}^d$ we obtain

$$\begin{aligned} \Gamma_r^{\mathscr{A}_{\varepsilon,\alpha}}(x,\xi) &= \int_{\mathbb{R}^d} \Gamma_r^{A_\alpha,b,c,V}(x-y,\xi)k_\varepsilon(y)\,\mathrm{d}y \\ &= \int_{\{y\in B(0,\varepsilon)\,:\,x-y\,\in\,\mathrm{supp}V\}} \Gamma_r^{\mathscr{A}}(x-y,\xi)k_\varepsilon(y)\,\mathrm{d}y \\ &\quad + \int_{\{y\in B(0,\varepsilon)\,:\,x-y\,\notin\,\mathrm{supp}V\}} \mathrm{Re}\,\langle A_\alpha(x-y)\xi, \mathcal{J}_r\xi\rangle\, k_\varepsilon(y)\,\mathrm{d}y \\ &=: I_1 + I_2. \end{aligned} \tag{4.72}$$

Since $\mathscr{A} \in \mathcal{S}_r(\Omega)$, we estimate I_1 by

$$\begin{aligned} I_1 &\geqslant \mu_r(\mathscr{A}) \int_{\{y\in B(0,\varepsilon)\,:\,x-y\,\in\,\mathrm{supp}V\}} \Big(|\xi|^2 + V(x-y)\Big)\, k_\varepsilon(y)\,\mathrm{d}y \\ &= \mu_r(\mathscr{A})|\xi|^2 \int_{\{y\in B(0,\varepsilon)\,:\,x-y\,\in\,\mathrm{supp}V\}} k_\varepsilon(y)\,\mathrm{d}y + \mu_r(\mathscr{A})V_\varepsilon(x). \end{aligned} \tag{4.73}$$

On the other hand, since $A \in \mathcal{A}_r(\Omega)$ and $\Delta_r(I_{\mathbb{R}^d}) = 2/r^*$, where $r^* = \max\{r, r'\}$, we estimate I_2 by

$$\begin{aligned} I_2 &\geqslant \min\left\{\frac{r}{2}\Delta_r(A), \frac{r}{r^*}\alpha\right\}|\xi|^2 \int_{\{y\in B(0,\varepsilon)\,:\,x-y\notin \operatorname{supp} V\}} k_\varepsilon(y)\,\mathrm{d}y \\ &\geqslant \min\left\{\mu_r(\mathscr{A}), \frac{r}{r^*}\alpha\right\}|\xi|^2 \int_{\{y\in B(0,\varepsilon)\,:\,x-y\notin \operatorname{supp} V\}} k_\varepsilon(y)\,\mathrm{d}y, \end{aligned} \tag{4.74}$$

where in the last inequality we used (4.11). Hence, by combining (4.72), (4.73) and (4.74) we get

$$\Gamma_r^{\mathscr{A}_{\varepsilon,\alpha}}(x,\xi) \geqslant \min\left\{\mu_r(\mathscr{A}), \frac{r}{r^*}\alpha\right\}(|\xi|^2 + V_\varepsilon(x)), \tag{4.75}$$

for all $x \in \Omega$, $\xi \in \mathbb{C}^d$, $\alpha > 0$ and $\varepsilon \in (0, \varepsilon_0)$. So far, we have proved that $\mathscr{A}_{\varepsilon,\alpha} \in \mathcal{S}_r(\Omega)$ for every $\alpha > 0$ and $\varepsilon \in (0, \varepsilon_0)$, provided that $\mathscr{A} \in \mathcal{S}_r(\Omega)$. Moreover, by taking $r = 2$ in (4.75), we deduce that

$$\Gamma_2^{\mathscr{A}_{\varepsilon,\alpha}}(x,\xi) \geqslant \mu_2(\mathscr{A})(|\xi|^2 + V_\varepsilon(x)),$$

for all $x \in \Omega$, $\xi \in \mathbb{C}^d$, $\alpha \geqslant \mu_2(\mathscr{A})$ and $\varepsilon \in (0, \varepsilon_0)$, concluding the proof of (ii). On the other hand, by defining $\alpha_p := (p^*/p)\mu_p(\mathscr{A})$ and by taking $r = p$ in (4.75), we prove that

$$\mu_p(\mathscr{A}_{\varepsilon,\alpha}) \geqslant \mu_p(\mathscr{A}), \tag{4.76}$$

for all $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \geqslant \alpha_p$. Moreover, by definition of μ_p and the continuity of $x \mapsto (A_{\varepsilon,\alpha_p}(x), b_\varepsilon(x), c_\varepsilon(x), V_\varepsilon(x))$, we get $\Gamma_p^{\mathscr{A}_{\varepsilon,\alpha}}(x,\xi) \geqslant \mu_p(\mathscr{A}_{\varepsilon,\alpha})(|\xi|^2 + V_\varepsilon(x))$ for all $x \in \Omega$, $\xi \in \mathbb{C}^d$, $\alpha > 0$ and $\varepsilon \in (0, \varepsilon_0)$. The limit in (iii) now follows from (i) and (4.76).

Item (iv) follows by the definition of convolution and by the boundedness of A, b, c, V. □

4.7.3 Truncations of first- and zero-order terms

Let $\{K_n\}_{n\in\mathbb{N}_+}$ be a family of compact subsets in Ω such that $K_n \subset \operatorname{int}(K_{n+1})$ for all $n \in \mathbb{N}_+$ and $\bigcup_{n=1}^\infty \operatorname{int}(K_n) = \Omega$. Let $\{\psi_n\}_{n\in\mathbb{N}_+}$ be a family of compactly supported C^∞ functions in Ω such that $0 \leqslant \psi_n \leqslant 1$, $\psi_n = 1$ on a neighborhood of K_n and $\operatorname{supp}\psi_n \subset \operatorname{int}(K_{n+1})$ for all $n \in \mathbb{N}_+$. If $A \in \mathcal{A}(\Omega)$, $b, c \in L^\infty(\Omega, \mathbb{C}^d)$ and $V \in L^\infty(\Omega, \mathbb{R}_+)$, we define

$$\begin{aligned} b_n &:= \psi_n b, \\ c_n &:= \psi_n c, \\ V_n &:= \psi_n V, \\ \mathscr{A}_n &:= (A, b_n, c_n, V_n). \end{aligned} \tag{4.77}$$

Lemma 4.33. *For every $\mathscr{A} = (A, b, c, V) \in \mathcal{B}_{\mu,M}(\Omega)$, $\mathscr{A}_n$ as above and $p \in (1, \infty)$ the following assertions hold true.*

(i) *For a.e. $x \in \Omega$,*

$$\begin{aligned} \lim_{n\to\infty} \|b_n(x) - b(x)\|_{\mathcal{B}(\mathbb{C}^d,\mathbb{C})} &= 0, \\ \lim_{n\to\infty} \|c_n(x) - c(x)\|_{\mathcal{B}(\mathbb{C},\mathbb{C}^d)} &= 0, \\ \lim_{n\to\infty} \|V_n(x) - V(x)\|_{\mathcal{B}(\mathbb{C})} &= 0. \end{aligned}$$

(ii) *For all $n \in \mathbb{N}_+$ we have $\mathscr{A}_n \in \mathcal{B}_{\mu,M}(\Omega)$.*

(iii) *Suppose that* $\mathscr{A} \in \mathcal{S}_p(\Omega)$. *Then*

$$\begin{aligned} \mathscr{A}_n &\in \mathcal{S}_p(\Omega), \qquad \forall n \in \mathbb{N}_+, \\ \mu_p(\mathscr{A}) &= \lim_{n\to\infty} \mu_p(\mathscr{A}_n). \end{aligned}$$

In particular, if $\mathscr{A} \in \mathcal{B}_p(\Omega)$ *then*

$$\mathscr{A}_n \in \mathcal{B}_p(\Omega), \quad \forall n \in \mathbb{N}_+.$$

(iv) *For all* $n \in \mathbb{N}_+$ *we have* $\operatorname{supp} b_n, \operatorname{supp} c_n \subseteq \operatorname{supp} V_n$.

(v) *There exists* $C > 0$ *such that* $\max\{\|b_n\|_\infty, \|c_n\|_\infty, \|V_n\|_\infty\} \leqslant C$ *for all* $n \in \mathbb{N}_+$.

Proof. Items (i) and (v) follow by the definition (4.77).

Clearly, for all $\xi \in \mathbb{C}^d$ and all $r \in (1,\infty)$,

$$\begin{aligned} &\Gamma_r^{\mathscr{A}_n}(\cdot,\xi) = (1-\psi_n)\mathrm{Re}\,\langle A\xi, \mathcal{J}_r\xi\rangle + \psi_n\Gamma_r^{\mathscr{A}}(\cdot,\xi), \\ &|\overline{b_n} - c_n|^2 \leqslant \psi_n M^2\, V_n. \end{aligned}$$

By taking $r = 2$ and by combining (4.11) with the fact that $0 \leqslant \psi_n \leqslant 1$, we get (ii).

Similarly, by taking $r = p$ and by definition of μ_p, we obtain $\mathscr{A}_n \in \mathcal{S}_p(\Omega)$ and $\mu_p(\mathscr{A}_n) \geqslant \mu_p(\mathscr{A})$, for all $n \in \mathbb{N}_+$. Since $\mathscr{A}_n \in \mathcal{S}_p(\Omega)$, we have $\Gamma_p^{\mathscr{A}_n}(x,\xi) \geqslant \mu_p(\mathscr{A}_n)(|\xi|^2 + V_n(x))$ for almost every $x \in \Omega$, all $\xi \in \mathbb{C}^d$ and $n \in \mathbb{N}_+$. The limit in (iii) now follows by (i).

Item (iv) follows by the definition (4.77) and Proposition 4.4(ii). □

4.7.4 Bilinear embedding for bounded potentials

Let $\Omega \subseteq \mathbb{R}^d$ be open, $p > 1$ and $\mathscr{V}$ and $\mathscr{W}$ satisfy Assumption BE. Denote by $q = p/(p-1)$ the conjugate exponent of p. We prove now the bilinear embedding for arbitrary $A, B \in \mathcal{A}(\Omega)$, $b, c, \beta, \gamma \in L^\infty(\Omega, \mathbb{C}^d)$ and $V, W \in L^\infty(\Omega, \mathbb{R}_+)$ such that $(A, b, c, V), (B, \beta, \gamma, W) \in \mathcal{B}_p(\Omega)$. We will do it in two steps.

- First, we will prove the bilinear embedding for all $\mathscr{C} = (C, d, e, U), \mathscr{D} = (D, l, m, Z) \in \mathcal{B}_p(\Omega)$ satisfying the assumptions of Section 4.7.2, that is,

$$\begin{aligned} &\operatorname{supp} d, \operatorname{supp} e \subseteq \operatorname{supp} U, \\ &\operatorname{supp} l, \operatorname{supp} m \subseteq \operatorname{supp} Z, \end{aligned}$$

and d, e, l, m, U, Z are compactly supported. Fix $f, g \in (L^p \cap L^q)(\Omega)$ and $\alpha \geqslant \max\{\mu(\mathscr{A}, \mathscr{B}), (p^*/p)\mu_p(\mathscr{A}), (p^*/q)\mu_q(\mathscr{B})\}$. Let $\mathscr{C}_\varepsilon := \mathscr{C}_{\varepsilon,\alpha}$ and $\mathscr{D}_\varepsilon := \mathscr{D}_{\varepsilon,\alpha}$ be the smooth approximations introduced in Section 4.7.2. By Lemma 4.31, Lemma 4.32 and a standard theorem in measure theory, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\nabla T_t^{\mathscr{C}_{\varepsilon_n}} f \to \nabla T_t^{\mathscr{C}} f$ and $U_{\varepsilon_n}^{1/2} T_t^{\mathscr{C}_{\varepsilon_n}} f \to U^{1/2} T_t^{\mathscr{C}} f$ as $n \to \infty$ almost everywhere on Ω and for all $t > 0$; the same for $(\mathscr{D}, g)$. Recall that we have already established in Section 4.6 the bilinear embedding for smooth matrices and smooth and compactly supported first- and zero-order coefficients. Consequently, Fatou's lemma and Lemma 4.32(ii),(iii) give

$$\begin{aligned} &\int_0^\infty \int_\Omega \sqrt{\left|\nabla T_t^{\mathscr{C}} f\right|^2 + U\left|T_t^{\mathscr{C}} f\right|^2}\sqrt{\left|\nabla T_t^{\mathscr{D}} g\right|^2 + Z\left|T_t^{\mathscr{D}} g\right|^2} \\ &\quad\leqslant \liminf_{n\to\infty} \int_0^\infty \int_\Omega \sqrt{\left|\nabla T_t^{\mathscr{C}_{\varepsilon_n}} f\right|^2 + U_{\varepsilon_n}\left|T_t^{\mathscr{C}_{\varepsilon_n}} f\right|^2}\sqrt{\left|\nabla T_t^{\mathscr{D}_{\varepsilon_n}} g\right|^2 + Z_{\varepsilon_n}\left|T_t^{\mathscr{D}_{\varepsilon_n}} g\right|^2} \\ &\quad\leqslant \liminf_{n\to\infty} C_n \|f\|_p \|g\|_q, \end{aligned}$$

where C_n continuously depends on $p, \lambda, \Lambda, \mu, M, \mu_p(\mathscr{C}_{\varepsilon_n,\alpha}), \mu_q(\mathscr{D}_{\varepsilon_n,\alpha})$. Therefore, Lemma 4.32(iii) implies that

$$\int_0^\infty \int_\Omega \sqrt{\left|\nabla T_t^{\mathscr{C}} f\right|^2 + U\left|T_t^{\mathscr{C}} f\right|^2}\sqrt{\left|\nabla T_t^{\mathscr{D}} g\right|^2 + Z\left|T_t^{\mathscr{D}} g\right|^2} \leqslant C\|f\|_p\|g\|_q, \tag{4.78}$$

where C continuously depends on $p, \lambda, \Lambda, \mu, M, \mu_p(\mathscr{C}), \mu_q(\mathscr{D})$.

- Let $\mathscr{A} = (A, b, c, V), \mathscr{B} = (B, \beta, \gamma, W)$ be as above. Fix $f, g \in (L^p \cap L^q)(\Omega)$. Let $\mathscr{A}_n, \mathscr{B}_n$ be as in (4.77). By arguing just as in the previous case, only replacing Lemma 4.32 by Lemma 4.33, we deduce that there exists a sequence $(n_j)_{j\in\mathbb{N}}$ such that $\nabla T_t^{\mathscr{A}_{n_j}} f \to \nabla T_t^{\mathscr{A}} f$ and $V_{n_j}^{1/2} T_t^{\mathscr{A}_{n_j}} f \to V^{1/2} T_t^{\mathscr{A}} f$ as $j \to \infty$ almost everywhere on Ω and for all $t > 0$; the same for $(\mathscr{B}, g)$. Consequently, Fatou's lemma, (4.78) applied with $\mathscr{C} = \mathscr{A}_{n_j}$ and $\mathscr{D} = \mathscr{B}_{n_j}$, and Lemma 4.33 give

$$\begin{aligned}
&\int_0^\infty \int_\Omega \sqrt{\left|\nabla T_t^{\mathscr{A}} f\right|^2 + V\left|T_t^{\mathscr{A}} f\right|^2}\sqrt{\left|\nabla T_t^{\mathscr{B}} g\right|^2 + W\left|T_t^{\mathscr{B}} g\right|^2} \\
&\quad\leqslant \liminf_{n\to\infty} \int_0^\infty \int_\Omega \sqrt{\left|\nabla T_t^{\mathscr{A}_{n_j}} f\right|^2 + V_{n_j}\left|T_t^{\mathscr{A}_{n_j}} f\right|^2}\sqrt{\left|\nabla T_t^{\mathscr{B}_{n_j}} g\right|^2 + W_{n_j}\left|T_t^{\mathscr{B}_{n_j}} g\right|^2} \\
&\quad\leqslant C\|f\|_p\|g\|_q.
\end{aligned}$$

4.8 Bilinear embedding for operators with unbounded potentials

In order to treat the general case with unbounded potentials, we will follow the argument used by Carbonaro and Dragičević in [21, Section 3.4] when they proved [21, Theorem 1.4]. Like in their case, Theorem 4.26 will follow from the special case of bounded potentials already proved in Section 4.7.4, once we prove the following approximation result.

Let $V \in L^1_{\mathrm{loc}}(\Omega)$ be a nonnegative unbounded function. For each $m \in \mathbb{N}_+$ define

$$V_m := \min\{V, m\}. \tag{4.79}$$

We also set $V_\infty = V$.

Lemma 4.34. *For every $\mathscr{A} = (A, b, c, V) \in \mathcal{B}_{\mu,M}(\Omega)$, $\mathscr{A}_m = (A, b, c, V_m)$ and $p \in (1, \infty)$ the following assertions hold true.*

(i) *For every $\varepsilon \in (0, \mu)$ there exists $m_\varepsilon \geqslant 0$ such that $\mathscr{A}_m \in \mathcal{B}_{\mu-\varepsilon,M}$ for all $m \geqslant m_\varepsilon$.*

(ii) *Suppose that $\mathscr{A} \in \mathcal{S}_p(\Omega)$. Then there exists $m_p \geqslant 0$ such that*

$$\begin{aligned}
&\mathscr{A}_m \in \mathcal{S}_p(\Omega), \quad \forall m \geqslant m_p, \\
&\lim_{m\to\infty} \mu_p(\mathscr{A}_m) = \mu_p(\mathscr{A}).
\end{aligned}$$

In particular, if $\mathcal{A} \in \mathcal{B}_p(\Omega)$ then

$$\mathcal{A}_m \in \mathcal{B}_p(\Omega), \quad \forall m \geqslant \max\{m_p, m_q\}.$$

Proof. Let $r \in (1, \infty)$ and $\mathscr{A} \in \mathcal{S}_r(\Omega)$. Clearly, for almost all $x \in \Omega$,

$$|\overline{b}(x) - c(x)| \leqslant M\sqrt{m}, \quad \forall m \geqslant \|\overline{b} - c\|_\infty^2 / M^2.$$

We claim that for every $\varepsilon \in (0, \mu_r(\mathscr{A}))$ there exists $m_{\varepsilon,r}$ such that

$$\Gamma_r^{A,b,c,m}(x,\zeta) \geqslant (\mu_r(\mathscr{A}) - \varepsilon)(|\zeta|^2 + m), \tag{4.80}$$

for almost every $x \in \Omega$, all $\zeta \in \mathbb{C}^d$, and all $m \geqslant m_{\varepsilon,r}$.

Once (4.80) is established, taking $r = 2$ yields (i) with $m_\varepsilon := \max\{m_{\varepsilon,2}, \|\bar{b} - c\|_\infty^2/M^2\}$. By taking instead $r = p$, we obtain

$$\mu_p(\mathscr{A}_m) \geqslant \mu_p(\mathscr{A}) - \varepsilon, \tag{4.81}$$

for all $\varepsilon \in (0, \mu_p(\mathscr{A}))$ and all $m \geqslant m_{\varepsilon,p}$. In particular, by choosing for instance $\varepsilon = \varepsilon_p := \mu_p(\mathscr{A})/2$, we have

$$\mathscr{A}_m \in \mathcal{S}_p(\Omega), \qquad \forall m \geqslant m_p := \max\{m_{\varepsilon_p,p}, \|\bar{b} - c\|_\infty^2/M^2\}.$$

Hence, $\Gamma_p^{\mathscr{A}_m}(x,\xi) \geqslant \mu_p(\mathscr{A}_m)(|\xi|^2 + V_m(x))$ for almost every $x \in \Omega$, all $\xi \in \mathbb{C}^d$ and all $m \geqslant m_p$. Therefore, since $V_m(x) \to V(x)$ for almost all $x \in \Omega$, as $m \to \infty$, we get

$$\mu_p(A) \geqslant \limsup_{m\to\infty} \mu_p(\mathscr{A}_m),$$

which, combined with (4.81), implies that

$$\varepsilon + \liminf_{m\to\infty} \mu_p(\mathscr{A}_m) \geqslant \mu_p(\mathscr{A}) \geqslant \limsup_{m\to\infty} \mu_p(\mathscr{A}_m),$$

for all $\varepsilon \in (0, \mu_p(\mathscr{A}))$. By sending $\varepsilon \to 0$, we deduce (ii).

It remains to prove the claim. For almost every $x \in \Omega$,

$$\Gamma_r^{A,b,c,m}(x,\xi) \geqslant \frac{r}{2}\Delta_r(A)|\xi|^2 + \left(\operatorname*{ess\,inf}_{x\in\Omega} \min_{|\sigma|=1} \operatorname{Re}\left\langle \bar{b}(x) + \mathcal{J}_r c(x), \sigma\right\rangle\right)|\xi| + m,$$

for all $\xi \in \mathbb{C}^d$ and all $m \in \mathbb{N}$. Therefore, it suffices to show that for every $\varepsilon \in (0, \mu_r(\mathscr{A}))$ there exists $m_{\varepsilon,r}$ such that

$$\frac{r}{2}\Delta_r(A)\, t^2 + \left(\operatorname*{ess\,inf}_{x\in\Omega} \min_{|\sigma|=1} \operatorname{Re}\left\langle \bar{b}(x) + \mathcal{J}_r c(x), \sigma\right\rangle\right) t + m \geqslant (\mu_r(\mathscr{A}) - \varepsilon)(t^2 + m), \tag{4.82}$$

for all $t \in [0, +\infty)$ and all $m \geqslant m_{\varepsilon,r}$. Since V is unbounded, in particular $V \neq 0$. Thus, by (4.12), for every $\varepsilon > 0$ we obtain

$$\begin{aligned} \frac{r}{2}\Delta_r(A) - \mu_r(\mathscr{A}) + \varepsilon &> 0, \\ 1 - \mu_r(\mathscr{A}) + \varepsilon &> 0. \end{aligned}$$

Therefore, (4.82) holds for all $t \in [0, +\infty)$ if

$$\left(\operatorname*{ess\,inf}_{x\in\Omega} \min_{|\sigma|=1} \operatorname{Re}\left\langle \bar{b}(x) + \mathcal{J}_r c(x), \sigma\right\rangle\right)^2 - 4m\left(\frac{r}{2}\Delta_r(A) - \mu_r(\mathscr{A}) + \varepsilon\right)(1 - \mu_r(\mathscr{A}) + \varepsilon) \leqslant 0.$$

Thus, we get the desired property by taking

$$m_{\varepsilon,r} := \frac{\left(\operatorname*{ess\,inf}_{x\in\Omega} \min_{|\sigma|=1} \operatorname{Re}\left\langle \bar{b}(x) + \mathcal{J}_r c(x), \sigma\right\rangle\right)^2}{4((r/2)\Delta_r(A) - \mu_r(\mathscr{A}) + \varepsilon)\cdot(1 - \mu_r(\mathscr{A}) + \varepsilon)},$$

which is a nonnegative finite constant by the boundedness of b and c. □

For every $m \in \mathbb{N}_+ \cup \{\infty\}$, denote by $\mathfrak{a}_m = \mathfrak{a}_{\mathscr{A}_m}$ and $\mathscr{L}_m = \mathscr{L}^{\mathscr{A}_m}$ the form and the operator associated with $\mathscr{A}_m = (A, b, c, V_m)$, respectively. By Lemma 4.34(i), there exist $\widetilde{\mu} \in (0, \mu)$ and $\widetilde{m} \geqslant 0$ such that $(A, b, c, V_m) \in \mathcal{B}_{\widetilde{\mu},M}(\Omega)$ for all $m \in \mathbb{N}_+ \cup \infty$ with $m \geqslant \widetilde{m}$. Consequently, for every such m, Corollary 4.3 ensures that $\mathscr{L}_m$ is sectorial of angle $\omega = \omega(\widetilde{\mu}, M, \Lambda)$ and that $-\mathscr{L}_m$ generates an analytic and contractive semigroup $(T_z^{\mathscr{A}_m})_{z \in \mathbf{S}_\vartheta}$ on $L^2(\Omega)$, with $\vartheta := \pi/2 - \omega$.

Proposition 4.35. *For all $f \in L^2(\Omega)$, $(A, b, c, V) \in \mathcal{B}_{\mu,M}(\Omega)$ and all $z \in \mathbf{S}_\vartheta$ we have*

$$\nabla T_z^{\mathscr{A}_m} f \to \nabla T_z^{\mathscr{A}} f \qquad \text{in } L^2(\Omega, \mathbb{C}^d),$$
$$V_m^{1/2} T_z^{\mathscr{A}_m} f \to V^{1/2} T_z^{\mathscr{A}} f \quad \text{in } L^2(\Omega),$$

as $m \to \infty$.

Proof. Consider the sesquilinear form $\mathfrak{b}$ on $L^2(\Omega)$ defined by

$$\mathfrak{b}(u, v) = \int_\Omega \langle A\nabla u, \nabla v\rangle + \left\langle \nabla u, \overline{b} \right\rangle \overline{v} + u \langle c, \nabla v\rangle, \quad \mathrm{D}(\mathfrak{b}) = \mathscr{V}.$$

Clearly, for all $m \in \mathbb{N}_+ \cup \{\infty\}$ we have

$$\mathfrak{a}_m(u, v) = \mathfrak{b}(u, v) + \int_\Omega V_m u \overline{v}.$$

The boundedness of the coefficients A, b, c implies that $\left\langle \nabla u, \overline{b} \right\rangle, \langle c, \nabla u\rangle \in L^2(\Omega)$ and $A\nabla u$, $A^*\nabla u, \overline{b}u, cu \in L^2(\Omega; \mathbb{C}^d)$ for all $u \in L^2(\Omega)$. Therefore, $\mathfrak{b}$ satisfies condition (B1) of Section 2.4.3.

Moreover, V and $\{V_m\}_{m \in \mathbb{N}_+}$ satisfy (V1) and (V2), by construction.

Finally, since $(A, b, c, V_m) \in \mathcal{B}_{\widetilde{\mu},M}(\Omega)$ for all $m \in \mathbb{N}_+ \cup \{\infty\}$ such that $m \geqslant \widetilde{m}$, Lemma 4.2 ensures that the sequence $\{\mathfrak{a}_m\}_{m \in \mathbb{N} \cup \{\infty\}, m \geqslant \widetilde{m}}$ satisfies (A1) and (A2). Therefore, we fall into the assumption of Theorem 2.43 and we conclude by applying it with $\varrho = 1$. □

By combining Lemma 4.34 and Proposition 4.35 and by applying the same limit argument of Section 4.7.4, we obtain the bilinear embedding (4.46) for unbounded potentials.

4.9 Application: maximal regularity and functional calculus

In case when $b = c = V = 0$, let $A \in \mathcal{A}_p(\Omega)$ and denote by $-\mathscr{L}_p^A$ the generator of $(T_t^{A,\mathscr{V}})_{t>0}$ on $L^p(\Omega)$. Then $\mathscr{L}_p^A$ admits a bounded holomorphic functional calculus of angle $\vartheta < \pi/2$ and has parabolic maximal regularity [19, Theorem 3].

Following the same line of reasoning as in [19, Theorem 3], that is, relying on

- elementary properties of the classes $\mathcal{S}_p(\Omega)$ (see Proposition 4.4(iv) and Proposition 4.5(i)),
- a well-known sufficient condition for a bounded holomorphic functional calculus (Corollary 2.22),
- the Dore-Venni theorem (Theorem 2.29),
- Theorem 4.26 applied with $B = A^*$, $\beta = \overline{c}$, $\gamma = \overline{b}$, $W = V$ and $\mathscr{W} = \mathscr{V}$,

we can deduce the following result, which generalizes [19, Theorem 3].

Theorem 4.36. *Let $\Omega, \mathscr{A} = (A, b, c, V)$ satisfy the standard assumptions of Section 4.1.1. Suppose that $\mathscr{V}$ falls into any of the special cases (a)-(d) of Section 3.1.1. Assume that $p > 1$ and $\mathscr{A} \in \mathcal{B}_p(\Omega)$. Let $-\mathscr{L}_p$ be the generator of $(T_t^{\mathscr{A}})_{t>0}$ on $L^p(\Omega)$. Then $\mathscr{L}_p$ admits a bounded holomorphic functional calculus of angle $\vartheta < \pi/2$. As a consequence, $\mathscr{L}_p$ has parabolic maximal regularity.*

The idea is to apply the bilinear embedding to suitably small complex rotations of the coefficient $\mathscr{A}$ and of its adjoint $\mathscr{A}^* := (A^*, \overline{c}, \overline{b}, V)$. After such rotations, however, the resulting potentials are no longer real, but rather of the form $e^{\pm i\vartheta}V$, $\vartheta \in (-\pi/2, \pi/2)$. For this reason, as a preliminary step we establish the bilinear embedding in the presence of complex potentials of the type ϱV, where $\varrho \in \mathbb{C}$ satisfies $\mathrm{Re}\,(\varrho) > 0$.

Although one could in principle consider more general complex potentials, we restrict to this particular case, since our ultimate aim is to apply the bilinear embedding to complex rotations of the coefficients in order to deduce the boundedness of the H^∞ functional calculus of $\mathscr{L}$ on L^p.

4.9.1 Complex potentials

Let $\Omega, A, b, c, V, \mathscr{V}$ satisfy the standard assumptions of Section 4.1.1. Let $\varrho \in \mathbb{C}$ such that $\mathrm{Re}\,(\varrho) > 0$. Assume that $(A, b, c, \mathrm{Re}\,(\varrho)V) \in \mathcal{B}(\Omega)$. Then, it can be shown in the same way as above that the sesquilinear form $\mathfrak{a}$, defined by

$$\mathrm{D}(\mathfrak{a}) = \left\{ u \in \mathscr{V} : \int_\Omega V|u|^2 < \infty \right\},$$
$$\mathfrak{a}(u,v) = \int_\Omega \langle A\nabla u, \nabla v\rangle_{\mathbb{C}^d} + \left\langle \nabla u, \overline{b} \right\rangle_{\mathbb{C}^d} \overline{v} + u \langle c, \nabla v\rangle_{\mathbb{C}^d} + \varrho V u \overline{v},$$

is densely defined, closed and sectorial. Therefore, $-\mathscr{L}$, the operator on $L^2(\Omega)$ associated with $\mathfrak{a}$, generates an analytic and contractive semigroup $(T_t^{\mathscr{A},\mathscr{V}})_{t>0}$ on $L^2(\Omega)$, where $\mathscr{A} = (A, b, c, \varrho V)$. More generally, all the results mentioned above hold also for these types of complex potentials and are proved as in the case where the potentials are real, provided that their real part belongs to the new class introduced before. For instance, Theorem 4.26 for complex potentials now reads as follows:

Theorem 4.37. *Let $\mathscr{V}, \mathscr{W}$ satisfy Assumption BE and $\mathscr{A} = (A, b, c, \varrho V)$ and $\mathscr{B} = (B, \beta, \gamma, \sigma W)$ be of the type described above. Let $p > 1$ and denote by $q = p/(p-1)$ its conjugate exponent. Assume that $\tilde{\mathscr{A}} = (A, b, c, \mathrm{Re}\,(\varrho)V)$, $\tilde{\mathscr{B}} = (B, \beta, \gamma, \mathrm{Re}\,(\sigma)W) \in \mathcal{B}_p(\Omega)$. Then there exists $C > 0$ independent of the dimension d such that*

$$\int_0^\infty \int_\Omega \sqrt{\left|\nabla T_t^{\mathscr{A},\mathscr{V}} f\right|^2 + \mathrm{Re}\,(\varrho) V \left|T_t^{\mathscr{A},\mathscr{V}} f\right|^2} \sqrt{\left|\nabla T_t^{\mathscr{B},\mathscr{W}} g\right|^2 + \mathrm{Re}\,(\sigma) W \left|T_t^{\mathscr{B},\mathscr{W}} g\right|^2} \leqslant C \|f\|_p \|g\|_q,$$

for all $f, g \in (L^p \cap L^q)(\Omega)$. The constant $C > 0$ can be chosen so as to (continuously) depend only on $p, \lambda(A,B), \Lambda(A,B), \mu(\tilde{\mathscr{A}}, \tilde{\mathscr{B}}), M(\tilde{\mathscr{A}}, \tilde{\mathscr{B}}), \mu_p(\tilde{\mathscr{A}})$ and $\mu_q(\tilde{\mathscr{B}})$.

To avoid burdening the notation, we have treated in detail only the case with real potentials. If V is bounded, then the heat flow-method of Section 4.6 and the approximation argument of Section 4.7 can be applied exactly in the same way. If V is unbounded, we argue as in Section 4.8 and apply Theorem 2.43.

4.9.2 Maximal regularity and functional calculus

For convenience, let us recall that the notions and results related to the H^∞-functional calculus and to maximal regularity have been introduced in Sections 2.3.4–2.3.6. The next proposition is modeled after [19, Proposition 20].

Proposition 4.38. *Let $\Omega, \mathscr{A} = (A, b, c, V)$ satisfy the standard assumptions of Section 4.1.1. Suppose that $\mathscr{V}$ satisfies (3.5). Choose $p > 1$ and $\mathscr{A} \in (\mathcal{S}_p \cap \mathcal{S}_2)(\Omega)$. Let $-\mathscr{L}_p$ be the generator of $(T_t^{\mathscr{A}})_{t>0}$ on $L^p(\Omega)$. If $\omega_{H^\infty}(\mathscr{L}_p) < \pi/2$, then $\mathscr{L}_p$ has parabolic maximal regularity.*

Proof. Since $\mathscr{A} \in (\mathcal{S}_p \cap \mathcal{S}_2)(\Omega)$, Corollary 4.24 and Theorem 2.9 imply that $\mathscr{L}_p$ is sectorial on $L^p(\Omega)$. As $L^p(\Omega)$ is reflexive, we can construct the H^∞-calculus as in Section 2.3.4. Moreover, since

$$\vartheta((\mathscr{L}_p)_{||}) \leqslant \omega_{H^\infty}((\mathscr{L}_p)_{||}) = \omega_{H^\infty}(\mathscr{L}_p) < \tfrac{\pi}{2},$$

and $L^p(\Omega)$ is a UMD space, Theorem 2.29 ensures that $(\mathscr{L}_p)_{||}$ has parabolic maximal regularity. Finally, the conclusion follows from Remark 2.27. □

We now prove Theorem 4.36 following the same strategy as Carbonaro and Dragičević in the proof of [19, Theorem 3], see [19, Section 7.3], in the case $b = c = V = 0$.

Proof of Theorem 4.36. Without loss of generality we suppose $p \geq 2$. Let $\mathscr{A} = (A, b, c, V)$ satisfy the standard assumptions of Section 4.1.1. In light of Proposition 4.38 it suffices to show that

$$\mathscr{A} \in \mathcal{B}_p(\Omega) \implies \omega_{H^\infty}(\mathscr{L}_p) < \pi/2.$$

Set $\mathscr{A}^* = (A^*, \bar{c}, \bar{b}, V)$. From Proposition 2.38 it follows that $\mathscr{L}_2^{\mathscr{A}^*} = \left(\mathscr{L}^{\mathscr{A}}\right)_2^*$, so $T_t^{\mathscr{A}^*,\mathscr{V}} = \left(T_t^{\mathscr{A},\mathscr{V}}\right)^*$ for all $t > 0$. Set $T_t = T_t^{\mathscr{A},\mathscr{V}}$ and $T_t^* = T_t^{\mathscr{A}^*,\mathscr{V}}$ for all $t > 0$.

By Proposition 4.4(iv) and Proposition 4.5(i), there exists $\vartheta \in (0, \pi/2)$ such that

$$(e^{\pm i\vartheta}A, e^{\pm i\vartheta}b, e^{\pm i\vartheta}c, (\cos\vartheta)V) \in \mathcal{B}_p(\Omega),$$
$$(e^{\mp i\vartheta}A^*, e^{\mp i\vartheta}c, e^{\mp i\vartheta}b, (\cos\vartheta)V) \in \mathcal{B}_p(\Omega).$$

Moreover, for every $r \in [q, p]$ both $(T_t)_{t>0}$ and $(T_t^*)_{t>0}$ are analytic (and contractive) in $L^r(\Omega)$ in the cone $\mathbf{S}_\vartheta$; see Corollary 4.24.

By Theorem 4.37 and (4.42) there exists $C > 0$ such that

$$\int_0^\infty \int_\Omega \sqrt{|\nabla T_{te^{\pm i\vartheta}} f|^2 + (\cos\vartheta)V\,|T_{te^{\pm i\vartheta}} f|^2}\,\sqrt{|\nabla T^*_{te^{\mp i\vartheta}} g|^2 + (\cos\vartheta)V\,|T^*_{te^{\mp i\vartheta}} g|^2}\;\mathrm{d}x\,\mathrm{d}t \leqslant C\|f\|_p\|g\|_q,$$

for all $f, g \in (L^p \cap L^q)(\Omega)$. Therefore,

$$\int_0^\infty \int_\Omega \sqrt{|\nabla T_{te^{\pm i\vartheta}} f|^2 + V\,|T_{te^{\pm i\vartheta}} f|^2}\,\sqrt{|\nabla T^*_{te^{\mp i\vartheta}} g|^2 + V\,|T^*_{te^{\mp i\vartheta}} g|^2}\;\mathrm{d}x\,\mathrm{d}t \leqslant \frac{C}{\cos\vartheta}\|f\|_p\|g\|_q, \tag{4.83}$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

On the other hand, the integration by parts (4.6) and Proposition 4.4(ii) give

$$\left|\int_\Omega \mathscr{L}_2 T_{te^{\pm i\vartheta}} f\;\overline{T^*_{te^{\mp i\vartheta}} g}\,\mathrm{d}x\right| \lesssim \int_\Omega \sqrt{|\nabla T_{te^{\pm i\vartheta}} f|^2 + V\,|T_{te^{\pm i\vartheta}} f|^2}\,\sqrt{|\nabla T^*_{te^{\mp i\vartheta}} g|^2 + V\,|T^*_{te^{\mp i\vartheta}} g|^2}\;\mathrm{d}x, \tag{4.84}$$

By combining (4.83) and (4.84), we deduce that

$$\int_0^\infty \left|\int_\Omega \mathscr{L}_p T_{2te^{\pm i\vartheta}} f\,\overline{g}\,\mathrm{d}x\right|\,\mathrm{d}t \lesssim \|f\|_p\|g\|_q,$$

for all $f, g \in (L^p \cap L^q)(\Omega)$. Analyticity of $(T_t)_{t>0}$ in $L^p(\Omega)$, Fatou's lemma and a density argument show that

$$\int_0^\infty \left|\int_\Omega \mathscr{L}_p T_{te^{\pm i\vartheta}} f\,\overline{g}\,\mathrm{d}x\right|\,\mathrm{d}t \lesssim \|f\|_p\|g\|_q, \tag{4.85}$$

for all $f \in L^p(\Omega)$ and all $g \in L^q(\Omega)$.

We have shown in Remark 2.24 that $\left\langle \overline{\mathrm{R}}(\mathscr{L}_p), \overline{\mathrm{R}}(\mathscr{L}_q^*)\right\rangle$ forms a dual subpair of $\langle L^p(\Omega), L^q(\Omega)\rangle$. We now apply Corollary 2.22 to the operator $(\mathscr{L}_p)_{||}$ and the dual pair $\left\langle \overline{\mathrm{R}}(\mathscr{L}_p), \overline{\mathrm{R}}(\mathscr{L}_q^*)\right\rangle$, and deduce from (4.85) that $\omega_{H^\infty}(\mathscr{L}_p) \leqslant \pi/2 - \vartheta$. □

Recent results regarding the holomorphic functional calculus for the operator $\mathscr{L}_p$ have been obtained by Egert [40], Bechtel [9] and Böhnlein, Egert and Rehberg [12]. In [40] the author considered elliptic systems of second order in divergence form with bounded coefficients and subject to mixed boundary conditions on bounded and connected open sets Ω whose boundary is Lipschitz regular around the Neumann part $\overline{\partial\Omega \setminus D}$. We recall that D is a closed subset of $\partial\Omega$; see Section 3.1.1. In [40, Theorem 1.3] he provided the optimal interval of p's for the bounded H^∞-calculus on L^p. More precisely, after defining the interval $\mathcal{J}(\mathscr{L}) = \mathcal{J}(\mathscr{L}^{\mathscr{A}})$ by

$$\mathcal{J}(\mathscr{L}) := \{p \in (1,\infty) : \sup_{t>0} \|T_t^{\mathscr{A}}\|_{\mathcal{B}(L^p)} < \infty\},$$

he proved that, given $p_0 \in \mathcal{J}(\mathscr{L})$, the operator $\mathscr{L}_p$ admits a bounded H^∞-calculus on L^p for all $p \in (p_0, 2) \cup (2, p_0)$. He used the regularity assumption on the domain Ω to prove that the semigroup satisfies specific off-diagonal estimates [40, Definition 1.6 and Proposition 4.4], which imply the boundedness of the H^∞-calculus [40, Lemma 5.3]. For this last implication, he did not need the previous further assumptions on Ω. In particular, he exploited the existence of a bounded linear Sobolev extension operator that extends

$$\begin{aligned} W_D^{1,2}(\Omega) &\to W_D^{1,2}(\mathbb{R}^d), \\ L^p(\Omega) &\to L^p(\mathbb{R}^d), \end{aligned}$$

for every $p \in (1,\infty)$. Since in generality that we consider, the domain Ω may be completely irregular and/or unbounded, we only deduce that our interval of p's for the bounded H^∞-calculus on L^p is contained in Egert's. In fact, Theorem 4.22 and Proposition 4.4(iii) imply that

$$\{p \in (1,\infty) : \mathscr{A} \in \mathcal{B}_p(\Omega)\} \subseteq \operatorname{int}(\mathcal{J}(\mathscr{L})).$$

See also [19, Section 1.5] for a similar discussion regarding unperturbed divergence form operators.

More recently, Bechtel [9, Proposition 3.6] improved the aforementioned [40, Theorem 1.3] by only assuming that Ω is open and locally uniform near $\partial\Omega \setminus D$; see [9, Section 2.1] for the definition. In both cases, the bounded H^∞-calculus of $\mathscr{L}_p$ in L^p was exploited to establish L^p-estimates for the square root of $\mathscr{L}_p$ [40, Theorem 1.2 and Theorem 1.4] and [9, Theorem 1.2].

Regarding unperturbed elliptic operators in divergence form, Böhnlein, Egert and Rehberg have recently investigated in [12] the boundedness of the H^∞-calculus in L^p when the operator is subject to dynamical boundary conditions. From the bilinear embedding [12, Theorem 1.4], they deduced that it admits a bounded H^∞-calculus in L^p, provided that the coefficient matrix is p-elliptic [12, Theorem 1.3].

CHAPTER 5

Negative potentials

In the previous chapter, we extended the dimension-free bilinear embedding and the associated L^p-semigroup theory to a class of divergence-form operators with general first-order perturbations. A key feature of that analysis was the introduction of a structural condition on the coefficients, reducing to p-ellipticity in the absence of first-order terms. The heat-flow method developed therein proved to be a robust tool in treating the presence of such perturbations.

In this chapter, we focus on Schrödinger-type operators with *negative* potentials, that is, operators of the form

$$\mathscr{L}u = -\mathrm{div}(A\nabla u) + Vu,$$

where the potential V is not assumed to be non-negative. Instead, we consider a class of form-bounded potentials, which allows for a controlled amount of negativity in V through the subcritical inequality (5.1).

Our main goal is to prove that the fundamental L^p-results obtained for non-negative potentials in [19, 21]—including the dimension-free bilinear embedding, L^p-contractivity and analyticity of the associated semigroup, bounded H^∞-functional calculus, and L^p-maximal regularity—can be extended to this broader class of operators with negative potentials. To this end, we introduce a new structural condition that generalizes p-ellipticity to the present setting, as it reduces to p-ellipticity when the negative part of the potential vanishes. Notably, this condition recovers the same range of L^p-contractivity obtained by [10, 74] in the case $A = I_d$ e $\Omega = \mathbb{R}^d$.

As in the previous chapter, our approach to proving the bilinear embedding is based on the heat-flow method developed in Section 3.5, suitably adapted to accommodate the complications introduced by the negative part of the potential. The main difficulty lies in the rigorous justification of the use of the subcriticality inequality (5.1) for specific functions u associated with the Bellman function. In particular, we must verify that such u belong to an appropriate function space $\mathscr{V}$ in order to apply (5.1), and subsequently justify the validity of a chain rule for computing ∇u. Moreover, working with potentials that satisfy the subcriticality inequality (5.1) prevents us from applying the truncation argument used in [21] and in Section 4.8, since this class of potentials is not closed under truncation of the positive part. This forces us to prove the bilinear embedding directly for potentials whose positive part is unbounded.

The chapter is organized as follows. In Section 5.1 we introduce the class of negative potentials under consideration and define the corresponding Schrödinger-type operators. Section 5.2 presents the new structural condition generalizing p-ellipticity and develops its key properties. In Section 5.3.1, we prove L^p-contractivity and analyticity results for semigroups. Section 5.4 is devoted to outline the heat-flow method in this setting at a formal level. In Section 5.5 we prove a chain rule necessary to justify the application of the heat-flow method. Section 5.6 establishes a stronger convexity property for the Bellman function $\mathcal{Q}$ than that in Theorem 3.22, under the

new (stronger) condition (5.11). The proof of the bilinear embedding is given in Sections 5.7 and 5.8. In Section 5.9, we prove the boundedness of the H^∞-functional calculus on L^p and the validity of L^p-maximal regularity. Finally, in Section 5.10 we provide some examples of strongly subcritical potentials.

5.1 Schrödinger-type operators with negative potentials

Strongly subcritical potentials

Let $\Omega \subseteq \mathbb{R}^d$ be a nonempty open set, and fix a closed subspace $\mathscr{V} \subseteq W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$.

Definition 5.1. A real locally integrable function V is called a *strongly subcritical potential* if there exist $\alpha \geqslant 0$ and $\sigma \in [0,1)$ such that

$$\int_\Omega V_-|v|^2 \leqslant \alpha \int_\Omega |\nabla v|^2 + \sigma \int_\Omega V_+|v|^2, \qquad \forall v \in \mathscr{V}. \tag{5.1}$$

We denote by $\mathcal{P}(\Omega,\mathscr{V})$ the class of all strongly subcritical potentials for $\mathscr{V}$. When $\mathscr{V}$ is clear from the context, we simply write $\mathcal{P}(\Omega)$.

For fixed $\alpha \geqslant 0$ and $\sigma \in [0,1)$ we write $\mathcal{P}_{\alpha,\sigma}(\Omega)$ for the subclass of $\mathcal{P}(\Omega)$ in which (5.1) holds with these constants. Hence,

$$\mathcal{P}(\Omega) = \bigcup_{\alpha\geqslant 0,\, \sigma\in[0,1)} \mathcal{P}_{\alpha,\sigma}(\Omega).$$

Given $V \in \mathcal{P}(\Omega,\mathscr{V})$, we define $\alpha(V,\mathscr{V})$ (or simply $\alpha(V)$ if no ambiguity arises) as the smallest admissible α for which (5.1) holds with some $\sigma \in [0,1)$. The corresponding constant σ will be denoted by $\sigma(V,\mathscr{V})$, or simply $\sigma(V)$.

In [10, 74], the restriction $\alpha \in (0,1)$ is imposed in the definition of strongly subcritical potentials. Here, however, we also allow $\alpha = 0$, since we want $\alpha(V,\mathscr{V}) = 0$ whenever V is nonnegative, so that the new structural conditions introduced later on the coefficients coincide with those in [21] when V is nonnegative; see (5.3) and Section 5.2. Indeed, if $V_- = 0$, then clearly $V \in \mathcal{P}(\Omega,\mathscr{V})$ and $\alpha(V,\mathscr{V}) = 0$ for every $\mathscr{V}$, and conversely, if $\alpha(V,\mathscr{V}) = 0$, then necessarily $V_- = 0$. In fact, suppose that

$$\int_\Omega V_-|v|^2 \leqslant \sigma \int_\Omega V_+|v|^2, \qquad \forall v \in \mathscr{V}, \tag{5.2}$$

and assume that $S_k := \{V_- > 1/k\}$ has positive measure for some $k \in \mathbb{N}$. Take $K \subseteq S_k$ with $0 < |K| < \infty$ and $\overline{K} \subset \Omega$. Then $V_\pm \in L^1(K)$. Let $(\varphi_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers with $0 \leqslant \varphi_\varepsilon \leqslant 1$, and set $v_\varepsilon := \mathbb{1}_K * \varphi_\varepsilon$. Then, for ε small enough, $v_\varepsilon \in C_c^\infty(\Omega) \subseteq \mathscr{V}$. By (5.2) and the dominated convergence theorem,

$$0 < \frac{1}{k}|K| < \int_K V_- = \lim_{\varepsilon\to 0}\int_\Omega V_-|v_\varepsilon|^2 \leqslant \sigma \lim_{\varepsilon\to 0}\int_\Omega V_+|v_\varepsilon|^2 = \int_K V_+ = 0,$$

a contradiction. Hence $|S_k| = 0$ for all $k \in \mathbb{N}$, which implies $V_- = 0$.

On the other hand, the restriction $\alpha < 1$ in [5, 10, 74] stems from the fact that the authors study the classical Schrödinger operator $-\Delta + V$. In our setting, we deal instead with the more general Schrödinger-type operator $-\operatorname{div}(A\nabla\cdot) + V$, where A is an elliptic matrix-valued function. In this case, rather than assuming $\alpha < 1$, we impose an upper bound on α that depends on A; see again (5.3). When $A = I_d$, this bound precisely reduces to $\alpha < 1$.

It is immediate to observe that if $\mathscr{V} \subseteq \mathscr{W}$, then $\mathcal{P}(\Omega,\mathscr{W}) \subseteq \mathcal{P}(\Omega,\mathscr{V})$. This inclusion may be strict. In Section 5.10, we will provide examples of strongly subcritical potentials and show that, for suitable open subsets Ω,

$$\mathcal{P}(\Omega, W^{1,2}(\Omega)) \subsetneq \mathcal{P}(\Omega, W_D^{1,2}(\Omega)) \subsetneq \mathcal{P}(\Omega, W_0^{1,2}(\Omega)).$$

The operator $-\mathrm{div}(A\nabla u) + Vu$

Let $\Omega \subseteq \mathbb{R}^d$ be a nonempty open set and $\mathscr{V}$ be a closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$. Suppose that $A \in \mathcal{A}(\Omega)$ and $V \in \mathcal{P}(\Omega, \mathscr{V})$. Consider the sesquilinear form $\mathfrak{a} = \mathfrak{a}_{A,V,\mathscr{V}}$ defined by

$$\mathrm{D}(\mathfrak{a}) = \left\{ u \in \mathscr{V} : \int_\Omega V_+ |u|^2 < \infty \right\},$$
$$\mathfrak{a}(u,v) = \int_\Omega \langle A\nabla u, \nabla v\rangle_{\mathbb{C}^d} + Vu\overline{v}.$$

Clearly, $\mathfrak{a}$ is densely defined in $L^2(\Omega)$.

Lemma 5.2. *Let $\mathcal{A} \in \mathcal{A}(\Omega)$ and $V \in \mathcal{P}_{\alpha,\sigma}(\Omega)$. Suppose that*

$$A - \alpha I_d \in \mathcal{A}(\Omega). \tag{5.3}$$

Then $\mathfrak{a}$ is accretive, closed and sectorial of angle

$$\omega := \arctan\left(\frac{\sqrt{\Lambda^2(A) - \lambda^2(A)}}{\lambda(A - \alpha I_d)} \right) \in (0, \pi/2).$$

Moreover, there exists $C > 0$ such that

$$\mathrm{Re}\,\mathfrak{a}(u,u) \geqslant C \left(\|\nabla u\|_2^2 + \||V|^{1/2} u\|_2^2 \right), \tag{5.4}$$

for all $u \in \mathrm{D}(\mathfrak{a})$.

Proof. Using the abbreviation $\mathfrak{a}(u) = \mathfrak{a}(u,u)$, (5.1) and (5.3), for all $u \in \mathrm{D}(\mathfrak{a})$ we have

$$\begin{aligned} \mathrm{Re}\,\mathfrak{a}(u) &= \int_\Omega \mathrm{Re}\,\langle A\nabla u, \nabla u\rangle + V_+|u|^2 - V_-|u|^2 \\ &\geqslant \int_\Omega \mathrm{Re}\,\langle (A - \alpha I_d)\nabla u, \nabla u\rangle + (1-\sigma)V_+|u|^2 \\ &\geqslant \int_\Omega \lambda(A - \alpha I_d)|\nabla u|^2 + (1-\sigma)V_+|u|^2. \end{aligned} \tag{5.5}$$

Therefore, $\mathfrak{a}$ is accretive.

Denote $\mathfrak{b} = \mathfrak{a}_{I_d, V_+, \mathscr{V}}$. We know from [72, Proposition 4.30] that $\mathfrak{b}$ is closed. On the other hand, (5.1) and (5.3) give

$$\begin{aligned} \mathrm{Re}\,\mathfrak{a}(u) &\geqslant \min\{\lambda(A - \alpha I_d), (1-\sigma)\}\mathrm{Re}\,\mathfrak{b}(u), \\ \mathrm{Re}\,\mathfrak{a}(u) &\leqslant \max\{\Lambda(A + \alpha I_d), (1+\sigma)\}\mathrm{Re}\,\mathfrak{b}(u), \end{aligned} \tag{5.6}$$

for all $u \in \mathrm{D}(\mathfrak{a}) = \mathrm{D}(\mathfrak{b})$. Hence, $\mathfrak{a}$ is closed.

By (3.1), (3.2) and (5.5) we also have

$$|\mathrm{Im}\,\mathfrak{a}(u)| \leqslant \frac{\sqrt{\Lambda^2(A) - \lambda^2(A)}}{\lambda(A - \alpha I_d)} \mathrm{Re}\,\mathfrak{a}(u), \quad \forall u \in \mathrm{D}(\mathfrak{a}), \tag{5.7}$$

which implies that $\mathfrak{a}$ is sectorial of angle ω.

Finally, suppose that $\alpha \neq 0 \neq \sigma$. Then, from (5.5) and (5.1) we obtain

$$\begin{aligned} \mathrm{Re}\,\mathfrak{a}(u) &\geqslant \min\left\{ \frac{\lambda(A - \alpha I_d)}{\alpha}, \frac{1-\sigma}{\sigma} \right\} \left(\alpha \int_\Omega |\nabla u|^2 + \sigma \int_\Omega V_+|u|^2 \right) \\ &\geqslant \min\left\{ \frac{\lambda(A - \alpha I_d)}{\alpha}, \frac{1-\sigma}{\sigma} \right\} \int_\Omega V_-|u|^2. \end{aligned} \tag{5.8}$$

Therefore, (5.4) follows by combining the firs inequality in (5.6) and (5.8). If either $\alpha = 0$ or $\sigma = 0$, the argument can be adapted in a straightforward way. □

Let $\alpha \geqslant 0$ and $\sigma \in [0,1)$. Denote by $\mathcal{AP}_{\alpha,\sigma}(\Omega,\mathscr{V})$ the class of all $(A,V) \in \mathcal{A}(\Omega)\times\mathcal{P}_{\alpha,\sigma}(\Omega,\mathscr{V})$ for which (5.3) holds. Moreover, we set

$$\mathcal{AP}(\Omega,\mathscr{V}) = \bigcup_{\alpha>0,\sigma\in[0,1)} \mathcal{AP}_{\alpha,\sigma}(\Omega,\mathscr{V}).$$

Clearly, for $(A,V) \in \mathcal{A}(\Omega)\times\mathcal{P}(\Omega,\mathscr{V})$, we have

$$(A,V) \in \mathcal{AP}(\Omega,\mathscr{V}) \iff A - \alpha(V,\mathscr{V})I_d \in \mathcal{A}(\Omega).$$

Quantitatively, (5.3) is equivalent to $\alpha < \lambda(A)$, where $\lambda(A)$ is the ellipticity constant of A. In the specific case $A = I_d$, this condition simply becomes $\alpha < 1$.

For every $(A,V) \in \mathcal{AP}(\Omega)$ we denote by $\mathscr{L} = \mathscr{L}_2^{A,V,\mathscr{V}}$ the unbounded operator on $L^2(\Omega)$ associated with the sesquilinear form $\mathfrak{a}$. That is,

$$\mathrm{D}(\mathscr{L}) := \Big\{u \in \mathrm{D}(\mathfrak{a}) : \exists w \in L^2(\Omega) :\ \mathfrak{a}(u,v) = \langle w, v\rangle_{L^2(\Omega)}\ \ \forall v \in \mathrm{D}(\mathfrak{a})\Big\}$$

and

$$\langle \mathscr{L}u, v\rangle_{L^2(\Omega)} = \mathfrak{a}(u,v), \quad \forall u \in \mathrm{D}(\mathscr{L}), \quad \forall v \in \mathrm{D}(\mathfrak{a})\,. \tag{5.9}$$

Formally, $\mathscr{L}$ is given by the expression

$$\mathscr{L}u = -\mathrm{div}\,(A\nabla u) + Vu.$$

Corollary 5.3. *Suppose that $(A,V) \in \mathcal{AP}_{\alpha,\sigma}(\Omega)$ for some $\alpha > 0$ and $\sigma \in [0,1)$. Then $\mathscr{L}$ is sectorial of angle ω and $-\mathscr{L}$ is the generator of a strongly continuous semigroup on $L^2(\Omega)$*

$$T_t = T_t^{A,V,\mathscr{V}}, \quad t > 0,$$

which is analytic and contractive in the cone $\mathbf{S}_{\pi/2-\omega}$.

Proof. It follows by combining Lemma 5.2 with Theorem 2.37. □

5.2 Perturbed p-ellipticity: the class $\mathcal{AP}_p(\Omega,\mathscr{V})$

In the spirit of Chapter 4, we aim to introduce a new condition on the coefficients of the operator $\mathscr{L}^{A,V,\mathscr{V}}$ that generalizes p-ellipticity and plays an analogous role in the context of bilinear embeddings, semigroup contractivity and bounded H^∞-functional calculus on L^p.

Let $p > 1$ and let q denote its conjugate exponent, i.e., $1/p + 1/q = 1$. Let $A \in \mathcal{A}_p(\Omega)$, $\mathscr{V}$ be a closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$ and $V \in \mathcal{P}(\Omega,\mathscr{V})$. We say that

- $(A,V) \in \widetilde{\mathcal{AP}}_p(\Omega,\mathscr{V})$ if
$$\Delta_p\left(A - \alpha(V,\mathscr{V})\frac{pq}{4}I_d\right) \geqslant 0; \tag{5.10}$$
- $(A,V) \in \mathcal{AP}_p(\Omega,\mathscr{V})$ if
$$\Delta_p\left(A - \alpha(V,\mathscr{V})\frac{pq}{4}I_d\right) > 0, \tag{5.11}$$
that is, if $A - \alpha(V,\mathscr{V})(pq)/4\, I_d$ is p-elliptic.

When $V_- = 0$, clearly $V \in \mathcal{P}(\Omega, \mathscr{V})$ for all closed subspaces $\mathscr{V}$ of $W^{1,2}(\Omega)$ containing $W^{1,2}(\Omega)$ and $\alpha(V, \mathscr{V}) = 0$. Hence, in this case (5.10) coincides with the weak p-ellipticity ($\Delta_p(A) \geqslant 0$), while (5.11) with p-ellipticity, namely,

$$A \in \mathcal{A}_p(\Omega) \iff (A, V) \in \mathcal{AP}_p(\Omega, \mathscr{V}) \text{ for all } \mathscr{V}.$$

Moreover, the class $\mathcal{AP}_p$ retains a lot of properties that the classes $\mathcal{A}_p$ possess, such as an invariance under conjugation, a decrease with respect to p, an invariance under adjointness; see Proposition 5.5(i),(ii),(vi).

Finally, observe that (5.11) coincides with (5.3) when $p = 2$, that is, $\mathcal{AP}(\Omega, \mathscr{V}) = \mathcal{AP}_2(\Omega, \mathscr{V})$.

If there is no ambiguity, $\widetilde{\mathcal{AP}}_p(\Omega)$ and $\mathcal{AP}_p(\Omega)$ will stand for $\widetilde{\mathcal{AP}}_p(\Omega, \mathscr{V})$ and $\mathcal{AP}_p(\Omega, \mathscr{V})$, respectively.

We list now some elementary properties of the class $\mathcal{AP}_p(\Omega)$, which will be used later on. These properties closely resemble those of the class $\mathcal{A}_p(\Omega)$ [20], stated in Lemma 3.7. The following lemma is a consequence of the inequality

$$|\Delta_p(A) - \Delta_p(B)| \leqslant \frac{\|A - B\|_\infty}{\min\{p, q\}},$$

showed in [20, p. 3204] for all $A, B \in L^\infty(\Omega; \mathbb{C}^{d,d})$.

Lemma 5.4. *Let $k \in \mathbb{N}$, $U \subseteq \mathbb{R}^k$ be open and $f : U \to \mathbb{C}$ be Lipschitz continuous. Then $U \ni \omega \mapsto \Delta_p(A - f(\omega)B)$ is Lipschitz continuous for all $p \in (1, \infty)$ and $A, B \in L^\infty(\Omega; \mathbb{C}^{d,d})$.*

Proposition 5.5. *Let $p \in (1, \infty)$, $A \in \mathcal{A}_p(\Omega)$ and $V \in \mathcal{P}(\Omega)$ such that $(A, V) \in \mathcal{AP}_p(\Omega)$. Then*

(i) *$(A, V) \in \mathcal{AP}_q(\Omega)$, where q is the conjugate exponent of p;*

(ii) *$(A, V) \in \mathcal{AP}_r(\Omega)$ for all exponents r satisfying $|1/2 - 1/r| \leqslant |1/2 - 1/p|$;*

(iii) *there exists $\varepsilon > 0$ such that $(A, V) \in \mathcal{A}_{p+\varepsilon}(\Omega)$;*

(iv) *there exists $\vartheta \in (0, \pi/2)$ such that $(e^{i\phi}A, V\cos\phi) \in \mathcal{AP}_p(\Omega)$ for all $\phi \in [-\vartheta, \vartheta]$;*

(v) *there exists $\varepsilon > 0$ such that $A - \mu(qp/4)I_d \in \mathcal{A}_p(\Omega)$ for all $\mu \in [\alpha(V) - \varepsilon, \alpha(V) + \varepsilon]$;*

(vi) *$(A^*, V) \in \mathcal{AP}_p(\Omega)$.*

Proof. Item (i) follows by Lemma 3.7(i).

Since I_d is r-ellptic for all $r \in (1, \infty)$ and $rr' \leqslant pq$ for all exponents r satisfying $|1/2 - 1/r| \leqslant |1/2 - 1/p|$, we have

$$\operatorname{Re}\left\langle \left(A - \alpha\frac{rr'}{4}I_d\right)\xi, \mathcal{I}_r\xi\right\rangle \geqslant \operatorname{Re}\left\langle \left(A - \alpha\frac{pq}{4}I_d\right)\xi, \mathcal{I}_r\xi\right\rangle \geqslant \Delta_r\left(A - \alpha\frac{pq}{4}I_d\right)|\xi|^2,$$

for all $\xi \in \mathbb{C}^d$. Then item (ii) follows from the r-ellipticity of $A - (\alpha pq/4)I_d$, which is a consequence of the fact that $\{\mathcal{A}_p(\Omega) : p \in [2, \infty)\}$ is a decreasing chain of matrix classes (see Lemma 3.7(iv)).

Item (iii) follows from the continuity of $p \mapsto \Delta_p$ (see Lemma 3.7(iv)) and $p \mapsto \alpha pq/4$, together with Lemma 5.4.

Since $\alpha(V\cos\phi) \leqslant \alpha(V)$ for all $\phi \in [-\pi/2, \pi/2]$ and I_d is p-ellptic, in order to prove item (iv) it suffices to show that for all $A, B \in L^\infty(\Omega; \mathbb{C}^{d,d})$ such that $A - B \in \mathcal{A}_p(\Omega)$ there exists $\vartheta \in (0, \pi/2)$ for which

$$e^{i\phi}A - B \in \mathcal{A}_p(\Omega),$$

for all $\phi \in [-\vartheta, \vartheta]$. This holds true since $\phi \mapsto \Delta_p(e^{i\phi}A - B)$ is Lipschitz continuous on $(-\pi/2, \pi/2)$ by Lemma 5.4.

Item (v) is a consequence of the Lipschitzianity of $\mu \mapsto \Delta_p(A-\mu I)$ on $\mathbb{R}$, which is guaranteed by Lemma 5.4.

Item (vi) follows by Lemma 3.7(iii). □

Remark 5.6. In analogy with Remark 4.7, we derive a sufficient condition ensuring the p-ellipticity of $A-(\alpha pq/4)I_d$, where $A \in \mathcal{A}(\Omega)$ and $\alpha < \lambda(A)$. This condition is more explicit and easier to verify in practice, as it relies on a suitable lower bound for $\Delta_p(A)$ with respect to the subcritical parameter α.

By Lemma 3.7(ii), we have

$$\operatorname{Re}\left\langle \left(A - \alpha\tfrac{pq}{4} I_d\right)\xi, \mathcal{I}_p\xi\right\rangle \geq \left(\Delta_p(A) - \alpha\tfrac{pq}{4}\|\mathcal{I}_p\|\right)|\xi|^2, \qquad \xi \in \mathbb{C}^d.$$

Set $p^* = \max\{p, q\}$. Recalling that $\|\mathcal{I}_p\| = 2/\min\{p, q\}$ (see (c) and (f)), it follows that $A - (\alpha pq/4)I_d$ is p-elliptic provided that

$$\Delta_p(A) > \frac{\alpha}{2}p^*. \tag{5.12}$$

As in Remark 4.7, by decoupling p and A by means of the constant $\mu(A)$ defined in (3.12), one can obtain an explicit range of exponent p for which (5.12) holds. Indeed, (4.24) implies that (5.12) is satisfied for all p such that

$$\lambda(A) > \frac{\alpha}{2}p^* + \frac{\lambda(A)}{\mu(A)}\,|1 - 2/p|\,.$$

Setting $\lambda = \lambda(A)$ and $\mu = \mu(A)$, this inequality holds precisely for all $p \in (\tilde{p}', \tilde{p})$, where

$$\tilde{p} := \frac{\lambda}{\alpha\mu}\left(\mu - 1 + \sqrt{(\mu-1)^2 + 4\alpha\mu/\lambda}\right).$$

The condition $\alpha < \lambda$ ensures that $\tilde{p} > 2$.

If A is real-valued, $\mu(A) = 1$ and the admissible interval reduces to

$$\left(\frac{2\sqrt{\lambda(A)}}{2\sqrt{\lambda(A)} - \sqrt{\alpha}}, \frac{2\sqrt{\lambda(A)}}{\sqrt{\alpha}}\right).$$

Although (5.12) gives a universal sufficient condition, it is generally far from capturing the exact range of p for which $A-(\alpha pq/4)I_d$ is p-elliptic. For matrices with a simpler structure, one can determine the precise range of p without relying on such estimate.

Consider first the case $A = I_d$ and $\alpha < \lambda(I_d) = 1$. Since the identity matrix is p-elliptic for all $p \in (1, \infty)$, the p-ellipticity of $(1 - \alpha pq/4)I_d$ reduces to the scalar condition $1 - \alpha pq/4 > 0$. This inequality is satisfied if and only if p lies in the interval (p_-, p_+), where

$$p_- = \frac{2}{1 + \sqrt{1-\alpha}}, \quad p_+ = \frac{2}{1 - \sqrt{1-\alpha}}.$$

Notice that (5.12) instead yields the interval $(2/(2-\sqrt{\alpha}), 2/\sqrt{\alpha})$, which is strictly contained in (p_-, p_+).

Next, consider complex rotations of the identity, namely $A = e^{i\phi}I_d$ and $\alpha < \lambda(e^{i\phi}I_d) = \cos\phi$, with $\phi \in (-\pi/2, \pi/2)$. It is known that $e^{i\phi}I_d$ is p-elliptic if and only if $\cos\phi > \cos\phi_p$, where the critical angle is $\phi_p = \arccos|1 - 2/p|$ [20]. Therefore, since $\Delta_p(A)$ is $\mathbb{R}$-homogeneous of degree 1 in A, for the perturbed matrix $B := e^{i\phi}I_d - (\alpha pq/4)I_d$, p-ellipticity is equivalent to the condition

$$\cos\vartheta > \cos\phi_p,$$

where $\vartheta := \arg\left(e^{i\phi} - \alpha\frac{pq}{4}\right)$. For B to be p-elliptic, its real part must be strictly positive, which implies the necessary condition $\cos\phi - \alpha\frac{pq}{4} > 0$. Under this requirement, the argument is given by

$$\vartheta = \arctan\left(\frac{\sin\phi}{\cos\phi - \alpha\frac{pq}{4}}\right).$$

Using the identity $\cos(\arctan(t)) = (1+t^2)^{-1/2}$, the condition $\cos\vartheta > \cos\phi_p$ is equivalent to

$$\frac{\cos\phi - \alpha\frac{pq}{4}}{\sqrt{1 + \alpha^2\left(\frac{pq}{4}\right)^2 - 2\alpha\frac{pq}{4}\cos\phi}} > \cos\phi_p.$$

For any fixed $\phi \in (-\pi/2, \pi/2)$ and $\alpha < \cos\phi$, this inequality characterizes precisely the range of exponents p for which $e^{i\phi}I_d - (\alpha pq/4)I_d$ is p-elliptic.

5.3 L^p-estimates of $(T_t^{A,V,\mathscr{V}})_{t>0}$ and $\mathscr{L}^{A,V,\mathscr{V}}$

As a first result, we aim to extend Theorem 3.17, as done in Section 4.5. Also in this case, we shall apply Nittka's Theorem 2.39. In this framework, the condition $\Delta_p(e^{i\phi}A) \geqslant 0$ is replaced by

$$(e^{i\phi}A, (\cos\phi)V) \in \widetilde{\mathcal{AP}}_p(\Omega, \mathscr{V}),$$

where $\mathscr{V}$ satisfies both (3.5) and (3.6). The additional assumption (3.6) will be required in order to guarantee that the subcritical inequality can be applied to a suitable function, as explained below.

As we have already seen in Section 5.1, working with strongly subcritical potentials and controlling the subcritical constant through the matrix A ensures the accretivity of the sesquilinear form $\mathfrak{a} = \mathfrak{a}_{A,V,\mathscr{V}}$, namely, $\operatorname{Re}\mathfrak{a}(u,u) \geqslant 0$ for all $u \in \mathrm{D}(\mathfrak{a})$. In that setting, one could directly apply inequality (5.1) with $v = u$, and the nonnegativity of $\operatorname{Re}\mathfrak{a}(u,u)$ followed from (5.3).

In the present case, however, our goal is to establish the L^p-contractivity of the semigroup $(T_t^{A,V,\mathscr{V}})_{t>0}$, which requires verifying that the form is L^p-dissipative, i.e. $\operatorname{Re}\mathfrak{a}(u, |u|^{p-2}u) \geqslant 0$ for all $u \in \mathrm{D}(\mathfrak{a})$ such that $|u|^{p-2}u \in \mathrm{D}(\mathfrak{a})$. To apply the subcritical inequality in this context, one needs to express $|u|^p$ as the modulus squared of a suitable function, for instance $|u|^{p/2-1}u$. The fact that $\mathscr{V}$ satisfies condition (3.6) ensures that $|u|^{p/2-1}u \in \mathscr{V}$ (see Corollary 5.7), and therefore the subcritical inequality (5.1) can be applied with $v = |u|^{p/2-1}u$. After applying it, one must control the negative term $-\alpha|\nabla(|u|^{p/2-1}u)|^2$ through the matrix A to obtain the nonnegativity of $\operatorname{Re}\mathfrak{a}(u, |u|^{p-2}u)$. Condition (5.10) guarantees precisely this, and the factor $pq/4$ naturally appears from the computation of the gradient of $v = |u|^{p/2-1}u$.

Once Theorem 3.17 is generalized, we will be able to extend Corollary 3.18 and [41, Theorem 1].

5.3.1 L^p-contractivity and analyticity of $(T_t^{A,V,\mathscr{V}})_{t>0}$

Before proving the L^p-contractivity of $(T_t^{A,V,\mathscr{V}})_{t>0}$, we begin by presenting a corollary of Lemma 3.15, which will allow us to apply (5.1) with $v = |u|^{p/2-1}u$.

Corollary 5.7. *Suppose that $\mathscr{V}$ satisfies (3.5) and (3.6). Let $u \in \mathscr{V}$ and $p \in (1,\infty)$ such that $|u|^{p-2}u \in W^{1,2}(\Omega)$. Then $u \in L^p(\Omega)$, $|u|^{p/2-1}u \in \mathscr{V}$ and*

$$|\nabla(|u|^{\frac{p}{2}-1}u)|^2 = \frac{p}{2}|u|^{p-2}\left(\frac{p}{2}|\operatorname{Re}(\operatorname{sign}\overline{u}\cdot\nabla u)|^2 + \frac{2}{p}|\operatorname{Im}(\operatorname{sign}\overline{u}\cdot\nabla u)|^2\right)\mathbb{1}_{\{u\neq 0\}}. \tag{5.13}$$

Proof. Let us prove first that $u \in L^p(\Omega)$, $|u|^{p/2-1}u \in W^{1,2}(\Omega)$ and (5.13). By assumption $|u|^{p-1} \in L^2(\Omega)$, hence $|u|^p = |u| \cdot |u|^{p-1} \in L^1(\Omega)$, namely $|u|^{p/2-1}u \in L^2(\Omega)$. By Lemma 3.15,

$$|u|^{p-2}|\nabla u| \sim |\nabla(|u|^{p-2}u)| \in L^2(\Omega),$$

which implies that $|u|^{p-2}|\nabla u|^2 \in L^1(\Omega)$. Therefore, $|u|^{p/2-1}\nabla u \in L^2(\Omega;\mathbb{C}^d)$. We conclude by applying Lemma 3.15 with $p/2+1$ instead of p.

Finally, we prove that $|u|^{p/2-1}u \in \mathscr{V}$. For all $n \in \mathbb{N}$ the function $\Phi_n(\zeta) = \zeta(|\zeta|^{p/2-1} \wedge n)$ is Lipschitz continuous with $\Phi_n(0) = 0$. Therefore, from Proposition 3.2 we have $u(|u|^{p/2-1} \wedge n) \in \mathscr{V}$ with gradient given by [41, (10)]. Hence,

$$\|u(|u|^{p/2-1} \wedge n)\|_{\mathscr{V}} \lesssim \||u|^{p/2-1}u\|_{W^{1,2}(\Omega)} < \infty, \tag{5.14}$$

for all $n \in \mathbb{N}$. On the other hand, Lebesgue's dominated convergence theorem and the fact that $u \in L^p$ give

$$\|u(|u|^{p/2-1} \wedge n)\|_2 \to \||u|^{p/2-1}u\|_2, \tag{5.15}$$

and $u(|u|^{p/2-1} \wedge n) \to |u|^{p/2-1}u$ in $\mathcal{D}'(\Omega)$, as $n \to \infty$. From the latter convergence, the density of $C_c^\infty(\Omega)$ in $L^2(\Omega)$ and (5.14) we deduce that

$$u(|u|^{p/2-1} \wedge n) \rightharpoonup |u|^{p/2-1}u \tag{5.16}$$

in $L^2(\Omega)$, as $n \to \infty$. By combining (5.15) and (5.16) we obtain $u(|u|^{p/2-1} \wedge n) \to |u|^{p/2-1}u$ strongly in $L^2(\Omega)$, as $n \to \infty$. Thus, (5.14) and Lemma 3.3 yield $|u|^{p/2-1}u \in \mathscr{V}$. □

Let $(A, V) \in \mathcal{AP}(\Omega, \mathscr{V})$. We prove now the L^p-contractivity of the semigroup $(T_t^{A,V,\mathscr{V}})_{t>0}$ by adapting the argument used in the proof of Theorem 3.17 to the present setting.

Theorem 5.8. *Suppose that $\mathscr{V}$ satisfies (3.5) and (3.6). Choose $p > 1$, $(A, V) \in \mathcal{AP}(\Omega, \mathscr{V})$ and $\phi \in \mathbb{R}$ such that $|\phi| < \pi/2 - \omega$ and $(e^{i\phi}A, (\cos\phi)V) \in \widetilde{\mathcal{AP}}_p(\Omega, \mathscr{V})$. Then*

$$\left(e^{-te^{i\phi}\mathscr{L}}\right)_{t>0}$$

extends to a strongly continuous semigroup of contractions on $L^p(\Omega)$.

If $V_- = 0$, the same conclusion holds under milder assumptions on $\mathscr{V}$, namely, when $\mathscr{V}$ only satisfies (3.5).

Proof. We will apply Nittka's invariance criterion (Theorem 2.39) with the sesquilinear form $\mathfrak{b} := e^{i\phi}\mathfrak{a}$, whose associated operator is $e^{i\phi}\mathscr{L}^{A,V}$.

The invariance of $\mathrm{D}(\mathfrak{b}) = \mathrm{D}(\mathfrak{a}_{A,V,\mathscr{V}}) = \mathrm{D}(\mathfrak{a}_{A,V_+,\mathscr{V}}) = \mathrm{D}(\mathfrak{a}_{I,V_+,\mathscr{V}})$ under $\mathbf{P}_{B^p}$ was established in Theorem 4.22, assuming only condition (3.5) on $\mathscr{V}$. Thus, it remains to prove the L^p-dissipativity of $\mathfrak{b}$. To this end, we will make use of the fact that $\mathscr{V}$ also satisfies condition (3.6).

Recall the definition (2.25) of $\mathrm{D}_p(\mathfrak{b})$. Let $u \in \mathrm{D}_p(\mathfrak{b})$. By Lemma 3.15, we get

$$\begin{aligned}\operatorname{Re}\mathfrak{b}(u, |u|^{p-2}u) \geqslant \int_\Omega \frac{p}{2}|u|^{p-2}\operatorname{Re}\left\langle e^{i\phi}A(\operatorname{sign}\overline{u}\cdot\nabla u), \mathcal{I}_p(\operatorname{sign}\overline{u}\cdot\nabla u)\right\rangle \mathbb{1}_{\{u\neq 0\}} + (\cos\phi)V_+|u|^p \\ - \int_\Omega (\cos\phi)V_-\left||u|^{\frac{p}{2}-1}u\right|^2.\end{aligned} \tag{5.17}$$

If $V_- = 0$, the assumption $(e^{i\phi}A, V\cos\phi) \in \widetilde{\mathcal{AP}}_p(\Omega, \mathscr{V})$ is equivalent to the condition $\Delta_p(e^{i\phi}A) \geqslant 0$. Hence, the integrand in the right-hand side term of (5.17) is nonnegative and we conclude. Notice that in this case we did not need the assumption (3.6) on $\mathscr{V}$.

Suppose now that $V_- \neq 0$. We would like to use in (5.17) the subcritical inequality (5.1) applied with the potential $(\cos\phi)V$ and $v = |u|^{p/2-1}u$. To this purpose, we can rely on Corollary 5.7 as it guarantees that $|u|^{p/2-1}u \in \mathscr{V}$. We highlight that only in this step we are making use of the fact that $\mathscr{V}$ also satisfies (3.6).

So, the subcritical inequality (5.1) gives

$$\begin{aligned}&\operatorname{Re}\mathfrak{b}(u,|u|^{p-2}u)\\&\geqslant \int_\Omega \frac{p}{2}|u|^{p-2}\operatorname{Re}\left\langle e^{i\phi}A(\operatorname{sign}\overline{u}\cdot\nabla u),\mathcal{I}_p(\operatorname{sign}\overline{u}\cdot\nabla u)\right\rangle \mathbb{1}_{\{u\neq 0\}} - \alpha(V\cos\phi)|\nabla(|u|^{\frac{p}{2}-1}u)|^2\\&\qquad + (1-\sigma)(\cos\phi)V_+|u|^p.\end{aligned}$$

By combining (5.13) with the fact that $pq/4 \geqslant 1$ for all $p \in (1,\infty)$, we get

$$\begin{aligned}|\nabla(|u|^{\frac{p}{2}-1}u)|^2 &\leqslant \frac{p}{2}\cdot\frac{pq}{4}|u|^{p-2}\left(\frac{2}{q}|\operatorname{Re}(\operatorname{sign}\overline{u}\cdot\nabla u)|^2 + \frac{2}{p}|\operatorname{Im}(\operatorname{sign}\overline{u}\cdot\nabla u)|^2\right)\mathbb{1}_{\{u\neq 0\}}\\&= \frac{p}{2}\cdot\frac{pq}{4}|u|^{p-2}\operatorname{Re}\langle\operatorname{sign}\overline{u}\cdot\nabla u,\mathcal{I}_p(\operatorname{sign}\overline{u}\cdot\nabla u)\rangle\,\mathbb{1}_{\{u\neq 0\}}.\end{aligned}$$

Hence, we may continue as

$$\begin{aligned}\operatorname{Re}\mathfrak{b}(u,|u|^{p-2}u) \geqslant \int_\Omega \frac{p}{2}|u|^{p-2}\operatorname{Re}\left\langle\left(e^{i\phi}A - \alpha(V\cos\phi)\frac{pq}{4}I_d\right)(\operatorname{sign}\overline{u}\cdot\nabla u),\mathcal{I}_p(\operatorname{sign}\overline{u}\cdot\nabla u)\right\rangle\mathbb{1}_{\{u\neq 0\}}\\ + (1-\sigma)(\cos\phi)V_+|u|^p.\end{aligned}$$

Now, the integrand is nonnegative since $(e^{i\phi}A, V\cos\phi) \in \widetilde{\mathcal{AP}}_p(\Omega,\mathscr{V})$ and $\sigma \in [0,1)$. □

The next corollary extends [21, Corollary 1.3] which was in turn a generalization of Corollary 3.18.

Corollary 5.9. *Suppose that $\mathscr{V}$ satisfies (3.5) and (3.6). Choose $p>1$ and $(A,V)\in\mathcal{AP}_p(\Omega,\mathscr{V})$. Then there exists $\vartheta=\vartheta(p,A,V,\mathscr{V})>0$ such that if $|1-2/r|\leqslant|1-2/p|$, then $\{T_z : z\in\mathbf{S}_\vartheta\}$ is analytic and contractive in $L^r(\Omega)$.*

If $V_-=0$, the same conclusion holds under milder assumptions on $\mathscr{V}$, namely, when $\mathscr{V}$ only satisfies (3.5).

Proof. By Proposition 5.5(i),(ii),(iv) there exists $\vartheta=\vartheta(p,A,V)>0$ such that

$$\Delta_r(e^{i\phi}A - \alpha(V\cos\phi)(rr'/4)I_d) > 0$$

for all $\phi\in[-\vartheta,\vartheta]$ and all r satisfying $|1/2-1/r|\leqslant|1/2-1/p|$. The contractivity part now follows from Theorem 5.8 and the relation

$$T_{te^{i\phi}} = \exp\left(-te^{i\phi}\mathscr{L}\right),$$

whereupon analyticity is a consequence of Theorem 2.9. □

5.3.2 L^p-estimates of $\mathscr{L}^{A,V,\mathscr{V}}$

We now establish some L^p-estimates for the operator $\mathscr{L}^{A,V}$. Besides being of independent interest, these results serve as auxiliary tools for extending [41, Theorem 1] through Corollary 5.12, and for applying the chain rule in Proposition 5.23.

Let $r \in (1,\infty)$. Recall that F_r stands for the power function of exponent r on $\mathbb{C}$. From Lemma 3.14 applied with $A = I_d$ we have

$$\begin{aligned} H_{F_r}^{I_d}[\zeta;X] &= \frac{r^2}{2}|\zeta|^{r-2}\mathrm{Re}\left\langle \mathrm{sign}\overline{\zeta}\cdot X, \mathcal{I}_r(\mathrm{sign}\overline{\zeta}\cdot X)\right\rangle, \\ &= \frac{r^2}{2}|\zeta|^{r-2}\left(\frac{2}{r'}|\mathrm{Re}\,(\mathrm{sign}\overline{\zeta}\cdot X)|^2 + \frac{2}{r}|\mathrm{Im}\,(\mathrm{sign}\overline{\zeta}\cdot X)|^2\right), \end{aligned}$$

for all $\zeta \in \mathbb{C}\setminus\{0\}$ and $X \in \mathbb{C}^d$. Therefore, since $rr'/4 \geqslant 1$ for all $r \in (1,\infty)$,

$$\frac{r'}{4}H_{F_r}^{I_d}[\zeta;X] \geqslant \frac{r}{2}|\zeta|^{r-2}\left(\frac{r}{2}|\mathrm{Re}\,(\mathrm{sign}\overline{\zeta}\cdot X)|^2 + \frac{2}{r}|\mathrm{Im}\,(\mathrm{sign}\overline{\zeta}\cdot X)|^2\right), \tag{5.18}$$

for all $\zeta \in \mathbb{C}\setminus\{0\}$ and $X \in \mathbb{C}^d$. In particular, from (5.13) we deduce that

$$\left|\nabla(|u|^{\frac{r}{2}-1}u)\right|^2 \leqslant \frac{r'}{4}H_{F_r}^{I_d}[u;\nabla u]\mathbb{1}_{\{u\neq 0\}}, \tag{5.19}$$

for all $u \in W^{1,2}(\Omega)$ such that $|u|^{\frac{r}{2}-1}u \in W^{1,2}(\Omega)$.

The next result was already proved by Egert in [41, Proposition 11] for the case $p \geq 2$ and $V = 0$. Here we extend it to the general setting.

Proposition 5.10. *Suppose that $\mathscr{V}$ satisfies (3.5) and (3.6). Let $p \in (1,\infty)$ and $(A,V) \in \mathcal{AP}_p(\Omega,\mathscr{V})$. If $u \in \mathrm{D}(\mathscr{L}^{A,V}) \cap L^p(\Omega)$ is such that $\mathscr{L}^{A,V}u \in L^p(\Omega)$, then $|u|^{p/2-1}u \in \mathscr{V}$ and*

$$\|\nabla(|u|^{\frac{p}{2}-1}u)\|_2^2 \lesssim \|\mathscr{L}^{A,V}u\|_p\,\|u\|_p^{p-1}. \tag{5.20}$$

In particular, $|u|^{p-2}|\nabla u|^2\mathbb{1}_{\{u\neq 0\}} \in L^1(\Omega)$.

Furthermore,

$$\int_\Omega V_-|u|^p \leqslant \alpha\int_\Omega \frac{q}{4}H_{F_p}^{I_d}[u;\nabla u]\mathbb{1}_{\{u\neq 0\}} + \sigma\int_\Omega V_+|u|^p,$$

where $q = p/(p-1)$.

Proof. Let $u \in \mathrm{D}(\mathscr{L}^{A,V}) \subseteq \mathrm{D}(\mathfrak{a}_{A,V,\mathscr{V}})$ be such that $u, \mathscr{L}^{A,V}u \in L^p(\Omega)$. Then, by Lebesgue's dominated convergence theorem we deduce that

$$\mathrm{Re}\int_\Omega \mathscr{L}^{A,V}(u)\cdot\overline{u}|u|^{p-2} = \lim_{n\to\infty}\mathrm{Re}\int_\Omega \mathscr{L}^{A,V}(u)\cdot\overline{u}\left(|u|^{p-2}\wedge n\right). \tag{5.21}$$

Proposition 3.2 and the fact that $u \in \mathrm{D}(\mathfrak{a}_{A,V,\mathscr{V}})$ give $u\left(|u|^{p-2}\wedge n\right) \in \mathrm{D}(\mathfrak{a}_{A,V,\mathscr{V}})$. Therefore, (5.9) yields

$$\begin{aligned} \int_\Omega \mathscr{L}^{A,V}(u)\cdot\overline{u}\left(|u|^{p-2}\wedge n\right) &= \int_\Omega \left\langle A\nabla u, \nabla\left[u\left(|u|^{p-2}\wedge n\right)\right]\right\rangle + V_+|u|^2\left(|u|^{p-2}\wedge n\right) \\ &\quad - \int_\Omega V_-\left|u(|u|^{p/2-1}\wedge\sqrt{n})\right|^2 \end{aligned} \tag{5.22}$$

Again from Proposition 3.2 we have $u(|u|^{p/2-1}\wedge\sqrt{n}) \in \mathscr{V}$. Hence

$$\int_\Omega V_-\left|u(|u|^{p/2-1}\wedge\sqrt{n})\right|^2 \leqslant \alpha\int_\Omega|\nabla[u\left(|u|^{p/2-1}\wedge\sqrt{n}\right)]|^2 + \sigma\int_\Omega V_+|u|^2\left(|u|^{p-2}\wedge n\right). \tag{5.23}$$

From [82, Lemma 5.2], the identity $\nabla|u| = \mathrm{Re}\,(\mathrm{sign}\overline{u}\cdot\nabla u)\,\mathbb{1}_{\{u\neq 0\}}$ and the fact that $\nabla u = 0$ almost everywhere on $\{u = 0\}$, we obtain

$$\begin{aligned} \nabla\left[u\left(|u|^{p-2}\wedge n\right)\right] &= \left(|u|^{p-2}\wedge n\right)\left(\nabla u + (p-2)\mathbb{1}_{\{|u|^{p-2}<n\}}\,\mathrm{sign}u\nabla|u|\right) \\ &= n\nabla u\mathbb{1}_{\{|u|^{p-2}\geqslant n\}} + |u|^{p-2}\left(\nabla u + (p-2)\,\mathrm{sign}u\nabla|u|\right)\mathbb{1}_{\{|u|^{p-2}<n\}} \\ &= n\nabla u\mathbb{1}_{\{|u|^{p-2}\geqslant n\}} + \frac{p}{2}|u|^{p-2}\mathrm{sign}u\cdot\mathcal{I}_p(\mathrm{sign}\overline{u}\cdot\nabla u)\mathbb{1}_{\{|u|^{p-2}<n,u\neq 0\}}, \end{aligned} \tag{5.24}$$

and

$$
\begin{aligned}
&|\nabla[u\left(|u|^{p/2-1}\wedge\sqrt{n}\right)]|^2\\
&\quad= n|\nabla u|^2\mathbb{1}_{\{|u|^{p-2}\geqslant n\}}+|u|^{p-2}\left|\nabla u+\left(\frac{p}{2}-1\right)\mathrm{sign}u\nabla|u|\right|^2\mathbb{1}_{\{|u|^{p-2}<n\}}\\
&\quad= n|\nabla u|^2\mathbb{1}_{\{|u|^{p-2}\geqslant n\}}\\
&\qquad+\frac{p}{2}|u|^{p-2}\left(\frac{p}{2}|\mathrm{Re}\,(\mathrm{sign}\overline{u}\cdot\nabla u)|^2+\frac{2}{p}|\mathrm{Im}\,(\mathrm{sign}\overline{u}\cdot\nabla u)|^2\right)\mathbb{1}_{\{|u|^{p-2}<n,u\neq0\}}\\
&\quad\leqslant n|\nabla u|^2\mathbb{1}_{\{|u|^{p-2}\geqslant n\}}+\frac{q}{4}H^{I_d}_{F_p}[u;\nabla u]\mathbb{1}_{\{|u|^{p-2}<n,u\neq0\}},
\end{aligned}
\tag{5.25}
$$

where in the last inequality we used (5.18).

From Proposition 5.5(ii) it follows that $(A,V)\in\mathcal{AP}(\Omega,\mathscr{V})$, that is, $A-\alpha I_d$ is elliptic. Therefore, by combining Lemma 3.14, (5.22), (5.23) (5.24) and (5.25), we obtain

$$
\begin{aligned}
&\mathrm{Re}\int_\Omega\mathscr{L}^{A,V}(u)\cdot\overline{u}\left(|u|^{p-2}\wedge n\right)\\
&\qquad\geqslant n\int_{\{|u|^{p-2}\geqslant n\}}\mathrm{Re}\,\langle A\nabla u,\nabla u\rangle-\alpha|\nabla u|^2\\
&\qquad\quad+p^{-1}\int_{\{|u|^{p-2}<n,u\neq0\}}H^{A}_{F_p}[u;\nabla u]-\alpha\frac{pq}{4}H^{I_d}_{F_p}[u;\nabla u]\\
&\qquad\quad+(1-\sigma)\int_\Omega V_+|u|^2(|u|^{p-2}\wedge n)\\
&\qquad\geqslant\int_{\{|u|^{p-2}<n,u\neq0\}}p^{-1}H^{A-\alpha(pq/4)I_d}_{F_p}[u;\nabla u]+(1-\sigma)V_+|u|^p.
\end{aligned}
\tag{5.26}
$$

Finally, $(A,V)\in\mathcal{AP}_p(\Omega,\mathscr{V})$, Lemma 3.14, Fatou's Lemma, (5.21) and (5.26) give

$$
\mathrm{Re}\int_\Omega\mathscr{L}^{A,V}(u)\cdot\overline{u}|u|^{p-2}\gtrsim\int_\Omega|u|^{p-2}|\nabla u|^2\mathbb{1}_{\{u\neq0\}}+V_+|u|^p.
\tag{5.27}
$$

By combining (5.27) with the assumptions on u and Lemma 3.15 applied with $p/2+1$ instead of p, we obtain $|u|^{p/2-1}u\in W^{1,2}(\Omega)$ and

$$
|\nabla(|u|^{p/2-1}u)|^2\sim|u|^{p-2}|\nabla u|^2\mathbb{1}_{\{u\neq0\}}.
\tag{5.28}
$$

In order to deduce that $|u|^{p/2-1}u\in\mathscr{V}$ we may argue as in the proof of Corollary 5.7, while (5.20) follows from (5.27), (5.28) and Hölder inequality.

The final statement is a consequence of (5.19), which holds as $|u|^{p/2-1}u\in W^{1,2}(\Omega)$. □

5.3.3 Egert's extrapolation

Let $p\in(1,\infty)$. We have shown that the semigroup $(T^{A,V,\mathscr{V}}_t)_{t>0}$ extrapolates to $L^r(\Omega)$ for all exponents r satisfying $|1/2-1/r|\leqslant|1/2-1/p|$, whenever $(A,V)\in\mathcal{AP}_p(\Omega,\mathscr{V})$ and $\mathscr{V}$ satisfies (3.5) and (3.6); see Corollary 5.9. In this section, we extend the extrapolation range under additional assumptions on $\mathscr{V}$. We adapt the argument of Egert in [41]: the key idea is to combine the extrapolation of the semigroup on $L^p(\Omega)$, already at our disposal, with ultracontracitvity techniques which rely on L^2 off-diagonal bounds for the semigroup. The additional assumptions on $\mathscr{V}$ are required precisely to ensure that these techniques can be applied. The argument from [41] carries over to our setting thanks to elliptic inequalities still satisfied by the underlying sesquilinear form, together with Proposition 5.10, which generalizes [41, Proposition 11]. For

this reason, we shall not reproduce all details of the proofs but rather highlight the key properties that allow us to adapt the results of [41, Section 3.3. & Section 5] to the present framework. For a deeper understanding of the method and its technical underpinnings, we refer the reader to the original exposition in [41].

For $d \geqslant 3$, let $2^* := 2d/(d-2)$ denote the Sobolev conjugate of 2. In the same spirit as [41], we introduce the following definition.

Definition 5.11. Let $d \geqslant 3$ and $\mathscr{V}$ be a closed subspace of $W^{1,2}(\Omega)$ containing $W_0^{1,2}(\Omega)$. If $\|v\|_{2^*} \lesssim \|v\|_{1,2}$ holds for all $v \in \mathscr{V}$, then $\mathscr{V}$ has the *embedding property*. It has the *homogeneous embedding property* if $\|v\|_{2^*} \lesssim \|\nabla v\|_2$ for all $v \in \mathscr{V}$.

In order to apply Davies' perturbation method, we will also require invariance under multiplication by bounded Lipschitz functions, that is,

$$u \in \mathscr{V} \Longrightarrow \varphi u \in \mathscr{V}, \tag{5.29}$$

for every bounded and Lipschitz function $\varphi : \mathbb{R}^d \to \mathbb{R}$.

Since we shall work with a single triple $(A, V, \mathscr{V})$, we simplify the notation by writing

$$T_t := T_t^{A,V,\mathscr{V}}, \qquad \mathscr{L} := \mathscr{L}^{A,V,\mathscr{V}}.$$

We now generalize [41, Theorem 1] as follows. Recall that ω is the angle defined in page 107.

Corollary 5.12. *Let $d \geqslant 3$ and assume that $\mathscr{V}$ has the embedding property and satisfies (3.5), (3.6) and (5.29). If $(A, V) \in \mathcal{AP}_p(\Omega, \mathscr{V})$, then for every $\varepsilon > 0$ the semigroup $(e^{-\varepsilon t} T_t)_{t>0}$ generated by $-\mathscr{L} - \varepsilon$ extrapolates to a strongly continuous semigroup on $L^r(\Omega)$ provided that*

$$|1/2 - 1/r| \leqslant 1/d + (1 - 2/d)|1/2 - 1/p|.$$

This semigroup is bounded holomorphic of angle $\pi/2 - \omega$. If $\mathscr{V}$ has the homogeneous embedding property, then the same result also holds for $\varepsilon = 0$.

The proof follows step by step [41, Section 3.3 & Section 5]. The first ingredient are L^2 off-diagonal estimates for the semigroup. Thanks to (5.29), for any bounded Lipschitz function φ with $\|\nabla\varphi\|_\infty \leqslant 1$ and any $\varrho > 0$, we may define the perturbed sesquilinear form

$$\mathfrak{b}(u, v) := \mathfrak{a}(e^{\varrho\varphi} u, e^{-\varrho\varphi} v), \qquad u, v \in \mathrm{D}(\mathfrak{a}),$$

Hence, in view of the elliptic inequalities

$$\begin{aligned} |\mathfrak{a}(u,u)| &\lesssim \|\nabla u\|_2^2 + \|V_+^{1/2} u\|_2^2, && u \in \mathrm{D}(\mathfrak{a}), \\ \operatorname{Re} \mathfrak{a}(u,u) &\geqslant c \left(\|\nabla u\|_2^2 + \|V_+^{1/2} u\|_2^2 \right), && u \in \mathrm{D}(\mathfrak{a}), \end{aligned} \tag{5.30}$$

we can apply Davies' perturbation method and prove the following result by choosing appropriate function φ and parameter ϱ. For more details, see, for example, the proof of [41, Proposition 7].

Proposition 5.13. *Let $\mathscr{V}$ satisfy (5.29), $(A, V) \in \mathcal{AP}(\Omega, \mathscr{V})$ and $\psi \in [0, \pi/2 - \omega)$. For all measurable sets $E, F \subseteq \Omega$, all $z \in \mathbf{S}_\psi$, and all $f \in L^2(\Omega)$ with support in E it follows*

$$\|T_z f\|_{L^2(F)} \leqslant e^{-\frac{d(E,F)}{4C|z|}} \|f\|_{L^2(E)},$$

where $C = \Lambda + (\Lambda^2 \cos(\omega))/(c \cos(\psi + \omega))$. Here c is the constant which appears in (5.30).

Once the L^2 off-diagonal bounds are established, we can develop the ultracontracitivity techniques of [41, Section 5].

Definition 5.14. Let $\psi \in [0,\pi)$. Given $p,r \in (1,\infty)$ with $p \leqslant r$, a family of operators $(S_z)_{z\in\mathbf{S}_\psi} \subset \mathcal{L}(L^2(\Omega))$ is said to be $p \to r$ *bounded* if

$$\|S_z f\|_r \leqslant C|z|^{\frac{d}{2r}-\frac{d}{2p}}\|f\|_p$$

holds for some constant C and all $z \in \mathbf{S}_\psi$ and all $f \in L^p(\Omega) \cap L^2(\Omega)$.

We now reproduce, in our setting, the analogues of [41, Lemma 15 & Lemma 16].

Lemma 5.15. *Assume $d \geqslant 3$ and that $\mathscr{V}$ has the embedding property. Suppose that $(T_t)_{t>0}$ is $p \to p$ bounded and let $\varepsilon > 0$. If $p < 2$, then the shifted semigroup $(e^{-\varepsilon t}T_t)_{t>0}$ is $p \to 2$ bounded, and if $p > 2$, then it is $2 \to p$ bounded. If $\mathscr{V}$ has the homogeneous embedding property, then the conclusion also holds for $\varepsilon = 0$.*

Proof. The proof follows the lines of [41, Lemma 15]. In particular, we can argue in the same way thanks to the elliptic inequality

$$\operatorname{Re}\mathfrak{a}(u,u) \gtrsim \|\nabla u\|_2^2, \qquad u \in \mathrm{D}(\mathfrak{a}).$$

which follows from (5.30). □

Lemma 5.16. *Let $\varepsilon \geqslant 0$ and $\mathscr{V}$ satisfy (5.29). Suppose either $p < 2$ and that $(e^{-\varepsilon t}T_t)_{t>0}$ is $p \to 2$ bounded, or suppose $p > 2$ and that it is $2 \to p$ bounded. Then for every $\psi \in [0, \frac{\pi}{2} - \omega)$ and every r between 2 and p, the holomorphic extension $(e^{-\varepsilon z}T_z)_{z\in\mathbf{S}_\psi}$ is $r \to r$ bounded.*

Proof. In the proof of [41, Lemma 16], the peculiarity that the semigroup is generated by a divergence-form operator with an elliptic coefficient matrix is used solely to apply [41, Proposition 7]. Apart from this, the argument applies to any semigroup satisfying the assumptions of the present lemma. Therefore, the claim follows from the proof of [41, Lemma 16] together with Proposition 5.13, which takes the role of [41, Proposition 7]. □

The following proposition is modeled after [41, Proposition 18].

Proposition 5.17. *Assume $d \geqslant 3$ and that $\mathscr{V}$ has the embedding property and satisfies (3.5), (3.6) and (5.29). Let $p > 2$ and assume that $(A,V) \in \mathcal{AP}_p(\Omega,\mathscr{V})$. Then for every $\psi \in [0, \frac{\pi}{2} - \omega)$ and every $\varepsilon > 0$ the semigroup $(e^{-\varepsilon z}T_z)_{z\in\mathbf{S}_\psi}$ is $r \to r$ bounded for $r \in (2, \frac{dp}{d-2})$. If $\mathscr{V}$ has the homogeneous embedding property, then the same result holds for $\varepsilon = 0$.*

Proof. The result follows by adapting the proof of [41, Proposition 18], replacing

- [41, Proposition 11] with Proposition 5.10;
- [41, Theorem 2] with Corollary 4.24;
- [41, Lemma 15] with Lemma 5.15;
- [41, Lemma 16] with Lemma 5.16. □

Proof of Corollary 5.12. By duality and Proposition 5.5(i),(vi), it suffices to consider the case $p, r > 2$. For r satisfying

$$\left|\frac{1}{2} - \frac{1}{r}\right| < \frac{1}{d} + \left(1 - \frac{2}{d}\right)\left|\frac{1}{2} - \frac{1}{p}\right|,$$

the uniform boundedness follows from Proposition 5.17, whereas the holomorphy follows by combining Proposition 5.17 with Theorem 2.9, applied to any $e^{\pm i\psi}(\mathscr{L} + \varepsilon)$ for $\psi \in [0, \pi/2 - \omega)$.

The endpoint case for r is immediate, since the perturbed p-ellipticity is an open-ended condition; see Proposition 5.5(iii). □

Remark 5.18.

- Theorem 5.8, Corollary 5.9 and Corollary 5.12 remain valid under the more general assumption that V satisfies

$$\int_\Omega V_-|v|^2 \leqslant \alpha \int_\Omega |\nabla v|^2 + \sigma \int_\Omega V_+|v|^2 + c(\alpha,\sigma)\int_\Omega |v|^2, \qquad v \in \mathscr{V},$$

  for some $\alpha \geq 0$, $\sigma \in [0,1)$, and $c(\alpha,\sigma) \in \mathbb{R}$. In this case, all assertions hold for the operator $\mathscr{L} + c(\alpha,\sigma)$ instead of $\mathscr{L}$. For convenience, we have always assumed $c(\alpha,\sigma) = 0$.

- When $A = I_d$, the condition $(I_d, V) \in \widetilde{\mathcal{AP}}_p(\Omega, \mathscr{V})$ (resp. $(I_d, V) \in \mathcal{AP}_p(\Omega, \mathscr{V})$) corresponds to p lying in the interval $[p_-, p_+]$ (resp. (p_-, p_+)), where $p_\mp = 2/(1 \pm \sqrt{1 - \alpha(V,\mathscr{V})})$; see Remark 5.6. Therefore, Theorem 5.8 and Corollary 5.9 generalize [10, Theorem 1] and [74, Theorem 6]. Similarly, Corollary 5.9 and Corollary 5.12 should be compared with [5, Proposition 3.3 & Theorem 3.4], where analogous results are obtained for the Schrödinger operator $-\Delta + V$ on non-compact complete Riemannian manifolds of homogeneous type.

5.4 Heat-flow monotonicity

We now extend the heat-flow method introduced in Section 3.5.2 to the case of operators with negative potential terms. As before, the idea is to study the monotonicity of certain functionals associated with semigroups, exploiting convexity with respect to complex matrices of appropriate functions [12, 17–21, 78]. Some of the formal identities obtained here coincide with those in Section 3.5.2, while certain steps are new due to the presence of the potential terms..

Let $\Omega \subseteq \mathbb{R}^d$, $A, B \in \mathcal{A}(\Omega)$, $\mathscr{V}, \mathscr{W}$ as in Sect. 3.1.1, and $V \in \mathcal{P}(\Omega, \mathscr{V})$, $W \in \mathcal{P}(\Omega, \mathscr{W})$. Denote

$$\sigma := \max\{\sigma(V, \mathscr{V}), \sigma(W, \mathscr{W})\}. \tag{5.31}$$

For $f, g \in L^2(\Omega)$ and a C^1 function $\Phi : \mathbb{C}^2 \to \mathbb{R}_+$, set

$$\mathcal{E}(t) = \int_\Omega \Phi(T_t^{A,V,\mathscr{V}} f,\, T_t^{B,W,\mathscr{W}} g), \qquad t > 0.$$

Arguing as in Section 3.5.2, we formally obtain

$$-\mathcal{E}'(t) = 2\mathrm{Re}\int_\Omega \Big[(\partial_\zeta \Phi)(u,v)\,\mathscr{L}^{A,V} u + (\partial_\eta \Phi)(u,v)\,\mathscr{L}^{B,W} v\Big], \tag{5.32}$$

where $(u, v) = (T_t^{A,V,\mathscr{V}} f,\, T_t^{B,W,\mathscr{W}} g)$.

(i) Suppose that we can split the operators $\mathscr{L}^{A,V}, \mathscr{L}^{B,W}$ as

$$\mathscr{L}^{A,V} = \mathscr{L}^{A,0} + V_+ - V_-,$$
$$\mathscr{L}^{B,W} = \mathscr{L}^{B,0} + W_+ - W_-.$$

Then

$$-\mathcal{E}'(t) = I_1 + I_2 - I_3, \tag{5.33}$$

where

$$\begin{aligned} I_1 &= 2\mathrm{Re}\int_\Omega (\partial_\zeta\Phi)(u,v)\mathscr{L}^{A,0}u + (\partial_\eta\Phi)(u,v)\mathscr{L}^{B,0}v,\\ I_2 &= 2\int_\Omega V_+\mathrm{Re}\Big[(\partial_\zeta\Phi)(u,v)u\Big] + W_+\mathrm{Re}\Big[(\partial_\eta\Phi)(u,v)v\Big],\\ I_3 &= 2\int_\Omega V_-\mathrm{Re}\Big[(\partial_\zeta\Phi)(u,v)u\Big] + W_-\mathrm{Re}\Big[(\partial_\eta\Phi)(u,v)v\Big]. \end{aligned}$$

As in Section 3.5.2, assume that $\Phi \in C^2$ and that $(\partial_\zeta\Phi)(u,v)$ and $(\partial_\eta\Phi)(u,v)$ belong to the form domains $\mathrm{D}(\mathfrak{a}_{A,0,\mathscr{V}})$ and $\mathrm{D}(\mathfrak{a}_{B,0,\mathscr{W}})$, respectively. Then, integrating by parts in I_1 yields

$$I_1 = \int_\Omega H_\Phi^{A,B}[(u,v);(\nabla u,\nabla v)], \tag{5.34}$$

in complete analogy with (3.37).

(ii) Suppose that there exist $k \in \mathbb{N}$, $\{\varphi_j,\psi_j : \mathbb{C}^2 \to \mathbb{C} : j = 1,\cdots,k\}$ and $\mu \in (\sigma,1]$ such that $\varphi_j(u,v) \in \mathscr{V}$ and $\psi_j(u,v) \in \mathscr{W}$ for all $j \in \{1,\cdots,k\}$ and

$$\begin{aligned} 2\mathrm{Re}\,[(\partial_\zeta\Phi)(u,v)u] &\leqslant \sum_{j=1}^k |\varphi_j(u,v)|^2 \leqslant \frac{2}{\mu}\mathrm{Re}\,[(\partial_\zeta\Phi)(u,v)u],\\ 2\mathrm{Re}\,[(\partial_\eta\Phi)(u,v)v] &\leqslant \sum_{j=1}^k |\psi_j(u,v)|^2 \leqslant \frac{2}{\mu}\mathrm{Re}\,[(\partial_\eta\Phi)(u,v)v]. \end{aligned} \tag{5.35}$$

Then, by means of the subcritical inequality (5.1), we get

$$I_3 \leqslant \frac{\sigma}{\mu}I_2 + \sum_{j=1}^k \int_\Omega \alpha(V,\mathscr{V})|\nabla[\varphi_j(u,v)]|^2 + \alpha(W,\mathscr{W})|\nabla[\psi_j(u,v)]|^2.$$

Hence, it follows from (5.33) and (5.34) that if

$$H_\Phi^{A,B}[(u,v);(\nabla u,\nabla v)] \geqslant \sum_{j=1}^k \Big(\alpha(V,\mathscr{V})|\nabla[\varphi_j(u,v)]|^2 + \alpha(W,\mathscr{W})|\nabla[\psi_j(u,v)]|^2\Big), \tag{5.36}$$

then the function $\mathcal{E}$ is nonincreasing on $(0,+\infty)$. Moreover, if a stronger inequality than (5.36) holds, that is,

$$\begin{aligned} H_\Phi^{A,B}[(u,v);(\nabla u,\nabla v)] \geqslant\ & \tau(u,v)|\nabla u|^2 + \tau^{-1}(u,v)|\nabla v|^2\\ &+ \sum_{j=1}^k \Big(\alpha(V,\mathscr{V})|\nabla[\varphi_j(u,v)]|^2 + \alpha(W,\mathscr{W})|\nabla[\psi_j(u,v)]|^2\Big), \end{aligned} \tag{5.37}$$

for some positive function τ on $\mathbb{C}^2$, this formal method with $\Phi = \mathcal{Q}$ can be used for proving bilinear inequalities in the spirit of Section 3.5.4 and Section 4.6.

As explained in [21], item (i) follows easily when the potentials V and W are bounded. Dealing with unbounded potentials requires greater care. In [21] such decomposition was not established for unbounded nonnegative potentials; instead, the authors deduced the bilinear embedding for such potentials via a truncation argument. As we will show in Section 5.7,

attempting to follow the same approach reveals that the negative part of the potentials can be truncated, whereas the positive part cannot. Therefore, in Section 5.7.2 we will justify the decomposition in item (i) for potentials with bounded negative part and possibly unbounded positive part, and subsequently adapt the truncation argument to remove the boundedness assumption on the negative part.

A further novelty of this method concerns item (ii), namely, finding functions φ_j and ψ_j such that

$$\varphi_j\left(T_t^{A,V,\mathscr{V}}f, T_t^{B,W,\mathscr{W}}g\right) \in \mathscr{V},$$
$$\psi_j\left(T_t^{A,V,\mathscr{V}}f, T_t^{B,W,\mathscr{W}}g\right) \in \mathscr{W},$$

and satisfying (5.35) and (5.37) for the Bellman function $\mathcal{Q}$. The candidate functions φ_j and ψ_j are given by the following lemma. More precisely, up to multiplicative constants, they are

$$\begin{aligned} \varphi_1(\zeta,\eta) &= \zeta|\zeta|^{p/2-1}, & \varphi_2(\zeta,\eta) &= \zeta\max\{|\zeta|^{p/2-1}, |\eta|^{1-q/2}\},\\ \psi_1(\zeta,\eta) &= \eta|\eta|^{q/2-1}, & \psi_2(\zeta,\eta) &= 0, \end{aligned}$$

for $\zeta,\eta\in\mathbb{C}$.

Lemma 5.19. *Choose $p>2$ and $\mu\in(0,1)$. There exists $\delta>0$, sufficiently small and depending on μ, such that*

$$2\,\mathrm{Re}\,[\partial_\zeta\mathcal{Q}(\zeta,\eta)\zeta] = p|\zeta|^p + 2\delta\left|\zeta\max\{|\zeta|^{p/2-1}, |\eta|^{1-q/2}\}\right|^2,$$
$$2\,\mathrm{Re}\,[\partial_\eta\mathcal{Q}(\zeta,\eta)\eta] \leqslant [q+(2-q)\delta]|\eta|^q \leqslant \frac{2}{\mu}\mathrm{Re}\,[\partial_\eta\mathcal{Q}(\zeta,\eta)\eta],$$

for all $\zeta,\eta\in\mathbb{C}$.

Proof. The first equality holds for all $\delta>0$.

On the other hand, an easy calculation shows that

$$2\mathrm{Re}\,[\partial_\eta\mathcal{Q}(\zeta,\eta)\eta] \leqslant [q+(2-q)\delta]|\eta|^q,$$
$$2\mathrm{Re}\,[\partial_\eta\mathcal{Q}(\zeta,\eta)\eta] \geqslant q|\eta|^q,$$

for all $\zeta,\eta\in\mathbb{C}$ and all $\delta>0$. Therefore, it suffices to prove that there exists $\delta>0$ such that

$$\mu[q+(2-q)\delta] < q.$$

Since $\mu\in(0,1)$, this is verified for δ sufficiently small. □

In view of Lemma 5.19, estimate (5.37) for $\Phi=\mathcal{Q}$ turns in

$$\begin{aligned} &H_{\mathcal{Q}}^{A,B}[(u,v);(\nabla u,\nabla v)]\\ &\quad\geqslant \tau(u,v)|\nabla u|^2 + \tau^{-1}(u,v)|\nabla v|^2\\ &\qquad + \alpha(V,\mathscr{V})\Big(p|\nabla(|u|^{p/2-1}u)|^2 + 2\delta|\nabla(u\max\{|u|^{p/2-1},|v|^{q/2-1}\})|^2\Big)\\ &\qquad + \alpha(W,\mathscr{W})[q+(2-q)\delta]|\nabla(|v|^{q/2-1}v)|^2. \end{aligned} \tag{5.38}$$

In the same spirit as Section 3.5.1 and Section 4.4.2, we aim to establish a pointwise estimate of $H_{\mathcal{Q}}^{A,B}$ on $\mathbb{C}^2\times\mathbb{C}^{2d}$ in a such way that ensures the validity of (5.38) for $u=T_t^{A,V,\mathscr{V}}f$ and $v=T_t^{B,W,\mathscr{W}}g$, where $f,g\in L^p\cap L^q$. To this end, we first need to show that the functions $|u|^{p/2-1}u$ and $u\max\{|u|^{p/2-1},|v|^{q/2-1}\}$ belong to $\mathscr{V}$ and $|v|^{q/2-1}v$ to $\mathscr{W}$ and that their gradients

can be computed using the chain rule. Regarding the terms $|u|^{p/2-1}u$ and $|v|^{q/2-1}v$, we can rely on [21, 41, 82] for this purpose. However, justifying the chain rule for $u\max\{|u|^{p/2-1}, |v|^{q/2-1}\}$ requires more effort. This is the focus of Section 5.5.

Subsequently, under the assumption that $(A, V) \in \mathcal{AP}_p(\Omega, \mathscr{V})$ and $(B, W) \in \mathcal{AP}_p(\Omega, \mathscr{W})$ we establish the desired lower pointwise estimate of $H_{\mathcal{Q}}^{A,B}$ in Section 5.6, which will implies (5.38).

5.5 Chain rule

As explained at the very end of Section 5.4 we would like to apply the chain rule to compute the gradient of $u\max\{|u|^{p/2-1}, |v|^{1-q/2}\}$. In general, justifying the chain rule is not a trivial problem. For real-valued functions belonging to $W^{1,2}(\Omega)$, it is known that the chain rule holds for composition with Lipschitz functions, see [91, Theorem 2.1.11]. However, this does not hold with the same generality for complex-valued or vector-valued functions, see [58].

In [4] mapping theorems for Sobolev spaces of vector-valued functions are provided. Given two Banach spaces $X \neq \{0\}$ and Y, it has been proved that each Lipschitz continuous mapping $\Phi : X \to Y$ gives rise to a mapping $u \mapsto F \circ u$ from $W^{1,p}(\Omega, X)$ to $W^{1,p}(\Omega, Y)$ if and only if Y has the Radon-Nikodým Property. Moreover, if in addition Φ is one-side Gateaux differentiable, no condition on the space is needed and a chain rule can even be proved.

We recall that a function $\Phi : X \to Y$ is said to be *one-side Gateaux differentiable at* x if the right-hand limit

$$D_v^+\Phi(x) := \lim_{t\to 0^+} \frac{1}{t}\left(\Phi(x+tv) - \Phi(x)\right)$$

exists for every direction $v \in X$. In this case, the left-hand limit

$$D_v^-\Phi(x) := \lim_{t\to 0^+} \frac{1}{-t}\left(\Phi(x-tv) - \Phi(x)\right)$$

exists as well and is given by

$$D_v^-\Phi(x) = -D_{-v}^+\Phi(x). \tag{5.39}$$

We say that Φ is one-side Gateaux differentiable on X if it is one-side Gateaux differentiable at x for all $x \in X$.

As special case of [4, Theorem 4.2] we have the following.

Theorem 5.20. *Let $1 \leqslant p \leqslant \infty$ and $u \in W^{1,p}(\Omega, \mathbb{R}^2)$. Suppose that $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous and one-sided Gateaux differentiable, and assume furthermore that Ω is bounded or $\Phi(0) = 0$. Then $\Phi \circ u \in W^{1,p}(\Omega)$ and we have the chain rule*

$$\partial_j(\Phi \circ u) = D^+_{\partial_j u}\Phi(u) = D^-_{\partial_j u}\Phi(u).$$

5.5.1 Chain rule in the heat-flow method

In this section we will justify the chain rule for $u\max\{|u|^{p/2-1}, |v|^{1-q/2}\}$ by means of Theorem 5.20.

Let $p \geqslant 2$, $q = p/(p-1)$, $\delta \in (0,1)$ and $n \in \mathbb{N}$. Define the functions $\Phi_\delta, \Phi_{\delta,n} : (-\delta/2, +\infty)^2 \to [0, +\infty)$ by

$$\Phi_{\delta,n}(s,t) = \max\{(s+\delta)^{p/2-1}, (t+\delta)^{1-q/2}\} \wedge n,$$
$$\Phi_n(s,t) = \max\{s^{p/2-1}, t^{1-q/2}\} \wedge n.$$

Clearly, after defining $\varphi, \psi : (-\delta/2, +\infty) \to [0, +\infty)$ as

$$\varphi(s) = (s+\delta)^{p/2-1},$$
$$\psi(t) = (t+\delta)^{1-q/2},$$

we can rewrite $\Phi_{\delta,n}$ as

$$\Phi_{\delta,n}(s,t) = n \wedge \begin{cases} \varphi(s), & \text{if } (s+\delta)^p \geqslant (t+\delta)^q, \\ \psi(t), & \text{if } (s+\delta)^p \leqslant (t+\delta)^q, \end{cases}$$
$$= n \wedge \begin{cases} \varphi(s), & \text{if } g(s) \geqslant t, \\ \psi(t), & \text{if } g(s) \leqslant t, \end{cases}$$

with $g : ((\delta/2)^{1/p} - \delta, +\infty) \to (-\delta/2, +\infty)$ being defined by

$$g(s) = (s+\delta)^{p-1} - \delta.$$

Finally, for all $(s,t) \in \operatorname{graf}(g)$ define

$$\Pi_0(s,t) = \{(x,y) \in \mathbb{R}^2 : y = g'(s)x\},$$
$$\Pi_\pm(s,t) = \{(x,y) \in \mathbb{R}^2 : \pm y > \pm g'(s)x\}.$$

Lemma 5.21. *Let $p \geqslant 2$, $q = p/(p-1)$, $\delta \in (0,1)$ and $n \in \mathbb{N}$. Then $\Phi_{\delta,n}$ is Lipschitz continuous and one-side Gateaux differentiable. In particular,*

- *if $n \neq (s+\delta)^p > (t+\delta)^q$,*

  $$D_v^+ \Phi_{\delta,n}(s,t) = \partial_s \Phi_{\delta,n}(s,t) \cdot v_1,$$

  for all $v = (v_1, v_2) \in \mathbb{R}^2$;

- *if $(s+\delta)^p < (t+\delta)^q \neq n$,*

  $$D_v^+ \Phi_{\delta,n}(s,t) = \partial_t \Phi_{\delta,n}(s,t) \cdot v_2,$$

  for all $v = (v_1, v_2) \in \mathbb{R}^2$;

- *if $n \neq (s+\delta)^p = (t+\delta)^q$,*

  $$D_v^+ \Phi_{\delta,n}(s,t) = \begin{cases} \varphi'(s)\mathbb{1}_{\{(s+\delta)^{p/2-1} < n\}} \cdot v_1, & \text{if } v \in \overline{\Pi_-(s,t)}, \\ \psi'(t)\mathbb{1}_{\{(t+\delta)^{1-q/2} < n\}} \cdot v_2, & \text{if } v \in \overline{\Pi_+(s,t)}; \end{cases}$$

- *if $n = (s+\delta)^p > (t+\delta)^q$,*

  $$D_v^+ \Phi_{\delta,n}(s,t) = \begin{cases} \varphi'(s) \cdot v_1, & \text{if } v_1 < 0, \\ 0, & \text{if } v_1 \geqslant 0; \end{cases}$$

- *if $(s+\delta)^p < (t+\delta)^q = n$,*

  $$D_v^+ \Phi_{\delta,n}(s,t) = \begin{cases} \psi'(t) \cdot v_2, & \text{if } v_2 < 0, \\ 0, & \text{if } v_2 \geqslant 0; \end{cases}$$

- *if* $n = (s+\delta)^p = (t+\delta)^q$,

$$D_v^+\Phi_{\delta,n}(s,t) = \begin{cases} \varphi'(s)\cdot v_1, & \text{if } v \in \overline{\Pi_-(s,t)}, v_1, v_2 < 0, \\ \psi'(t)\cdot v_2, & \text{if } v \in \Pi_+(s,t), v_1, v_2 < 0, \\ 0, & \text{if either } v_1 \geqslant 0 \text{ or } v_2 \geqslant 0. \end{cases}$$

Proof. Set

$$\begin{aligned} \Xi_{n,\delta} = &\{(s,t) \in (-\delta/2, +\infty)^2 : n = (s+\delta)^p > (t+\delta)^q\} \\ &\cup \{(s,t) \in (-\delta/2, +\infty)^2 : (s+\delta)^p < (t+\delta)^q = n\} \\ &\cup \{(s,t) \in (-\delta/2, +\infty)^2 : (s+\delta)^p = (t+\delta)^q \leqslant n\}. \end{aligned}$$

Clearly, $\Phi_{\delta,n} \in C^1\left((-\delta/2,+\infty)^2 \setminus \Xi_n\right)$ with bounded first order derivatives. Moreover, for every two different elements in $(-\delta/2,+\infty)^2$ the connecting line segment intersects Ξ_n at most finitely times. Therefore, $\Phi_{\delta,n}$ is Lipschitz on $(-\delta/2,+\infty)^2$.

In order to verify that $\Phi_{\delta,n}$ is one-side Gateaux differentiable, we have to show that

$$D_v^+\Phi_{\delta,n}(s,t) := \lim_{h\to 0^+} \frac{\Phi_{\delta,n}((s,t)+hv) - \Phi_{\delta,n}(s,t)}{h}$$

exists for all $(s,t) \in (-\delta/2,+\infty)^2$ and $v \in \mathbb{R}^2$. We will only consider the case when $(s+\delta)^p = (t+\delta)^q \neq n$. The other cases are simpler and will not written down here.

Since g is convex, for all $(x,y) \in (s,t) + \overline{\Pi_-(s,t)}$, with $y \neq t$, we have $g(x) > y$. Therefore, for all $v \in \overline{\Pi_-(s,t)}$ we get

$$(s,t) + hv \in \{(x,y) \in \mathbb{R}^2 : y < g(x)\},$$

for any $h > 0$. Hence,

$$\lim_{h\to 0^+} \frac{\Phi_{\delta,n}((s,t)+hv) - \Phi(s,t)}{h} = \lim_{h\to 0^+} \frac{\varphi(s+hv_1) - \varphi(s)}{h} = \varphi'(s)\cdot v_1.$$

On the other hand, if $v \in \Pi_+(s,t)$ there exists $h_0 \in (0,1)$ such that

$$(s,t) + hv \in \{(x,y) \in \mathbb{R}^2 : y > g(x)\},$$

for all $h \in (0,h_0)$. Thus,

$$\lim_{h\to 0^+} \frac{\Phi_{\delta,n}((s,t)+hv) - \Phi(s,t)}{h} = \lim_{h\to 0^+} \frac{\psi(t+hv_2) - \psi(t)}{h} = \psi'(t)\cdot v_2.$$

We conclude by observing that

$$\varphi'(s)v_1 = \psi'(t)v_2,$$

for all $(s+\delta)^p = (t+\delta)^q$ and $v \in \Pi_0(s,t)$. □

Lemma 5.22. *Assume that $\mathscr{V}$ and $\mathscr{W}$ are as in the statement of Proposition 3.4. Let $p \geqslant 2$, $q = p/(p-1)$ and $n \in \mathbb{N}$. Suppose that $u \in \mathscr{V}$ and $v \in \mathscr{W}$ are such that*

$$|v|^{q-2}|\nabla v|^2 \mathbb{1}_{\{v\neq 0\}} \in L^1(\Omega).$$

Then

(i) $u\left(\max\{|u|^{p/2-1}, |v|^{1-q/2}\} \wedge n\right) \in \mathscr{V}$;

(ii) *almost everywhere on* $\{n^{2p/(p-2)} > |u|^p = |v|^q\}$ *we have*

$$\left(\frac{p}{2}-1\right)\frac{u}{|u|}\mathrm{Re}\left(\frac{\overline{u}}{|u|}\nabla u\right)\mathbb{1}_{\{0<|u|^{p/2-1}<n\}} = \left(1-\frac{q}{2}\right)\frac{u}{|v|}\mathrm{Re}\left(\frac{\overline{v}}{|v|}\nabla v\right)\mathbb{1}_{\{0<|v|^{1-q/2}<n\}};$$

(iii)

$$\nabla\left[u\left(\max\{|u|^{p/2-1},|v|^{1-q/2}\}\wedge n\right)\right] = z_n,$$

where

$$\begin{aligned} z_n &:= \left(\max\{|u|^{p/2-1},|v|^{1-q/2}\}\wedge n\right)\nabla u \\ &+ \begin{cases} \left(\frac{p}{2}-1\right)|u|^{p/2-1}\frac{u}{|u|}\mathrm{Re}\left(\frac{\overline{u}}{|u|}\nabla u\right)\mathbb{1}_{\{0<|u|^{p/2-1}<n\}}, & \text{if } |u|^p \geqslant |v|^q, \\ \left(1-\frac{q}{2}\right)|v|^{1-q/2}\frac{u}{|v|}\mathrm{Re}\left(\frac{\overline{v}}{|v|}\nabla v\right)\mathbb{1}_{\{0<|v|^{1-q/2}<n\}}, & \text{if } |u|^p \leqslant |v|^q. \end{cases} \end{aligned} \tag{5.40}$$

Proof. Fix $n\in\mathbb{N}$. For all $\delta\in(0,1)$ define $\Psi,\Psi_\delta:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$ by

$$\begin{aligned} \Psi(\zeta,\eta) &= \zeta\cdot\Phi_n(|\zeta|,|\eta|), \\ \Psi_\delta(\zeta,\eta) &= \zeta\cdot\Phi_{\delta,n}(|\zeta|,|\eta|). \end{aligned}$$

By combining Lemma 5.21 with Theorem 5.20, we obtain $\Phi_{\delta,n}(|u|,|v|)\in W^{1,2}_{\mathrm{loc}}(\Omega)$ and

$$\partial_j[\Phi_{\delta,n}(|u|,|v|)] = D^+_{(\partial_j|u|,\partial_j|v|)}\Phi_{\delta,n}(|u|,|v|) = D^-_{(\partial_j|u|,\partial_j|v|)}\Phi_{\delta,n}(|u|,|v|), \tag{5.41}$$

for all $j\in\{1,\cdots,d\}$ and $\delta\in(0,1)$. In particular, from (5.39), (5.41) and Lemma 5.21, for all $\delta\in(0,1)$ we deduce that

$$\left(\frac{p}{2}-1\right)(|u|+\delta)^{p/2-2}\nabla|u|\mathbb{1}_{\{(|u|+\delta)^{p/2-1}<n\}} = \left(1-\frac{q}{2}\right)(|v|+\delta)^{-q/2}\nabla|v|\mathbb{1}_{\{(|v|+\delta)^{1-q/2}<n\}} \tag{5.42}$$

almost everywhere on $\{n^{2p/(p-2)} > (|u|+\delta)^p = (|v|+\delta)^q\}$ and

$$\begin{aligned} &\nabla[\Phi_{\delta,n}(|u|,|v|)] \\ &= \begin{cases} \left(\frac{p}{2}-1\right)\frac{(|u|+\delta)^{p/2-1}}{|u|+\delta}\nabla|u|\mathbb{1}_{\{(|u|+\delta)^{p/2-1}<n\}}, & \text{if } (|u|+\delta)^p \geqslant (|v|+\delta)^q, \\ \left(1-\frac{q}{2}\right)\frac{(|v|+\delta)^{1-q/2}}{|v|+\delta}\nabla|v|\mathbb{1}_{\{(|v|+\delta)^{1-q/2}<n\}}, & \text{if } (|u|+\delta)^p \leqslant (|v|+\delta)^q. \end{cases} \end{aligned} \tag{5.43}$$

Item (ii) follows now by the identity

$$\nabla|u| = \mathrm{Re}\left(\frac{\overline{u}}{|u|}\nabla u\right)\mathbb{1}_{\{u\neq 0\}}$$

and by sending $\delta\to 0$ in (5.42).

Let prove now item (i), that is, $\Psi(u,v)\in\mathscr{V}$. The function Ψ_δ is Lipschitz continuous and $\Psi_\delta(0)=0$ for all $\delta\in(0,1)$. Therefore, $\Psi_\delta(u,v)\in\mathscr{V}$ by Proposition 3.4. Moreover, for all $\delta\in(0,1)$ we have

$$|\Psi_\delta(u,v)| \leqslant n|u| \in L^2(\Omega). \tag{5.44}$$

Thus, from Lebesgue's dominated convergence theorem we obtain

$$\|\Psi_\delta(u,v)\|_{L^2} \to \|\Psi(u,v)\|_{L^2}, \tag{5.45}$$

and

$$\Psi_\delta(u,v) \to \Psi(u,v) \in L^2(\Omega) \tag{5.46}$$

in $\mathcal{D}'(\Omega)$, as $\delta \to 0$. From (5.46), the density of $C_c^\infty(\Omega)$ in $L^2(\Omega)$ and (5.44) we deduce that

$$\Psi_\delta(u,v) \rightharpoonup \Psi(u,v) \tag{5.47}$$

in $L^2(\Omega)$, as $\delta \to 0$. By combining (5.45) and (5.47) we get

$$\Psi_\delta(u,v) \to \Psi(u,v) \tag{5.48}$$

strongly in $L^2(\Omega)$, as $\delta \to 0$. Moreover, (5.43) and the product rule imply the existence of a positive constant C, not depending on δ, such that

$$|\nabla[\Psi_\delta(u,v)]| \leqslant C\left(|\nabla u| + |v|^{q/2-1}|\nabla v|\mathbb{1}_{\{v\neq 0\}}\right), \tag{5.49}$$

which belongs to $L^2(\Omega)$ by the assumptions on u and v. Thus,

$$\|\Psi_\delta(u,v)\|_{\mathscr{V}} \lesssim \|u\|_{W^{1,2}} + \||v|^{q/2-1}|\nabla v|\mathbb{1}_{\{v\neq 0\}}\|_{L^2} < \infty, \tag{5.50}$$

for all $\delta \in (0,1)$. Therefore, (5.48), (5.50) and Lemma 3.3 yield $\Psi(u,v) \in \mathscr{V}$.

Finally, prove the chain rule in (iii). From (5.49) and Lebesgue's dominated convergence theorem we get

$$\nabla[\Psi_\delta(u,v)] \to z_n \in L^2(\Omega) \tag{5.51}$$

in $\mathcal{D}'(\Omega)$, as $\delta \to 0$. Therefore, by combining (5.46) and (5.51) we infer that $\nabla[\Psi(u,v)] = z_n$. □

Proposition 5.23. *Assume that $\mathscr{V}$ and $\mathscr{W}$ are as in the statement of Proposition 3.4. Let $p \geqslant 2$, $q = p/(p-1)$ and $n \in \mathbb{N}$. Suppose that $u \in \mathscr{V}$ and $v \in \mathscr{W}$ are such that*

$$u \in L^p(\Omega), \qquad v \in L^q(\Omega), \qquad \left(|u|^{p-2} + |v|^{2-q}\right)|\nabla u|^2 + |v|^{q-2}|\nabla v|^2\mathbb{1}_{\{v\neq 0\}} \in L^1(\Omega).$$

Then

(i) $u\max\{|u|^{p/2-1}, |v|^{1-q/2}\} \in \mathscr{V}$;

(ii) *almost everywhere on $\{|u|^p = |v|^q\}$ we have*

$$\left(\frac{p}{2}-1\right)\frac{u}{|u|}\mathrm{Re}\left(\frac{\overline{u}}{|u|}\nabla u\right)\mathbb{1}_{\{u\neq 0\}} = \left(1-\frac{q}{2}\right)\frac{u}{|v|}\mathrm{Re}\left(\frac{\overline{v}}{|v|}\nabla v\right)\mathbb{1}_{\{v\neq 0\}};$$

(iii)

$$\nabla\left[u\max\{|u|^{p/2-1}, |v|^{1-q/2}\}\right] = \begin{cases} |u|^{p/2-1}\left(\nabla u + \left(\frac{p}{2}-1\right)\frac{u}{|u|}\mathrm{Re}\left(\frac{\overline{u}}{|u|}\nabla u\right)\mathbb{1}_{\{u\neq 0\}}\right), & \text{if } |u|^p \geqslant |v|^q, \\ |v|^{1-q/2}\left(\nabla u + \left(1-\frac{q}{2}\right)\frac{u}{|v|}\mathrm{Re}\left(\frac{\overline{v}}{|v|}\nabla v\right)\mathbb{1}_{\{v\neq 0\}}\right), & \text{if } |u|^p \leqslant |v|^q. \end{cases}$$

Proof. Item (ii) follows by Lemma 5.22(ii).

Let $n \in \mathbb{N}$. Recall the definition of z_n in (5.40) and define

$$w = u\max\{|u|^{p/2-1}, |v|^{1-q/2}\},$$
$$w_n = u\left(\max\{|u|^{p/2-1}, |v|^{1-q/2}\} \wedge n\right),$$
$$z = \begin{cases} |u|^{p/2-1}\left(\nabla u + \left(\frac{p}{2}-1\right)\frac{u}{|u|}\mathrm{Re}\left(\frac{\overline{u}}{|u|}\nabla u\right)\mathbb{1}_{\{u\neq 0\}}\right), & \text{if } |u|^p \geqslant |v|^q, \\ |v|^{1-q/2}\left(\nabla u + \left(1-\frac{q}{2}\right)\frac{u}{|v|}\mathrm{Re}\left(\frac{\overline{v}}{|v|}\nabla v\right)\mathbb{1}_{\{v\neq 0\}}\right), & \text{if } |u|^p \leqslant |v|^q. \end{cases}$$

By Lemma 5.22(i), (iii), $w_n \in \mathscr{V}$ and $\nabla w_n = z_n$ for all $n \in \mathbb{N}$. Moreover, Lemma 5.22(iii) and the assumptions on u and v give

$$\begin{aligned} &|w_n| \leqslant |w| \leqslant \max\{|u|^{p/2}, |v|^{q/2}\} \in L^2(\Omega), \\ &|\nabla w_n| \leqslant |z| \leqslant \left(|u|^{p/2-1} + |v|^{1-q/2}\right) |\nabla u| \mathbb{1}_{\{u\neq 0\}} + |v|^{q/2-1} |\nabla v| \mathbb{1}_{\{v\neq 0\}} \in L^2(\Omega), \end{aligned} \tag{5.52}$$

for all $n \in \mathbb{N}$. Thus, as in the proof of Lemma 5.22, we can prove that $w_n \to w$ strongly in $L^2(\Omega)$ and $(w_n)_{n\in\mathbb{N}}$ is bounded in $\mathscr{V}$. Therefore, Lemma 3.3 yields $\Psi(u,v) \in \mathscr{V}$.

Moreover, from (5.52) and Lebesgue's dominated convergence theorem we have

$$\begin{aligned} w_n &\to w \in L^2(\Omega), \\ \nabla w_n &\to z \in L^2(\Omega), \end{aligned} \tag{5.53}$$

in $\mathcal{D}'(\Omega)$, as $n \to \infty$. Hence, $\nabla w = z$. □

In Section 5.6 we will provide a pointwise lower estimate of the generalized Hessian of $\mathcal{Q}$. To this purpose, it is useful introduce the following functions.

For any $(X,Y) \in \mathbb{C}^d \times \mathbb{C}^d$ we define the nonnegative functions $b_p[\,\cdot\,;(X,Y)]$ on $\mathbb{C}^2 \setminus \{(\zeta,\eta) \in \mathbb{C}^2 \,:\, \eta = 0\}$, $g_p[\,\cdot\,;X]$ on $\mathbb{C}$ and $h_p[\,\cdot\,;(X,Y)]$ on $\mathbb{C}^2$ by the following rules

$$\begin{aligned} b_p[(\zeta,\eta);(X,Y)] &= |\eta|^{2-q}|X|^2 + \left(1-\frac{q}{2}\right)^2 |\zeta|^2 |\eta|^{-q} |\mathrm{Re}\,(e^{-i\arg\eta} Y)|^2 \\ &\quad + 2\left(1-\frac{q}{2}\right) |\zeta||\eta|^{1-q} \left\langle \mathrm{Re}\,(e^{-i\arg\zeta} X), \mathrm{Re}\,(e^{-i\arg\eta} Y)\right\rangle, \\ g_p[\zeta;X] &= \frac{p}{2}|\zeta|^{p-2}\left(\frac{p}{2}|\mathrm{Re}\,(e^{-i\arg\zeta}X)|^2 + \frac{2}{p}|\mathrm{Im}\,(e^{-i\arg\zeta}X)|^2\right), \\ h_p[(\zeta,\eta);(X,Y)] &= \begin{cases} b_p[(\zeta,\eta);(X,Y)] & |\zeta|^p < |\eta|^q; \\ g_p[\zeta;X] & |\zeta|^p \geqslant |\eta|^q. \end{cases} \end{aligned} \tag{5.54}$$

For any $X, Y \in \mathbb{C}^d$, the function $b_p[\,\cdot\,;(X,Y)]$ is continuous in $\mathbb{C}^2 \setminus \{\eta = 0\}$, while $g_p[\,\cdot\,;X]$ is continuous in all $\mathbb{C}$.

The definitions of g_p and b_p are motivated by the fact that

$$|\nabla[u \max\{|u|^{p/2-1}, |v|^{1-q/2}\}]|^2 = \begin{cases} b_p[(u,v);(\nabla u, \nabla v)] \mathbb{1}_{\{v\neq 0\}} & |u|^p \leqslant |v|^q; \\ g_p[u;\nabla u] & |u|^p \geqslant |v|^q, \end{cases} \tag{5.55}$$

for any u, v as in the assumption of Proposition 5.23.

5.6 Generalized convexity of the Bellman function

As explained in Section 5.4, at a certain point, we need the estimate (5.38) in order to apply the heat-flow method and prove the bilinear inequality (5.61). From (5.19), applied with $r = p$ and $r = q$, and (5.55), it follows that to deduce (5.38) it suffices to establish a lower bound on the generalized Hessian of $\mathcal{Q}$ on $\mathbb{C}^2 \times \mathbb{C}^{2d}$ of the form

$$\begin{aligned} H_{\mathcal{Q}}^{(A,B)}[(\zeta,\eta);(X,Y)] &\geqslant \tau(\zeta,\eta)|X|^2 + \tau(\zeta,\eta)^{-1}|Y|^2 \\ &\quad + \mu\left(\frac{pq}{4} H_{F_p}^{I_d}[\zeta;X] + 2\delta h_p[(\zeta,\eta);(X,Y)]\right) \\ &\quad + \gamma[q + (2-q)\delta]\frac{p}{4} H_{F_q}^{I_d}[\eta;Y], \end{aligned}$$

whenever $A-\mu(pq/4)I_d, B-\gamma(pq/4)I_d \in \mathcal{A}_p(\Omega)$, for some $\mu,\gamma>0$.

The proof of this result will be given in Theorem 5.25, by taking the parameter δ in the definition of the Bellman function $\mathcal{Q}$ sufficiently small and appealing to Theorem 3.22. The argument adapts that of Theorem 3.22, itself inspired by the proofs of [36, Theorem 3], [17, Theorem 15], [18, Theorem 5.2], and [20, Theorem 5.2]; see also Theorem 4.15.

To enhance clarity, we begin with a lemma. Clearly, for $|\zeta|^p \leqslant |\eta|^q$ we have

$$b_p[\omega;(X,Y)] \leqslant K_q[\eta;(X,Y)], \tag{5.56}$$

for all $X,Y\in\mathbb{C}^d$, where

$$K_q[\eta;(X,Y)] := |\eta|^{2-q}|X|^2 + 2\left(1-\frac{q}{2}\right)|X||Y| + \left(1-\frac{q}{2}\right)^2|\eta|^{q-2}|Y|^2.$$

Lemma 5.24. *Let $p\geqslant 2$, $q=p/(p-1)$ and $\mu,\gamma>0$. For all $\delta\in(0,1)$ consider the positive constants $C(\delta)$ and $D(\delta)$ of Remark 3.23. Then there exists $\delta_0\in(0,1)$ such that for all $\delta\in(0,\delta_0)$ we have $\widetilde{C}(\delta)>0$ such that*

$$\begin{aligned}
&2C(\delta)\Big(D(\delta)|\eta|^{2-q}|X|^2 + D(\delta)^{-1}|\eta|^{q-2}|Y|^2\Big) + \frac{pq}{4}H^{(\mu I_d,\gamma I_d)}_{F_2\otimes F_{2-q}}[(\zeta,\eta);(X,Y)]\\
&\quad\geqslant \widetilde{C}(\delta)\Big(|\eta|^{2-q}|X|^2 + |\eta|^{q-2}|Y|^2\Big)\\
&\qquad + \gamma(2-q)\frac{p}{4}H^{I_d}_{F_q}[\eta;Y] + 2\mu K_q[\eta;(X,Y)],
\end{aligned}$$

for all $|\zeta|^p<|\eta|^q$ and $X,Y\in\mathbb{C}^d$.

Proof. For all $\delta\in(0,1)$, $|\zeta|^p<|\eta|^q$ and $X,Y\in\mathbb{C}^d$, denote

$$\begin{aligned}
L_\delta[\omega;(X,Y)] :=\ &2C(\delta)\Big(D(\delta)|\eta|^{2-q}|X|^2 + D(\delta)^{-1}|\eta|^{q-2}|Y|^2\Big)\\
&+ \frac{pq}{4}H^{(\mu I_d,\gamma I_d)}_{F_2\otimes F_{2-q}}[(\zeta,\eta);(X,Y)]\\
&- \gamma(2-q)\frac{p}{4}H^{I_d}_{F_q}[\eta;Y] - 2\mu K_q[\omega;(X,Y)].
\end{aligned}$$

We want to prove the existence of $\delta_0\in(0,1)$ such that for all $\delta\in(0,\delta_0)$ we have $\widetilde{C}(\delta)>0$ such that

$$L_\delta[\omega;(X,Y)] \geqslant \widetilde{C}(\delta)\Big(|\eta|^{2-q}|X|^2 + |\eta|^{q-2}|Y|^2\Big), \tag{5.57}$$

for any $\omega=(\zeta,\eta)\in\{|\zeta|^p<|\eta|^q\}$ and $X,Y\in\mathbb{C}^d$.

By combining [20, Corollary 5.12] applied with $A=\mu I_d$ and $B=\gamma I_d$ and the inequality

$$H^{I_d}_{F_q}[\eta;Y] \lesssim_q |\eta|^{q-2}|Y|^2, \quad \eta\in\mathbb{C}\setminus\{0\}, Y\in\mathbb{C}^d,$$

we obtain, for some constant $\Gamma=\Gamma(p,\mu,\gamma)\in\mathbb{R}$,

$$\begin{aligned}
&L_\delta[\omega;(X,Y)]\\
&\geqslant 2\cdot\Bigg[\left(C(\delta)D(\delta)+\mu\left(\frac{pq}{4}-1\right)\right)|\eta|^{2-q}|X|^2 - \max\{\mu,\gamma\}(2-q)\left(\frac{pq}{2}+1\right)|X||Y|\\
&\qquad + \Big(C(\delta)D(\delta)^{-1}-\Gamma\Big)|\eta|^{q-2}|Y|^2\Bigg],
\end{aligned}$$

for all $\omega = (\zeta, \eta) \in \{|\zeta|^p < |\eta|^q\}$ and $X, Y \in \mathbb{C}^d$. From (3.31) and the fact that $(pq)/4 \geqslant 1$, there exists $\delta_0 \in (0,1)$ such that

$$4\left(C(\delta)D(\delta) + \mu\left(\frac{pq}{4} - 1\right)\right)\left(C(\delta)D(\delta)^{-1} - \Gamma\right) > \left[\max\{\mu,\gamma\}(2-q)\left(\frac{pq}{2}+1\right)\right]^2,$$

for all $\delta \in (0, \delta_0)$. Therefore, [21, Corollary 3.4] applied for all $\delta \in (0,\delta_0)$ implies the existence of $\widetilde{C}(\delta) > 0$ for which (5.57) holds. □

Theorem 5.25. *Choose $p \geqslant 2$ and $A, B \in \mathcal{A}_p(\Omega)$. Suppose that there exist $\mu, \gamma > 0$ such that $A - \mu(pq/4)I_d, B - \gamma(pq/4)I_d \in \mathcal{A}_p(\Omega)$. Then there exists a continuous function $\tau : \mathbb{C}^2 \to [0, +\infty)$ such that $\tau^{-1} = 1/\tau$ is locally integrable on $\mathbb{C}^2 \setminus \{(0,0)\}$, and $\delta \in (0,1)$ such that $\mathcal{Q} = \mathcal{Q}_{p,\delta}$ as in (11) admits the following property:*

- *there exists $\widetilde{C} > 0$ such that for any $\omega = (\zeta,\eta) \in \mathbb{C}^2 \setminus \Upsilon$, $X, Y \in \mathbb{C}^d$, and a.e. $x \in \Omega$, we have*

$$\begin{aligned} H_{\mathcal{Q}}^{(A(x),B(x))}[\omega;(X,Y)] \geqslant{}& \widetilde{C}(\tau|X|^2 + \tau^{-1}|Y|^2) \\ &+ \mu\left(\frac{pq}{4}H_{F_p}^{I_d}[\zeta;X] + 2\delta h_p[\omega;(X,Y)]\right) \\ &+ \gamma[q + (2-q)\delta]\frac{p}{4}H_{F_q}^{I_d}[\eta;Y]. \end{aligned}$$

The implied constants depending on A, B, p, μ and γ, but not on the dimension d.

We may take $\tau(\zeta,\eta) = \max\{|\zeta|^{p-2}, |\eta|^{2-q}\}$.

Proof. Since $A - \mu(pq/4)I_d$, $B - \gamma(pq/4)I_d \in \mathcal{A}_p(\Omega)$, by Theorem 3.22 and Remark 3.23 there exists $\delta_0 \in (0,1)$ such that for all $\delta \in (0,\delta_0)$ we have $C = C(\delta) > 0$ such that, a.e. $x \in \Omega$,

$$H_{\mathcal{Q}}^{(A,B)}[\omega;(X,Y)] \geqslant 2\delta C(\rho|X|^2 + \rho^{-1}|Y|^2) + \frac{pq}{4}H_{\mathcal{Q}}^{(\mu I_d,\gamma I_d)}[\omega;(X,Y)], \tag{5.58}$$

for all $\omega = (\zeta,\eta) \in \mathbb{C}^2 \setminus \Upsilon$ and $X, Y \in \mathbb{C}^d$. Here, C is the constant of Remark 3.23 and ρ is the function defined in (3.30).

First assume that $|\zeta|^p > |\eta|^q > 0$. Then, by (5.18) we have for all $\delta \in (0,1)$

$$\begin{aligned} &\frac{pq}{4}H_{\mathcal{Q}}^{(\mu I_d,\gamma I_d)}[(\zeta,\eta);(X,Y)] \\ &\quad= \mu(p+2\delta)\frac{q}{4}H_{F_p}^{I_d}[\zeta;X] + \gamma[q+\delta(2-q)]\frac{p}{4}H_{F_q}^{I_d}[\eta;Y] \\ &\quad\geqslant \mu\frac{pq}{4}H_{F_p}^{I_d}[\zeta;X] + 2\delta\mu\frac{p}{2}|\zeta|^{p-2}\left(\frac{p}{2}|\mathrm{Re}\,(e^{-i\arg\zeta}X)|^2 + \frac{2}{p}|\mathrm{Im}\,(e^{-i\arg\zeta}X)|^2\right) \\ &\qquad+ \gamma[q+\delta(2-q)]\frac{p}{4}H_{F_q}^{I_d}[\eta;Y]. \end{aligned}$$

Hence, from (5.58) and the definitions of ρ and h_p, in this region we get

$$\begin{aligned} H_{\mathcal{Q}}^{(A,B)}[(\zeta,\eta);(X,Y)] \geqslant{}& 2\delta C(p-1)^{-1}\left(|\zeta|^{p-2}|X|^2 + |\zeta|^{2-p}|Y|^2\right) \\ &+ \mu\left(\frac{pq}{4}H_{F_p}^{I_d}[\zeta;X] + 2\delta h_p[\omega;(X,Y)]\right) \\ &+ \gamma[q+(2-q)\delta]\frac{p}{4}H_{F_q}^{I_d}[\eta;Y], \end{aligned}$$

for all $\delta \in (0,\delta_0)$.

Suppose now that $|\zeta|^p < |\eta|^q$. In this region $\rho(\zeta,\eta) = D(\delta)|\eta|^{2-q}$, where $D(\delta)$ is the constant of Remark 3.23. In view of (5.56) it suffices to prove the inequality with K_q instead of h_p. From the definition of the Bellman function $\mathcal{Q}$ we have

$$H_{\mathcal{Q}}^{(\mu I_d,\gamma I_d)}[(\zeta,\eta);(X,Y)] = \mu H_{F_p}^{I_d}[\zeta;X] + \gamma H_{F_q}^{I_d}[\eta;Y] + \delta H_{F_2\otimes F_{2-q}}^{(\mu I_d,\gamma I_d)}[(\zeta,\eta);(X,Y)].$$

Therefore, we conclude by combining (5.58) and Lemma 5.24. □

5.6.1 Regularization of $\mathcal{Q}$

Recall notation (3.32). The following theorem is modeled after [21, Theorem 3.5]. See also [36, Theorem 4], Corollary 3.25 and Corollary 4.20.

Theorem 5.26. *Choose $p \geqslant 2$ and $A, B \in \mathcal{A}_p(\Omega)$. Suppose that there exist $\mu, \gamma > 0$ such that $A - \mu(pq/4)I_d, B - \gamma(pq/4)I_d \in \mathcal{A}_p(\Omega)$. Let $\delta \in (0,1)$, $\widetilde{C} > 0$ and function $\tau : \mathbb{C}^2 \to (0,\infty)$ be as in Theorem 5.25. Then for $\mathcal{Q} = \mathcal{Q}_{p,\delta}$ and any $\omega = (\zeta, \eta) \in \mathbb{C}^2$ we have, for a.e. $x \in \Omega$ and every $(X, Y) \in \mathbb{C}^d \times \mathbb{C}^d$,*

$$\begin{aligned} H^{(A(x),B(x))}_{\mathcal{Q}*\varphi_\nu}[\omega;(X,Y)] \geqslant C\Big((\tau * \varphi_\nu)(\omega)|X|^2 + (\tau^{-1} * \varphi_\nu)(\omega)|Y|^2\Big) \\ + \mu\frac{pq}{4}\left(H^{I_d}_{F_p}[\cdot;X] * \varphi_\nu\right)(\zeta) \\ + \gamma[q + (2-q)\delta]\frac{p}{4}\left(H^{I_d}_{F_q}[\cdot;Y] * \varphi_\nu\right)(\eta) \\ + 2\delta\mu(h_p[\,\cdot\,;(X,Y)] * \varphi_\nu)(\omega). \end{aligned}$$

with the implied constant depending on A, B, p, μ and γ, but not on the dimension d.

Proof. As in the proof of Corollary 3.25, we get

$$H^{(A,B)}_{\mathcal{Q}*\varphi_\nu}[\omega;(X,Y)] = \int_{\mathbb{R}^4} H^{(A,B)}_{\mathcal{Q}}[\omega - \mathcal{W}^{-1}_{2,1}(z);(X,Y)]\varphi_\nu(z)\,\mathrm{d}z.$$

Hence, Theorem 5.25 gives the desired inequality. □

The next two results are auxiliary for Proposition 5.29 below. Let $F \in C^1(\mathbb{R}^n;\mathbb{R})$. Define

$$\begin{aligned} E_\pm &:= \{y \in \mathbb{R}^n \,:\, \pm F(y) > 0\}, \\ E_0 &:= \{y \in \mathbb{R}^n \,:\, F(y) = 0\}, \\ \widetilde{E}_0 &:= \{y \in E_0 \,:\, DF(y) \neq 0\}. \end{aligned}$$

Lemma 5.27. *Let $y_0 \in \widetilde{E}_0$ and $\varphi \in L^1(\mathbb{R}^n)$. Then,*

$$\lim_{\nu\to 0}\int_{E_\pm}\varphi(y_0 - y)\,\mathrm{d}y = c_\pm(y_0),$$

with $c_+(y_0) + c_-(y_0) = \int_{\mathbb{R}^n}\varphi$.

More precisely,

$$c_\pm(y_0) = \int_{\Pi_\pm(y_0)}\varphi(y_0 - y)\,\mathrm{d}y = \int_{\widetilde{\Pi}_\mp(y_0)}\varphi(y)\,\mathrm{d}y,$$

where

$$\begin{aligned} \Pi_\pm(y_0) &= \{y \in \mathbb{R}^n \,:\, \pm DF(y_0)\cdot(y - y_0) > 0\}, \\ \widetilde{\Pi}_\pm(y_0) &= \{y \in \mathbb{R}^n \,:\, \pm DF(y_0)\cdot y > 0\}. \end{aligned}$$

Proof. Observe that

$$\int_{E_+}\varphi_\nu(y_0 - y)\,\mathrm{d}y = \int_{\mathbb{R}^n}\varphi(-y)\mathbb{1}_{E_+}(y_0 + \nu y)\,\mathrm{d}y. \tag{5.59}$$

For all $y \in \mathbb{R}^n$, define

$$g_{y_0,y}(\nu) := F(y_0 + \nu y), \qquad \nu \in \mathbb{R}.$$

Notice that $g_{y_0}(0) = 0$ implies

$$\lim_{\nu\to 0} \mathbb{1}_{E_+}(y_0 + \nu y) = \begin{cases} 1, & \text{if } g'_{y_0,y}(0) > 0, \\ 0, & \text{if } g'_{y_0,y}(0) < 0. \end{cases}$$

But $g'_{y_0}(\nu) = DF(y_0 + \nu y) \cdot y$. Hence,

$$\lim_{\nu\to 0} \mathbb{1}_{E_+}(y_0 + \nu y) = \begin{cases} 1, & \text{if } DF(y_0) \cdot y > 0, \\ 0, & \text{if } DF(y_0) \cdot y < 0, \end{cases}$$

and, since $|\{y \in \mathbb{R}^n \,:\, DF(y_0) \cdot y = 0\}| = 0$, Lebesgue's dominated convergence theorem and (5.59) give

$$\begin{aligned} \lim_{\nu\to 0} \int_{E_+} \varphi_\nu(y_0 - y)\,\mathrm{d}y &= \int_{\widetilde{\Pi}_+(y_0)} \varphi(-y)\,\mathrm{d}y \\ &= \int_{\widetilde{\Pi}_-(y_0)} \varphi(y)\,\mathrm{d}y \\ &= \int_{\Pi_+(y_0)} \varphi(y_0 - y)\,\mathrm{d}y = c_+(y). \end{aligned}$$

Analogously, $\lim_{\nu\to 0} \int_{E_+} \varphi_\nu(y_0 - y)\,\mathrm{d}y = c_-(y_0)$. Clearly, $c_+(y_0) + c_-(y_0) = \int_{\mathbb{R}^n} \varphi$. □

Corollary 5.28. *Let g_+, g_- be functions on $\mathbb{R}^n$ which are continuous at y for all $y \in \widetilde{E}_0$. Define*

$$g(y) = \begin{cases} g_+(y), & \text{if } y \in \overline{E_+}, \\ g_-(y), & \text{if } y \in E_-. \end{cases}$$

Suppose that $|E_0| = 0$. Let $\varphi \in L^1(\mathbb{R}^n)$ be such that $\varphi \geqslant 0$, $\int_{\mathbb{R}^n} \varphi = 1$, $\operatorname{supp}\varphi \subseteq B(0,1)$. Then for all $y_0 \in \widetilde{E}_0$ there exists $c_\pm(y_0) \geqslant 0$ such that

(i) $c_+(y_0) + c_-(y_0) = 1$,

(ii) $\lim_{\nu\to 0}(g * \varphi_\nu)(y_0) = c_+(y_0)g_+(y_0) + c_-(y_0)g_-(y_0)$.

Moreover, if φ is radial then $c_+(y_0) = c_-(y_0) = 1/2$.

Proof. Fix $y_0 \in \widetilde{E}_0$. As $|E_0| = 0$, we have

$$\begin{aligned} (g * \varphi_\nu)(y_0) &= \int_{\mathbb{R}^n} \varphi_\nu(y_0 - y) g(y)\,\mathrm{d}y \\ &= \int_{E_+} \varphi_\nu(y_0 - y) g_+(y)\,\mathrm{d}y + \int_{E_-} \varphi_\nu(y_0 - y) g_-(y)\,\mathrm{d}y \\ &= \int_{E_+} \varphi_\nu(y_0 - y)[g_+(y) - g_+(y_0)]\,\mathrm{d}y + g_+(y_0) \int_{E_+} \varphi_\nu(y_0 - y)\,\mathrm{d}y \\ &\quad + \int_{E_-} \varphi_\nu(y_0 - y)[g_-(y) - g_-(y_0)]\,\mathrm{d}y + g_-(y_0) \int_{E_-} \varphi_\nu(y_0 - y)\,\mathrm{d}y. \end{aligned}$$

Since

$$\int_{\pm E} \varphi_\nu(y_0 - y)\,\mathrm{d}y \leqslant \int_{\mathbb{R}^n} \varphi_\nu(y_0 - y)\,\mathrm{d}y = 1,$$

$g_\pm$ are continuous at y_0 and $\operatorname{supp}\varphi_\nu \subseteq B(0,\nu)$, we obtain

$$\left| \int_{E_\pm} \varphi_\nu(y_0 - y)[g_\pm(y) - g_\pm(y_0)]\,\mathrm{d}y \right| \leqslant \sup_{|y-y_0|<\nu} |g_\pm(y) - g_\pm(y_0)| \to 0, \qquad \text{as } \nu \to 0.$$

On the other hand, Lemma 5.27 yields

$$g_\pm(y_0)\int_{E_\pm}\varphi_\nu(y_0-y)\,\mathrm{d}y\to g_\pm(y_0)c_\pm(y_0),\qquad\text{as }\nu\to 0.$$

Now suppose that φ is radial. Therefore, from Lemma 5.27 we have

$$c_\pm(y_0)=\int_{\tilde\Pi_\mp(y_0)}\varphi(y)\,\mathrm{d}y=\frac12\int_{\mathbb{R}^n}\varphi(y)\,\mathrm{d}y=\frac12.$$ □

Proposition 5.29. *Let $p\geqslant 2$. Then for all $X,Y\in\mathbb{C}^d$ the followings hold:*

(i) *for any $\omega=(\zeta,\eta)\in\mathbb{C}^2\setminus\{|\zeta|^p=|\eta|^q\}$,*

$$\lim_{\nu\to0}(h_p[\,\cdot\,;(X,Y)]*\varphi_\nu)(\omega)=h_p[\omega;(X,Y)];$$

(ii) *for any $\omega=(\zeta,\eta)\in\{|\zeta|^p=|\eta|^q\}\setminus\{(0,0)\}$,*

$$\lim_{\nu\to0}(h_p[\,\cdot\,;(X,Y)]*\varphi_\nu)(\omega)=\frac12\left(b_p[\omega;(X,Y)]+g_p[\zeta;X]\right).$$

Proof. The first assertion follows by the continuity of $h_p[\,\cdot\,;(X,Y)]$ in $\mathbb{C}^2\setminus\{|\zeta|^p=|\eta|^q\}$ for all $X,Y\in\mathbb{C}^d$.

The second one follows by applying Corollary 5.28 with $g_+=(g_p)_{\mathcal{W}}$, $g_-=(b_p)_{\mathcal{W}}$ and $F=(F_p\otimes\mathbf{1}-\mathbf{1}\otimes F_q)_{\mathcal{W}}$. Notice that in this case $\tilde E_0=E_0\setminus\{(0,0)\}$, which gives $|E_0|=|\tilde E_0|=0$ because $DF(\omega)\neq0$ for all $\omega\in\tilde E_0$. □

5.7 Proof of the bilinear embedding for potentials with bounded negative part

In case when the potentials are assumed to be nonnegative, in [21, Theorem 1.4] Carbonaro and Dragičević proved that there exists $C>0$ independent on the dimension d such that

$$\int_0^\infty\int_\Omega\sqrt{\left|\nabla T_t^{A,V,\mathscr{V}}f\right|^2+V\left|T_t^{A,V,\mathscr{V}}f\right|^2}\sqrt{\left|\nabla T_t^{B,W,\mathscr{W}}g\right|^2+W\left|T_t^{B,W,\mathscr{W}}g\right|^2}\leqslant C\|f\|_p\|g\|_q,\tag{5.60}$$

for all $A,B\in\mathcal{A}_p(\Omega)$, $V,W\in L^1_{\mathrm{loc}}(\Omega,\mathbb{R}_+)$ and all $f,g\in(L^p\cap L^q)(\Omega)$, where $\mathscr{V}$ and $\mathscr{W}$ are two closed subspaces of $W^{1,2}(\Omega)$ satisfying Assumption BE and $q=p/(p-1)$ is the conjugate exponent of p.

Given $V\in\mathcal{P}(\Omega,\mathscr{V})$ and $W\in\mathcal{P}(\Omega,\mathscr{W})$, we extend the bilinear embedding in (5.60) to the semigroups $(T_t^{A,V,\mathscr{V}})_{t>0}$ and $(T_t^{B,W,\mathscr{W}})_{t>0}$. In accordance with Theorem 3.19 and Theorem 4.26, we need a stronger condition than the one which implies the L^p contractivity of such semigroups. In the next two sections we shall prove the following result.

Theorem 5.30. *Suppose that $(\mathscr{V},\mathscr{W})$ satisfies Assumption BE. Choose $p>1$. Let q be its conjugate exponent, i.e., $1/p+1/q=1$. Assume that $(A,V)\in\mathcal{AP}_p(\Omega,\mathscr{V})$ and $(B,W)\in\mathcal{AP}_p(\Omega,\mathscr{W})$.*

There exists $C>0$ such that for any $f,g\in(L^p\cap L^q)(\Omega)$ we have

$$\int_0^\infty\int_\Omega\sqrt{\left|\nabla T_t^{A,V,\mathscr{V}}f\right|^2+\left|V\right|\left|T_t^{A,V,\mathscr{V}}f\right|^2}\sqrt{\left|\nabla T_t^{B,W,\mathscr{W}}g\right|^2+\left|W\right|\left|T_t^{B,W,\mathscr{W}}g\right|^2}\leqslant C\|f\|_p\|g\|_q.\tag{5.61}$$

We may choose $C>0$ which depends on $p,A,B,\alpha(V,\mathscr{V}),\alpha(W,\mathscr{W})$, but not on the dimension d.

This result incorporates several earlier theorems as special cases, including:

- $V = W$ nonnegative, $\Omega = \mathbb{R}^d$, A, B equal and real [35, Theorem 1]
- $V = W = 0$, $\Omega = \mathbb{R}^d$ [20, Theorem 1.1] (Theorem 3.19)
- $V = W = 0$ [19, Theorem 2] (Theorem 3.19)
- V, W nonnegative [21, Theorem 1.4].

Suppose that $(\mathscr{V}, \mathscr{W})$ satisfies Assumption BE. Take $p > 1$, $A, B \in \mathcal{A}_p(\Omega)$ and $V \in \mathcal{P}(\Omega, \mathscr{V})$, and $W \in \mathcal{P}(\Omega, \mathscr{W})$ such that $(A, V) \in \mathcal{AP}_p(\Omega, \mathscr{V})$ and $(B, W) \in \mathcal{AP}_p(\Omega, \mathscr{W})$. Denote by q the conjugate exponent of p. It is enough to consider the case $p \geqslant 2$.

For $f, g \in (L^p \cap L^q)(\Omega)$ define

$$\mathcal{E}(t) = \int_\Omega \mathcal{Q}\left(T_t^{A,V,\mathscr{V}} f, T_t^{B,W,\mathscr{V}} g\right), \ t > 0. \tag{5.62}$$

In the case when $V_- = W_- = 0$, in [21] the authors supposed the potentials to be bounded in order to study the monotonicity of the flow $\mathcal{E}$. In this case the operator $\mathscr{L}^{A,V}$ turns out to be the sum of the second-order operator $L^{A,0}$ and the multiplication operator V. More precisely, $\mathrm{D}(\mathscr{L}^{A,V}) = \mathrm{D}(\mathscr{L}^{A,0})$ and for $u \in \mathrm{D}(\mathscr{L}^{A,V})$ we have

$$\mathscr{L}^{A,V} u = \mathscr{L}^{A,0} u + Vu.$$

The same holds for B, W. See [21, Section 3.3].

Once the bilinear inequality was proved for bounded potentials, they deduced the general one by approximating the unbounded potentials with their truncations. The key point to apply that argument was the uniform sectoriality (in $n \in \mathbb{N}$) of the operators $\mathscr{L}^{A,V\wedge n}$, in the sense of [21, (3.15)]. See [21, Section 3.4].

Let us try to repeat this approach by cutting both the negative and the positive part of the potentials. Let $A \in \mathcal{A}(\Omega)$ and $\{V_j\}_{j\in J}$ be a family of potentials. Denote by $\mathscr{L}_j$ the operator associated with the sesquilinear form $\mathfrak{a}_j := \mathfrak{a}_{A,V_j}$. By Theorem 2.37(i), the uniform sectoriality of such a family of operators is ensured provided that (5.7) holds for all $j \in J$ with a constant independent of the parameter j. More precisely, $\{\mathscr{L}_j\}_{j\in J}$ is a family of uniformly sectorial operators if there exists $\alpha > 0$ such that

- for all $j \in J$ there exists $\sigma_j \in [0, 1]$ such that

$$\int_\Omega (V_j)_- |u|^2 \leqslant \alpha \int_\Omega |\nabla u|^2 + \sigma_j \int_\Omega (V_j)_+ |u|^2,$$

  for all $u \in \mathscr{V}$;
- $A - \alpha I_d \in \mathcal{A}(\Omega)$.

Given $V \in \mathcal{P}_{\alpha,\sigma}(\Omega)$, it is clear that

$$V_n := V_+ - V_- \wedge n \in \mathcal{P}_{\alpha,\sigma}(\Omega)$$

for all $n \in \mathbb{N}$. Therefore, if $(A, V) \in \mathcal{AP}(\Omega, \mathscr{V})$, the above properties are satisfied by the family of potentials $\{V_n\}_{n\in\mathbb{N}}$.

Now, if we also try to truncate the positive part, we immediately notice that some problems come out. In fact, fixed $n \in \mathbb{N}$, it is not so clear at which altitude we have to cut the positive part

of V_n in order to obtain a family of potentials for which the existence of such uniform constant is guaranteed. If we first truncate the positive part and then the negative one is even worse. In both cases, the point is that we might need all V_+ in order to control either $V_- \wedge n$ or V_-.

These observations force us to work with potentials whose positive part is unbounded. On the other hand, as noted before, we are allowed to truncate the negative part. We will thus first assume that our potentials have a bounded negative part. Once the bilinear embedding is established for this class of potentials, we will then extend the result to potentials with unbounded negative part by approximating them with their truncations, as done in [21].

In Section 5.7.2 we will exhibit an alternative proof of [21, Theorem 1.4], studying the monotonicity of $\mathcal{E}$ directly working with possibly unbounded potentials. In addition to being of independent interest, this new proof represents an auxiliary step in establishing the bilinear embedding in the case where the potentials may have a nontrivial (bounded) negative part. In particular, it will enable us to apply the chain rule through Proposition 5.23. In fact, we will prove that

$$\begin{aligned} 2\operatorname{Re}\int_\Omega \Big(\partial_\zeta \mathcal{Q}(u,v)\mathscr{L}^{A,V_+}u + \partial_\eta \mathcal{Q}(u,v)\mathscr{L}^{B,W_+}v\Big) \\ \geqslant \liminf_{\nu\to 0}\int_\Omega H^{(A,B)}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)] \\ + 2\int_\Omega V_+(\partial_\zeta\mathcal{Q})(u,v)\cdot u + W_+(\partial_\eta\mathcal{Q})(u,v)\cdot v, \end{aligned} \tag{5.63}$$

for all $u \in \mathrm{D}(\mathfrak{a}_{A,V_+,\mathscr{V}})$, $v \in \mathrm{D}(\mathfrak{a}_{B,W_+,\mathscr{W}})$ such that $u, v, \mathscr{L}^{A,V_+}u, \mathscr{L}^{B,W_+}v \in (L^p \cap L^q)(\Omega)$. As showed in [21, Sections 3.3.1 and 3.3.2], (5.63) implies

$$\begin{aligned} 2\operatorname{Re}\int_\Omega \Big(\partial_\zeta \mathcal{Q}(u,v)\mathscr{L}^{A,V_+}u + \partial_\eta \mathcal{Q}(u,v)\mathscr{L}^{B,W_+}v\Big) \\ \gtrsim \int_{\Omega\setminus\{u=0,v=0\}} \tau(u,v)\left(|\nabla u|^2 + V_+|u|^2\right) + \tau(u,v)^{-1}\left(|\nabla v|^2 + W_+|v|^2\right), \end{aligned} \tag{5.64}$$

with $\tau(u,v) = \max\{|u|^{p-2}, |v|^{2-q}\}$. In particular, (5.64) and the last two estimates of (3.29) give

$$|v|^{2-q}|\nabla u|^2 \in L^1(\Omega), \tag{5.65}$$

for all $u \in \mathrm{D}(\mathfrak{a}_{A,V_+,\mathscr{V}})$, $v \in \mathrm{D}(\mathfrak{a}_{B,W_+,\mathscr{W}})$ such that $u, v, \mathscr{L}^{A,V_+}u, \mathscr{L}^{B,W_+}v \in (L^p \cap L^q)(\Omega)$. Therefore, if V_- and W_- are bounded, then Proposition 5.10 and (5.65) guarantee that such u, v fall into the assumptions of Proposition 5.23; see Remark 5.31.

5.7.1 Potentials with bounded negative part

We prove now the bilinear embedding assuming that V_-, W_- are *essentially bounded*. In that case, $\mathrm{D}(\mathscr{L}^{A,V}) = \mathrm{D}(\mathscr{L}^{A,V_+})$ and for $u \in \mathrm{D}(\mathscr{L}^{A,V})$ we have

$$\mathscr{L}^{A,V}u = \mathscr{L}^{A,V_+}u - V_-u. \tag{5.66}$$

The same holds true for B, W.

Remark 5.31. From (5.66) it follows that

$$u \in \mathrm{D}(\mathfrak{a}_{A,V,\mathscr{V}}) \quad \text{and} \quad u, \mathscr{L}^{A,V}u \in (L^p \cap L^q)(\Omega),$$

if and only if

$$u \in \mathrm{D}(\mathfrak{a}_{A,V_+,\mathscr{V}}) \quad \text{and} \quad u, \mathscr{L}^{A,V_+}u \in (L^p \cap L^q)(\Omega),$$

provided that V_- is bounded.

In Proposition 5.33 we will show how we may deduce the bilinear estimate (5.61) from (5.63). First, we will start with a reduction, in the same spirit as Proposition 3.29 and Proposition 4.27.

Proposition 5.32. *Suppose that $A, B, V, W, \mathscr{V}, \mathscr{W}, p, q$ are as in the formulation of Theorem 5.30 and $V_-, W_- \in L^\infty(\Omega)$. Assume that*

$$\int_\Omega \sqrt{|\nabla u|^2 + |V||u|^2}\sqrt{|\nabla v|^2 + |W||v|^2} \lesssim \operatorname{Re}\int_\Omega \left(\partial_\zeta \mathcal{Q}(u,v)\mathscr{L}^{A,V}u + \partial_\eta \mathcal{Q}(u,v)\mathscr{L}^{B,W}v\right), \tag{5.67}$$

for all $u \in \mathrm{D}(\mathfrak{a}_{A,V,\mathscr{V}})$, $v \in \mathrm{D}(\mathfrak{a}_{B,W,\mathscr{W}})$ such that $u, v, \mathscr{L}^{A,V}u, \mathscr{L}^{B,W}v \in (L^p \cap L^q)(\Omega)$. Then (5.61) holds.

Proof. Arguing as in the proof of Proposition 3.29, by studying the monotonicity of the flow $\mathcal{E}$ defined in (5.62), we deduce from (5.67) the bilinear inequality

$$\int_0^\infty \int_\Omega \sqrt{|\nabla T_t^{A,V,\mathscr{V}} f|^2 + |V||T_t^{A,V,\mathscr{V}} f|^2}\sqrt{|\nabla T_t^{B,W,\mathscr{W}} g|^2 + |W||T_t^{B,W,\mathscr{W}} g|^2} \lesssim \|f\|_p^p + \|g\|_q^q,$$

for all $f, g \in L^p \cap L^q(\Omega)$. In the present setting, the analyticity of the semigroups $(T_t^{A,V,\mathscr{V}})_{t>0}$ and $(T_t^{B,W,\mathscr{W}})_{t>0}$ both on L^p and L^q is ensured by Corollary 5.9. This plays the same role as in Proposition 3.29, where the argument relied on analyticity in the case without potentials.

At this point, as usual (5.61) follows by replacing f and g with sf and $s^{-1}g$ and minimizing the right-hand side with respect to $s > 0$. □

Proposition 5.33. *Suppose that $A, B, V, W, \mathscr{V}, \mathscr{W}, p, q$ are as in the formulation of Theorem 5.30 and $V_-, W_- \in L^\infty(\Omega)$. Assume that (5.63) is satisfied for all $u \in \mathrm{D}(\mathfrak{a}_{A,V_+,\mathscr{V}})$, $v \in \mathrm{D}(\mathfrak{a}_{B,W_+,\mathscr{W}})$ such that $u, v, \mathscr{L}^{A,V_+}u, \mathscr{L}^{B,W_+}v \in (L^p \cap L^q)(\Omega)$. Then (5.61) holds.*

Proof. By Proposition 5.32 it suffices to prove (5.67) for all $u \in \mathrm{D}(\mathfrak{a}_{A,V,\mathscr{V}})$, $v \in \mathrm{D}(\mathfrak{a}_{B,W,\mathscr{W}})$ such that $u, v, \mathscr{L}^{A,V}u, \mathscr{L}^{B,W}v \in (L^p \cap L^q)(\Omega)$. Given such u and v, denote

$$\mathcal{O}(\mathcal{Q})(u,v) = 2\operatorname{Re}\int_\Omega \left(\partial_\zeta \mathcal{Q}(u,v)\mathscr{L}^{A,V}u + \partial_\eta \mathcal{Q}(u,v)\mathscr{L}^{B,W}v\right). \tag{5.68}$$

By (5.66), we get

$$\begin{aligned}\mathcal{O}(\mathcal{Q})(u,v) = {} & 2\operatorname{Re}\int_\Omega \left(\partial_\zeta \mathcal{Q}(u,v)\mathscr{L}^{A,V_+}u + \partial_\eta \mathcal{Q}(u,v)\mathscr{L}^{B,W_+}v\right) \\ & - 2\int_\Omega \left[V_-(\partial_\zeta \mathcal{Q})(u,v)\cdot u + W_-(\partial_\eta \mathcal{Q})(u,v)\cdot v\right].\end{aligned}$$

Hence, from Remark 5.31 and (5.63) we have

$$\mathcal{O}(\mathcal{Q})(u,v) \geqslant I_1 + I_2 - I_3, \tag{5.69}$$

where

$$\begin{aligned} I_1 &= \liminf_{\nu\to 0}\int_\Omega H^{(A,B)}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)], \\ I_2 &= 2\int_\Omega \left[V_+(\partial_\zeta \mathcal{Q})(u,v)\cdot u + W_+(\partial_\eta \mathcal{Q})(u,v)\cdot v\right], \\ I_3 &= 2\int_\Omega \left[V_-(\partial_\zeta \mathcal{Q})(u,v)\cdot u + W_-(\partial_\eta \mathcal{Q})(u,v)\cdot v\right]. \end{aligned}$$

Estimating I_1, I_2, I_3 separately, we will prove first that

$$\mathcal{O}(\mathcal{Q})(u,v) \gtrsim \int_{\Omega\setminus\{u=0,v=0\}} \tau(u,v)\left(|\nabla u|^2 + V_+|u|^2\right) + \tau^{-1}(u,v)\left(|\nabla v|^2 + W_+|v|^2\right), \tag{5.70}$$

and then that

$$\mathcal{O}(\mathcal{Q})(u,v) \gtrsim \int_{\Omega\setminus\{u=0,v=0\}} \tau(u,v)\cdot V_-|u|^2 + \tau^{-1}(u,v)\cdot W_-|v|^2, \tag{5.71}$$

which, along with the fact that for $w \in W^{1,2}(\Omega)$ we have $\nabla w = 0$ almost everywhere on $\{w=0\}$, imply (5.67). Recall that $\tau(u,v) = \max\{|u|^{p-2}, |v|^{2-q}\}$.

To verify both of the above estimates, we will make use of Proposition 5.23, whose assumptions are fulfilled by such functions u and v thanks to Proposition 5.10 and (5.65).

Proof of (5.70):

We start estimating I_1. Recall definition (5.54). Set $\omega = (u,v)$ and $\nabla\omega = (\nabla u, \nabla v)$ and denote

$$G_p(u,v) = u\max\{|u|^{p/2-1}, |v|^{1-q/2}\}.$$

By Theorem 5.26 applied with

$$\begin{aligned}\mu &= \alpha(V,\mathscr{V}),\\ \gamma &= \alpha(W,\mathscr{W}),\end{aligned}$$

we obtain

$$\begin{aligned} I_1 \geqslant \liminf_{\nu\searrow 0} \int_\Omega \widetilde{C}\Big(&(\tau * \varphi_\nu)(u,v)|\nabla u|^2 + (\tau^{-1} * \varphi_\nu)(u,v)|\nabla v|^2\Big)\\ &+ \mu\frac{pq}{4}\left(H^{I_d}_{F_p}[\cdot;\nabla u] * \varphi_\nu\right)(u)\\ &+ \gamma[q+(2-q)\delta]\frac{p}{4}\left(H^{I_d}_{F_q}[\cdot;\nabla v] * \varphi_\nu\right)(v)\\ &+ 2\delta\mu(h_p[\,\cdot\,;\nabla\omega] * \varphi_\nu)(\omega). \end{aligned} \tag{5.72}$$

Proposition 5.23 and (5.55) shows that for a.e. $x \in \Omega \cap \{|u|^p = |v|^q\}$,

$$b_p[\omega;\nabla\omega]\mathbb{1}_{\{v\neq 0\}} = g_p[u;\nabla u] = h_p[\omega;\nabla\omega] = |\nabla[G_p(u,v)]|^2.$$

Therefore, Proposition 5.29 gives that for almost all $x \in \Omega\setminus\{u=0,v=0\}$,

$$\lim_{\nu\to 0}(h_p[\,\cdot\,;\nabla\omega] * \varphi_\nu])(\omega) = |\nabla[G_p(u,v)]|^2. \tag{5.73}$$

Hence, by combining (5.72) and (5.73) with Fatou's Lemma and the facts that $F_p \in C^2(\mathbb{C})$, $F_q \in C^2(\mathbb{C}\setminus\{0\})$ and $\nabla[G_p(u,v)] = 0$ almost everywhere in $\{u=0,v=0\}$, we get

$$I_1 \geqslant J(\mu,\gamma) + \widetilde{C}\int_{\Omega\setminus\{u=0,v=0\}} \tau(u,v)|\nabla u|^2 + \tau^{-1}(u,v)|\nabla v|^2, \tag{5.74}$$

where

$$J(\mu,\gamma) := \int_\Omega \mu\frac{pq}{4}H^{I_d}_{F_p}[u;\nabla u] + \gamma[q+(2-q)\delta]\frac{p}{4}H^{I_d}_{F_q}[v;\nabla v]\mathbb{1}_{\{v\neq 0\}} + 2\delta\mu|\nabla[G_p(u,v)]|^2.$$

Let estimate now I_2. By Theorem 3.22(ii),

$$I_2 \gtrsim \int_\Omega \tau(u,v)V_+|u|^2 + \tau^{-1}(u,v)W_+|v|^2. \tag{5.75}$$

Finally we estimate I_3. Lemma 5.19 gives

$$I_3 \leqslant \int_\Omega pV_-|u|^p + [q+(2-q)\delta]W_-|v|^q + 2\delta V_-\,|G_p(u,v)|^2\,. \tag{5.76}$$

Recall definition (5.31). Let $\kappa \in (\sigma, 1)$. By combining the fact that $V \in \mathcal{P}(\Omega, \mathscr{V})$ and $W \in \mathcal{P}(\Omega, \mathscr{W})$ with Proposition 5.10, Proposition 5.23 and Lemma 5.19 applied with $\mu = \sigma/\kappa \in (\sigma, 1)$, we get

$$\begin{aligned} I_3 &\leqslant J(\mu,\gamma) + \sigma \int_\Omega \left[pV_+|u|^p + [q+(2-q)\delta]W_+|v|^q + 2\delta V_+|G_p(u,v)|^2 \right] \\ &\leqslant J(\mu,\gamma) + 2\int_\Omega \sigma\, V_+(\partial_\zeta \mathcal{Q})(u,v)\cdot u + \kappa\, W_+(\partial_\eta \mathcal{Q})(u,v)\cdot v \\ &\leqslant J(\mu,\gamma) + \kappa I_2. \end{aligned} \tag{5.77}$$

The estimates (5.74), (5.77) yield, together with (5.69),

$$\mathcal{O}(\mathcal{Q})(u,v) \geqslant (1-\kappa)I_2 + \widetilde{C}\int_{\Omega\setminus\{u=0,v=0\}} \tau(u,v)|\nabla u|^2 + \tau^{-1}(u,v)|\nabla v|^2.$$

Therefore, (5.75) and the fact that $\kappa \in (0,1)$ give (5.70).

Proof of (5.71):

By Proposition 5.5(v) there exist $\mu > \alpha(V) = \alpha(V, \mathscr{V})$ and $\gamma > \alpha(W) = \alpha(W, \mathscr{W})$ such that $A - \mu(pq/4)I_d, B - \gamma(pq/4)I_d \in \mathcal{A}_p(\Omega)$. Therefore, by repeating the same argument to get (5.74), we obtain

$$I_1 \geqslant J(\mu,\gamma).$$

Moreover, since $\sigma < 1$, we may choose such μ and γ so that

$$\begin{aligned} 1 &> \frac{\alpha(V)}{\mu} > \sigma, \\ 1 &> \frac{\alpha(W)}{\gamma} > \kappa\cdot\frac{\alpha(W)}{\gamma} > \sigma, \end{aligned} \tag{5.78}$$

for some $\kappa \in (0,1)$. Lemma 5.19 applied with such κ gives

$$I_2 \geqslant \int_\Omega pV_+|u|^p + \kappa[q+(2-q)\delta]W_+|v|^q + 2\delta V_+|G_p(u,v)|^2.$$

Therefore,

$$\begin{aligned} I_1 + I_2 \geqslant\, & J(\mu,\gamma) + \int_\Omega pV_+|u|^p + \kappa[q+(2-q)\delta]W_+|v|^q + 2\delta V_+|G_p(u,v)|^2 \\ =\, & p\frac{\mu}{\alpha(V)}\left[\alpha(V)\int_\Omega \frac{q}{4} H^{I_d}_{F_p}[u;\nabla u] + \frac{\alpha(V)}{\mu}\int_\Omega V_+|u|^p\right] \\ & + [q+(2-q)\delta]\frac{\gamma}{\alpha(W)}\left[\alpha(W)\int_\Omega \frac{p}{4} H^{I_d}_{F_q}[v;\nabla v]\mathbb{1}_{\{v\neq 0\}} + \kappa\frac{\alpha(W)}{\gamma}\int_\Omega W_+|v|^q\right] \\ & + 2\delta\frac{\mu}{\alpha(V)}\left[\alpha(V)\int_\Omega |\nabla[G_p(u,v)]|^2 + \frac{\alpha(V)}{\mu}\int_\Omega V_+|G_p(u,v)|^2\right]. \end{aligned}$$

By combining again the fact that $V, W \in \mathcal{P}(\Omega)$ with Proposition 5.10, Proposition 5.23 and (5.78), we get

$$I_1 + I_2 \geqslant \int_\Omega p\frac{\mu}{\alpha(V)}V_-|u|^p + [q+(2-q)\delta]\frac{\gamma}{\alpha(W)}W_-|v|^q + 2\delta\frac{\mu}{\alpha(V)}V_-|G_p(u,v)|^2,$$

which, together with (5.76) and the facts that $\mu > \alpha(V)$ and $\gamma > \alpha(W)$, gives

$$\begin{aligned} I_1 + I_2 - I_3 &\gtrsim I_3 \\ &\gtrsim \int_\Omega \tau(u,v)\cdot V_-|u|^2 + \tau^{-1}(u,v)\cdot W_-|v|^2, \end{aligned}$$

where in the last inequality we used Theorem 3.22(ii). □

5.7.2 Proof of (5.63): alternative proof of [21, Theorem 1.4] for unbounded nonnegative potentials

In this section we will prove (5.63) and, consequently, the bilinear embedding by means of Proposition 5.33.

Recall notation (5.68). Since (5.63) only involves the positive part of the potentials V and W, we will assume that $V_- = W_- = 0$ and prove

$$\begin{aligned}\mathcal{O}(\mathcal{Q})(u,v) \geqslant \liminf_{\nu\to 0} \int_\Omega H^{(A,B)}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)] \\ + 2\int_\Omega V(\partial_\zeta \mathcal{Q})(u,v)\cdot u + W(\partial_\eta \mathcal{Q})(u,v)\cdot v,\end{aligned} \tag{5.79}$$

for all $u \in \mathrm{D}(\mathfrak{a}_{A,V,\mathscr{V}})$, $v \in \mathrm{D}(\mathfrak{a}_{B,W,\mathscr{W}})$ such that $u, v, \mathscr{L}^{A,V}u, \mathscr{L}^{B,W}v \in (L^p \cap L^q)(\Omega)$.

Such estimate was already proved in [21, Sections 3.3] for bounded potentials. In the right-hand side of (5.79), the matrices are separated from the potentials. This suggests that we must somehow separate the matrices from the potentials in $\mathcal{O}(u,v)$ as well. Given the generality of our setting, where the potentials are unbounded, such separation cannot be achieved directly within the operators themselves. Instead, it is more convenient to work with sesquilinear forms, where matrices and potentials naturally appear as separate terms. To transition to this framework, an integration by parts argument, as formulated in (5.9), is required. To this end, we will employ the sequence $(\mathcal{R}_{n,\nu})_{n\in\mathbb{N},\nu\in(0,1)}$ of Section 3.5.3, following the approach of Carbonaro and Dragičević in the case where $V = W = 0$ [19].

Analogously to (3.45) and (3.46), respectively, we would like to show

$$\begin{aligned}\mathcal{O}(\mathcal{Q})(u,v) &= \lim_{\nu\to 0}\lim_{n\to\infty} \mathcal{O}(\mathcal{R}_{n,\nu})(u,v), \\ \mathcal{O}(\mathcal{R}_{n,\nu})(u,v) &= \int_\Omega H^{(A,B)}_{\mathcal{R}_{n,\nu}}[(u,v);(\nabla u,\nabla v)] \\ &\quad + 2\int_\Omega V \operatorname{Re}\left(u\cdot(\partial_\zeta \mathcal{R}_{n,\nu})(u,v)\right) + W\operatorname{Re}\left(v\cdot(\partial_\eta \mathcal{R}_{n,\nu})(u,v)\right).\end{aligned} \tag{5.80}$$

In this way, thanks to Theorem 3.26, Corollary 3.25, the fact that $\mathcal{Q} \in C^1(\mathbb{C}^2)$, the nonnegativity of V and W and Fatou's Lemma (twice), if we had

$$\begin{aligned}\operatorname{Re}\left(\zeta\cdot\partial_\zeta \mathcal{R}_{n,\nu}(\zeta,\eta)\right) &\geqslant 0, \\ \operatorname{Re}\left(\eta\cdot\partial_\eta \mathcal{R}_{n,\nu}(\zeta,\eta)\right) &\geqslant 0, \\ \operatorname{Re}\left(\zeta\cdot\partial_\zeta (\mathcal{Q}*\varphi_\nu)(\zeta,\eta)\right) &\geqslant 0, \\ \operatorname{Re}\left(\eta\cdot\partial_\eta (\mathcal{Q}*\varphi_\nu)(\zeta,\eta)\right) &\geqslant 0,\end{aligned} \tag{5.81}$$

for all $n \in \mathbb{N}$, $\nu \in (0,1)$ and $\zeta, \eta \in \mathbb{C}$, we would deduce (5.79) from (5.80).

So, here is the plan of this section. First we will prove (5.81) and finally we will demonstrate (5.79).

Since $V_- = W_- = 0 \in L^\infty(\Omega)$, in view of Proposition 5.33 this method provides an alternative proof of [21, Theorem 1.4] for unbounded nonnegative potentials.

Proof of (5.81)

Recall the definition (3.40) of $\mathcal{R}_{n,\nu}$.

Outside the annulus $\{3n \leqslant |\omega| \leqslant 4n\}$, the (A,B)-convexity of $\mathcal{R}_{n,\nu}$ follows from the (A,B)-convexity of $\mathcal{Q}*\varphi_\nu$ and $\mathcal{P}_n*\varphi_\nu$. Within the annulus, a lower bound of the generalized Hessian of $\mathcal{P}_n*\varphi_\nu$ suggests how large the constant C_1 needs to be chosen in order to compensate for the

lack of convexity of $\mathcal{Q}_{n,\nu}$ in this region; see [19, Proposition 15(ii) & Theorem 16]. We will do the same to prove the first two inequalities of (5.81): we will initially prove the corresponding inequalities for $\mathcal{Q} * \varphi_\nu$ and $\mathcal{P}_n * \varphi_\nu$, then establish in the annulus a lower bound for the terms involving $\mathcal{P}_n * \varphi_\nu$, and finally choose C_1 large enough.

We will start with two lemmas. For all $a \in \mathbb{R}^2$ define the function $P_a : \mathbb{R}^2 \to \mathbb{R}$ as

$$P_a(x) := \langle a, a - x \rangle , \qquad x \in \mathbb{R}^2.$$

When $a \neq (0,0)$ we can define the reflection $R_a : \mathbb{R}^2 \to \mathbb{R}^2$ with respect the line $\{x \in \mathbb{R}^2 : P_a(x) = 0\}$ as

$$R_a(x) := x + 2P_a(x)\frac{a}{|a|^2}, \qquad x \in \mathbb{R}^2.$$

Lemma 5.34. *For all $a \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $x \in \mathbb{R}^2$ we have*

(i) $P_a(R_a(x)) = -P_a(x)$,

(ii) $|a - R_a(x)| = |a - x|$,

(iii) $|R_a(x)|^2 = |x|^2 + 4P_a(x)$.

Proof. Item (i) follows by a trivial computation.

To prove item (ii) we observe that

$$\begin{aligned} |a - R_a(x)|^2 &= |a - x|^2 + \frac{4}{|a|^2}P_a^2(x) - \frac{4}{|a|^2}P_a(x)\,\langle a - x, a\rangle \\ &= |a - x|^2. \end{aligned}$$

Finally, we have

$$\begin{aligned} |R_a(x)|^2 &= |x|^2 + \frac{4}{|a|^2}P_a^2(x) + \frac{4}{|a|^2}P_a(x)\,\langle x, a\rangle \\ &= |x|^2 + \frac{4}{|a|^2}P_a^2(x) + \frac{4}{|a|^2}P_a(x)\,\langle x - a, a\rangle + 4P_a(x) \\ &= |x|^2 + 4P_a(x). \end{aligned}$$

□

For our purposes, we further assume that the function φ, appearing in the definition of the approximate identity $(\varphi_\nu)_{\nu>0}$, is radially decreasing. This assumption will be used in the next lemma.

Lemma 5.35. *Let $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^4)$ and φ as above. Suppose that*

$$F(x,y) = G(|x|,|y|) \tag{5.82}$$

for some nonnegative function G on $[0,\infty) \times [0,\infty)$ and all $x, y \in \mathbb{R}^2$. Then

$$\begin{aligned} \int_{\mathbb{R}^4} P_a(x)F(a - x, b - y)\varphi_\nu(x,y)\ \mathrm{d}(x,y) \geqslant 0, \\ \int_{\mathbb{R}^4} P_b(y)F(a - x, b - y)\varphi_\nu(x,y)\ \mathrm{d}(x,y) \geqslant 0, \end{aligned}$$

for all $(a,b) \in \mathbb{R}^2 \times \mathbb{R}^2$ and $\nu \in (0,1)$.

Proof. Fix $(a,b) \in \mathbb{R}^2 \times \mathbb{R}^2$ and $\nu \in (0,1)$. It suffices to prove the first inequality. If $a = (0,0)$ the assertion clearly holds. So suppose that $a \neq (0,0)$. Since the support of the integrand is contained in $B_{\mathbb{R}^4}(0,\nu)$, we have

$$\begin{aligned}&\int_{\mathbb{R}^4} P_a(x)F(a-x,b-y)\varphi_\nu(x,y)\ \mathrm{d}(x,y)\\ &\quad= \int_{B_{\mathbb{R}^2}(0,\nu)} P_a(x) \int_{\{y\,:\,|y|^2<\nu^2-|x|^2\}} F(a-x,b-y)\varphi_\nu(x,y)\ \mathrm{d}y\ \mathrm{d}x.\end{aligned}$$

Set $\Theta_a := \{x \in B_{\mathbb{R}^2}(0,\nu)\,:\,P_a(x) < 0\}$. If $\Theta_a = \emptyset$, we conclude. Otherwise, from Lemma 5.34(iii) we have $R_a(\Theta_a) \subseteq B_{\mathbb{R}^2}(0,\nu)$ and hence

$$B_{\mathbb{R}^2}(0,\nu) = \overline{\Theta_a} \sqcup R_a(\Theta_a) \sqcup \left(B_{\mathbb{R}^2}(0,\nu) \setminus \left(\overline{\Theta_a} \cup R_a(\Theta_a)\right)\right).$$

Therefore, by the nonnegativity of F and φ on $\mathbb{R}^4$ and of P_a on $B_{\mathbb{R}^2}(0,\nu) \setminus (\overline{\Theta_a} \cup R_a(\Theta_a)$,

$$\begin{aligned}&\int_{\mathbb{R}^4} P_a(x)F(a-x,b-y)\varphi_\nu(x,y)\ \mathrm{d}(x,y)\\ &\qquad\geqslant \int_{\Theta_a} P_a(x) \int_{\{|y|^2<\nu^2-|x|^2\}} F(a-x,b-y)\varphi_\nu(x,y)\ \mathrm{d}y\ \mathrm{d}x\\ &\qquad\quad+ \int_{R_a(\Theta_a)} P_a(x) \int_{\{|y|^2<\nu^2-|x|^2\}} F(a-x,b-y)\varphi_\nu(x,y)\ \mathrm{d}y\ \mathrm{d}x.\end{aligned} \tag{5.83}$$

By combining Lemma 5.34(i) and a change of variable in (5.83) by means of R_a, we get

$$\begin{aligned}&\int_{\mathbb{R}^4} P_a(x)F(a-x,b-y)\varphi_\nu(x,y)\ \mathrm{d}(x,y)\\ &\qquad\geqslant \int_{\Theta_a} P_a(x) \int_{\{|y|^2<\nu^2-|x|^2\}} F(a-x,b-y)\varphi_\nu(x,y)\ \mathrm{d}y\ \mathrm{d}x\\ &\qquad\quad- \int_{\Theta_a} P_a(x) \int_{\{|y|^2<\nu^2-|R_a(x)|^2\}} F\Big(a-R_a(x),b-y\Big)\varphi_\nu\Big(R_a(x),y\Big)\ \mathrm{d}y\ \mathrm{d}x.\end{aligned} \tag{5.84}$$

Lemma 5.34(iii) gives $|R_a(x)| \leqslant |x|$ for any $x \in \Theta_a$. Thus, by also using that φ is radially decreasing, we obtain

$$\begin{gathered}\{y\,:\,|y|^2 < \nu^2 - |R_a(x)|^2\} \supseteq \{y\,:\,|y|^2 < \nu^2 - |x|^2\},\\ \varphi_\nu(R_a(x),y) \geqslant \varphi_\nu(x,y).\end{gathered}$$

Therefore, by merging these with (5.82), Lemma 5.34(ii) and the nonnegativity of F and φ, for every $x \in \Theta_a$ we have

$$\begin{aligned}&\int_{\{|y|^2<\nu^2-|R_a(x)|^2\}} F(a-R_a(x),b-y)\varphi_\nu(R_a(x),y)\ \mathrm{d}y\\ &\qquad\geqslant \int_{\{|y|^2<\nu^2-|x|^2\}} F(a-x,b-y)\varphi_\nu(x,y)\ \mathrm{d}y.\end{aligned} \tag{5.85}$$

We conclude by combining (5.84) and (5.85) with the fact that P_ζ is negative on Θ_ζ. □

Proposition 5.36. *Let $\nu \in (0,1)$. Then*

(i) *For all $\omega = (\zeta,\eta) \in \mathbb{C} \times \mathbb{C}$ we have*

$$\begin{aligned}\mathrm{Re}\left(\zeta \cdot \partial_\zeta(\mathcal{Q} * \varphi_\nu)(\omega)\right) &\geqslant 0,\\ \mathrm{Re}\left(\eta \cdot \partial_\eta(\mathcal{Q} * \varphi_\nu)(\omega)\right) &\geqslant 0.\end{aligned}$$

(ii) *For all* $\omega = (\zeta, \eta) \in \mathbb{C} \times \mathbb{C}$ *and* $n \in \mathbb{N}_+$ *we have*

$$\begin{aligned} \operatorname{Re}\left(\zeta \cdot \partial_\zeta (\mathcal{P}_n * \varphi_\nu)(\omega)\right) &\geqslant 0, \\ \operatorname{Re}\left(\eta \cdot \partial_\eta (\mathcal{P}_n * \varphi_\nu)(\omega)\right) &\geqslant 0. \end{aligned}$$

Moreover, for all $n \in \mathbb{N}_+$ *and all* $|\omega| \geqslant 2n$,

$$2 \operatorname{Re}\left(\sigma \cdot (\partial_\sigma \mathcal{P}_n * \varphi_\nu)(\omega)\right) \geqslant (p+\varepsilon) n^{p-2} |\sigma|^2, \quad \sigma = \zeta, \eta.$$

Proof. An easy computation shows that

$$\begin{aligned} \partial_\zeta \mathcal{Q}(\omega) &= \frac{\overline{\zeta}}{2} \left(p|\zeta|^{p-2} + 2\delta \begin{cases} |\eta|^{2-q}, & \text{if } |\zeta|^p \leqslant |\eta|^q, \\ |\zeta|^{p-2}, & \text{if } |\zeta|^p \geqslant |\eta|^q \end{cases} \right), \\ \partial_\eta \mathcal{Q}(\omega) &= \frac{\overline{\eta}}{2} \left(q|\eta|^{q-2} + (2-q)\delta \begin{cases} |\zeta|^2 |\eta|^{-q}, & \text{if } |\zeta|^p \leqslant |\eta|^q, \\ |\eta|^{q-2}, & \text{if } |\zeta|^p \geqslant |\eta|^q \end{cases} \right), \\ \partial_\sigma \mathcal{P}_n(\omega) &= \frac{\overline{\sigma}}{2} \Big(g_n(|\omega|) + K_{p+\varepsilon}\, g_n(|\sigma|) \Big), \quad \sigma = \zeta, \eta, \end{aligned}$$

where

$$g_n(t) = (p+\varepsilon) \begin{cases} n^{-\varepsilon} t^{p+\varepsilon-2}, & 0 \leqslant t \leqslant n, \\ n^{p-2}, & t \geqslant n. \end{cases} \tag{5.86}$$

Therefore, we conclude the proof of (i) and the first part of (ii) by recalling the convention (3.32) and by combining Lemma 5.35 with the identity

$$\operatorname{Re}\left(\zeta \cdot \overline{\zeta - \mathcal{V}_1^{-1}(x)}\right) = P_{\mathcal{V}_1(\zeta)}(x), \tag{5.87}$$

which holds for all $\zeta \in \mathbb{C}$, $x \in \mathbb{R}^2$.

For the second part of (ii), we restrict ourselves to the case $\sigma = \zeta$, as the remaining case follows by an entirely analogous argument. We have

$$\begin{aligned} &2 \operatorname{Re}\left(\zeta \cdot \partial_\zeta (\mathcal{P}_n * \varphi_\nu)(\omega)\right) \\ &\quad = \int_{B_{\mathbb{R}^4}(0,\nu)} \operatorname{Re}\left(\zeta \cdot \overline{\zeta - \mathcal{V}_1^{-1}(x)}\right) \times \\ &\qquad\qquad \times \left[g_n\left(\left|\omega - \mathcal{W}_{2,1}^{-1}(x,y)\right|\right) + K_{p+\varepsilon}\, g_n\left(\left|\zeta - \mathcal{V}_1^{-1}(x)\right|\right) \right] \varphi_\nu(x,y)\, \mathrm{d}(x,y). \end{aligned}$$

If we assume that $|\omega| > 2n$, then $|\omega - \mathcal{W}_{2,1}^{-1}(x,y)| > 2n - \nu > n$. Therefore, by (5.86) we get

$$\begin{aligned} &2 \operatorname{Re}\left(\zeta \cdot \partial_\zeta (\mathcal{P}_n * \varphi_\nu)(\omega)\right) \\ &\quad = (p+\varepsilon) n^{p-2} \int_{\mathbb{R}^4} \operatorname{Re}\left(\zeta \cdot \overline{\zeta - \mathcal{V}_1^{-1}(x)}\right) \varphi_\nu(x,y)\, \mathrm{d}(x,y) \\ &\qquad + K_{p+\varepsilon} \int_{\mathbb{R}^4} \operatorname{Re}\left(\zeta \cdot \overline{\zeta - \mathcal{V}_1^{-1}(x)}\right) g_n\left(\left|\zeta - \mathcal{V}_1^{-1}(x)\right|\right) \varphi_\nu(x,y)\, \mathrm{d}(x,y) \\ &\quad = (p+\varepsilon) n^{p-2} |\zeta|^2 - (p+\varepsilon) n^{p-2} \int_{\mathbb{R}^4} \operatorname{Re}\left(\zeta \cdot \overline{\mathcal{V}_1^{-1}(x)}\right) \varphi_\nu(x,y)\, \mathrm{d}(x,y) \\ &\qquad + K_{p+\varepsilon} \int_{\mathbb{R}^4} \operatorname{Re}\left(\zeta \cdot \overline{\zeta - \mathcal{V}_1^{-1}(x)}\right) g_n\left(\left|\zeta - \mathcal{V}_1^{-1}(x)\right|\right) \varphi_\nu(x,y)\, \mathrm{d}(x,y). \end{aligned}$$

The first integral in the right-hand side of the last equality is zero since the integrand is odd for every $\zeta \in \mathbb{C}$. Finally, Lemma 5.35 and (5.87) imply that the second integral is nonnegative. Thus, we conclude. □

Corollary 5.37. *Let $p > 2$. There exists $C_1 > 0$ in the definition (3.40) of $\mathcal{R}_{n,\nu}$ such that*

$$\begin{aligned} \operatorname{Re}\left(\zeta \cdot (\partial_\zeta \mathcal{R}_{n,\nu})(\omega)\right) &\geqslant 0, \\ \operatorname{Re}\left(\eta \cdot (\partial_\eta \mathcal{R}_{n,\nu})(\omega)\right) &\geqslant 0, \end{aligned}$$

for all $n \in \mathbb{N}$, $\nu \in (0,1)$ and $\omega \in \mathbb{C}^2$.

Proof. The nonnegativity of both terms in the region $\{|\omega| < 3n\} \cup \{|\omega| > 4n\}$ follows, for any $C_1 > 0$, from Proposition 5.36(i) and the first part of (ii). Prove now the nonnegativity in the annulus $\{3n \leqslant |\omega| \leqslant 4n\}$. Since ψ is even in each variable, we have

$$\partial_{\zeta_j} \psi_n(0,\eta) = 0, \quad \partial_{\eta_j} \psi_n(\zeta,0) = 0, \tag{5.88}$$

for all $\zeta = \zeta_1 + i\zeta_2$, $\eta = \eta_1 + i\eta_2 \in \mathbb{C}$ and $j \in \{1,2\}$. Therefore, the mean value theorem implies that there exists $C = C(\psi) > 0$, independent of n, such that

$$\begin{aligned} |\partial_\zeta \psi_n(\zeta,\eta)| &\leqslant \frac{C}{n^2}|\zeta|, \\ |\partial_\eta \psi_n(\zeta,\eta)| &\leqslant \frac{C}{n^2}|\eta|, \end{aligned} \tag{5.89}$$

for all $n \in \mathbb{N}_+$ and all $\zeta, \eta \in \mathbb{C}$. Hence, by combining (5.89) with the first estimate of [19, Lemma 14] and by applying the product rule, we infer that there exists $C_0 = C_0(p,\psi) > 0$ such that

$$\begin{aligned} 2\operatorname{Re}\left(\zeta \cdot (\partial_\zeta \mathcal{Q}_{n,\nu})(\omega)\right) &\geqslant 2\psi_n(\omega) \operatorname{Re}\Big(\zeta \cdot (\partial_\zeta \mathcal{Q} * \varphi_\nu)(\omega)\Big) - C_0 n^{p-2}|\zeta|^2, \\ 2\operatorname{Re}\left(\eta \cdot (\partial_\eta \mathcal{Q}_{n,\nu})(\omega)\right) &\geqslant 2\psi_n(\omega) \operatorname{Re}\Big(\eta \cdot (\partial_\eta \mathcal{Q} * \varphi_\nu)(\omega)\Big) - C_0 n^{p-2}|\eta|^2, \end{aligned}$$

for every $\omega \in \mathbb{C}^2$ with $|\omega| \leqslant 5n$ and all $n \in \mathbb{N}_+$ and $\nu \in (0,1)$. Moreover, by the nonnegativity of ψ and by Proposition 5.36(i), we get

$$\begin{aligned} 2\operatorname{Re}\left(\zeta \cdot (\partial_\zeta \mathcal{Q}_{n,\nu})(\omega)\right) &\geqslant -C_0 n^{p-2}|\zeta|^2, \\ 2\operatorname{Re}\left(\eta \cdot (\partial_\eta \mathcal{Q}_{n,\nu})(\omega)\right) &\geqslant -C_0 n^{p-2}|\eta|^2, \end{aligned} \tag{5.90}$$

for every $\omega \in \mathbb{C}^2$ with $|\omega| \leqslant 5n$ and all $n \in \mathbb{N}_+$ and $\nu \in (0,1)$. In order to conclude, we choose C_1 large enough and combine (5.90) with the second part of Proposition 5.36(ii). □

Proof of (5.79)

By using Theorem 3.26 (ii) and (iv), Lemma 3.24(ii), the fact that $\mathcal{Q} \in C^1(\mathbb{R}^4)$ and Lebesgue's dominated convergence theorem twice, we deduce that

$$\mathcal{O}(\mathcal{Q})(u,v) = \lim_{\nu \to 0} \lim_{n \to +\infty} \mathcal{O}(\mathcal{R}_{n,\nu})(u,v). \tag{5.91}$$

Combining Theorem 3.26 (i) and (v) with the mean value theorem, we get

$$\begin{aligned} |(\partial_\zeta \mathcal{R}_{n,\nu})(\zeta,\eta)| &\leqslant C(n,\nu)|\zeta|, \\ |(\partial_\eta \mathcal{R}_{n,\nu})(\zeta,\eta)| &\leqslant C(n,\nu)|\eta|, \end{aligned}$$

for any $\zeta, \eta \in \mathbb{C}$. These estimates, together with Lemma 3.27, imply that

$$\begin{aligned} (\partial_\zeta \mathcal{R}_{n,\nu})(u,v) &\in \mathrm{D}(\mathfrak{a}_{A,V,\mathscr{V}}), \\ (\partial_\eta \mathcal{R}_{n,\nu})(u,v) &\in \mathrm{D}(\mathfrak{a}_{B,W,\mathscr{W}}), \end{aligned}$$

for all $u \in \mathrm{D}(\mathfrak{a}_{A,V,\mathscr{V}})$, $v \in \mathrm{D}(\mathfrak{a}_{B,W,\mathscr{W}})$. Hence we can integrate by parts the integral on the right-hand side of (5.91) and, by means of the chain rule for the composition of smooth functions with vector-valued Sobolev functions, deduce that

$$\begin{aligned}\mathcal{O}(\mathcal{R}_{n,\nu})(u,v) = \int_\Omega H^{(A,B)}_{\mathcal{R}_{n,\nu}}[(u,v);(\nabla u,\nabla v)] \\ + 2\int_\Omega V \operatorname{Re}\left(u\cdot(\partial_\zeta \mathcal{R}_{n,\nu})(u,v)\right) + W\operatorname{Re}\left(v\cdot(\partial_\eta \mathcal{R}_{n,\nu})(u,v)\right).\end{aligned} \tag{5.92}$$

By Theorem 3.26 and Corollary 5.37, the integral on the right-hand side of (5.92) is nonnegative for all $n \in \mathbb{N}_+$. Hence, by Fatou's lemma and Theorem 3.26 (ii),

$$\begin{aligned}&\lim_{n\to+\infty} \mathcal{O}(\mathcal{R}_{n,\nu})(u,v) \\ &\quad\geqslant \int_\Omega H^{(A,B)}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)] \\ &\qquad + 2\int_\Omega V \operatorname{Re}\left(u\cdot(\partial_\zeta(\mathcal{Q}*\varphi_\nu))(u,v)\right) + W\operatorname{Re}\left(v\cdot(\partial_\eta(\mathcal{Q}*\varphi_\nu))(u,v)\right).\end{aligned} \tag{5.93}$$

By Proposition 5.36(i), the integrand of the second integral on the right-hand side of (5.93) is nonnegative for all $\nu \in (0,1)$. Hence, by Fatou's lemma and the fact that $\mathcal{Q} \in C^1(\mathbb{R}^4)$,

$$\begin{aligned}\liminf_{\nu\to 0} \int_\Omega V \operatorname{Re}\left(u\cdot(\partial_\zeta(\mathcal{Q}*\varphi_\nu))(u,v)\right) + W\operatorname{Re}\left(v\cdot(\partial_\eta(\mathcal{Q}*\varphi_\nu))(u,v)\right) \\ \geqslant \int_\Omega V(\partial_\zeta\mathcal{Q})(u,v)\cdot u + W(\partial_\eta\mathcal{Q})(u,v)\cdot v.\end{aligned} \tag{5.94}$$

Therefore, combining (5.91), (5.93) and (5.94), we get (5.79).

5.8 The general case: unbounded potentials

As in Section 4.8, to handle the general case with unbounded potentials, we follow the argument of Carbonaro and Dragičević in [21, Section 3.4], where they proved [21, Theorem 1.4]. Similarly, Theorem 5.30 will follow from the special case of potentials with bounded negative part, already established in Section 5.7, once we prove the following approximation result, which is a special case of Theorem 2.43. The only remaining task is to verify that the hypotheses of that theorem are satisfied in the present context.

Let $\mathscr{V}$ be a closed subspace of $W^{1,2}(\Omega)$ containing $W^{1,2}_0(\Omega)$ and $(A,V) \in \mathcal{AP}_{\alpha,\sigma}(\Omega,\mathscr{V})$ for some $\alpha > 0$ and $\sigma \in [0,1)$. For each $n \in \mathbb{N}$ define

$$V_n := V_+ - V_- \wedge n,$$

We also set $V_\infty = V$.

Clearly, for each $n \in \mathbb{N}\cup\{\infty\}$ one has

$$\int_\Omega (V_n)_-|u|^2 \leqslant \alpha\int_\Omega |\nabla u|^2 + \sigma\int_\Omega V_+|u|^2, \qquad \forall u \in \mathscr{V}.$$

Hence, $(A,V_n) \in \mathcal{AP}_{\alpha,\sigma}(\Omega,\mathscr{V})$ for all $n \in \mathbb{N}\cup\{\infty\}$. As a consequence, by Corollary 5.3, each operator $\mathscr{L}_n := \mathscr{L}^{A,V_n}$ is sectorial of angle ω, and $-\mathscr{L}_n$ generates an analytic and contractive semigroup $(T^{A,V_n}_z)_{z\in\mathbf{S}_\vartheta}$ on $L^2(\Omega)$, with $\vartheta := \pi/2 - \omega$. Here, ω denotes the angle introduced in page 107.

Theorem 5.38. *For all $f \in L^2(\Omega)$ and all $z \in \mathbf{S}_\vartheta$ we have*

$$\begin{aligned} \nabla T_z^{A,V_n} f &\to \nabla T_z^{A,V} f \qquad \text{in} \quad L^2(\Omega, \mathbb{C}^d), \\ |V_n|^{1/2} T_z^{A,V_n} f &\to |V|^{1/2} T_z^{A,V} f \qquad \text{in} \quad L^2(\Omega), \end{aligned}$$

as $n \to \infty$.

Proof. For every $n \in \mathbb{N}_+ \cup \{\infty\}$, denote by $\mathfrak{a}_n = \mathfrak{a}_{A,V_n}$ the form associated with (A, V_n). Consider the sesquilinear form $\mathfrak{b}$ on $L^2(\Omega)$ defined by

$$\mathfrak{b}(u, v) = \int_\Omega \langle A\nabla u, \nabla v \rangle, \quad \mathrm{D}(\mathfrak{b}) = \mathscr{V}.$$

Clearly, for all $n \in \mathbb{N}_+ \cup \{\infty\}$ we have

$$\mathfrak{a}_n(u, v) = \mathfrak{b}(u, v) + \int_\Omega V_n u \overline{v}.$$

The boundedness of A implies $A\nabla u, A^*\nabla u \in L^2(\Omega; \mathbb{C}^d)$ for all $u \in L^2(\Omega)$. Therefore, $\mathfrak{b}$ satisfies condition (B1) of Section 2.4.3.

Moreover, V and $\{V_n\}_{n\in\mathbb{N}_+}$ satisfy (V1) and (V2), by construction.

Finally, since $(A, V_n) \in \mathcal{AP}_{\alpha,\sigma}(\Omega, \mathscr{V})$ for all $n \in \mathbb{N}_+ \cup \{\infty\}$, Lemma 5.2 ensures that the sequence $\{\mathfrak{a}_n\}_{n\in\mathbb{N}\cup\{\infty\}}$ satisfies (A1) and (A2). Therefore, we fall into the assumption of Theorem 2.43 and we conclude by applying it with $\varrho = 1$. □

5.9 Application: maximal regularity and functional calculus

It was recalled in Section 4.9 that a divergence-form operator without potentials and with a p-elliptic coefficient matrix admits a bounded H^∞-calculus on $L^p(\Omega)$ and enjoys parabolic maximal regularity [19, Theorem 3]. There, the argument of [19] was adapted to include first-order perturbations and nonnegative potentials under the assumption $\mathscr{A} \in \mathcal{B}_p(\Omega)$.

Here, we apply the same approach to operators with coefficients in $\mathcal{AP}_p(\Omega, \mathscr{V})$, further extending [19, Theorem 3]. The reasoning again relies on the Dore-Venni theorem (Theorem 2.29) and on the sufficient condition for a bounded holomorphic functional calculus from Corollary 2.22 leading to the following result.

Theorem 5.39. *Suppose that $\mathscr{V}$ falls into any of the special cases (a)-(d) of Section 3.1.1. Assume that $p > 1$ and $(A, V) \in \mathcal{AP}_p(\Omega, \mathscr{V})$. Let $-\mathscr{L}_p$ be the generator of $(T_t)_{t>0}$ on $L^p(\Omega)$. Then $\mathscr{L}_p$ admits a bounded holomorphic functional calculus of angle $\vartheta < \pi/2$. As a consequence, $\mathscr{L}_p$ has parabolic maximal regularity.*

As a preliminary step, we will establish in Section 5.9.1 a bilinear estimate with complex time, analogous to [19, (42)] and (4.83); see (5.95).

5.9.1 Bilinear embedding with complex time

Let $\vartheta, \phi \in (-\pi/2, \pi/2)$ be such that $(e^{i\vartheta}A, (\cos\vartheta)V) \in \mathcal{A}_p(\Omega, \mathscr{V})$ and $(e^{i\phi}B, (\cos\phi)W) \in \mathcal{A}_p(\Omega, \mathscr{W})$. We will show that

$$\int_0^\infty \int_\Omega \sqrt{\left|\nabla T_{te^{i\vartheta}}^{A,V} f\right|^2 + |V| \left|T_{te^{i\vartheta}}^{A,V} f\right|^2} \sqrt{\left|\nabla T_{te^{i\phi}}^{B,W} g\right|^2 + |W| \left|T_{te^{i\phi}}^{B,W} g\right|^2} \, \mathrm{d}x \, \mathrm{d}t \leqslant C \|f\|_p \|g\|_q, \quad (5.95)$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

First, assume that V and W have bounded negative part. We repeat the same argument of Section 5.7, summarized as follows:

- Define $\gamma_{\vartheta,\phi} : [0,\infty) \to \mathbb{C}^2$ by

$$\gamma_{\vartheta,\phi}(t) := \left(T^{A,V}_{e^{i\vartheta}t} f, T^{B,W}_{e^{i\phi}t} g\right)$$

and $\mathcal{E}_{\vartheta,\phi} : [0,\infty) \to [0,\infty)$ by

$$\mathcal{E}_{\vartheta,\phi}(t) = \int_\Omega \mathcal{Q}\left(T^{A,V}_{e^{i\vartheta}t} f, T^{B,W}_{e^{i\phi}t} g\right) , \ t > 0.$$

We have

$$-\mathcal{E}'_\vartheta(t) = 2\,\mathrm{Re}\int_\Omega \left(e^{i\vartheta}\partial_\zeta \mathcal{Q}(\gamma_{\vartheta,\phi}(t))\mathscr{L}^{A,V} T^{A,V}_{e^{i\vartheta}t} f + e^{i\phi}\partial_\eta \mathcal{Q}(\gamma_{\vartheta,\phi}(t))\mathscr{L}^{B,W} T^{B,W}_{e^{i\phi}t} g\right).$$

As in Proposition 5.32, it suffices to show

$$\begin{aligned} 2\,\mathrm{Re}\int_\Omega \left(e^{i\vartheta}\partial_\zeta \mathcal{Q}(u,v)\mathscr{L}^{A,V} u + e^{i\phi}\partial_\eta \mathcal{Q}(u,v)\mathscr{L}^{B,W} v\right) \\ \geqslant \int_\Omega \sqrt{|\nabla u|^2 + |V||u|^2}\sqrt{|\nabla v|^2 + |W||v|^2} \end{aligned} \tag{5.96}$$

for all $u \in \mathrm{D}(\mathfrak{a}_{A,V_+,\mathscr{V}})$, $v \in \mathrm{D}(\mathfrak{a}_{B,W_+,\mathscr{W}})$ such that $u, v, \mathscr{L}^{A,V_+}u, \mathscr{L}^{B,W_+}v \in (L^p \cap L^q)(\Omega)$.

- Recall definition (3.40). As done in Section 5.7.2, we can prove that there exists $C_1 = C_1(\vartheta,\phi,p) > 0$ in the definition of $\mathcal{R}_{n,\nu}$ such that

$$\begin{aligned} \mathrm{Re}\left(e^{i\vartheta}\zeta \cdot (\partial_\zeta \mathcal{R}_{n,\nu})(\omega)\right) &\geqslant 0, \\ \mathrm{Re}\left(e^{i\phi}\eta \cdot (\partial_\eta \mathcal{R}_{n,\nu})(\omega)\right) &\geqslant 0, \\ \mathrm{Re}\left(e^{i\vartheta}\zeta \cdot \partial_\zeta(\mathcal{Q} * \varphi_\nu)(\omega)\right) &\geqslant 0, \\ \mathrm{Re}\left(e^{i\phi}\eta \cdot \partial_\eta(\mathcal{Q} * \varphi_\nu)(\omega)\right) &\geqslant 0, \end{aligned}$$

for all $n \in \mathbb{N}$, $\nu \in (0,1)$ and $\omega = (\zeta,\eta) \in \mathbb{C}^2$.

- Consequently (see Section 5.7.2), since $e^{i\vartheta}A, e^{i\phi}B \in \mathcal{A}_p(\Omega)$ we have

$$\begin{aligned} 2\,\mathrm{Re}\int_\Omega \left(e^{i\vartheta}\partial_\zeta \mathcal{Q}(u,v)\mathscr{L}^{A,V_+} u + e^{i\phi}\partial_\eta \mathcal{Q}(u,v)\mathscr{L}^{B,W_+} v\right) \\ \geqslant \liminf_{\nu\to 0}\int_\Omega H^{(e^{i\vartheta}A, e^{i\phi}B)}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u,\nabla v)] \\ + 2\int_\Omega (\cos\vartheta)V_+(\partial_\zeta\mathcal{Q})(u,v)\cdot u + (\cos\phi)W_+(\partial_\eta\mathcal{Q})(u,v)\cdot v, \end{aligned} \tag{5.97}$$

for all $u \in \mathrm{D}(\mathfrak{a}_{A,V_+,\mathscr{V}})$, $v \in \mathrm{D}(\mathfrak{a}_{B,W_+,\mathscr{W}})$ such that $u, v, \mathscr{L}^{A,V_+}u, \mathscr{L}^{B,W_+}v \in (L^p \cap L^q)(\Omega)$

- As described in the proof of Proposition 5.33, from (5.97) and the fact that $(e^{i\vartheta}A, (\cos\vartheta)V) \in \mathcal{A}_p(\Omega,\mathscr{V})$ and $(e^{i\phi}B, (\cos\phi)W) \in \mathcal{A}_p(\Omega,\mathscr{W})$ we deduce (5.96), which in turn implies (5.95).

Finally, the bilinear estimate (5.95) in the general case is obtained by combining Theorem 5.38 with the previously established estimate for potentials with bounded negative part.

5.9.2 Maximal regularity and functional calculus

For convenience, let us recall that the notions and results related to the H^∞-functional calculus and to maximal regularity have been introduced in Sections 2.3.4–2.3.6. The next proposition is modeled after [19, Proposition 20] and serves as an analogue of Proposition 4.38.

Proposition 5.40. *Suppose that $\mathscr{V}$ satisfies (3.5) and (3.6). Choose $p > 1$ and $(A, V) \in \mathcal{AP}_p(\Omega, \mathscr{V})$. Let $-\mathscr{L}_p$ be the generator of $(T_t)_{t>0}$ on $L^p(\Omega)$. If $\omega_{H^\infty}(\mathscr{L}_p) < \pi/2$, then $\mathscr{L}_p$ has parabolic maximal regularity.*

Proof. Since $(A, V) \in \mathcal{AP}_p(\Omega)$, Corollary 5.9 and Theorem 2.9 imply that $\mathscr{L}_p$ is sectorial on $L^p(\Omega)$. As $L^p(\Omega)$ is reflexive, we can construct the H^∞-calculus as in Section 2.3.4. Moreover, since

$$\vartheta((\mathscr{L}_p)_{||}) \leqslant \omega_{H^\infty}((\mathscr{L}_p)_{||}) = \omega_{H^\infty}(\mathscr{L}_p) < \tfrac{\pi}{2},$$

and $L^p(\Omega)$ is a UMD space, Theorem 2.29 ensures that $(\mathscr{L}_p)_{||}$ has parabolic maximal regularity. Finally, the conclusion follows from Remark 2.27. □

We now prove Theorem 5.39 by following the same strategy as in Theorem 4.36, which itself is modeled on [19, Theorem 3].

Proof of Theorem 5.39. Without loss of generality we suppose $p \geqslant 2$. In light of Proposition 5.40 it suffices to show that

$$(A, V) \in \mathcal{AP}_p(\Omega, \mathscr{V}) \implies \omega_{H^\infty}(\mathscr{L}_p) < \pi/2.$$

From Proposition 2.38 it follows that $\mathscr{L}_2^{A^*,V} = \left(\mathscr{L}^{A,V}\right)_2^*$, so $T_t^{A^*,V} = \left(T_t^{A,V}\right)^*$ for all $t > 0$. Set $T_t = T_t^{A,V}$ and $T_t^* = T_t^{A^*,V}$ for all $t > 0$.

By Proposition 5.5(iv),(vi), there exists $\vartheta \in (0, \pi/2)$ such that $(e^{\pm i\vartheta}A, (\cos\vartheta)V)$, $(e^{\mp i\vartheta}A^*, (\cos\vartheta)V) \in \mathcal{AP}_p(\Omega, \mathscr{V})$. Moreover, for every $r \in [q, p]$ both $(T_t)_{t>0}$ and $(T_t^*)_{t>0}$ are analytic (and contractive) in $L^r(\Omega)$ in the cone $\mathbf{S}_\vartheta$; see Corollary 5.9.

From (5.95) there exists $C > 0$ such that

$$\int_0^\infty \int_\Omega \sqrt{|\nabla T_{te^{\pm i\vartheta}} f|^2 + |V|\,|T_{te^{\pm i\vartheta}} f|^2}\, \sqrt{|\nabla T^*_{te^{\mp i\vartheta}} g|^2 + |W|\,|T^*_{te^{\mp i\vartheta}} g|^2}\ \mathrm{d}x\ \mathrm{d}t \leqslant C\,\|f\|_p\,\|g\|_q, \tag{5.98}$$

for all $f, g \in (L^p \cap L^q)(\Omega)$.

It follows from (5.98) and the inequality

$$\left|\int_\Omega \mathscr{L}_2 T_{te^{\pm i\vartheta}} f\ \overline{T^*_{te^{\mp i\vartheta}} g}\,\mathrm{d}x\right| \lesssim \int_\Omega \sqrt{|\nabla T_{te^{\pm i\vartheta}} f|^2 + |V|\,|T_{te^{\pm i\vartheta}} f|^2}\, \sqrt{|\nabla T^*_{te^{\mp i\vartheta}} g|^2 + |W|\,|T^*_{te^{\mp i\vartheta}} g|^2}\ \mathrm{d}x,$$

that

$$\int_0^\infty \left|\int_\Omega \mathscr{L}_p T_{2te^{\pm i\vartheta}} f\,\overline{g}\,\mathrm{d}x\right| \mathrm{d}t \lesssim \|f\|_p \|g\|_q,$$

for all $f, g \in (L^p \cap L^q)(\Omega)$. Analyticity of $(T_t)_{t>0}$ in $L^p(\Omega)$, Fatou's lemma and a density argument show that

$$\int_0^\infty \left|\int_\Omega \mathscr{L}_p T_{te^{\pm i\vartheta}} f\,\overline{g}\,\mathrm{d}x\right| \mathrm{d}t \lesssim \|f\|_p \|g\|_q, \tag{5.99}$$

for all $f \in L^p(\Omega)$ and all $g \in L^q(\Omega)$.

We have shown in Remark 2.24 that $\left\langle \overline{\mathrm{R}}(\mathscr{L}_p), \overline{\mathrm{R}}(\mathscr{L}_q^*)\right\rangle$ forms a dual subpair of $\langle L^p(\Omega), L^q(\Omega)\rangle$. We now apply Corollary 2.22 to the operator $(\mathscr{L}_p)_{||}$ and the dual pair $\left\langle \overline{\mathrm{R}}(\mathscr{L}_p), \overline{\mathrm{R}}(\mathscr{L}_q^*)\right\rangle$, and deduce from (5.99) that $\omega_{H^\infty}(\mathscr{L}_p) \leqslant \pi/2 - \vartheta$. □

5.10 Strongly subcritical potentials

This final section is devoted to presenting strongly subcritical potentials associated with different choices of $\mathscr{V}$. Given the definition, it is natural to begin with Hardy-type inequalities, which provide canonical examples of subcritical potentials on $W_0^{1,2}(\Omega)$ and, more generally, on $W_D^{1,2}(\Omega)$. In contrast, Hardy's inequality does not yield such potentials on the full space $W^{1,2}(\Omega)$, where a different line of argument is required.

5.10.1 Hardy's inequality on domain

For every closed $D \subsetneq \partial\Omega$ we define the function $\operatorname{dist}_D = \operatorname{dist}(\cdot, D)$ on Ω. In the special case when $D = \partial\Omega$, we simply write $\operatorname{dist}_\Omega = \operatorname{dist}_{\partial\Omega}$. The classical p-Hardy inequality on Ω takes the form

$$\int_\Omega \left| \frac{u}{\operatorname{dist}_\Omega} \right|^p \leqslant c \int_\Omega |\nabla u|^p, \qquad u \in W_0^{1,p}(\Omega), \tag{5.100}$$

where $p \in (1,\infty)$ and $c = c(d,p) > 0$. This inequality was first investigated in the one-dimensional setting by Hardy (see [49, Sect. 33] and the references therein). Necas [68] subsequently extended the p-Hardy inequality to higher dimensions, proving that (5.100) holds for every $p \in (1,\infty)$ on any bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, with a constant $c = c(\Omega, d, p) > 0$. Later developments showed that a domain $\Omega \subset \mathbb{R}^d$ satisfies the p-Hardy inequality under the weaker assumption that the complement of Ω is uniformly p-fat [2, 54, 59, 87]. As a consequence, by taking $p = 2$, for such domains Ω one obtains

$$-(\operatorname{dist}_\Omega)^{-2} \in \mathcal{P}\left(\Omega, W_0^{1,2}(\Omega)\right).$$

Without imposing any geometric restriction on $\Omega \subset \mathbb{R}^d$, the space $W_0^{1,p}(\Omega)$ is the largest subspace of $W^{1,p}(\Omega)$ on which Hardy's inequality (5.100) holds. More precisely, if $u \in W^{1,p}(\Omega)$ and $u/\operatorname{dist}_\Omega \in L^p(\Omega)$, then necessarily $u \in W_0^{1,p}(\Omega)$ [38, p. 223]. An even stronger statement is true: it suffices to assume that $u/\operatorname{dist}_\Omega$ belongs to the weak $L^p(\Omega)$ [54]. In particular, for every $\mathscr{V}$ satisfying $W_0^{1,2}(\Omega) \subsetneq \mathscr{V} \subseteq W^{1,2}(\Omega)$ and for any domain $\Omega \subset \mathbb{R}^d$, we have the strict inclusion

$$\mathcal{P}(\Omega, \mathscr{V}) \subsetneq \mathcal{P}\left(\Omega, W_0^{1,2}(\Omega)\right).$$

More recently, Egert, Haller-Dintelmann and Rehberg developed a geometric framework for Hardy's inequality on bounded domains Ω, in the setting where functions vanish only on a closed portion D of the boundary, i.e., when they belong to $W_D^{1,2}(\Omega)$ [42]. We refer to their work for the underlying geometric definitions. For every $p \in (1,\infty)$ they proved in [42, Theorem 3.1] the existence of a constant $c > 0$ such that

$$\int_\Omega \left| \frac{u}{\operatorname{dist}_D} \right|^p \leqslant c \int_\Omega |\nabla u|^p, \qquad u \in W_D^{1,p}(\Omega), \tag{5.101}$$

provided the following conditions are satisfied:

(i) The set D is l-thick for some $l \in (d-p, d)$.

(ii) The space $W_D^{1,p}(\Omega)$ admits an equivalent norm given by $\|\nabla \cdot\|_{L^p(\Omega)}$.

(iii) There exists a continuous linear extension operator $E : W^{1,p}(\Omega) \to W_D^{1,p}(\mathbb{R}^d)$.

In particular, by taking $p = 2$ one has

$$-(\mathrm{dist}_D)^{-2} \in \mathcal{P}\left(\Omega, W^{1,2}_D(\Omega)\right).$$

Conditions (i) and (ii) are automatically fulfilled if, for every $x \in \overline{\partial\Omega \setminus D}$, there exists an open neighborhood U_x such that $\Omega \cap U_x$ is a $W^{1,2}$-extension domain [42, Theorem 3.2].

Moreover, if in addition D is porous, then $W^{1,2}_D(\Omega)$ is the largest subspace of $W^{1,2}(\Omega)$ on which Hardy's inequality (5.101) holds. More precisely, if $u \in W^{1,2}(\Omega)$ and $u/\mathrm{dist}_D \in L^2(\Omega)$, then necessarily $u \in W^{1,2}_D(\Omega)$ [42]. As a consequence, it follows that

$$-(\mathrm{dist}_D)^{-2} \notin \mathcal{P}\left(\Omega, W^{1,2}(\Omega)\right),$$

and thus we obtain the strict inclusion

$$\mathcal{P}\left(\Omega, W^{1,2}(\Omega)\right) \subsetneq \mathcal{P}\left(\Omega, W^{1,2}_D(\Omega)\right).$$

So far, we have provided examples of subcritical potentials for $W^{1,2}_0(\Omega)$ and for $W^{1,2}_D(\Omega)$. What remains is to give an example of a potential belonging to $\mathcal{P}(\Omega, W^{1,2}(\Omega))$, and hence to $\mathcal{P}(\Omega, \widetilde{W^{1,2}_D}(\Omega))$. In the next section we shall address this, applying an argument we learned from [62]. This approach will also allow us to construct further examples within the classes $\mathcal{P}(\Omega, W^{1,2}_0(\Omega))$ and $\mathcal{P}(\Omega, W^{1,2}_D(\Omega))$.

5.10.2 Potentials on homogeneous open subsets

Let $\Omega \subseteq \mathbb{R}^d$ be open and $\mathscr{V} = \mathscr{V}(\Omega)$ be a closed subspace of $W^{1,2}(\Omega)$ containing $W^{1,2}_0(\Omega)$. Denote by $\Delta_{\mathscr{V}} = \mathscr{L}^{I,0,\mathscr{V}}$ the Laplacian on $L^2(\Omega)$ with domain $\mathrm{D}(\Delta_{\mathscr{V}}) \subseteq \mathscr{V}$. For all $x \in \Omega$ and $r > 0$, define

$$v(x,r) := |\Omega \cap B(x,r)|.$$

The aim of this section is to establish a sufficient condition for strong subcriticality, thereby providing examples of potentials in $\mathcal{P}(\Omega, \mathscr{V})$, and in particular in $\mathcal{P}(\Omega, W^{1,2}(\Omega))$. The argument is not new: we shall follow and adapt the method of [62, Section 5], proving under suitable assumptions on Ω that the finiteness of

$$\|V_-^{1/2}\|_{vol} := \int_0^1 \left\| \frac{V_-^{1/2}}{v(\cdot,\sqrt{t})^{\frac{1}{r_1}}} \right\|_{r_1} \frac{\mathrm{d}t}{\sqrt{t}} + \int_1^\infty \left\| \frac{V_-^{1/2}}{v(\cdot,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \frac{\mathrm{d}t}{\sqrt{t}}, \tag{5.102}$$

for some $r_1, r_2 > 2$, is sufficient for V to be a strongly subcritical potential on Ω. The quantity (5.102) has been introduced by Assad and Ouhabaz in [5] for studying the boundedness on L^p of Riesz transforms of Schrödinger operators on complete Riemannian manifolds. It subsequently appeared in [62] in the context of L^p-boundedness of Riesz transforms of the Hodge-de Rham Laplacian on complete Riemannian manifolds. There, the negative part R_- of the Ricci curvature plays the role of V_- and it has been proved to be subcritical whenever $\|R_-\|_{vol}$ is small enough [62, Proposition 5.9]. In both papers, two structural assumptions on the manifold are required: the volume doubling property and Gaussian upper estimates for the heat kernel of the Laplace–Beltrami operator. To replicate the argument of [62] in our setting, we impose analogous conditions on the pair $(\Omega, \mathscr{V}(\Omega))$: we assume that

(i) there exists $C > 0$ such that $v(x, 2r) \leqslant C v(x,r)$ for all $x \in \Omega$ and $r > 0$;

(ii) the semigroup $(e^{-t\Delta_{\mathscr{V}}})_{t>0}$ has a Gaussian (upper) bound, that is, there exist $k_t(x,y) \in L^\infty(\Omega\times\Omega)$ and $C, c > 0$ satisfying

$$|k_t(x,y)| \leqslant \frac{Ce^{-c\frac{|x-y|^2}{t}}}{v(x,\sqrt{t})}, \quad a.e. x, y, \forall t > 0,$$

such that

$$e^{-t\Delta_{\mathscr{V}}} f(x) = \int_\Omega K_t(x,y) f(y)\,\mathrm{d}y,$$

for almost all $x \in \Omega$, all $t > 0$ and $f \in L^2(\Omega)$.

Assumptions (i) and (ii) allow us to apply [5, Proposition 2.9] for $\Delta_{\mathscr{V}}$ and obtain

$$\|v(\cdot,\sqrt{t})^{\frac{1}{p}-\frac{1}{q}} e^{-t\Delta_{\mathscr{V}}}\|_{p-q} \leqslant C, \quad \forall\, 1 < p \leqslant q < \infty, \tag{5.103}$$

where C is a non-negative constant depending on p, q and on the constants appearing in (i) and (ii). For every $\varepsilon > 0$ we have the domination $|e^{-t(\Delta_{\mathscr{V}}+\varepsilon)} f| \leqslant e^{-t\Delta_{\mathscr{V}}}|f|$ for all $f \in C_c^\infty(\Omega)$. Therefore, (5.103) yields

$$\|v(\cdot,\sqrt{t})^{\frac{1}{p}-\frac{1}{q}} e^{-t(\Delta_{\mathscr{V}}+\varepsilon)}\|_{p-q} \leqslant C, \quad \forall\, 1 < p \leqslant q < \infty, \tag{5.104}$$

with C as in (5.103), thus not depending on ε. Such estimate is the key ingredient for the next lemma, modeled after [62, Lemma 5.4].

Lemma 5.41. *Assume thath (i) and (ii) are satisfied. Let $V \in L^1_{\mathrm{loc}}(\Omega)$ be non-negative. Then there exists a constant $C \geqslant 0$, depending on the constants appearing in (i) and (ii), such that*

$$\|V^{1/2}\,(\Delta_{\mathscr{V}} + \varepsilon)^{-1/2}\,\|_{2-2} \leqslant C\|V^{1/2}\|_{vol},$$

for all $\varepsilon > 0$.

Proof. Set $H = \Delta_{\mathscr{V}} + \varepsilon$. Writing

$$H^{-1/2} = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-tH} \frac{\mathrm{d}t}{\sqrt{t}},$$

and using the Hölder inequality, we obtain

$$\begin{aligned}
&\|V^{1/2}H^{-1/2}\|_{2-2} \\
&\leqslant C\int_0^1 \left\| \frac{V^{1/2}}{v(\cdot,\sqrt{t})^{\frac{1}{r_1}}} v(\cdot,\sqrt{t})^{\frac{1}{r_1}} e^{-tH} \right\|_{2-2} \frac{\mathrm{d}t}{\sqrt{t}} + \int_1^\infty \left\| \frac{V^{1/2}}{v(\cdot,\sqrt{t})^{\frac{1}{r_2}}} v(\cdot,\sqrt{t})^{\frac{1}{r_2}} e^{-tH} \right\|_{2-2} \frac{\mathrm{d}t}{\sqrt{t}} \\
&\leqslant C\int_0^1 \left\| \frac{V^{1/2}}{v(\cdot,\sqrt{t})^{\frac{1}{r_1}}} \right\|_{r_1} \left\| v(\cdot,\sqrt{t})^{\frac{1}{r_1}} e^{-tH} \right\|_{2-\frac{2r_1}{r_1-2}} \frac{\mathrm{d}t}{\sqrt{t}} \\
&\quad + C\int_1^\infty \left\| \frac{V^{1/2}}{v(\cdot,\sqrt{t})^{\frac{1}{r_2}}} \right\|_{r_2} \left\| v(\cdot,\sqrt{t})^{\frac{1}{r_2}} e^{-tH} \right\|_{2-\frac{2r_2}{r_2-2}} \frac{\mathrm{d}t}{\sqrt{t}}.
\end{aligned}$$

Since for $i = 1, 2$ we have

$$\frac{1}{r_i} = \frac{1}{2} - \frac{r_i - 2}{2r_i},$$

we conclude by invoking (5.104). □

The following corollary is modeled after [62, Proposition 5.8].

Corollary 5.42. *Assume thath (i) and (ii) are satisfied. Let $V \in L^1_{\mathrm{loc}}(\Omega)$ be nonnegative such that $\|V^{1/2}\|_{vol} < \infty$. Then there exists a nonnegative constant α such that $-V \in \mathcal{P}_{\alpha,0}(\Omega, \mathscr{V})$. The constant α can be chosen equal to $C\|V^{1/2}\|_{vol}$, with $C \geqslant 0$ depending on the constant appearing in (i) and (ii).*

In particular, for every nonnegative $W \in L^1_{\mathrm{loc}}(\Omega)$, the potential $W - V$ belongs to $\mathcal{P}_{\alpha,\sigma}(\Omega, \mathscr{V})$ for all $\sigma \in [0, 1)$.

Proof. Let $\varepsilon > 0$ and set $H = \Delta_{\mathscr{V}} + \varepsilon$. We have

$$\begin{aligned}\int_\Omega V|u|^2 = \|V^{1/2}u\|_2^2 &= \|V^{1/2}H^{-1/2}H^{1/2}u\|_2^2 \\ &\leqslant \|V^{1/2}H^{-1/2}\|_{2-2}^2 \|H^{1/2}u\|_2^2 \\ &= \|V^{1/2}H^{-1/2}\|_{2-2}^2 \langle Hu, u\rangle \\ &= \|V^{1/2}H^{-1/2}\|_{2-2}^2 \left(\int_\Omega |\nabla u|^2 + \varepsilon|u|^2\right).\end{aligned}$$

Applying Lemma 5.41 and sending $\varepsilon \to 0$ yield the claim.

The last assertion follows from the facts that $-V \in \mathcal{P}_{\alpha,0}(\Omega, \mathscr{V})$ and $(W - V)_- \leqslant V$ for every nonnegative $W \in L^1_{\mathrm{loc}}(\Omega)$. □

Remark 5.43. (i) If the volume on Ω is polynomial, that is, $cr^d \leqslant v(\cdot, r) \leqslant Cr^d$, then $\|V_-^{1/2}\|_{vol} < \infty$ if and only if $V_- \in L^{d/2-\eta} \cap L^{d/2+\eta}$ for some $\eta > 0$.

(ii) If $\|V_-^{1/2}\|_{vol} < \infty$ then Ω is unbounded.

An immediate consequence of Corollary 5.42 is the following (compare with [62, Proposition 5.9]).

Corollary 5.44. *Assume that (i) and (ii) are satisfied. Suppose that there exists $W \in L^1_{\mathrm{loc}}(\Omega)$ nonnegative such that $\|W^{1/2}\|_{vol} < \infty$. Then, for all $A \in \mathcal{A}(\Omega)$ there exists $V \in \mathcal{P}(\Omega, \mathscr{V})$ such that $(A, V) \in \mathcal{AP}(\Omega, \mathscr{V})$.*

Let us return to the assumptions (i) and (ii) and discuss situations in which they are satisfied. It is well known that the semigroup associated with the Dirichlet Laplacian $\Delta_{W_0^{1,2}(\Omega)}$ admits a Gaussian upper bound for every open set $\Omega \subseteq \mathbb{R}^d$ [29]. Furthermore, the semigroup associated with $\Delta_{\mathscr{V}}$ also enjoys a Gaussian upper bound provided that the space $\mathscr{V}$ satisfies (3.5) and the following two conditions:

(a) $\mathscr{V}$ enjoys the homogeneous Sobolev embedding property; see Definition 5.11;

(b) $u \in \mathscr{V} \Rightarrow e^{\psi}u \in \mathscr{V}$ for every real-valued function $\psi \in C^\infty(\mathbb{R}^d)$ such that both ψ and $\nabla\psi$ are bounded on $\mathbb{R}^d$.

For details, see [72, Chapter 6.3]. Condition (a) always holds when $\mathscr{V} = W_0^{1,2}(\Omega)$ [91, Theorem 2.4.1]. On the other hand, condition (b) and (3.5) are always verified when $\mathscr{V}$ falls into any of the special cases (a)-(d) of Section 3.1.1; see for example [41, Lemma 4(ii)].

In some cases the heat kernel can be written explicitly. For instance, when $\Omega = \mathbb{R}^d_+ := \{x \in \mathbb{R}^d : x_d > 0\}$ the heat kernel k_t of the Neumann Laplacian $\Delta_{W^{1,2}(\mathbb{R}^d_+)}$ is given by

$$k_t(x, y) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}} + \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y'|^2}{4t}},$$

where $y' = (y_1, \ldots, y_{d-1}, -y_d)$ if $y = (y_1, \ldots, y_d)$ [46]. Hence, $(\mathbb{R}^d_+, W^{1,2}(\mathbb{R}^d_+))$ satisfies (ii). Moreover, the volume on $\mathbb{R}^d_+$ is polynomial, so $\mathbb{R}^d_+$ also satisfies (i). Therefore, by Corollary 5.42 and Remark 5.43(i), we obtain

$$\emptyset \neq (L^{d/2-\eta} \cap L^{d/2+\eta})(\mathbb{R}^d_+, (-\infty, 0]) \subseteq \mathcal{P}(\mathbb{R}^d_+, W^{1,2}(\mathbb{R}^d_+)),$$

for some $\eta > 0$.

CHAPTER 6

A new approximation argument for the bilinear embedding

This final chapter departs slightly from the framework of the previous two. We return to the setting of unperturbed divergence-form operators, with the aim of providing a new proof of Theorem 3.19.

Why seek a new proof? In [22], a trilinear embedding was established for divergence-form operators (without lower-order terms), distinguishing between two cases: when $\Omega = \mathbb{R}^d$, and when Ω is an arbitrary open subset of $\mathbb{R}^d$. In the former, the result was obtained by adapting the heat-flow method from [20]; in the latter, the authors relied on the approximation techniques developed in [19]. However, unlike the bilinear case—where both methods could be applied under the same structural assumptions on the coefficient matrices—when Ω is an arbitrary open subset of $\mathbb{R}^d$ the trilinear setting imposes stronger assumptions on the coefficients in order to apply the approximation procedure of [19] within this context.

The ultimate goal of this chapter is to bridge this gap by presenting a unified proof of the trilinear embedding that avoids such additional assumptions. The key idea is to first revisit the bilinear setting and develop a new technique for proving the bilinear embedding, which we then aim to adapt and extend to the trilinear context.

The chapter is organized as follows. In Section 6.1, we recall the trilinear embedding theorem and clarify the origin of the stronger assumptions needed in the general-domain case. Section 6.2 introduces the new strategy at a formal level, highlighting the main ideas behind the approach. In Section 6.3, we construct a new approximating sequence that replaces $\mathcal{R}_{n,\nu}$, introduced in [19] and used in subsequent works [12, 77, 78]. Finally, Section 6.4 is devoted to proving the bilinear embedding using this new method, which serves as the foundation for the trilinear result.

6.1 Toward the Trilinear Embedding

In [22], the authors aimed to establish an inequality of the type presented in Theorem 3.19 for a triplet of matrices. Analogously to the bilinear case, [22, Lemma 7.11], which corresponds to [19, Lemma 19], does not hold in general for an arbitrary choice of triplet $(\mathscr{U}, \mathscr{V}, \mathscr{W})$. We therefore impose the following assumption, analogous to Assumption BE.

Assumption TE. We say that the triple $(\mathscr{U}, \mathscr{V}, \mathscr{W})$ satisfies the Assumption TE if either of the following holds:

- $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}$ fall into any of the special cases (a)-(c), or
- $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}$ are of the type described in (a) or (d).

Under this assumption, they proved the following result.

Theorem 6.1. ([22, Theorem 1.1]) *Let $\Omega \subseteq \mathbb{R}^d$ be an open set and let the spaces $\mathscr{U}, \mathscr{V}, \mathscr{W}$ satisfy Assumption TE. Take $p, q, r \in (1, \infty)$ such that*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

Suppose that accretive matrices $A, B, C : \Omega \to \mathbb{C}^{d\times d}$ are $\max\{p, q, r\}$-elliptic. Then for $f \in (L^p \cap L^2)(\Omega)$, $g \in (L^q \cap L^2)(\Omega)$ and $h \in (L^r \cap L^2)(\Omega)$ we have

$$\int_0^\infty \int_\Omega |\nabla T_t^{A,\mathscr{U}} f||\nabla T_t^{B,\mathscr{V}} g||T_t^{C,\mathscr{W}} h| \,\mathrm{d}x \,\mathrm{d}t \lesssim \|f\|_p \|g\|_q \|h\|_r. \tag{6.1}$$

When $\Omega = \mathbb{R}^d$, the same conclusion holds under milder assumptions, namely, when

- *A is p-elliptic and $(1 + p/q)$-elliptic,*
- *B is q-elliptic and $(1 + q/p)$-elliptic,* $(\star)$
- *C is r-elliptic.*

The implied embedding constants only depend on p, q, r and $*$-ellipticity constants of A, B, C alluded to in the theorem's assumptions.

We recall that conditions $(\star)$ are weaker than $\max\{p, q, r\}$-ellipticity of A, B, C, thanks to Lemma 3.7(iv) and [22, Lemma 5.6]. In order to prove such theorem when Ω is an arbitrary open set of $\mathbb{R}^d$, the authors adapted the method that Carbonaro and Dragičević used to establish Theorem 3.19. In both situations, the main problem was justifying an integration by parts. However, while in the bilinear case it was possible to deduce (3.27) starting from the same assumptions on the matrices as in the case where $\Omega = \mathbb{R}^d$, in the trilinear setting the authors had to assume stronger properties on the matrices to overcome this technical issue, namely the integration by parts.

Our goal is to prove the trilinear embedding (6.1) under $(\star)$ even when Ω is not the entire space $\mathbb{R}^d$. To this end, we believe that the key is appropriately modifying the sequence $\mathfrak{X}_{n,\nu}$, introduced in [22, Section 7.5] for the heat-flow method; see [22, Section 7] and Section 6.2. To prove Theorem 6.1, the authors closely followed the method applied in [19] to get the bilinear inequality (3.27), adapting it to their context step by step. Therefore, it makes sense to revisit the bilinear case (which is technically easier to handle) and modify the parts of the proof where the stronger assumptions on the matrices were heavily exploited in the corresponding trilinear proof. Once these modifications are made, we can return to the trilinear setting, adapting the new technique to this context. In this thesis we will focus only on the bilinear case.

The idea is to apply the heat-flow method again as in Section 3.5.4, but replacing the approximating sequence $\mathcal{R}_{n,\nu}$, defined in (3.40), with a new one. We will heavily rely on Proposition 5.10 applied with $V = 0$, which, in this case, takes the following specific form.

Proposition 6.2. *Let $p \in (1, \infty)$. If A is p-elliptic and $u \in \mathrm{D}(\mathscr{L}^A) \cap L^p(\Omega)$ is such that $\mathscr{L}^A u \in L^p(\Omega)$, then it follows that*

$$\int_\Omega |u|^{p-2} |\nabla u|^2 \mathbb{1}_{\{u\neq 0\}} \lesssim \|\mathscr{L}^A u\|_p \|u\|_p^{p-1} < \infty.$$

Another key point, although simple, is to preserve as much information as possible when estimating the quadratic form of the generalized Hessian of a function or of a product of functions. For this purpose, for all $j,k=1,2$ and $\Phi:\mathbb{C}^2\to\mathbb{R}$ of class C^2, it is useful to write

$$|D^2_{\omega_j\omega_k}\Phi| = \left(\sum_{h,l=1}^{2} |\partial^2_{\omega_j^h\omega_k^l}\Phi_{\mathcal{W}}|^2\right)^{1/2},$$

where $\omega_j=\omega_j^1+i\omega_j^2$ for all $j=1,2$. In this way, the boundedness of the matrices A and B implies

$$\left|H_{\Phi}^{(A,B)}[\omega;\Xi]\right| \lesssim_{A,B} \sum_{j,k=1}^{2} \left|D^2_{\omega_j\omega_k}\Phi(\omega)\right| |\Xi_j||\Xi_k|, \tag{6.2}$$

for all $\omega\in\mathbb{C}^2$ and $\Xi\in\mathbb{C}^{2d}$.

Moreover, if $\Psi:\mathbb{C}^2\to\mathbb{R}$ is another C^2 function, applying Leibniz's rule twice gives

$$D^2_{jk}(\Psi\cdot\Phi) = \Psi\cdot D^2_{jk}\Phi + \Phi\cdot D^2_{jk}\Psi + D_k\Psi\cdot D_j\Phi + D_j\Psi\cdot D_k\Phi,$$

for all $j,k\in\{1,\dots,4\}$. Defining the matrix-valued function $L=L^{(\Psi,\Phi)}$ with entries

$$L_{jk}^{(\Psi,\Phi)} = D_j\Psi\cdot D_k\Phi,$$

for all $\omega\in\mathbb{C}^2$ and $\Xi\in\mathbb{C}^{2d}$ we obtain

$$H_{\Psi\cdot\Phi}^{(A,B)}[\omega;\Xi] = \Psi(\omega)H_{\Phi}^{(A,B)}[\omega;\Xi] + \Phi(\omega)H_{\Psi}^{(A,B)}[\omega;\Xi] + G_{(\Psi,\Phi)}^{(A,B)}[\omega;\Xi], \tag{6.3}$$

where

$$G_{(\Psi,\Phi)}^{(A,B)}[\omega;\Xi] := 2\left\langle [L_{\mathrm{s}}(\omega)\otimes I_{\mathbb{R}^d}]\mathcal{W}_{2,d}(\Xi), [\mathcal{M}(A)\oplus\mathcal{M}(B)]\,\mathcal{W}_{2,d}(\Xi)\right\rangle. \tag{6.4}$$

In this case, the boundedness of A and B implies

$$\left|G_{(\Psi,\Phi)}^{(A,B)}[\omega;\Xi]\right| \lesssim_{A,B} \sum_{j,k=1}^{2} \left|\partial_{\omega_j}\Psi(\omega)\cdot\partial_{\omega_k}\Phi(\omega)\right| |\Xi_j||\Xi_k|, \tag{6.5}$$

for all $\omega\in\mathbb{C}^2$ and $\Xi\in\mathbb{C}^{2d}$.

6.2 The new strategy: a different approximation sequence in the heat-flow method

Recall the definition (3.40) of $\mathcal{R}_{n,\nu}$ and that $\psi_n=\psi(\cdot/n)$, where ψ is a smooth and compactly supported function on $\mathbb{C}^2$ such that $\psi\geqslant 0$, $\psi=1$ on $\{|\omega|\leqslant 3\}$ and $\psi=0$ on $\{|\omega|\geqslant 4\}$. As we saw in Section 3.5.4, we have used Theorem 3.26(i)-(v) to deduce

$$-\mathcal{E}'(t) = \lim_{\nu\to 0}\lim_{n\to\infty}\int_{\Omega} H_{\mathcal{R}_{n,\nu}}^{(A,B)}\left[\left(T_t^{A,\mathscr{V}}f, T_t^{B,\mathscr{W}}g\right); \left(\nabla T_t^{A,\mathscr{V}}f, \nabla T_t^{B,\mathscr{W}}g\right)\right], \tag{6.6}$$

and exploited the (A,B)-convexity of $\mathcal{R}_{n,\nu}$ to apply Fatou's lemma and infer, by also using Theorem 3.26(ii), that

$$\begin{aligned}\lim_{n\to\infty}\int_{\Omega} H_{\mathcal{R}_{n,\nu}}^{(A,B)}&\left[\left(T_t^{A,\mathscr{V}}f, T_t^{B,\mathscr{W}}g\right); \left(\nabla T_t^{A,\mathscr{V}}f, \nabla T_t^{B,\mathscr{W}}g\right)\right]\\ &\geqslant \int_{\Omega} H_{\mathcal{Q}*\varphi_\nu}^{(A,B)}\left[\left(T_t^{A,\mathscr{V}}f, T_t^{B,\mathscr{W}}g\right); \left(\nabla T_t^{A,\mathscr{V}}f, \nabla T_t^{B,\mathscr{W}}g\right)\right].\end{aligned} \tag{6.7}$$

In order to compensate for the lack of (A,B)-convexity of $\psi_n \cdot (\mathcal{Q} * \varphi_\nu)$ in the ring $\{3n < |\omega| < 4n\}$ and achieve the (A,B)-convexity of $\mathcal{R}_{n,\nu}$, the authors exploited the (A,B)-convexity of a specific perturbation P_s of the two-complex variable power function F_s, defined by

$$\begin{aligned} F_s \ : \ \mathbb{C}^2 \ &\to \ \mathbb{R} \\ \omega \ &\mapsto \ |\omega|^s, \end{aligned}$$

provided that both A, B are s-elliptic [19, Proposition 11]. See [19, (24)] for the definition of P_s. An analogous result also holds in higher dimensions [22, Lemma 4.6]. However, while the assumptions on the matrices A, B are the same both in Theorem 3.19 and in [19, Proposition 11], in the trilinear setting they are not. More precisely, the assumptions $(\star)$ are not enough to apply [22, Lemma 4.6] with A, B, C. The authors needed A, B, C to be $\max\{p,q,r\}$-elliptic.

Notice that $\psi_n \cdot (\mathcal{Q} * \varphi_\nu)$ still satisfies properties (i)-(v) of Theorem 3.26; see the proof of [19, Theorem 16] or Proposition 6.4 below. So, (6.6) still holds for $\psi_n \cdot (Q * \varphi_\nu)$ instead of $\mathcal{R}_{n,\nu}$. Hence, in view of Proposition 6.2, a possible idea is not to focus on compensating for the lack of (A,B)-convexity of $\psi_n \cdot (\mathcal{Q} * \varphi_\nu)$ in the ring $\{3n < |\omega| < 4n\}$, but instead to estimate a part of the generalized Hessian of $\psi_n \cdot (\mathcal{Q} * \varphi_\nu)$ uniformly in n, and subsequently invoke Lebesgue's dominated convergence theorem in place of Fatou's lemma in order to infer the analogue of (6.7). For this reason, we will not add any sequence to $\psi_n \cdot (\mathcal{Q} * \varphi_\nu)$. Unluckily, ψ_n is not the right truncation for this purpose. We need to change it.

6.2.1 The admissible smooth truncations

In this section, we will present the new approach during the heat flow method. As mentioned above, the idea is to appropriately and smoothly truncate the regularized Bellman function $\mathcal{Q} * \varphi_\nu$ in order to prove the analogues of (6.6) and (6.7). In [19, 22] the authors used the sequence $\{\psi_n\}_{n\in\mathbb{N}}$, previously recalled. However, while the counterpart of (6.6) still holds for $\psi_n \cdot (\mathcal{Q} * \varphi_\nu)$, this truncation encounters difficulties in justifying the analogue of (6.7); see Section 6.2.2. Therefore, in constructing the new sequence of truncations $\{\Psi_n\}_{n\in\mathbb{N}}$, we must incorporate all the properties of $\{\psi_n\}_{n\in\mathbb{N}}$ that allow us to prove (6.6) and then impose new features enabling us to deduce (6.7).

Regarding the first issue, that is proving (6.6), we want our new approximating sequence to satisfy (i)-(v) of Theorem 3.26 and ensure that the analogue of Lemma 3.27 holds true. For this purpose, we introduce a family of admissible sequences of truncations which $\{\psi_n\}_{n\in\mathbb{N}}$ belongs to.

Definition 6.3. A sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ of smooth and nonnegative functions on $\mathbb{C}^2$ is said to be an *admissible sequence of truncations* if

a) $\Psi_n \lesssim 1$, uniformly in n,

b) $(\Psi_n)_{\mathcal{W}}$ is even in each of the variables $\zeta_1, \zeta_2, \eta_1, \eta_2$,

and there exist two sequences $\{K_{1,n}\}_{n\in\mathbb{N}}$ and $\{K_{2,n}\}_{n\in\mathbb{N}}$ of compact subsets of $\mathbb{C}^2$ such that

c) $K_{1,n} \subseteq \operatorname{int}(K_{2,n})$ for all $n \in \mathbb{N}$,

d) $K_{j,n} \subseteq K_{j,n+1}$ for all $n \in \mathbb{N}$ and $j \in \{1,2\}$,

e) $0 \in \operatorname{int}(K_{1,n})$ for all $n \in \mathbb{N}$,

f) $\bigcup_{n\in\mathbb{N}} K_{j,n} = \mathbb{C}^2$ for all $j \in \{1,2\}$,

g) $\Psi_n = 1$ on $K_{1,n}$,

h) $\mathrm{supp}\Psi_n \subseteq K_{2,n}$,

i) $|D\Psi_n(\omega)| \lesssim \min\{1, |\omega|^{-1}\}$ for all $\omega \in K_{2,n} \setminus K_{1,n}$, uniformly in n.

By f), it is not restrictive assuming that

$$|\omega| \geqslant 1, \quad \forall \omega \in K_{2,n} \setminus K_{1,n}, \forall n \in \mathbb{N}. \tag{6.8}$$

Clearly, d), f) and g) imply

$$\begin{aligned} \Psi_n &\to 1, \\ D\Psi_n &\to 0, \\ D^2\Psi_n &\to 0, \end{aligned} \tag{6.9}$$

pointwise on $\mathbb{C}^2$, as $n \to \infty$.

The sequence $\{\psi_n\}_{n\in\mathbb{N}}$ is an example of admissible sequence of truncations with $K_{1,n} = \overline{B(0,3n)}$ and $K_{2,n} = \overline{B(0,4n)}$.

For all $n \in \mathbb{N}$ and $\nu \in (0,1)$ define

$$\mathcal{Q}_{n,\nu} = \Psi_n \cdot (\mathcal{Q} * \varphi_\nu). \tag{6.10}$$

The following two results are modeled after Theorem 3.26 and Lemma 3.27, respectively.

Proposition 6.4. *Let $n \in \mathbb{N}$ and $\nu \in (0,1)$. The following statements hold.*

(i) $D^2\mathcal{Q}_{n,\nu} \in L^\infty(\mathbb{C}^2; \mathbb{R}^{4\times 4})$.

(ii) *We have*

$$\begin{aligned} D\mathcal{Q}_{n,\nu} &\to D(\mathcal{Q} * \varphi_\nu), \\ D^2\mathcal{Q}_{n,\nu} &\to D^2(\mathcal{Q} * \varphi_\nu) \end{aligned}$$

pointwise in $\mathbb{C}^2$ as $n \to \infty$.

(iii) *For any $n \in \mathbb{N}$ there exists $C(n,\nu) > 0$ such that*

$$|(D\mathcal{Q}_{n,\nu})(\omega)| \leqslant C(n,\nu)|\omega|, \quad \forall \omega \in \mathbb{C}^2.$$

(iv) *There exists $C(\nu) > 0$ that does not depend on n such that*

$$|(D\mathcal{Q}_{n,\nu})(\omega)| \leqslant C(\nu)(|\omega|^{p-1} + |\omega|^{q-1}),$$

for all $\omega \in \mathbb{C}^2$, $n \in \mathbb{N}$ and $\nu \in (0,1)$.

(v) *For any $n \in \mathbb{N}$ and $\nu > 0$ we have*

$$(\partial_\zeta \mathcal{Q}_{n,\nu})(0,\eta) = 0, \quad (\partial_\zeta \mathcal{Q}_{n,\nu})(\zeta,0) = 0,$$

for all $\zeta, \eta \in \mathbb{C}$.

Proof. The proof is the same as [19, Theorem 16 (i)-(v)]. For the sake of clarity, we will write all the details in order to understand why we need the required proprieties of Ψ_n.

Item (i) follows from the fact that $\mathcal{Q}_{n,\nu} \in C_c^2(\mathbb{C}^2)$.

Item (ii) is a consequence of (6.9).

From e) and g) we deduce that $\Psi_n \equiv 1$ in a neighborhood of 0. Therefore, by [19, (31)] we conclude that $(D\mathcal{Q}_{n,\nu})(0) = 0$. Hence, by the mean value theorem and the fact that $\mathcal{Q}_{n,\nu} \in C_c^\infty(\mathbb{C}^2)$, we get item (iii).

To prove item (iv), we observe that g) and h) imply that $D\Psi_n \equiv 0$ on $(K_{2,n} \setminus K_{1,n})^{\mathrm{c}}$, while i), Lemma 3.24(i) and (6.8) give

$$|(D\Psi_n)(\omega)| \cdot |(\mathcal{Q} * \varphi_\nu)(\omega)| \lesssim |\omega|^{p-1} + |\omega|^{q-1}, \quad \forall \omega \in K_{2,n} \setminus K_{1,n},$$

and a) and Lemma 3.24(ii) yield

$$|\Psi_n(\omega)| \cdot |D(\mathcal{Q} * \varphi_\nu)(\omega)| \lesssim |\omega|^{p-1} + |\omega|^{q-1}, \quad \forall \omega \in \mathbb{C}^2.$$

Since $\mathcal{Q}$ and φ_ν are even functions in each of the variables $\zeta_1, \zeta_2, \eta_1, \eta_2$, the function $\mathcal{Q} * \varphi_\nu$ also has this property. Hence, we get item (v) by b). □

Corollary 6.5. *Suppose that $(\mathscr{V}, \mathscr{W})$ satisfies Assumption BE. Let $u \in \mathscr{V}$ and $v \in \mathscr{W}$. Then*

$$(\partial_\zeta \mathcal{Q}_{n,\nu})(u, v) \in \mathscr{V} \quad \text{and} \quad (\partial_\eta \mathcal{Q}_{n,\nu})(u, v) \in \mathscr{W}$$

for all $n \in \mathbb{N}_+$ and $\nu > 0$.

Proof. It is a consequence of Proposition 6.4 (i), (iii) and (v); see the proof of Lemma 3.27. □

As we will see in the proof of the following proposition, Proposition 6.4 and Corollary 6.5 allow us to deduce (6.6) for $\mathcal{Q}_{n,\nu}$. In addition, next proposition provides us the new properties $\{\Psi_n\}_{n\in\mathbb{N}}$ must satisfy to prove (6.7) for $\mathcal{Q}_{n,\nu}$. Recall definition (6.4).

Proposition 6.6. *Let $p \geqslant 2$, $q = p/(p-1)$, A, B be p-elliptic and $\mathscr{V}$ and $\mathscr{W}$ satisfy Assumption BE. Suppose that there exist an admissible sequence of truncations $\{\Psi_n\}_{n\in\mathbb{N}}$ and $F : \mathbb{C}^2 \times \mathbb{C}^{2d} \to [0, \infty)$ such that the following assertions hold.*

(i) *For all $\nu \in (0, 1)$ there exists a positive constant $C = C(\nu)$ such that*

$$\left|(\mathcal{Q} * \varphi_\nu)(\omega) H^{(A,B)}_{\Psi_n}[\omega; \Xi] + G^{(A,B)}_{(\Psi_n, \mathcal{Q}*\varphi_\nu)}[\omega; \Xi]\right| \leqslant C\, F(\omega, \Xi), \tag{6.11}$$

for all $n \in \mathbb{N}$, $\omega \in \mathbb{C}^2$ and $\Xi \in \mathbb{C}^{2d}$.

(ii) *For all $u \in \mathscr{V}$ and $v \in \mathscr{W}$ such that $u, v, \mathscr{L}^A u, \mathscr{L}^B v \in (L^p \cap L^q)(\Omega)$ we have*

$$F(u, v, \nabla u, \nabla v) \in L^1(\Omega). \tag{6.12}$$

Then the statement of Theorem 3.19 holds.

Proof. By Proposition 3.29, for proving the bilinear embedding it suffices to show that

$$\int_\Omega |\nabla u||\nabla v| \lesssim 2\,\mathrm{Re} \int_\Omega \left((\partial_\zeta \mathcal{Q})(u, v)\mathscr{L}^A u + (\partial_\eta \mathcal{Q})(u, v)\mathscr{L}^B v\right), \tag{6.13}$$

for all $u \in \mathscr{V}$, $v \in \mathscr{W}$ such that $u, v, \mathscr{L}^A u, \mathscr{L}^B v \in L^p \cap L^q$.

By using Proposition 6.4 (i) and (iv), Lemma 3.24(ii), the fact that $\mathcal{Q} \in C^1(\mathbb{C}^2)$ and Lebesgue's dominated convergence theorem twice, we deduce that

$$\begin{aligned} &2\mathrm{Re} \int_\Omega \left((\partial_\zeta \mathcal{Q})(u, v)\mathscr{L}^A u + (\partial_\eta \mathcal{Q})(u, v)\mathscr{L}^B v\right) \\ &\quad = \lim_{\nu\to 0} \lim_{n\to\infty} \int_\Omega \left((\partial_\zeta \mathcal{Q}_{n,\nu})(u, v)\mathscr{L}^A u + (\partial_\eta \mathcal{Q}_{n,\nu})(u, v)\mathscr{L}^B v\right). \end{aligned} \tag{6.14}$$

By Corollary 6.5 we have $(\partial_\zeta \mathcal{Q}_{n,\nu})(u,v) \in \mathscr{V}$, $(\partial_\eta \mathcal{Q}_{n,\nu})(u,v) \in \mathscr{W}$. Hence we can integrate by parts the integral on the right-hand side of (6.14) and, by means of the chain-rule for the composition of smooth functions with vector-valued Sobolev functions, deduce that

$$\begin{aligned} 2\mathrm{Re} \int_\Omega \Big((\partial_\zeta \mathcal{Q}_{n,\nu})(u,v) \mathscr{L}^A u + (\partial_\eta \mathcal{Q}_{n,\nu})(u,v) \mathscr{L}^B v \Big) \\ = \int_\Omega H^{(A,B)}_{\mathcal{Q}_{n,\nu}}[(u,v);(\nabla u, \nabla v)]. \end{aligned}$$

Therefore, (6.3) implies

$$\begin{aligned} 2\mathrm{Re} \int_\Omega \Big((\partial_\zeta \mathcal{Q}_{n,\nu})(u,v) \mathscr{L}^A u + (\partial_\eta \mathcal{Q}_{n,\nu})(u,v) \mathscr{L}^B v \Big) \\ &= \int_\Omega \Psi_n(u,v) H^{(A,B)}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u, \nabla v)] \\ &\quad + \int_\Omega (\mathcal{Q} * \varphi_\nu)(u,v) H^{(A,B)}_{\Psi_n}[(u,v);(\nabla u, \nabla v)] \\ &\quad + \int_\Omega G^{(A,B)}_{(\Psi_n, \mathcal{Q}*\varphi_\nu)}[(u,v);(\nabla u, \nabla v)] \\ &=: I^1_{n,\nu} + I^2_{n,\nu} + I^3_{n,\nu}. \end{aligned} \tag{6.15}$$

Now, the nonnegativity of Ψ_n, the (A,B)-convexity of $\mathcal{Q} * \varphi_\nu$ (Corollary 3.25), (6.9) and Fatou's lemma give

$$\lim_{n\to+\infty} I^1_{n,\nu} \geqslant \int_\Omega H^{(A,B)}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u, \nabla v)], \tag{6.16}$$

for all $\nu \in (0,1)$.

On the other hand, by combining (6.9), (6.11), (6.12) and Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{n\to\infty} I^2_{n,\nu} = 0, \\ \lim_{n\to\infty} I^3_{n,\nu} = 0. \end{aligned} \tag{6.17}$$

Now, (6.15), (6.16), (6.17) and Corollary 3.25 yield

$$\begin{aligned} \lim_{n\to\infty} 2\mathrm{Re} \int_\Omega \Big((\partial_\zeta \mathcal{Q}_{n,\nu})(u,v) \mathscr{L}^A u + (\partial_\eta \mathcal{Q}_{n,\nu})(u,v) \mathscr{L}^B v \Big) \\ &\geqslant \int_\Omega H^{(A,B)}_{\mathcal{Q}*\varphi_\nu}[(u,v);(\nabla u, \nabla v)] \\ &\geqslant C_0 \int_\Omega |\nabla u||\nabla v|, \end{aligned}$$

for all $\nu \in (0,1)$, where $C_0 = C_0(p, \Delta_p(A,B), \lambda(A,B), \Lambda(A,B)) > 0$ does not depend on ν. The desired inequality (6.13) now follows from (6.14). □

6.2.2 Replacement of the sequence $\{\psi_n\}_{n\in\mathbb{N}}$

In view of Proposition 6.2, one candidate for F is the function

$$\mathbb{C}^2 \times \mathbb{C}^{2d} \ni (\zeta, \eta, X, Y) \mapsto \begin{cases} (|\zeta|^{p-2}+1)\,|X|^2 + (|\eta|^{q-2}+1)\,|Y|^2, & \text{if } \eta \neq 0, \\ (|\zeta|^{p-2}+1)\,|X|^2 + |Y|^2, & \text{if } \eta = 0. \end{cases} \tag{6.18}$$

We will explain now the reason why the sequence $\{\psi_n\}_n$ used by Carbonaro and Dragičević in [19, 22] is not suitable for invoking Proposition 6.2. The problem is that we are not able to obtain the estimate (6.11) with F defined by (6.18).

Let u and v be as in the statement of Proposition 6.6 and set

$$\omega = (u, v), \qquad \nabla\omega = (\nabla u, \nabla v).$$

Let us focus on estimating only the first term in the left-hand side of (6.11), that is,

$$(\mathcal{Q} * \varphi_\nu)(\omega) H^{(A,B)}_{\psi_n}[\omega; \nabla\omega]. \tag{6.19}$$

From the definition of $\{\psi_n\}_{n\in\mathbb{N}}$ it suffices estimating (6.19) in the region $\{3n \leqslant |(u, v)| \leqslant 4n\}$. By (6.2) we have

$$\left|H^{(A,B)}_{\psi_n}[\omega; \nabla\omega]\right| \lesssim |D^2_{\zeta\zeta}\psi_n(\omega)||\nabla u|^2 + |D^2_{\eta\eta}\psi_n(\omega)||\nabla v|^2 + |D^2_{\zeta\eta}\psi_n(\omega)||\nabla u||\nabla v|.$$

Suppose that $|v|^q \leqslant |u|^p$, then by the first estimate of (3.33), we get

$$|(\mathcal{Q} * \varphi_\nu)(\omega)| \lesssim_\nu |u|^p + 1.$$

Therefore, if we estimated (6.19) on $\{3n \leqslant |(u, v)| \leqslant 4n\} \cap \{|v|^q \leqslant |u|^p\}$ by combining the previous two inequalities, mixed terms would also appear, such as

$$|u|^p \left(|D^2_{\eta\eta}\psi_n(\omega)| + |D^2_{\zeta\eta}\psi_n(\omega)|\right) |\nabla v|^2, \tag{6.20}$$

which could be uniformly controlled in n by

$$|u|^{p-2}|\nabla v|^2,$$

as we are in the region $\{3n \leqslant |(u, v)| \leqslant 4n\}$. However, we cannot deduce that this term is an integrable function from Proposition 6.2 since there is no relation between u and v. In the region $\{|v|^q \geqslant |u|^p\}$ we would have an analogous problem.

In view of (6.20), we would like to replace $\{\psi_n\}_n$ with a new sequence $\{\Psi_n\}_n$ on $\mathbb{C}^2$ such that

$$\begin{aligned} |D^2_{\eta\eta}\Psi_n| &= 0 \quad \text{on} \quad \{(\zeta, \eta) \in \mathbb{C}^2 : |\eta|^q \leqslant |\zeta|^p\}, \\ |D^2_{\zeta\zeta}\Psi_n| &= 0 \quad \text{on} \quad \{(\zeta, \eta) \in \mathbb{C}^2 : |\eta|^q \geqslant |\zeta|^p\}, \\ |D^2_{\zeta\eta}\Psi_n| &= 0 \quad \text{on} \quad \mathbb{C}^2. \end{aligned} \tag{6.21}$$

6.2.3 The formal $\{\Psi_n\}_n$

In view of Proposition 6.2 and Proposition 6.6, to prove the bilinear embedding it suffices constructing an admissible sequence of truncations $\{\Psi_n\}_{n\in\mathbb{N}}$ such that

$$\left|(\mathcal{Q} * \varphi_\nu)(\omega) H^{(A,B)}_{\Psi_n}[\omega; \Xi] + G^{(A,B)}_{(\Psi_n, \mathcal{Q}*\varphi_\nu)}[\omega; \Xi]\right| \lesssim_\nu F(\omega; \Xi) \tag{6.22}$$

for all $\omega = (\zeta, \eta) \in \mathbb{C}^2$ and all $\Xi = (X, Y) \in \mathbb{C}^{2d}$, where F is defined in (6.18). In this section we will show the idea of such construction. The essential property that $\{\Psi_n\}_{n\in\mathbb{N}}$ needs to satisfy is (6.21).

Let $\phi \in C^\infty_c([0, \infty))$ be such that

- $0 \leqslant \phi \leqslant 1$;
- $\phi = 1$ on $[0, 3]$;
- $\phi = 0$ on $[4, \infty)$.

Let $p \geqslant 2$ and denote by q its conjugate exponent. Define the function $\Psi : \mathbb{C}^2 \to [0,1]$ by

$$\Psi(\zeta,\eta) := \begin{cases} \phi(|\eta|^q), & \text{if } |\zeta|^p \leqslant |\eta|^q; \\ \phi(|\zeta|^p), & \text{if } |\zeta|^p \geqslant |\eta|^q. \end{cases} \tag{6.23}$$

Clearly, $\Psi \in C_c(\mathbb{C}^2) \cap C^\infty(\mathbb{C}^2 \setminus \widetilde{\Upsilon}_0)$, where

$$\widetilde{\Upsilon}_0 := \{(\zeta,\eta) \in \mathbb{C}^2 \,:\, |\zeta|^p = |\eta|^q\}.$$

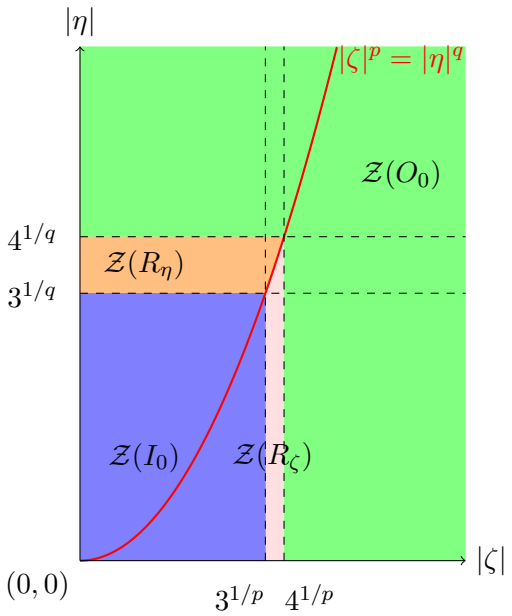
Moreover,

- $\Psi = 1$ on $I_0 := \{(\zeta,\eta) \in \mathbb{C}^2 \,:\, |\zeta|^p \leqslant 3, |\eta|^q \leqslant 3\}$;
- $\Psi = 0$ on $O_0 := \{(\zeta,\eta) \in \mathbb{C}^2 \,:\, |\zeta|^p \geqslant 4\} \cup \{(\zeta,\eta) \in \mathbb{C}^2 \,:\, |\eta|^q \geqslant 4\}$;
- $\partial_\eta \Psi = 0$ on $R_\zeta := \{(\zeta,\eta) \in \mathbb{C}^2 \,:\, |\zeta|^p > |\eta|^q, |\zeta|^p \in [3,4]\}$;
- $\partial_\zeta \Psi = 0$ on $R_\eta := \{(\zeta,\eta) \in \mathbb{C}^2 \,:\, |\zeta|^p < |\eta|^q, |\eta|^q \in [3,4]\}$.

Define the map $\mathcal{Z} : \mathbb{C}^2 \to \mathbb{R}^2_+$ by

$$\mathcal{Z}(\zeta,\eta) = (|\zeta|, |\eta|), \qquad \zeta,\eta \in \mathbb{C}. \tag{6.24}$$

We have the following partition of $\mathbb{R}^2_+$:



For all $n \in \mathbb{N}$ define the dilation $\mathcal{D}_n : \mathbb{C}^2 \to \mathbb{C}^2$ by

$$\mathcal{D}_n(\omega) = \left(\frac{\zeta}{n^{1/p}}, \frac{\eta}{n^{1/q}}\right), \qquad \omega = (\zeta,\eta) \in \mathbb{C}^2, \tag{6.25}$$

and the function $\Psi_n : \mathbb{C}^2 \to \mathbb{R}$ by

$$\Psi_n := \Psi \circ \mathcal{D}_n. \tag{6.26}$$

Explicitly,

$$\Psi_n(\zeta,\eta) = \begin{cases} \phi(|\eta|^q/n), & \text{if } |\zeta|^p \leqslant |\eta|^q; \\ \phi(|\zeta|^p/n), & \text{if } |\zeta|^p \geqslant |\eta|^q, \end{cases}$$

for any $\zeta, \eta \in \mathbb{C}$.

Each function Ψ_n is not C^2 on the entire $\mathbb{C}^2$. However, apart from the smoothness condition, it is straightforward to verify that $\{\Psi_n\}_{n\in\mathbb{N}}$ is an admissible sequence of truncations with

$$K_{1,n} = \mathcal{D}_n^{-1}(I_0),$$
$$K_{2,n} = \mathcal{D}_n^{-1}(\mathbb{C}^2 \setminus O_0),$$

for all $n \in \mathbb{N}$.

We now show why the sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ constitutes a natural candidate for proving the inequality in (6.22).

Fix

$$(\zeta,\eta) \in \mathbb{C}^2 \setminus \{(\zeta,\eta) \in \mathbb{C}^2 : |\zeta|^p = |\eta|^q, |\zeta|^p \in [3n,4n], |\eta|^q \in [3n,4n]\}.$$

We exclude this set since each function Ψ_n fails to be C^1 at such points. We have

$$|D^2_{\zeta\zeta}\Psi_n(\zeta,\eta)| \lesssim \begin{cases} \frac{1}{n^{2/p}}, & \text{if } |\zeta|^p > |\eta|^q,\ 3n \leqslant |\zeta|^p \leqslant 4n; \\ 0, & \text{otherwise,} \end{cases}$$
$$|D^2_{\eta\eta}\Psi_n(\zeta,\eta)| \lesssim \begin{cases} \frac{1}{n^{2/q}}, & \text{if } |\zeta|^p < |\eta|^q,\ 3n \leqslant |\eta|^q \leqslant 4n; \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand,

$$D^2_{\zeta\eta}\Psi_n(\zeta,\eta) = 0.$$

Therefore, from (6.2) we obtain

$$\left|H^{(A,B)}_{\Psi_n}[(\zeta,\eta);(X,Y)]\right| \lesssim \begin{cases} \frac{1}{n^{2/p}}|X|^2, & \text{if } |\zeta|^p > |\eta|^q,\ 3n \leqslant |\zeta|^p \leqslant 4n; \\ \frac{1}{n^{2/q}}|Y|^2, & \text{if } |\zeta|^p < |\eta|^q,\ 3n \leqslant |\eta|^q \leqslant 4n; \\ 0, & \text{otherwise.} \end{cases} \tag{6.27}$$

Hence, by combining the first estimate of (3.33) and (6.27), we get

$$\begin{aligned} &\left|(\mathcal{Q} * \varphi_\nu)(\zeta,\eta) H^{(A,B)}_{\Psi_n}[(\zeta,\eta);(X,Y)]\right| \\ &\qquad \lesssim_\nu \begin{cases} \left(|\zeta|^{p-2}+1\right)|X|^2, & \text{if } |\zeta|^p > |\eta|^q,\ 3n \leqslant |\zeta|^p \leqslant 4n; \\ \left(|\eta|^{q-2}+1\right)|Y|^2, & \text{if } |\zeta|^p < |\eta|^q,\ 3n \leqslant |\eta|^q \leqslant 4n; \\ 0, & \text{otherwise} \end{cases} \\ &\qquad \lesssim_\nu F(\zeta,\eta;X,Y), \end{aligned}$$

for all $X, Y \in \mathbb{C}^d$. The implied constants do not depend on $n \in \mathbb{N}$.

We would like the previous estimate to hold for *any* $(\zeta,\eta) \in \mathbb{C}^2$. The problem is that the function Ψ is not globally C^1, and thus neither is each Ψ_n. To address this, we will regularize Ψ by means of convolution with smooth approximation of identity. Once Ψ is regularized, the corresponding dilations from (6.26) are then considered. However, this regularization process implies that near $\widetilde{\Upsilon}_0$, the regularized version of Ψ might depend on both variables ζ and η. Nevertheless, this does not actually cause any issues because $|\zeta|^p$ and $|\eta|^q$ are comparable near $\widetilde{\Upsilon}_0$; see Lemma 6.8. This property allows us to establish the desired estimate (6.22) even in proximity to $\widetilde{\Upsilon}_0$.

Moreover, all the previous analysis has focused only on the first term on the left-hand side of (6.22). We also need to estimate the second one. It turns out the sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ remains well suited for estimating also such term, and no additional difficulties arise in this case.

6.3 The new truncation

In this section we will show the construction of an admissible sequence of truncations which will allow us to apply Proposition 6.6. As first step, we need to regularize the function Ψ defined in Section 6.2.3; see (6.28).

6.3.1 The regularized Ψ_κ

Let $p \geqslant 2$ and denote by q its conjugate exponent. Let Ψ be defined as in (6.23) and $\varphi \in C_c^\infty(\mathbb{R}^4)$ be radial such that $0 \leqslant \varphi \leqslant 1$, $\operatorname{supp}\varphi \subset B(0,1)$ and $\int \varphi = 1$. For all $\kappa \in (0,1)$ define the function $\varphi_\kappa(\cdot) = \kappa^{-4}\varphi(\cdot/\kappa)$.

Recall notation (3.32) and definition (6.24) of $\mathcal{Z}$. For all $\kappa \in (0,1)$ define the function $\Psi_\kappa : \mathbb{C}^2 \to [0,1]$ by

$$\Psi_\kappa := \Psi * \varphi_\kappa. \tag{6.28}$$

We aim to study the behavior of the regularized Ψ_κ in different regions of $\mathbb{C}^2$. To this end, it is convenient to work in $\mathbb{R}^2_+$ thanks to the identity

$$\Psi = \widetilde{\Psi} \circ \mathcal{Z},$$

with $\widetilde{\Psi}$ being defined in $\mathbb{R}^2_+$ by

$$\widetilde{\Psi}(s,t) := \begin{cases} \phi(s^p), & \text{if } s^p \geqslant t^q; \\ \phi(t^q), & \text{if } s^p \leqslant t^q. \end{cases}$$

Here, ϕ is the same function used for the construction of Ψ. Clearly, $\widetilde{\Psi} \in C_c(\mathbb{R}^2_+) \cap C^\infty(\mathbb{R}^2_+ \setminus \Upsilon_0)$, where

$$\Upsilon_0 := \left\{(s,t) \in \mathbb{R}^2_+ \,:\, s^p = t^q\right\}.$$

Moreover,

- $\widetilde{\Psi} = 1$ on $\{s^p \leqslant 3\} \cap \{t^q \leqslant 3\}$;
- $\widetilde{\Psi} = 0$ on $\{s^p \geqslant 4\} \cup \{t^q \geqslant 4\}$.

For all $\kappa \in (0,1)$ we define five regions of $\mathbb{R}^2_+$ as follows:

$$\begin{aligned} I_\kappa &:= [0, 3^{1/p} - \kappa) \times [0, 3^{1/q} - \kappa), \\ R_{\zeta,\kappa} &:= [3^{1/p} - \kappa, 4^{1/p} + \kappa] \times [0, 4^{1/q} + \kappa], \\ R_{\eta,\kappa} &:= [0, 4^{1/p} + \kappa] \times [3^{1/q} - \kappa, 4^{1/q} + \kappa], \\ R_\kappa &:= R_{\zeta,\kappa} \cup R_{\eta,\kappa}, \\ O_\kappa &:= \mathbb{R}^2_+ \setminus \left([0, 4^{1/p} + \kappa] \times [0, 4^{1/q} + \kappa]\right). \end{aligned} \tag{6.29}$$

It is clear that these regions form a partition of $\mathbb{R}^2_+$:

$$\mathbb{R}^2_+ = I_\kappa \sqcup R_\kappa \sqcup O_\kappa.$$

We now refine this partition further. For all $\kappa \in (0,1)$ define the following subsets of R_κ:

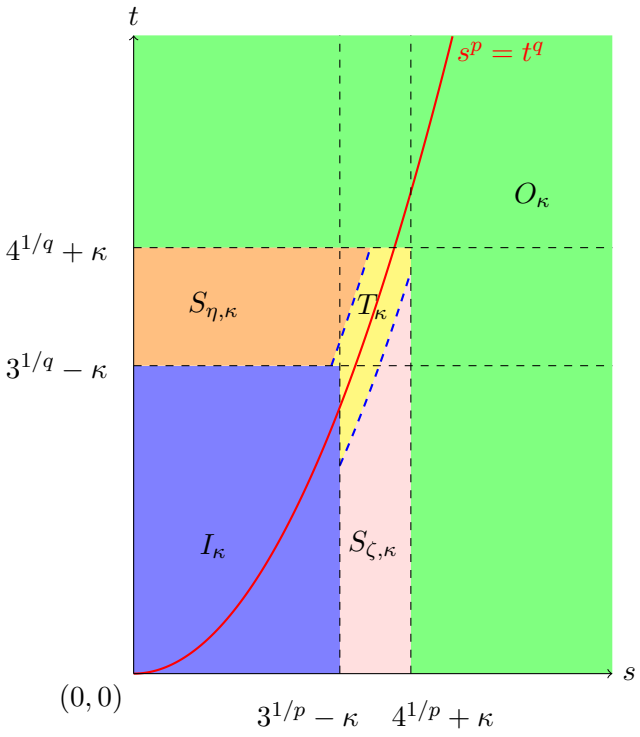
$$\begin{aligned} T_\kappa &:= \{(s,t) \in R_\kappa \,:\, \operatorname{dist}((s,t), \Upsilon_0) \leqslant \kappa\}, \\ S_{\zeta,\kappa} &:= (R_\kappa \cap \{s^p > t^q\}) \setminus T_\kappa, \\ S_{\eta,\kappa} &:= (R_\kappa \cap \{s^p < t^q\}) \setminus T_\kappa. \end{aligned} \tag{6.30}$$

They yield a partition of R_κ itself:

$$R_\kappa = S_{\zeta,\kappa} \sqcup S_{\eta,\kappa} \sqcup T_\kappa.$$

Therefore, for all $\kappa \in (0,1)$ we obtain the following full partition of $\mathbb{R}^2_+$,

$$\mathbb{R}^2_+ = I_\kappa \sqcup S_{\zeta,\kappa} \sqcup S_{\eta,\kappa} \sqcup T_\kappa \sqcup O_\kappa. \tag{6.31}$$



In what follows, we analyze the behavior of Ψ_k in each of the regions defined above. More precisely, we will show that:

- Ψ_k is constant on $\mathcal{Z}^{-1}(I_\kappa)$ and on $\mathcal{Z}^{-1}(O_\kappa)$;
- it depends only on the variable ζ on $\mathcal{Z}^{-1}(S_{\zeta,\kappa})$;
- it depends only on the variable η on $\mathcal{Z}^{-1}(S_{\eta,\kappa})$;
- and it may depend on both variables on $\mathcal{Z}^{-1}(T_\kappa)$.

Lemma 6.7. *Let $\kappa \in (0,1)$. Then, $\Psi_\kappa \in C^\infty_c(\mathbb{C}^2)$, $\Psi_\kappa = 1$ on $\mathcal{Z}^{-1}(I_\kappa)$ and $\Psi_\kappa = 0$ on $\mathcal{Z}^{-1}(O_\kappa)$. Moreover,*

(i) $\partial_\zeta \Psi_\kappa(\omega) = 0$ *if* $\mathcal{Z}(\omega) \notin S_{\zeta,\kappa} \cup T_\kappa$;

(ii) $\partial_\eta \Psi_\kappa(\omega) = 0$ *if* $\mathcal{Z}(\omega) \notin S_{\eta,\kappa} \cup T_\kappa$.

Proof. Recall notations (3.19) and (3.32). Since the support of φ_κ is contained in $B_{\mathbb{R}^4}(0,\kappa)$, we have

$$\begin{aligned} \Psi_\kappa(\omega) &= \int_{B(0,\kappa)} \Psi_{\mathcal{W}}(\mathcal{W}(\omega) - z)\varphi_\kappa(z)\, \mathrm{d}z \\ &= \int_{B(0,\kappa)} \Psi(\omega - \mathcal{W}^{-1}(z))\varphi_\kappa(z)\, \mathrm{d}z, \end{aligned} \tag{6.32}$$

for all $\omega \in \mathbb{C}^2$.

Assume now that $\omega \in \mathbb{C}^2$ is such that $\mathcal{Z}(\omega) \in I_\kappa$. Then, by triangle inequality,

$$|\zeta - \mathcal{V}^{-1}(x)| < 3^{1/p}, \qquad |\eta - \mathcal{V}^{-1}(y)| < 3^{1/q},$$

for all $z = (x, y) \in B_{\mathbb{R}^4}(0, \kappa)$. Therefore, $\Psi(\omega - \mathcal{W}^{-1}(z)) = 1$ for all such z, and so by (6.32) we obtain

$$\Psi_\kappa(\omega) = \int_{\mathbb{R}^4} \varphi_\kappa(z)\, \mathrm{d}z = 1.$$

A similar argument shows that $\Psi_\kappa(\omega) = 0$ for all $\omega \in \mathcal{Z}^{-1}(O_\kappa)$.

We now prove only (i); item (ii) follows analogously. First, we claim that

$$\partial_\zeta \Psi_\kappa(\omega) = 0, \qquad \forall \mathcal{Z}(\omega) \notin \{s^p > t^q\} \cup T_\kappa. \tag{6.33}$$

Fix $\omega \in \mathbb{C}^2$ such that $\mathcal{Z}(\omega) \notin \{s^p > t^q\} \cup T_\kappa$. Observe that $\mathcal{Z}(\omega) \notin T_\kappa$ implies

$$B_{\mathbb{R}^2}(\mathcal{Z}(\omega), \kappa) \cap \Upsilon_0 = \emptyset.$$

Indeed, if there existed (s, t) in that intersection, then

$$\begin{aligned} \kappa &< \operatorname{dist}(\mathcal{Z}(\omega), \Upsilon_0) \\ &\leqslant |\mathcal{Z}(\omega) - (s, t)| \\ &< \kappa, \end{aligned}$$

which is a contradiction. Therefore,

$$B_{\mathbb{R}^2}(\mathcal{Z}(\omega), \kappa) \subseteq \{s^p \leqslant t^q\}, \tag{6.34}$$

since $\mathcal{Z}(\omega) \in \{s^p \leqslant t^q\}$. On the other hand, for every $z = (x, y) \in B_{\mathbb{R}^4}(0, \kappa)$ we have

$$\mathcal{Z}\left(\omega - \mathcal{W}^{-1}(z)\right) \in B_{\mathbb{R}^2}(\mathcal{Z}(\omega), \kappa), \tag{6.35}$$

since

$$\Big||\zeta - \mathcal{V}^{-1}(x)| - |\zeta|\Big|^2 + \Big||\eta - \mathcal{V}^{-1}(y)| - |\eta|\Big|^2 \leqslant |\mathcal{V}^{-1}(x)|^2 + |\mathcal{V}^{-1}(y)|^2 < \kappa^2.$$

By combining definitions (6.23) and (6.28) with (6.32), (6.34) and (6.35), we obtain (6.33).

Moreover, we already showed that Ψ_κ is constant on $\mathcal{Z}^{-1}(I_\kappa \cup O_\kappa) = \mathcal{Z}^{-1}(\mathbb{R}^2_+ \setminus R_\kappa)$, so $\partial_\zeta \Psi_\kappa(\omega) = 0$ also when $\mathcal{Z}(\omega) \notin R_\kappa$. Therefore, $\partial_\zeta \Psi_\kappa(\omega) = 0$ if

$$\mathcal{Z}(\omega) \notin R_\kappa \cap (\{s^p > t^q\} \cup T_\kappa).$$

We conclude by observing that

$$\begin{aligned} R_\kappa \cap (\{s^p > t^q\} \cup T_\kappa) &= (R_\kappa \cap \{s^p > t^q\}) \cup T_\kappa \\ &= [(R_\kappa \cap \{s^p > t^q\}) \setminus T_\kappa] \cup T_\kappa \\ &= S_{\zeta,\kappa} \cup T_\kappa, \end{aligned}$$

where we used the fact that $T_\kappa \subseteq R_\kappa$ in the first equality and the definition (6.30) of $S_{\zeta,\kappa}$ in the last one. □

On $\mathcal{Z}^{-1}(T_\kappa)$ the function Ψ_κ might depend on both ζ and η. However, the next lemma ensures that $|\zeta|^p$ and $|\eta|^q$ are comparable in that region.

Lemma 6.8. *Let $p \geqslant 2$, $q = p/(p-1)$ and $\kappa \in (0,1)$. Then there exists a positive constant $\delta = \delta(p)$ such that*

$$s^p \sim_\kappa t^q,$$

for all $(s,t) \in T_\kappa$ and $\kappa \in (0,\delta)$.

In order to prove the previous lemma we need the following auxiliary result. For all $(s,t) \in T_\kappa$, with $\kappa \in (0,1)$ small enough, we denote by (s',t') the unique element in Υ_0 such that

$$|(s,t) - (s',t')| = \operatorname{dist}((s,t), \Upsilon_0) \leqslant \kappa. \tag{6.36}$$

Lemma 6.9. *Let $p \geqslant 2$, $q = p/(p-1)$ and $\kappa \in (0,1)$. Suppose that $(s,t) \in T_\kappa$. Then there exist two positive constants $\delta = \delta(p)$ and $C = C(p)$, not depending on κ, such that*

$$s, t \geqslant C$$

for all $\kappa \in (0,\delta)$.

Proof. Let $(s,t) \in T_\kappa$. Since $T_\kappa \subseteq R_\kappa$, either $s \geqslant 3^{1/p} - \kappa$ or $t \geqslant 3^{1/q} - \kappa$. Suppose we are in the first case. Then, by (6.36),

$$s' = s' - s + s \geqslant -\kappa + s \geqslant 3^{1/p} - 2\kappa. \tag{6.37}$$

Since $(s',t') \in \Upsilon_0$, we have $(s')^p = (t')^q$. Hence,

$$t = t - t' + t' = t - t' + (s')^{p/q} \geqslant -\kappa + (3^{1/p} - 2\kappa)^{p/q},$$

where in the last inequality we used (6.36) and (6.37). Choosing $\delta > 0$ sufficiently small, both s and t remain uniformly bounded from below by a positive constant $C = C(p,q)$.

The case $t \geqslant 3^{1/q} - \kappa$ is entirely analogous. □

Proof of Lemma 6.8. Let $(s,t) \in T_\kappa$. As $(s',t') \in \Upsilon_0$ we get

$$\begin{aligned} s - t^{q/p} &= s - (t')^{q/p} + (t')^{q/p} - t^{q/p} \\ &= s - s' + (t')^{q/p} - t^{q/p}. \end{aligned} \tag{6.38}$$

By Lemma 6.9 and (6.36) there exist $C, \delta > 0$, not depending on κ, such that

$$s, t, s', t' \in [C, 1/C], \tag{6.39}$$

for all $\kappa \in (0,\delta)$. Since the function $[C,1/C] \ni x \mapsto x^{q/p}$ is $L = L(C)$-Lipschitz continuous, we obtain for all $\kappa \in (0,\delta)$,

$$|(t')^{q/p} - t^{q/p}| \leqslant L|t' - t| \leqslant L\kappa, \tag{6.40}$$

where in the last inequality we used (6.36). Therefore, by combining (6.36), (6.38), (6.39) and (6.40), we get

$$\begin{aligned} |s - t^{q/p}| &\leqslant (1+L)\kappa \leqslant \frac{1+L}{C^{q/p}} \kappa \cdot t^{q/p}, \\ |s - t^{q/p}| &\leqslant (1+L)\kappa \leqslant \frac{1+L}{C} \kappa \cdot s. \end{aligned}$$

Thus,

$$\begin{aligned} s^p &\leqslant \left(1 + \frac{1+L}{C^{q/p}}\kappa\right)^p t^q, \\ t^q &\leqslant \left(1 + \frac{1+L}{C}\kappa\right)^p s^p, \end{aligned}$$

which imply that $s^p \sim_\kappa t^q$. □

Corollary 6.10. *Let $p \geqslant 2$, $q = p/(p-1)$, $\kappa \in (0,1)$ and $n \in \mathbb{N}_+$. Then there exists a positive constant $\delta = \delta(p)$, not depending on n, such that*

$$s^p \sim_\kappa t^q,$$

for all $(s,t) \in \{(u,v) \in \mathbb{R}^2_+ : (u/n^{1/p}, v/n^{1/q}) \in T_\kappa\}$ and $\kappa \in (0,\delta)$. The implied constants do not depend on n.

Proof. It follows by combining Lemma 6.8 with the identities

$$\left(\frac{s}{n^{1/p}}\right)^p = \frac{s^p}{n}, \qquad \left(\frac{t}{n^{1/q}}\right)^q = \frac{t^q}{n},$$

which holds for $n \in \mathbb{N}$ and $s,t \in \mathbb{R}^2_+$. □

6.3.2 The sequence $\{\Psi_n\}_{n\in\mathbb{N}}$

Let $p \geqslant 2$ and denote by q its conjugate exponent. Recall definitions (6.29) and (6.30). Fix $\kappa \in (0,\delta)$, where δ is the positive constant of Lemma 6.8 and Corollary 6.10. For all $n \in \mathbb{N}$ define the function $\Psi_{\kappa,n} : \mathbb{C}^2 \to [0,1]$ by

$$\Psi_{\kappa,n} := \Psi_\kappa \circ \mathcal{D}_n, \tag{6.41}$$

where Ψ_κ is the regularization of Ψ, both defined in (6.28), and $\mathcal{D}_n$ is the dilation operator introduced in (6.26). Clearly, each $\Psi_{\kappa,n}$ is a smooth function on the entire $\mathbb{C}^2$. Moreover, it is straightforward to verify that $\{\Psi_{\kappa,n}\}_{n\in\mathbb{N}}$ is an admissible sequence of truncations with

$$\begin{aligned} K_{1,n} &= (\mathcal{Z} \circ \mathcal{D}_n)^{-1}(I_\kappa), \\ K_{2,n} &= (\mathcal{Z} \circ \mathcal{D}_n)^{-1}(O_\kappa), \end{aligned}$$

for all $n \in \mathbb{N}$.

Since the regularization parameter κ is fixed, from now we will omit that and denote only by $\{\Psi_n\}_{n\in\mathbb{N}}$ the sequence of admissible truncations defined in (6.41), which should not be confused with the regularization Ψ_κ of Ψ. For the same reason, we set

$$\begin{aligned} I &= I_\kappa, & O &= O_\kappa, \\ S_\zeta &= S_{\zeta,\kappa}, & S_\eta &= S_{\eta,\kappa}, \\ T &= T_\kappa, & R &= R_\kappa. \end{aligned}$$

We are ready to estimate the first- and second-order derivatives of Ψ_n.

Corollary 6.11. *Let $n \in \mathbb{N}$. Then*

$$|\partial_\zeta \Psi_n(\omega)| \lesssim \begin{cases} \frac{1}{n^{1/p}}, & \text{if } (\mathcal{Z} \circ \mathcal{D}_n)(\omega) \in S_\zeta \cup T, \\ 0, & \text{otherwise,} \end{cases}$$

$$|\partial_\eta \Psi_n(\omega)| \lesssim \begin{cases} \frac{1}{n^{1/q}}, & \text{if } (\mathcal{Z} \circ \mathcal{D}_n)(\omega) \in S_\eta \cup T, \\ 0, & \text{otherwise.} \end{cases}$$

The implied constants do not depend on n.

Proof. Let us prove only the first estimate. Clearly,

$$\partial_\zeta \Psi_n(\omega) = \frac{1}{n^{1/p}} \partial_\zeta \Psi_\kappa(\mathcal{D}_n(\omega)), \tag{6.42}$$

for every $\omega \in \mathbb{C}^2$. Thus, by Lemma 6.7 we obtain that $\partial_\zeta \Psi_n(\omega) = 0$ if $(\mathcal{Z} \circ \mathcal{D}_n)(\omega) \notin S_\zeta \cup T$.

On the other hand, (6.42) implies that

$$|\partial_\zeta \Psi_n(\omega)| \lesssim \frac{1}{n^{1/p}},$$

if $(\mathcal{Z} \circ \mathcal{D}_n)(\omega) \in S_\zeta \cup T$. □

Corollary 6.12. *Let $n \in \mathbb{N}$. Then*

$$|D^2_{\zeta\zeta} \Psi_n(\omega)| \lesssim \begin{cases} \frac{1}{n^{2/p}}, & \text{if } (\mathcal{Z} \circ \mathcal{D}_n)(\omega) \in S_\zeta \cup T, \\ 0, & \text{otherwise,} \end{cases}$$

$$|D^2_{\eta\eta} \Psi_n(\omega)| \lesssim \begin{cases} \frac{1}{n^{2/q}}, & \text{if } (\mathcal{Z} \circ \mathcal{D}_n)(\omega) \in S_\eta \cup T, \\ 0, & \text{otherwise,} \end{cases}$$

$$|D^2_{\zeta\eta} \Psi_n(\omega)| \lesssim \begin{cases} \frac{1}{n^{1/p}} \cdot \frac{1}{n^{1/q}}, & \text{if } (\mathcal{Z} \circ \mathcal{D}_n)(\omega) \in T, \\ 0, & \text{otherwise.} \end{cases}$$

The implied constants do not depend on n.

Proof. The first two estimates can be proven in the same way as in Corollary 6.11. Let us prove only the third one. Clearly,

$$|D^2_{\zeta\eta} \Psi_n(\omega)| = \frac{1}{n^{1/p}} \cdot \frac{1}{n^{1/q}} |D_{\zeta\eta} \Psi_\kappa(\mathcal{D}_n(\omega))|, \tag{6.43}$$

for all $\omega \in \mathbb{C}^2$. Then, by Lemma 6.7, we obtain that $|D^2_{\zeta\eta} \Psi_n(\omega)| = 0$ if

$$(\mathcal{Z} \circ \mathcal{D}_n)(\omega) \notin (S_\zeta \cup T) \cap (S_\eta \cup T) = T,$$

where the last equality holds since $S_\zeta \cap S_\eta = \emptyset$.

On the other hand, (6.43) implies that

$$|D^2_{\zeta\eta} \Psi_n(\omega)| \lesssim \frac{1}{n^{1/p}} \cdot \frac{1}{n^{1/q}},$$

if $(\mathcal{Z} \circ \mathcal{D}_n)(\omega) \in T$. □

Corollary 6.13. *Let $A, B \in \mathcal{A}(\Omega)$ and $n \in \mathbb{N}$. Then*

$$|H^{(A,B)}_{\Psi_n}[\omega; (X, Y)]| \lesssim \begin{cases} \frac{1}{n^{2/p}} |X|^2, & \text{if } (\mathcal{Z} \circ \mathcal{D}_n)(\omega) \in S_\zeta \\ \frac{1}{n^{2/q}} |Y|^2, & \text{if } (\mathcal{Z} \circ \mathcal{D}_n)(\omega) \in S_\eta, \\ \frac{1}{n^{2/p}} |X|^2 + \frac{1}{n^{2/q}} |Y|^2, & \text{if } (\mathcal{Z} \circ \mathcal{D}_n)(\omega) \in T, \\ 0, & \text{otherwise,} \end{cases} \tag{6.44}$$

for all $X, Y \in \mathbb{C}^d$. The implied constant does not depend on n.

Proof. It follows by combining (6.2), applied with $\Phi = \Psi_n$, with Corollary 6.12, (6.31) and the inequality

$$\frac{1}{n^{1/p}} |X| \cdot \frac{1}{n^{1/q}} |Y| \lesssim \frac{1}{n^{2/p}} |X|^2 + \frac{1}{n^{2/q}} |Y|^2.$$ □

6.4 Proof of the bilinear embedding

Let $\Omega \subseteq \mathbb{R}^d$, $p \geqslant 2$ and denote by q its conjugate exponent. Suppose that $A, B \in \mathcal{A}(\Omega)$ are p-elliptic and $\mathscr{V}$ and $\mathscr{W}$ satisfy Assumption BE. From Proposition 6.2 we know that

$$\left(|u|^{p-2}+1\right)|\nabla u|^2 + \left(|v|^{q-2}\mathbb{1}_{\{v\neq 0\}}+1\right)|\nabla v|^2 \in L^1(\Omega),$$

for all $u \in \mathscr{V}$ and $v \in \mathscr{W}$ such that $u, v, \mathscr{L}^A u, \mathscr{L}^B v \in (L^p \cap L^q)(\Omega)$. Therefore, Proposition 6.6 ensures that to prove Theorem 3.19 it suffices to establish the validity of (6.22) for the sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ introduced in Section 6.3.2 and the function F defined in (6.18). That is, we must show that

$$\left|(\mathcal{Q}*\varphi_\nu)(\omega)H^{A,B}_{\Psi_n}[\omega;(X,Y)] + G^{(A,B)}_{(\Psi_n,\mathcal{Q}*\varphi_\nu)}[\omega;(X,Y)]\right| \lesssim_\nu F(\omega;(X,Y)),$$

for all $\omega = (\zeta,\eta) \in \mathbb{C}^2$ and $X, Y \in \mathbb{C}^d$.

We proceed by estimating the terms on the left-hand side of the inequality separately.

6.4.1 Estimate of $(\mathcal{Q}*\varphi_\nu)H^{A,B}_{\Psi_n}$

Proposition 6.14. *Let $A, B \in \mathcal{A}(\Omega)$, $\nu \in (0,1)$ and $n \in \mathbb{N}$. Then there exists $C = C(\nu) > 0$, not depending on n, such that*

$$\begin{aligned}&\left|(\mathcal{Q}*\varphi_\nu)(\omega)H^{A,B}_{\Psi_n}[\omega;(X,Y)]\right| \\ &\qquad\leqslant C\cdot\begin{cases}\left(|\zeta|^{p-2}+1\right)|X|^2, & \text{if } (\mathcal{Z}\circ\mathcal{D}_n)(\omega)\in S_\zeta,\\ \left(|\eta|^{q-2}+1\right)|Y|^2, & \text{if } (\mathcal{Z}\circ\mathcal{D}_n)(\omega)\in S_\eta\\ \left(|\zeta|^{p-2}+1\right)|X|^2+\left(|\eta|^{q-2}+1\right)|Y|^2, & \text{if } (\mathcal{Z}\circ\mathcal{D}_n)(\omega)\in T\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

for all $X, Y \in \mathbb{C}^d$.

Proof. Let $\omega = (\zeta,\eta) \in \mathbb{C}^2$. By Corollary 6.13 it suffices to consider the case when $(\mathcal{Z}\circ\mathcal{D}_n)(\omega) \in S_\zeta \cup S_\eta \cup T$.

Suppose that $\boxed{(\mathcal{Z}\circ\mathcal{D}_n)(\omega)\in S_\zeta}$. In particular, we have $|\zeta|^p > |\eta|^q$, which, combined with the first estimate of (3.33), yields

$$(\mathcal{Q}*\varphi_\nu)(\omega) \lesssim_\nu |\zeta|^p + 1. \tag{6.45}$$

Moreover, $(\mathcal{Z}\circ\mathcal{D}_n)(\omega) \in S_\zeta$ implies $|\zeta|^p \lesssim n$. Therefore, from Corollary 6.13 and (6.45) we deduce that

$$\begin{aligned}\left|(\mathcal{Q}*\varphi_\nu)(\omega)H^{A,B}_{\Psi_n}[\omega;(X,Y)]\right| &\lesssim_\nu \frac{\left(|\zeta|^p+1\right)|X|^2}{n^{2/p}}\\ &\lesssim_\nu \left(|\zeta|^{p-2}+1\right)|X|^2,\end{aligned}$$

for all $X, Y \in \mathbb{C}^d$.

If $\boxed{(\mathcal{Z}\circ\mathcal{D}_n)(\omega)\in S_\eta}$, we prove in the same way as before that

$$\left|(\mathcal{Q}*\varphi_\nu)(\omega)H^{A,B}_{\Psi_n}[\omega;(X,Y)]\right| \lesssim_\nu \left(|\eta|^{q-2}+1\right)|Y|^2,$$

for all $X, Y \in \mathbb{C}^d$.

Finally, suppose that $\boxed{(\mathcal{Z}\circ\mathcal{D}_n)(\omega)\in T}$. By Corollary 6.10, we have

$$|\zeta|^p \sim |\eta|^q. \tag{6.46}$$

Therefore, from Corollary 6.13, the first estimate of (3.33) and (6.46) we obtain

$$\begin{aligned}\left|(\mathcal{Q}*\varphi_\nu)(\omega)\cdot H^{A,B}_{\Psi_n}[\omega;(X,Y)]\right| &\lesssim_\nu (|\zeta|^p+|\eta|^q+1)\left(\frac{|X|^2}{n^{2/p}}+\frac{|Y|^2}{n^{2/q}}\right)\\ &\lesssim_\nu (|\zeta|^p+1)\frac{|X|^2}{n^{2/p}}+(|\eta|^q+1)\frac{|Y|^2}{n^{2/q}},\end{aligned}$$

for every $X,Y\in\mathbb{C}^d$. Now, we conclude by observing that

$$\frac{1}{n^{1/p}}\lesssim\frac{1}{|\zeta|},\qquad \frac{1}{n^{1/q}}\lesssim\frac{1}{|\eta|},$$

which hold since $T\subseteq R$. □

6.4.2 Estimate of $G^{(A,B)}_{(\Psi_n,\mathcal{Q}*\varphi_\nu)}$

Proposition 6.15. *Let $A,B\in\mathcal{A}(\Omega)$, $\nu\in(0,1)$ and $n\in\mathbb{N}$. Then there exists $C=C(\nu)>0$, not depending on n, such that*

$$\begin{aligned}&|G^{(A,B)}{}_{(\Psi_n,\mathcal{Q}*\varphi_\nu)}[\omega;(X,Y)]|\\ &\qquad\leqslant C\cdot\begin{cases}(|\zeta|^{p-2}+1)\,|X|^2+|X||Y|, & \text{if } (\mathcal{Z}\circ\mathcal{D}_n)(\omega)\in S_\zeta,\\ (|\eta|^{q-2}+1)\,|Y|^2+|X||Y|, & \text{if } (\mathcal{Z}\circ\mathcal{D}_n)(\omega)\in S_\eta,\\ (|\zeta|^{p-2}+1)\,|X|^2+(|\eta|^{q-2}+1)\,|Y|^2, & \text{if } (\mathcal{Z}\circ\mathcal{D}_n)(\omega)\in T,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

for all $X,Y\in\mathbb{C}^d$.

Proof. By (6.5) we need to estimate the following four terms

$$\begin{aligned}&|\partial_\zeta\Psi_n(\omega)\cdot\partial_\zeta(\mathcal{Q}*\varphi_\nu)(\omega)||X|^2,\\ &|\partial_\eta\Psi_n(\omega)\cdot\partial_\eta(\mathcal{Q}*\varphi_\nu)(\omega)||Y|^2,\\ &|\partial_\zeta\Psi_n(\omega)\cdot\partial_\eta(\mathcal{Q}*\varphi_\nu)(\omega)||X||Y|,\\ &|\partial_\eta\Psi_n(\omega)\cdot\partial_\zeta(\mathcal{Q}*\varphi_\nu)(\omega)||X||Y|,\end{aligned}\tag{6.47}$$

for all $\omega=(\zeta,\eta)\in\mathbb{C}^2$ and all $X,Y\in\mathbb{C}^d$.

We now estimate the first term. If $(\mathcal{Z}\circ\mathcal{D}_n)(\omega)\notin S_\zeta\cup T$, the first term vanishes by Corollary 6.11. Therefore, we may assume that $(\mathcal{Z}\circ\mathcal{D}_n)(\omega)\in S_\zeta\cup T$. In this region, from the definition of S_ζ and Corollary 6.10 we have either

$$|\zeta|^p\geqslant|\eta|^q\quad\text{ or }\quad|\zeta|^p\sim|\eta|^q.\tag{6.48}$$

Thus, the second estimate in (3.33) and Corollary 6.11 imply

$$\begin{aligned}|\partial_\zeta\Psi_n(\omega)\cdot\partial_\zeta(\mathcal{Q}*\varphi_\nu)(\omega)||X|^2&\lesssim_\nu\frac{1}{n^{1/p}}\left(|\zeta|^{p-1}+1\right)|X|^2\\ &\lesssim\left(|\zeta|^{p-2}+1\right)|X|^2,\end{aligned}$$

where in the last inequality we used that $|\zeta|^p \lesssim n$, which follows by the fact that $S_\zeta \cup T \subseteq R$. Therefore, we get

$$|\partial_\zeta \Psi_n(\omega) \cdot \partial_\zeta (\mathcal{Q} * \varphi_\nu)(\omega)||X|^2 \lesssim_\nu \begin{cases} (|\zeta|^{p-2}+1)\,|X|^2, & \text{if } (\mathcal{Z}\circ\mathcal{D}_n)(\omega) \in S_\zeta \cup T, \\ 0, & \text{otherwise.} \end{cases} \tag{6.49}$$

In the same way, one can prove that

$$|\partial_\eta \Psi_n(\omega) \cdot \partial_\eta (\mathcal{Q} * \varphi_\nu)(\omega)||Y|^2 \lesssim_\nu \begin{cases} (|\eta|^{q-2}+1)\,|Y|^2, & \text{if } (\mathcal{Z}\circ\mathcal{D}_n)(\omega) \in S_\eta \cup T, \\ 0, & \text{otherwise.} \end{cases} \tag{6.50}$$

We now estimate the third term in (6.47). As before, by Corollary 6.11 this term vanishes if $(\mathcal{Z}\circ\mathcal{D}_n)(\omega) \notin S_\zeta \cup T$. Hence, we may assume that $(\mathcal{Z}\circ\mathcal{D}_n)(\omega) \in S_\zeta \cup T$. Then, the third estimate in (3.33), Corollary 6.11 and (6.48) give

$$\begin{aligned} |\partial_\zeta \Psi_n(\omega) \cdot \partial_\eta (\mathcal{Q} * \varphi_\nu)(\omega)||X||Y| &\lesssim_\nu \frac{1}{n^{1/p}}\,(|\zeta|+1)\,|X||Y| \\ &\lesssim |X||Y|, \end{aligned}$$

where in the last step we used again the bound $|\zeta|^p \lesssim n$. Therefore, we conclude that

$$|\partial_\zeta \Psi_n(\omega) \cdot \partial_\eta (\mathcal{Q} * \varphi_\nu)(\omega)||X||Y| \lesssim_\nu \begin{cases} |X||Y|, & \text{if } (\mathcal{Z}\circ\mathcal{D}_n)(\omega) \in S_\zeta \cup T, \\ 0, & \text{otherwise.} \end{cases} \tag{6.51}$$

Similarly, one can show that

$$|\partial_\eta \Psi_n(\omega) \cdot \partial_\zeta (\mathcal{Q} * \varphi_\nu)(\omega)||X||Y| \lesssim_\nu \begin{cases} |X||Y|, & \text{if } (\mathcal{Z}\circ\mathcal{D}_n)(\omega) \in S_\eta \cup T, \\ 0, & \text{otherwise.} \end{cases} \tag{6.52}$$

Finally, combining (6.49)–(6.52) yields the desired result. □

By combining Propositions 6.14 and 6.15 with the fact that $\eta \neq 0$ whenever $(\mathcal{Z}\circ\mathcal{D}_n)(\zeta,\eta) \in S_\eta \cup T$, we obtain (6.22). Therefore, invoking Proposition 6.2 together with Proposition 6.6 we conclude the new proof of Theorem 3.19.

APPENDIX A

Interior elliptic regularity

In Section 4.1 we showed that we can associate a contractive and analytic semigroup $(T_t)_{t>0}$ on $L^2(\Omega)$ with the sesquilinear form defined in (4.1), provided that its coefficients satisfy conditions (4.3) and (4.5). In this section we will see that if we assume that the coefficients are also smooth, then $T_t f$ is also smooth for all $t > 0$ and $f \in L^2(\Omega)$.

For every function f on $\mathbb{R}^d$ and $y \in \mathbb{R}^d \setminus \{0\}$, we introduce the functions $\tau_y f$, $\delta_y f$ on $\mathbb{R}^d$ by the following rule:

$$\tau_y f(x) = f(x - y),$$
$$\delta_y f(x) = \frac{\tau_{-y} f(x) - f(x)}{|y|},$$

for every $x \in \mathbb{R}^d$.

We will use the following lemmas to prove Lemma A.4. See, for example, [44, Proposition 4.8] for their proofs.

Lemma A.1. *Let $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d)$, $\nu \in \mathbb{S}^{d-1}$, $R > 0$ and $x_0 \in \mathbb{R}^d$. Then*

$$\int_{B(x_0,R)} |\delta_{h\nu} f|^2 \leqslant \int_{B(x_0,R+h)} |\partial_\nu f|^2, \quad \forall h > 0.$$

Lemma A.2. *Let $f \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$. Then $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d)$ if and only if*

$$\limsup_{h \to 0} \|\delta_{h\nu} f\|_{L^2_{\mathrm{loc}}} < \infty, \quad \forall \nu \in \mathbb{S}^{d-1}.$$

In this case,

$$\|\partial_\nu f\|_{L^2(B)} \leqslant \limsup_{h \to 0} \|\delta_{h\nu} f\|_{L^2(B)},$$

for every compact set $B \subset \mathbb{R}^d$.

Let $\Omega \subseteq \mathbb{R}^d$ be open, A a complex $d \times d$ matrix-valued function on Ω, b, c two complex d-dimensional vector-valued functions on Ω and V a nonnegative function on Ω. Set $\mathscr{A} = (A, b, c, V)$. We will write

$$\mathfrak{a}_{\mathscr{A}}(u, v) = \int_\Omega \left[\langle A \nabla u, \nabla v \rangle_{\mathbb{C}^d} + \left\langle \nabla u, \overline{b} \right\rangle_{\mathbb{C}^d} \overline{v} + u \langle c, \nabla v \rangle_{\mathbb{C}^d} + V u \overline{v} \right]$$

for every $u, v \in W^{1,2}_{\mathrm{loc}}(\Omega)$ for which the integral on the right-hand side is finite. Note that now we are not assuming that the domain of the form is of the type described in Section 3.1.1.

Lemma A.3. *Let $\Omega \subseteq \mathbb{R}^d$ be open, $A \in L^\infty(\Omega, \mathbb{C}^{d\times d})$, $b, c \in L^\infty(\Omega, \mathbb{C}^d)$ and $V \in L^\infty(\Omega, \mathbb{C})$. Take $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Suppose that there exists $f \in L^2_{\mathrm{loc}}(\Omega)$ such that*

$$\mathfrak{a}_{\mathscr{A}}(u, \varphi) = \int_\Omega f\,\overline{\varphi}, \quad \forall \varphi \in C_c^\infty(\Omega). \tag{A.1}$$

Then

$$\mathfrak{a}_{\mathscr{A}}(u, \varphi) = \int_\Omega f\,\overline{\varphi}, \quad \forall \varphi \in W^{1,2}_c(\Omega),$$

where $W^{1,2}_c(\Omega)$ consists of all functions in $W^{1,2}(\Omega)$ having compact support in Ω.

Proof. Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $f \in L^2_{\mathrm{loc}}(\Omega)$ for which (A.1) holds. Let $\varphi \in W^{1,2}(\Omega)$ with compact support in Ω and $\Omega' \Subset \Omega$ such that $\operatorname{supp}\varphi \Subset \Omega'$. By [1, Lemma 2.18(b) and Lemma 3.15] there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega')$ such that

$$\varphi_n \to \varphi \ \text{ in } W^{1,2}(\Omega'). \tag{A.2}$$

Therefore, by using also the assumption on u and the boundedness of A, b, c and V (thanks to which we have, for instance, that $Vu \in L^2(\Omega')$), we get

$$\begin{aligned}
\mathfrak{a}_{\mathscr{A}}(u, \varphi) &= \int_{\Omega'} \left[\langle A\nabla u, \nabla\varphi\rangle + \left\langle \nabla u, \overline{b}\right\rangle \overline{\varphi} + u\,\langle c, \nabla\varphi\rangle + Vu\overline{\varphi}\right] \\
&= \lim_{n\to\infty} \int_{\Omega'} \left[\langle A\nabla u, \nabla\varphi_n\rangle + \left\langle \nabla u, \overline{b}\right\rangle \overline{\varphi_n} + u\,\langle c, \nabla\varphi_n\rangle + Vu\overline{\varphi_n}\right] \\
&= \lim_{n\to\infty} \mathfrak{a}_{\mathscr{A}}(u, \varphi_n).
\end{aligned} \tag{A.3}$$

Since $\varphi_n \in C_c^\infty(\Omega')$ for all $n \in \mathbb{N}$ and $\Omega' \Subset \Omega$, by (A.1) we have

$$\mathfrak{a}_{\mathscr{A}}(u, \varphi_n) = \int_\Omega f\,\overline{\varphi_n} = \int_{\Omega'} f\,\overline{\varphi_n},$$

for all $n \in \mathbb{N}$. On the other hand, $f \in L^2(\Omega')$. Hence by (A.2) we obtain

$$\lim_{n\to\infty} \mathfrak{a}_{\mathscr{A}}(u, \varphi_n) = \int_{\Omega'} f\,\overline{\varphi} = \int_\Omega f\,\overline{\varphi}. \tag{A.4}$$

We conclude by combining (A.3) and (A.4). □

The following lemma was proved in [44, Theorem 4.9] when $b, c, V = 0$. We adapt its proof to the general case when b, c and V are not null.

Lemma A.4. *Let $\Omega \subseteq \mathbb{R}^d$ be open, $A \in (\mathrm{Lip}_{\mathrm{loc}} \cap L^\infty)(\Omega, \mathbb{C}^{d\times d})$, $b, c \in (\mathrm{Lip}_{\mathrm{loc}} \cap L^\infty)(\Omega, \mathbb{C}^d)$ and $V \in (\mathrm{Lip}_{\mathrm{loc}} \cap L^\infty)(\Omega, \mathbb{C})$ such that $\mathscr{A} = (A, b, c, \operatorname{Re} V) \in \mathcal{B}(\Omega)$. Take $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Suppose that there exists $f \in L^2_{\mathrm{loc}}(\Omega)$ such that*

$$\mathfrak{a}_{\mathscr{A}}(u, \varphi) = \int_\Omega f\,\overline{\varphi}, \quad \forall \varphi \in C_c^\infty(\Omega). \tag{A.5}$$

Then $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$.

Proof. Fix $j \in \{1, \dots, d\}$ and set

$$\begin{aligned}
\tau_h &= \tau_{he_j}, \\
\delta_h &= \delta_{he_j}, \\
\mathscr{A}_h &= (\tau_{-h}A, \tau_{-h}b, \tau_{-h}c, \tau_{-h}V) = \tau_{-h}\mathscr{A}, \\
\delta_h(\mathscr{A}) &= (\delta_h A, \delta_h b, \delta_h c, \delta_h V),
\end{aligned}$$

where $e_j \in \mathbb{R}^d$ is the unit vector $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the j-th position.

Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $f \in L^2_{\mathrm{loc}}(\Omega)$ for which (A.5) holds. Let $\varphi \in W^{1,2}(\Omega)$ with compact support in Ω. If h is small enough, $\tau_h\varphi$ also belongs to $W^{1,2}(\Omega)$ and has compact support in Ω. Therefore, Lemma A.3 implies that

$$\mathfrak{a}_{\mathscr{A}}(u, \tau_h\varphi) = \int_\Omega f\, \tau_h\overline{\varphi}, \tag{A.6}$$

for $|h| \ll 1$.

On the other hand, by $\nabla\tau_h = \tau_h\nabla$, a change of variable and (A.6), we find that

$$\mathfrak{a}_{\mathscr{A}_h}(\tau_{-h}u, \varphi) = \mathfrak{a}_{\mathscr{A}}(u, \tau_h\varphi) = \int_\Omega f\, \tau_h\overline{\varphi}. \tag{A.7}$$

Subtracting (A.5) from equation (A.7), we get

$$\mathfrak{a}_{\mathscr{A}_h}(\tau_{-h}u, \varphi) - \mathfrak{a}_{\mathscr{A}}(u, \varphi) = \int_\Omega f(\tau_h\overline{\varphi} - \overline{\varphi}).$$

Observe that $\mathfrak{a}_{\mathscr{A}_h}(\tau_{-h}u, \varphi)$ is well-defined for small $|h|$, even though $\mathscr{A}$ is defined on Ω. This is due to the fact that φ has compact support.

By writing $\tau_{-h}u = \tau_{-h}u - u + u$ and by dividing by h both terms of the previous identity, we have

$$\mathfrak{a}_{\mathscr{A}_h}(\delta_h u, \varphi) = -\mathfrak{a}_{\delta_h(\mathscr{A})}(u, \varphi) + \int_\Omega f\, \overline{\delta_{-h}\varphi}. \tag{A.8}$$

Now we will choose an adequate φ. Let $x_0 \in \Omega$ and $R > 0$ such that $\overline{B_{2R}} := \overline{B(x_0, 2R)} \subset \Omega$. Fix $\eta \in C^\infty_c(\Omega)$ such that $0 \leqslant \eta \leqslant 1$, $\operatorname{supp}\eta \subseteq B_R$ and $\eta = 1$ on $B_{R/2}$. Define

$$\begin{aligned} v &:= \delta_h u, \\ \varphi &:= \eta^2 v \end{aligned}$$

for $|h| \ll 1$. By plugging this particular φ in (A.8), we obtain

$$\mathfrak{a}_{\mathscr{A}_h}(v, \eta^2 v) = -\mathfrak{a}_{\delta_h(\mathscr{A})}(u, \eta^2 v) + \int_\Omega f\, \overline{\delta_{-h}(\eta^2 v)}. \tag{A.9}$$

By writing $\nabla(\eta^2 v) = \eta^2\nabla v + 2\eta v\nabla\eta$ in the definition of $\mathfrak{a}_{\mathscr{A}_h}$ and $\mathfrak{a}_{\delta_h(\mathscr{A})}$, (A.9) yields

$$\begin{aligned} &\int_{B_R} \eta^2 \Big[\langle \tau_{-h}A\nabla v, \nabla v\rangle + \Big\langle \nabla v, \tau_{-h}\overline{b}\Big\rangle\, \overline{v} + v\, \langle \tau_{-h}c, \nabla v\rangle + \tau_{-h}V|v|^2 \Big] \\ &\quad = -2\int_{B_R} \eta \Big[\langle \tau_{-h}A\nabla v, \nabla\eta\rangle\, \overline{v} + |v|^2\, \langle \tau_{-h}c, \nabla\eta\rangle \Big] \\ &\qquad - \int_{B_R} \eta^2 \Big[\langle \delta_h(A)\nabla u, \nabla v\rangle + \Big\langle \nabla u, \delta_h(\overline{b})\Big\rangle\, \overline{v} + u\, \langle \delta_h(c), \nabla v\rangle + \delta_h(V)u\overline{v} \Big] \\ &\qquad - 2\int_{B_R} \eta\overline{v}\, [\langle \delta_h(A)\nabla u, \nabla\eta\rangle + u\, \langle \delta_h(c), \nabla\eta\rangle] \\ &\qquad + \int_{B_{R+h}} f\, \overline{\delta_{-h}(\eta^2 v)}. \end{aligned}$$

Here we also used that $\operatorname{supp}\eta \subseteq B_R$. Therefore, since $\mathscr{A} \in \mathcal{B}(\Omega)$, there exists $\mu \in (0, 1)$ such

that

$$\begin{aligned}\mu\int_{B_R}\eta^2|\nabla v|^2 \leqslant &-2\,\mathrm{Re}\int_{B_R}\eta\left[\langle\tau_{-h}A\nabla v,\nabla\eta\rangle\,\overline{v}+|v|^2\,\langle\tau_{-h}c,\nabla\eta\rangle\right]\\ &-\mathrm{Re}\int_{B_R}\eta^2\left[\langle\delta_h(A)\nabla u,\nabla v\rangle+\left\langle\nabla u,\delta_h(\overline{b})\right\rangle\overline{v}+u\,\langle\delta_h(c),\nabla v\rangle+\delta_h(V)u\overline{v}\right]\\ &-2\,\mathrm{Re}\int_{B_R}\eta\overline{v}\left[\langle\delta_h(A)\nabla u,\nabla\eta\rangle+u\,\langle\delta_h(c),\nabla\eta\rangle\right]\\ &+\mathrm{Re}\int_{B_{R+h}}f\,\overline{\delta_{-h}(\eta^2 v)}.\end{aligned}$$

The boundedness of A and c implies that $\tau_{-h}A$ and $\tau_{-h}c$ are bounded on B_R uniformly in $|h|\ll 1$. Moreover, as A is locally Lipschitz, $\delta_h(A)$ is also bounded on B_R uniformly in $|h|\ll 1$. The same holds true for $\delta_h(\overline{b}),\delta_h(c)$ and $\delta_h(V)$. Therefore, by using also that $\|\nabla\eta\|_\infty\lesssim_R 1$ and $0\leqslant\eta\leqslant 1$, we get

$$\begin{aligned}\int_{B_R}\eta^2|\nabla v|^2 &\lesssim_{\mu,R}\int_{B_R}\left(\eta|\nabla v||v|+\eta|v|^2\right)\\ &\quad+\int_{B_R}\eta^2\left(|\nabla u|+|u|\right)|\nabla v|\\ &\quad+\int_{B_R}\eta\left(|\nabla u|+|u|\right)|v|\\ &\quad+\int_{B_{R+h}}|f||\delta_{-h}(\eta^2 v)|\\ &=: I_1+I_2+I_3+I_4.\end{aligned}\tag{A.10}$$

By repeatedly applying the inequality $2yz\leqslant\varepsilon y^2+\varepsilon^{-1}z^2$ for all $y,z\in\mathbb{R}$, $\varepsilon>0$ and the fact that $0\leqslant\eta\leqslant 1$, for all $\varepsilon>0$ we obtain

$$\begin{aligned}I_1&\lesssim\varepsilon\int_{B_R}\eta^2|\nabla v|^2+\varepsilon^{-1}\int_{B_R}|v|^2+\int_{B_R}|v|^2,\\ I_2&\lesssim\varepsilon\int_{B_R}\eta^2|\nabla v|^2+\varepsilon^{-1}\int_{B_R}\left(|\nabla u|^2+|u|^2\right),\\ I_3&\lesssim\int_{B_R}\left(|\nabla u|^2+|u|^2+|v|^2\right),\\ I_4&\lesssim\varepsilon\int_{B_{R+h}}|\delta_{-h}(\eta^2 v)|^2+\varepsilon^{-1}\int_{B_{R+h}}|f|^2.\end{aligned}$$

These estimates combined with (A.10) give

$$\begin{aligned}\int_{B_R}\eta^2|\nabla v|^2\lesssim\;&\varepsilon\int_{B_R}\eta^2|\nabla v|^2\\ &+\varepsilon^{-1}\int_{B_R}\left(|\nabla u|^2+|u|^2+|v|^2\right)\\ &+\int_{B_R}\left(|\nabla u|^2+|u|^2+|v|^2\right)\\ &+\varepsilon\int_{B_{R+h}}|\delta_{-h}(\eta^2 v)|^2+\varepsilon^{-1}\int_{B_{R+h}}|f|^2,\end{aligned}\tag{A.11}$$

where the constant depends on μ and R, but not on h. On the other hand, by Lemma A.1,

$$\begin{aligned}\int_{B_{R+h}}|\delta_{-h}(\eta^2 v)|^2\leqslant\int_{B_{R+2h}}|\nabla(\eta^2 v)|^2&\lesssim\int_{B_{R+2h}}\eta^4|\nabla v|^2+\eta^2|v|^2|\nabla\eta|^2\\ &\lesssim\int_{B_R}\eta^2|\nabla v|^2+\int_{B_R}|v|^2,\end{aligned}\tag{A.12}$$

where in the last inequality we used that $\eta \leqslant 1$ and $\operatorname{supp}\eta \subseteq B_R$. Hence (A.11) and (A.12) give

$$\begin{aligned}\int_{B_R} \eta^2 |\nabla v|^2 \lesssim \varepsilon \int_{B_R} \eta^2 |\nabla v|^2 \\ + (\varepsilon^{-1}+1) \int_{B_R} \left(|\nabla u|^2 + |u|^2 + |v|^2\right) \\ + \varepsilon \int_{B_R} |v|^2 + \varepsilon^{-1} \int_{B_{R+h}} |f|^2,\end{aligned}$$

where the constant depends on μ and R, but not on h. Therefore, we may choose ε independent of h small enough such that the term with $\|\eta \nabla v\|^2_{L^2(B_R)}$ can be absorbed in the left-hand side of the inequality, obtaining

$$\int_{B_R} \eta^2 |\nabla v|^2 \lesssim \int_{B_R} |\nabla u|^2 + \int_{B_R} |u|^2 + \int_{B_R} |v|^2 + \int_{B_{R+h}} |f|^2,$$

which implies that

$$\int_{B_{R/2}} |\nabla v|^2 \lesssim \int_{B_R} |\nabla u|^2 + \int_{B_R} |u|^2 + \int_{B_R} |v|^2 + \int_{B_{R+h}} |f|^2. \tag{A.13}$$

By Lemma A.1 again,

$$\int_{B_R} |v|^2 = \int_{B_R} |\delta_h u|^2 \leqslant \int_{B_{R+h}} |\nabla u|^2.$$

Hence (A.13) yields

$$\int_{B_{R/2}} |\delta_h \nabla u|^2 = \int_{B_{R/2}} |\nabla v|^2 \lesssim \int_{B_{R+h}} |\nabla u|^2 + \int_{B_R} |u|^2 + \int_{B_{R+h}} |f|^2.$$

Since the constant in the previous estimate does not depend on h, we get

$$\limsup_{h\to 0} \int_{B_{R/2}} |\delta_h \nabla u|^2 \lesssim \int_{B_R} |\nabla u|^2 + \int_{B_R} |u|^2 + \int_{B_R} |f|^2.$$

From Lemma A.2 we infer that $u \in W^{2,2}(B_{R/2})$ and that

$$\int_{B_{R/2}} |D^2 u|^2 \lesssim \int_{B_R} |\nabla u|^2 + \int_{B_R} |u|^2 + \int_{B_R} |f|^2. \qquad \square$$

Lemma A.5. *Let $\Omega, A, b, c, V, \mathscr{V}$ satisfy the standard assumptions of Section 4.1.1. Suppose that $A \in C^\infty(\Omega, \mathbb{C}^{d\times d})$, $b, c \in C^\infty(\Omega, \mathbb{C}^d)$, $V \in C^\infty(\Omega, \mathbb{R}_+) \cap L^\infty(\Omega, \mathbb{R}_+)$ and that $\mathscr{A} = (A, b, c, V) \in \mathcal{B}(\Omega)$. Let $\mathscr{L}^{\mathscr{A}} = \mathscr{L}^{\mathscr{A},\mathscr{V}}$ be the operator on $L^2(\Omega)$ associated with the sesquilinear form $\mathfrak{a}_{\mathscr{A},\mathscr{V}}$ defined in (4.1). Then*

$$\mathrm{D}\left((\mathscr{L}^{\mathscr{A}})^k\right) \subseteq W^{2k,2}_{\mathrm{loc}}(\Omega) \tag{A.14}$$

for all $k \geqslant 1$. In particular,

$$\bigcap_{k\geqslant 1} \mathrm{D}\left((\mathscr{L}^{\mathscr{A}})^k\right) \subseteq C^\infty(\Omega). \tag{A.15}$$

Proof. When $k = 1$, (A.14) holds by Lemma A.4.

Assume that (A.14) holds for $k \in \mathbb{N}_+$. Then, for all $u \in \mathrm{D}\left((\mathscr{L}^{\mathscr{A}})^{k+1}\right)$ we have

$$\begin{aligned} u &\in \mathrm{D}\left((\mathscr{L}^{\mathscr{A}})^k\right) \subseteq W^{2k,2}_{\mathrm{loc}}(\Omega), \\ \mathscr{L}^{\mathscr{A}} u &\in \mathrm{D}\left((\mathscr{L}^{\mathscr{A}})^k\right) \subseteq W^{2k,2}_{\mathrm{loc}}(\Omega). \end{aligned}$$

We will first prove that $u \in W^{2k+1,2}_{\mathrm{loc}}(\Omega)$ and then that $u \in W^{2k+2,2}_{\mathrm{loc}}(\Omega)$, thus establishing (A.14) for all $k \in \mathbb{N}$.

Let $|\alpha| \leqslant 2k-1$. For all $\varphi \in C^\infty_c(\Omega)$ we have

$$\begin{aligned}\left\langle \partial^\alpha \mathscr{L}^{\mathscr{A}} u, \varphi \right\rangle &= (-1)^{|\alpha|} \left\langle \mathscr{L}^{\mathscr{A}} u, \partial^\alpha \varphi \right\rangle \\ &= (-1)^{|\alpha|} \int_\Omega \left[\langle A \nabla u, \partial^\alpha \nabla \varphi \rangle + \left\langle \nabla u, \overline{b} \right\rangle \overline{\partial^\alpha \varphi} + u \langle c, \partial^\alpha \nabla \varphi \rangle + V u \overline{\partial^\alpha \varphi} \right].\end{aligned}$$

By assumptions A, b, c, V are smooth. So, by integrating by parts and by applying the product rule, we get

$$\begin{aligned}\left\langle \partial^\alpha \mathscr{L}^{\mathscr{A}} u, \varphi \right\rangle &= \int_\Omega \langle A \nabla \partial^\alpha u, \nabla \varphi \rangle - \sum_{i,j=1}^d \int_\Omega \partial_i \left(\sum_{0 \leqslant \beta < \alpha} \binom{\alpha}{\beta} \left(\partial^{\alpha-\beta} a_{ij} \right) \left(\partial^\beta \partial_j u \right) \right) \overline{\varphi} \\ &\quad + \int_\Omega \left\langle \nabla \partial^\alpha u, \overline{b} \right\rangle \overline{\varphi} + \sum_{i=1}^d \int_\Omega \left(\sum_{0 \leqslant \beta < \alpha} \binom{\alpha}{\beta} \left(\partial^{\alpha-\beta} b_i \right) \left(\partial^\beta \partial_i u \right) \right) \overline{\varphi} \\ &\quad + \int_\Omega \partial^\alpha u \langle c, \nabla \varphi \rangle - \sum_{i=1}^d \int_\Omega \partial_i \left(\sum_{0 \leqslant \beta < \alpha} \binom{\alpha}{\beta} \left(\partial^{\alpha-\beta} c_i \right) \left(\partial^\beta u \right) \right) \overline{\varphi} \\ &\quad + \int_\Omega V \partial^\alpha u \overline{\varphi} + \int_\Omega \left(\sum_{0 \leqslant \beta < \alpha} \binom{\alpha}{\beta} \left(\partial^{\alpha-\beta} V \right) \left(\partial^\beta u \right) \right) \overline{\varphi} \\ &= \int_\Omega \left[\langle A \nabla \partial^\alpha u, \nabla \varphi \rangle + \int_\Omega \left\langle \nabla \partial^\alpha u, \overline{b} \right\rangle \overline{\varphi} + \int_\Omega \partial^\alpha u \langle c, \nabla \varphi \rangle + \int_\Omega V \partial^\alpha u \overline{\varphi} \right] \\ &\quad + \int_\Omega P(\partial) u \overline{\varphi},\end{aligned}$$

where P is a polynomial with smooth coefficients and $\deg P \leqslant 2k$. Therefore, by using that $u \in W^{2k,2}_{\mathrm{loc}}(\Omega)$, we obtain $P(\partial)u \in L^2_{\mathrm{loc}}(\Omega)$ and $\partial^\alpha u \in W^{1,2}_{\mathrm{loc}}(\Omega)$. Hence we may write

$$\mathfrak{a}_{\mathscr{A}}(\partial^\alpha u, \varphi) = \int_\Omega \left(\partial^\alpha \mathscr{L}^{\mathscr{A}} u - P(\partial) u \right) \overline{\varphi}.$$

On the other hand, $\partial^\alpha \mathscr{L}^{\mathscr{A}} u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, since $\mathscr{L}^{\mathscr{A}} u \in W^{2k,2}_{\mathrm{loc}}(\Omega)$. This shows that $\partial^\alpha \mathscr{L}^{\mathscr{A}} u \in L^2_{\mathrm{loc}}(\Omega)$ and thus $\partial^\alpha \mathscr{L}^{\mathscr{A}} u - P(\partial) u \in L^2_{\mathrm{loc}}(\Omega)$. Now we may apply Lemma A.4 which gives $\partial^\alpha u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ for all $|\alpha| \leqslant 2k-1$. Thus we proved $u \in W^{2k+1,2}_{\mathrm{loc}}(\Omega)$.

Let now $|\beta| \leqslant 2k$. As before, we get

$$\begin{aligned}&\int_\Omega \left\langle A \nabla \partial^\beta u, \nabla \varphi \right\rangle + \int_\Omega \left\langle \nabla \partial^\beta u, \overline{b} \right\rangle \overline{\varphi} + \int_\Omega \partial^\beta u \langle c, \nabla \varphi \rangle + \int_\Omega V \partial^\beta u \overline{\varphi} \\ &\qquad = \int_\Omega \left(\partial^\beta \mathscr{L}^{\mathscr{A}} u - Q(\partial) u \right) \overline{\varphi}, \quad \forall \varphi \in C^\infty_c(\Omega),\end{aligned}$$

where Q is a polynomial with smooth coefficients and $\deg Q \leqslant 2k+1$. We proved before that $u \in W^{2k+1,2}_{\mathrm{loc}}(\Omega)$, hence we have $\partial^\beta u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and $Q(\partial)u \in L^2_{\mathrm{loc}}(\Omega)$. Therefore, repeating the same procedure as before, we obtain $u \in W^{2k+2,2}_{\mathrm{loc}}(\Omega)$. By induction we conclude the proof of (A.14).

Prove now (A.15). Consider $R > 0$ and $x_0 \in \Omega$ such that $\overline{B_R} = \overline{B(x_0, R)} \subseteq \Omega$. Fix $\eta \in C^\infty_c(\Omega)$ such that $0 \leqslant \eta \leqslant 1$, $\operatorname{supp} \eta \subseteq B_R$ and $\eta = 1$ on $B_{R/2}$. Let $u \in \bigcap_{k \geqslant 1} \mathrm{D}\left((\mathscr{L}^{\mathscr{A}})^k \right)$. By (A.14), $u \in W^{2k,2}_{\mathrm{loc}}(\Omega)$, for all $k \geqslant 1$. Therefore, $\eta u \in W^{2k,2}(\mathbb{R}^d)$, for all $k \geqslant 1$. By the Sobolev embedding theorem, for example see [1, Theorem 4.12, Part I, Case A], $\eta u \in C^\infty(\mathbb{R}^d)$, hence $u \in C^\infty(B(x_0, R/2))$. Since $x_0 \in \Omega$ was arbitrary, we conclude that $u \in C^\infty(\Omega)$. □

Lemma A.6. *Let $\Omega, A, b, c, V, \mathscr{V}$ satisfy the standard assumptions of Section 4.1.1. Suppose that $A \in C^\infty(\Omega, \mathbb{C}^{d\times d})$, $b, c \in C^\infty(\Omega, \mathbb{C}^d)$, $V \in C^\infty(\Omega, \mathbb{R}_+) \cap L^\infty(\Omega, \mathbb{R}_+)$ and that $\mathscr{A} = (A, b, c, V) \in \mathcal{B}(\Omega)$. Let $(T_t^{\mathscr{A}})_{t>0}$ be the semigroup on $L^2(\Omega)$ associated with the sesquilinear form $\mathfrak{a}_{\mathscr{A},\mathscr{V}}$. Then $T_t^{\mathscr{A}} f \in C^\infty(\Omega)$ for all $t > 0$ and all $f \in L^2(\Omega)$.*

Proof. Since $\mathscr{A} \in \mathcal{B}(\Omega)$, we know that $(T_t^{\mathscr{A}})_{t>0}$ is analytic and is generated by $-\mathscr{L}^{\mathscr{A}}$, the operator on $L^2(\Omega)$ associated with the sesquilinear form $\mathfrak{a}_{\mathscr{A},\mathscr{V}}$; see Section 4.1.

Set $\mathscr{L} = \mathscr{L}^A$ and $T_t = T_t^{\mathscr{A}}$ for all $t > 0$. By induction we deduce that

$$T_t f \in \mathrm{D}\left(\mathscr{L}^k\right), \qquad \forall t > 0, \quad \forall f \in L^2(\Omega) \tag{A.16}$$

for all $k \geqslant 1$. In fact, for $k = 1$ this follows from the analyticity of $(T_t)_{t>0}$ and [43, Chapter II, Theorem 4.6(c)]. Assume that (A.16) holds for $k \in \mathbb{N}_+$. We want to prove that $\mathscr{L}^k T_t f \in \mathrm{D}(\mathscr{L})$ for all $t > 0$ and all $f \in L^2(\Omega)$. Fix $t > 0$, $f \in L^2(\Omega)$ and choose $\varepsilon \in (0, t)$. By using the fact that $\mathscr{L}$ commutes with $(T_t)_{t>0}$ on $\mathrm{D}(\mathscr{L})$ [43, Chapter II, Lemma 1.3(ii)], the inductive hypothesis on T_ε and (A.16) applied for $T_{t-\varepsilon}$ with $k = 1$, we get

$$\mathscr{L}^k T_t f = \mathscr{L}^k T_{t-\varepsilon} T_\varepsilon f = T_{t-\varepsilon} \mathscr{L}^k T_\varepsilon f \in \mathrm{D}(\mathscr{L}).$$

We conclude by invoking (A.15). □

Bibliography

[1] R. A. Adams and J. J. F. Fournier, *Sobolev spaces*, Second edition, Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003. ↑171, 175

[2] A. Ancona, *On strong barriers and an inequality of Hardy for domains in* $\mathbb{R}^n$, J. Lond. Math. Soc. (2) **34** (1986), no. 2, 274–290. ↑146

[3] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Second edition, Monographs in Mathematics, vol. 96, Birkhäuser/Springer Basel, 2011. ↑37, 39, 43

[4] W. Arendt and M. Kreuter, *Mapping theorems for Sobolev-spaces of vector-valued functions*, Studia Math. **240** (2016), no. 3, 275–299. ↑49, 121

[5] J. Assaad and E. M. Ouhabaz, *Riesz transforms of Schrödinger operators on manifolds*, J. Geom. Anal. **22** (2012), no. 4, 1108–1136. ↑106, 118, 147, 148

[6] P. Auscher, *On necessary and sufficient conditions for* L^p*-estimates of Riesz transforms associated to elliptic operators on* $\mathbb{R}^n$ *and related estimates*, Mem. Amer. Math. Soc. **186** (2007), no. 871. ↑12

[7] P. Auscher, S. Hofmann, and J.-M. Martell, *Vertical versus conical square functions*, Trans. Amer. Math. Soc. **364** (2012), no. 10, 5469–5489. ↑12, 183

[8] D. Bakry, *Étude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée*, Séminaire de Probabilités, XXI, 1987, pp. 137–172. ↑11

[9] S. Bechtel, L^p*-estimates for the square root of elliptic systems with mixed boundary conditions II*, J. Differential Equations **379** (2024), 104–124. ↑104

[10] A. G. Belyi and Y. A. Semenov, L^p*-theory of Schrödinger semigroups. II*, Sib. Math. J. **31** (1990), 540–549. ↑15, 105, 106, 118, 184

[11] J. J. Betancor, E. Dalmasso, J. C. Fariña, and R. Scotto, *Bellman Functions and Dimension Free* L^p*-estimates for the Riesz Transforms in Bessel settings*, Nonlinear Anal. **197** (2020), 111850, 24. ↑60

[12] T. Böhnlein, M. Egert, and J. Rehberg, *Bounded functional calculus for divergence form operators with dynamical boundary conditions*. https://arxiv.org/abs/2406.09583. ↑13, 59, 60, 64, 104, 118, 151

[13] D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Prob. **12** (1984), no. 3, 647–702. ↑60

[14] D. L. Burkholder, *A proof of Pełczyński's conjecture for the Haar system*, Studia Math. **1** (1988), 79–83. ↑60

[15] D. L. Burkholder, *Explorations in martingale theory and its applications*, École d'Été de Probabilités de Saint-Flour XIX —1989, 1991, pp. 1–66. ↑60

[16] A. Carbonaro and O. Dragičević, *Bellman function and dimension-free estimates in a theorem of Bakry*, J. Funct. Anal. **265** (2013), 1085–1104. ↑10, 11, 13, 14, 50, 60, 183, 186

[17] A. Carbonaro and O. Dragičević, *Functional calculus for generators of symmetric contraction semigroups*, Duke Math. J. **166** (2017), no. 5, 937–974. ↑11, 13, 14, 50, 51, 56, 59, 60, 62, 77, 118, 127, 183, 186, 187

[18] A. Carbonaro and O. Dragičević, *Bounded holomorphic functional calculus for nonsymmetric Ornstein-Uhlenbeck operators*, Ann. Sc. Norm. Super. Pisa Cl Sci. **4** (2019), 1497–1533. ↑11, 13, 14, 50, 51, 60, 62, 77, 118, 127, 183, 186, 187

[19] A. Carbonaro and O. Dragičević, *Bilinear embedding for divergence-form operators with complex coefficients on irregular domains*, Calc. Var. Partial Differential Equations **59** (2020), no. 3, 36 pp. Paper No. 104. ↑7, 12, 13, 14, 15, 16, 45, 46, 47, 48, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 68, 76, 77, 78, 82, 87, 88, 89, 101, 102, 103, 104, 105, 118, 132, 137, 138, 141, 143, 145, 151, 152, 154, 155, 156, 157, 183, 184, 185, 186, 187, 188, 192, 193, 194

[20] A. Carbonaro and O. Dragičević, *Convexity of power functions and bilinear embedding for divergence-form operators with complex coefficients*, J. Eur. Math. Soc. (JEMS) **22** (2020), no. 10, 3175–3221. ↑11, 12, 14, 16, 36, 45, 46, 50, 51, 52, 53, 55, 56, 57, 58, 59, 60, 61, 62, 67, 70, 76, 77, 78, 79, 80, 81, 82, 85, 86, 87, 88, 92, 93, 96, 109, 110, 118, 127, 132, 151, 183, 184, 186, 187, 188, 192, 193

[21] A. Carbonaro and O. Dragičević, *Bilinear embedding for Schrödinger-type operators with complex coefficients*, Publ. Mat. **69** (2025), no. 1, 83–108. ↑8, 12, 36, 37, 38, 39, 40, 45, 46, 47, 48, 56, 57, 58, 59, 60, 62, 67, 78, 81, 82, 84, 85, 87, 88, 99, 105, 106, 113, 118, 119, 121, 128, 129, 131, 132, 133, 137, 142, 183, 185, 186, 188, 192

[22] A. Carbonaro, O. Dragičević, V. Kovač, and K. A. Škreb, *Trilinear embedding for divergence-form operators with complex coefficients*, Advances in Mathematics **431** (2023), 109239. ↑16, 50, 60, 63, 89, 151, 152, 154, 157, 184, 186, 193

[23] R. Chill, H. Meinlschmidt, and J. Rehberg, *On the numerical range of second-order elliptic operators with mixed boundary conditions in L^p*, J. Evol. Equ. **21** (2021), 3267–3288. ↑47, 185

[24] A. Cialdea and V. Maz'ya, *Criterion for the L^p-dissipativity of second order differential operators with complex coefficients*, J. Math. Pures Appl. (9) **84** (2005), no. 8, 1067–1100. ↑8, 14, 15, 36, 51, 71, 85, 86, 87, 184

[25] M. Costabel, *On the limit Sobolev regularity for Dirichlet and Neumann problems on Lipschitz domains*, Math. Nachr. **292** (2019), no. 10, 2165–2173. ↑58

[26] M. Cowling, I. Doust, A. McIntosh, and A. Yagi, *Banach space operators with a bounded H^∞ functional calculus*, Austral. Math. Soc. **60** (1996), 51–89. ↑15, 24, 28, 29, 30, 193

[27] M. Cowling and S. Meda, *Harmonic analysis and ultracontractivity*, Trans. Amer. Math. Soc. **340** (1993), 733–752. ↑29

[28] K. Dahmani, *Sharp dimension free bound for the Bakry-Riesz vector*, arXiv e-prints (2016Nov), arXiv:1611.07696, available at 1611.07696. ↑60

[29] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989. ↑149

[30] M. Dindoš and J. Pipher, *Regularity theory for solutions to second order elliptic operators with complex coefficients and the L^p Dirichlet problem*, Adv. Math. **341** (2019), 255–298. ↑12, 50, 51, 186, 187

[31] K. Domelevo and S. Petermichl, *Sharp L^p estimates for discrete second order Riesz transforms*, Adv. Math. **262** (2014), 932–952. ↑60

[32] G. Dore and A. Venni, *On the closedness of the sum of two closed operators*, Math. Z. **196** (1987), no. 2, 189–201. ↑15, 31, 32, 33, 193

[33] O. Dragičević and A. Volberg, *Bellman function, Littlewood-Paley estimates and asymptotics for the Ahlfors-Beurling operator in $L^p(\mathbb{C})$*, Indiana Univ. Math. J. **54** (2005), no. 4, 971–995. ↑11, 60, 183

[34] O. Dragičević and A. Volberg, *Bellman functions and dimensionless estimates of Littlewood-Paley type*, J. Operator Theory **56** (2006), no. 1, 167–198. ↑10, 13, 50, 51, 60, 183, 186, 187

[35] O. Dragičević and A. Volberg, *Bilinear embedding for real elliptic differential operators in divergence form with potentials*, J. Funct. Anal. **261** (2011), no. 10, 2816–2828. ↑11, 13, 45, 50, 51, 58, 59, 60, 88, 132, 183, 186, 187, 191

[36] O. Dragičević and A. Volberg, *Linear dimension-free estimates in the embedding theorem for Schrödinger operators*, J. Lond. Math. Soc. (2) **85** (2012), no. 1, 191–222. ↑10, 13, 50, 51, 60, 127, 129, 183, 186, 187

[37] N. Dunford and J. T. Schwartz, *Linear operators. Part I*, Wiley Classics Library, John Wiley and Sons, Inc., New York, 1988. General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication. ↑24

[38] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*, Oxford University Press, Oxford, 1987. ↑146

[39] M. Egert, R. Haller, S. Monniaux, and P Tolksdorf, *Harmonic Analysis Techniques for Elliptic Operators*. https://www.mathematik.tu-darmstadt.de/media/analysis/lehrmaterial_anapde/ISem_complete_lecture_notes.pdf. ↑20, 24

[40] M. Egert, *L^p-estimates for the square root of elliptic systems with mixed boundary conditions*, J. Differential Equations **265** (2018), no. 4, 1279–1323. ↑104

[41] M. Egert, *On p-elliptic divergence form operators and holomorphic semigroups*, Journal of Evolution Equations **20** (2020), no. 3, 705–724. ↑14, 48, 57, 111, 112, 113, 114, 115, 116, 117, 121, 149, 184

[42] M. Egert, R. Haller-Dintelmann, and J. Rehberg, *Hardy's Inequality for Functions Vanishing on a Part of the Boundary*, Potential Anal. **43** (2015), 49–78. ↑146, 147

[43] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000. ↑20, 22, 23, 26, 176

[44] M. Giaquinta and L. Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, Second edition, Publications of the Scuola Normale Superiore, Edizioni Della Normale Pisa, 2012. ↑170, 171

[45] P. Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985. ↑58

[46] P. Gyrya and L. Saloff-Coste, *Neumann and Dirichlet heat kernels in inner uniform domains*, Astérisque, Société mathématique de France, 2011. ↑150

[47] M. Haase, *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications, vol. 169, Birkhäuser Verlag, Basel, 2006. ↑20, 22, 24, 25, 26, 27, 28, 29

[48] M. Haase, P. C. Kunstmann, and H. Vogt, *On the numerical range of generators of symmetric* L_∞*-contractive semigroups*, Arch. Math. (Basel) **107** (2016), no. 5, 553–559. ↑59

[49] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Mathematical Library, Cambridge University Press, 1952. ↑146

[50] E. Hewitt and K. Stromberg, *Real and abstract analysis: A modern treatment of the theory of functions of a real variable*, Springer-Verlag, New York, 1965. ↑41

[51] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis, *Analysis in banach spaces. vol. i: Martingales and littlewood–paley theory*, A Series of Modern Surveys in Mathematics, vol. 63, Springer, Cham, 2016. ↑33

[52] D. Jerison and C. E. Kenig, *The functional calculus for the Laplacian on Lipschitz domains*, Journées "équations aux Dérivées Partielles" (Saint Jean de Monts, 1989), 1989, pp. Exp. No. IV, 10. ↑58

[53] T. Kato, *Perturbation theory for linear operators*, Second edition, Springer-Verlag, Berlin-New York, 1976. Grundlehren der Mathematischen Wissenschaften, Band 132. ↑33, 37, 38

[54] J. Kinnunen and O. Martio, *Hardy's inequalities for Sobolev functions*, Math. Res. Lett. **4** (1997), no. 4, 489–500. ↑146

[55] V. Kovač and K. A. Škreb, *Bilinear embedding in Orlicz spaces for divergence-form operators with complex coefficients*, J. Funct. Anal. **284** (2023), no. 9, 109884. ↑60

[56] Ch. Kriegler, *Analyticity angle for non-commutative diffusion semigroups*, J. Lond. Math. Soc. (2) **83** (2011), no. 1, 168–186. ↑59

[57] G. Leoni, *A first course in Sobolev spaces*, Second edition, Graduate Studies in Mathematics, vol. 181, American Mathematical Society, Providence, RI, 2017. ↑62, 83

[58] G. Leoni and M. Morini, *Necessary and sufficient conditions for the chain rule in* $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^d)$ *and* $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^d)$, J. Eur. Math. Soc. (JEMS) **9** (2007), no. 2, 219–252. ↑121

[59] J. L. Lewis, *Uniformly fat sets*, Trans. Am. Math. Soc. **308** (1988), no. 1, 177–196. ↑146

[60] X.-D. Li, *Martingale transforms and* L^p*-norm estimates of Riesz transforms on complete Riemannian manifolds*, Probab. Theory Related Fields **141** (2008), no. 1-2, 247–281. ↑11

[61] V. A. Liskevich and M. A. Perel′muter, *Analyticity of sub-Markovian semigroups*, Proc. Amer. Math. Soc. **123** (1995), no. 4, 1097–1104. ↑59

[62] J. Magniez, *Riesz transforms of the Hodge-de Rham Laplacian on Riemannian manifolds*, Math. Nachr. **289(6-8)** (2016), 1021–1043. ↑147, 148, 149

[63] G. Mauceri and M. Spinelli, *Riesz transforms and spectral multipliers of the Hodge-Laguerre operator*, J. Funct. Anal. **269** (2015), no. 11, 3402–3457. ↑60

[64] A. McIntosh, *Operators which have an* H_∞ *functional calculus*, Miniconference on operator theory and partial differential equations (North Ryde, 1986), 1986, pp. 210–231. ↑28

[65] F. Nazarov, S. Treil, and A. Volberg, *The Bellman functions and two-weight inequalities for Haar multipliers*, J. Amer. Math. Soc. **12** (1999), no. 4, 909–928. ↑60

[66] F. Nazarov, S. Treil, and A. Volberg, *Bellman function in stochastic control and harmonic analysis*, Systems, approximation, singular integral operators, and related topics (Bordeaux, 2000), 2001, pp. 393–423. ↑60

[67] F. L. Nazarov and S. R. Treĭl′, *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis*, Algebra i Analiz **8** (1996), no. 5, 32–162. ↑13, 50, 51, 59, 60, 186, 187, 191

[68] J. Nečas, *Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationnelle*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze (3) **16** (1962), 305–326. ↑146

[69] R. Nittka, *Projections onto convex sets and L^p-quasi-contractivity of semigroups*, Arch. Math. (Basel) **98** (2012), no. 4, 341–353. ↑36, 37

[70] A. Osękowski, *Sharp martingale and semimartingale inequalities*, Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series), vol. 72, Birkhäuser/Springer Basel AG, Basel, 2012. ↑60

[71] E.-M. Ouhabaz, *Second order elliptic operators with essential spectrum $[0, \infty)$ on L^p*, Comm. Partial Differential Equations **20** (1995), 763–773. ↑38

[72] E. M. Ouhabaz, *Analysis of heat equations on domains*, London Mathematical Society Monographs Series, vol. 31, Princeton University Press, Princeton, NJ, 2005. ↑21, 22, 33, 34, 35, 36, 47, 58, 69, 84, 85, 107, 149, 185

[73] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York, 1983. ↑20, 22

[74] M. A. Perel′muter, *Schrödinger operators with form-bounded potentials in L^p-spaces*, Annales de l'I.H.P., section A **52** (1990), no. 2, 151–161. ↑15, 105, 106, 118, 184

[75] S. Petermichl, L. Slavin, and B. D. Wick, *New estimates for the Beurling-Ahlfors operator on differential forms*, J. Operator Theory **65** (2011), no. 2, 307–324. ↑60

[76] S. Petermichl and A. Volberg, *Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular*, Duke Math. J. **112** (2002), no. 2, 281–305. ↑11, 13

[77] A. Poggio, *Bilinear embedding for divergence-form operators with negative potentials*. https://arxiv.org/abs/2510.05714. ↑17, 151

[78] A. Poggio, *Bilinear embedding for perturbed divergence-form operator with complex coefficients on irregular domains*, J. Geom. Anal. **35** (2025), no. 58. ↑17, 48, 118, 151, 186

[79] J. Prüss and H. Sohr, *On operators with bounded imaginary powers in Banach spaces*, Math. Z. **203** (1990), no. 3, 429–452. MR1038710 ↑15, 31, 193

[80] L. Schwartz, *Théorie des distributions*, Publications de l'Institut de Mathématique de l'Université de Strasbourg, No. IX-X., Hermann, Paris, 1966. ↑62, 83

[81] B. Simon, *A canonical decomposition for quadratic forms with applications to monotone convergence theorems*, Journal of Functional Analysis **28** (1978), no. 3, 377–385. ↑38

[82] Z. Sobol and H. Vogt, *On the L_p-theory of C_0-semigroups associated with second-order elliptic operators. I*, J. Funct. Anal. **193** (2002), no. 1, 24–54. ↑114, 121

[83] E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory.*, Annals of Mathematics Studies, No. 63, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1970. ↑11

[84] H. Vogt and J. Voigt, *Holomorphic families of forms, operators and C_0-semigroups*, Monatshefte für Mathematik **187** (2018), no. 2, 375–380. ↑38

[85] A. Volberg and F. Nazarov, *Heat extension of the Beurling operator and estimates for its norm*, Algebra i Analiz **15** (2003), no. 4, 142–158. ↑11, 13

[86] A. Volberg, *Bellman approach to some problems in harmonic analysis,* Exposé n° XIX, Ecole Polytéchnique (2002), 14. ↑60

[87] A. Wannebo, *Hardy inequalities*, Proc. Am. Math. Soc. **109** (1990), no. 1, 85–95. ↑146

[88] J. Wittwer, *Survey article: a user's guide to Bellman functions*, Rocky Mountain J. Math. **41** (2011), no. 3, 631–661. ↑60

[89] B. Wróbel, *Dimension-free L^p estimates for vectors of Riesz transforms associated with orthogonal expansions*, Anal. PDE **11** (2018), no. 3, 745–773. ↑60

[90] A. Yagi, *Coïncidence entre des espaces d'interpolation et des domaines de puissances fractionnaires d'opérateurs*, C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), no. 6, 173–176. ↑28, 29

[91] W. P. Ziemer, *Weakly differentiable functions*, Graduate Texts in Mathematics, vol. 120, Springer-Verlag, New York, 1989. Sobolev spaces and functions of bounded variation. MR1014685 ↑121, 149

Data Management Plan (DMP)

Type of data and methods used if new data was collected or produced

The doctoral research is theoretical and belongs to the field of pure Mathematics, specifically to the *L^p-theory of differential operators*. No experimental, numerical, or observational datasets were collected. The research data consist exclusively of original theoretical results, proofs, and mathematical arguments.

1. What data have you collected and/or produced?

The research data produced during the doctoral studies consist of:

- original mathematical results and proofs;
- scientific articles and preprints corresponding to the main chapters of the doctoral dissertation.

These results are documented in:

- one article published in *The Journal of Geometric Analysis*;
- one preprint available on ArXiv and submitted to an international peer-reviewed journal;
- one additional manuscript corresponding to the final chapter of the dissertation, which will be uploaded to ArXiv and subsequently submitted to a scientific journal.

2. How did you collect and/or produce new data and how did you use existing data for your doctoral thesis?

The research results were produced through theoretical analysis, mathematical reasoning, and the development of original proofs. Existing scientific literature in the field was used as a theoretical foundation and is properly cited in the dissertation and related publications. No external datasets were reused or processed.

3. Have you been dealing with sensitive data?

No sensitive, personal, or confidential data were involved in the doctoral research. Therefore, no specific ethical or data protection measures were required.

How the data was stored and protected during research for a doctoral thesis

1. How did you store the data?

Research materials such as manuscripts, LaTeX source files, and drafts of articles were stored digitally on secure personal computers. Preprints were archived on the online repository arXiv.

2. If you have worked with sensitive data, how have you ensured that it was kept safe and secure?

Not applicable.

Long-term data availability and storage

1. In which data repository will you store the data for the long term after the completion of the research work and make it accessible in accordance with the requirement of Article 50 of the Regulations on Doctoral Studies of the UL?

The research outputs are made openly accessible through:

- **ArXiv**, as an international open-access repository for mathematical research;
- the publishers' platforms of the respective peer-reviewed journals;
- the **UL Repository** and the **IRIS UniGe platform**, where the doctoral dissertation and metadata referring to the published articles will be deposited.

These platforms ensure visibility, accessibility, and long-term preservation of the research results.

2. Do you plan to restrict access to the data for a certain period?

No restriction of access is planned. The research results are openly accessible through ArXiv and scientific journals. In the case of journal embargo policies, access is regulated by the publishers' standard conditions, while preprints remain freely available.

Razširjeni povzetek

Naj bo $\Omega \subseteq \mathbb{R}^d$ neprazna odprta množica. Z $\mathcal{A}(\Omega)$ označimo razred vseh kompleksnih, enakomerno strogo eliptičnih $d \times d$ matričnih funkcij na Ω in s koeficienti v L^∞. Natančneje, $\mathcal{A}(\Omega)$ vsebuje vse merljive preslikave $A : \Omega \to \mathbb{C}^{d\times d}$, za katere obstajajo konstante $\lambda, \Lambda > 0$, da za skoraj vsak $x \in \Omega$ velja

$$\begin{aligned} \operatorname{Re}\langle A(x)\xi, \xi\rangle &\geq \lambda|\xi|^2, \qquad && \forall\, \xi \in \mathbb{C}^d, \\ |\langle A(x)\xi, \sigma\rangle| &\leq \Lambda|\xi|\,|\sigma|, && \forall\, \xi, \sigma \in \mathbb{C}^d. \end{aligned} \tag{1}$$

Naj bosta $b, c \in L^\infty(\Omega; \mathbb{C}^d)$ in $V \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R})$.

Glavni cilj te disertacije je razširitev izreka o bilinearni vložitvi, dokazanega v [21, Theorem 1.4], na eliptične operatorje v divergenčni formi, to je, na operatorje oblike

$$\mathscr{L}u = -\operatorname{div}(A\nabla u) + \langle \nabla u, \overline{b}\rangle - \operatorname{div}(cu) + Vu,$$

kjer bodisi

- b in c nista identično enaki nič in je potencial V je nenegativen; $(\star)$
- $b = c = 0$ in V ima netrivialni negativni del.

Ti operatorji so definirani na $L^2(\Omega)$ v šibkem smislu, preko seskvilinearne forme, pri čemer se različne vrste robnih pogojev odrazijo v izbiri domene forme; glej razdelek 1.1.

Izreki o bilinearni vložitvi so postali osnovno orodje v harmonični analizi, z aplikacijami od brezdimenzijskih ocen za Rieszove transformacije [16, 33, 34, 36] do natančnih rezultatov za spektralne množitelje [17, 18]. Teorija je bila kasneje razširjena na operatorje v divergenčni formi, pri čemer sta Dragičević in Volberg [35] prva prispevala z dokazom brezdimenzijske bilinearne vložitve na *celotnem* prostoru $\mathbb{R}^d$, in sicer za Schrödingerjeve operatorje, povezane z *realnimi* eliptičnimi matrikami in *nenegativnimi* potenciali. Odtlej je bila teorija postopoma posploševana v dveh smereh: najprej s prehodom iz realnih na *kompleksne* eliptične matrike, in nato s prehodom s celotnega prostora $\mathbb{R}^d$ na *poljubne* odprte podmnožice $\mathbb{R}^d$. Kompleksni primer na $\mathbb{R}^d$ je bil prvič obravnavan v [7], kjer je bila bilinearna vložitev dokazana z drugačnimi tehnikami kot v [35], čeprav brez dimenzijsko neodvisnih konstant; to sta kasneje dokazala Carbonaro in Dragičević v [20] z uvedbo pojma p-eliptičnosti za matrike koeficientov. Kasneje so razvili metodo aproksimacije za razširitev bilinearne vložitve na poljubna odprta področja in pokazali, kako bilinearna vložitev implicira omejenost H^∞ funkcijskega računa in L^p maksimalno regularnost operatorja $\mathscr{L}$ [19]. Nedavno so bili nenegativni potenciali vključeni v kompleksni okvir, pri čemer so bilinearno vložitev dokazali zgolj pod predpostavko p-eliptičnosti matrik [21].

V nasprotju s tem je manj znano, kaj se zgodi v prisotnosti členov nižje stopnje ali ko potencial lahko zavzame negativne vrednosti. Prva novost te disertacije je obravnava operatorjev s perturbacijami prvega reda, torej z $b, c \neq 0$, v kombinaciji z nenegativnimi potenciali.

V tem okviru pokažemo, da bilinearna vložitev še vedno velja pod ustreznimi pogoji, ki nadzorujejo interakcijo med perturbacijami prvega reda in glavno komponento, s čimer posplošimo p-eliptičnost.

Drugi prispevek se nanaša na Schrödingerjeve operaterje z potenciali, ki smejo zabzeti tudi negativne vrednosti. Obravnavamo poseben razred takih potencialov, katerih negativni del je nadzorovan z subkritično neenakostjo; glej (8). Za dokaz bilinearne vložitve za te operatorje uvedemo nov pogoj, ki se ujema s standardno p-eliptičnostjo, kadar je potencial nenegativen, sicer pa upošteva njegov negativni del. Natančneje, ustrezne perturbacije koeficientnih matrik — ki odražajo ta negativni del — morajo ostati p-eliptične.

Pogoj p-eliptičnosti se je prav tako izkazal za ključnega pri L^p-kontraktivnosti polgrupe $(T_t^A)_{t>0}$. V [20] je bilo pokazano, da L^p-kontraktivnost velja pod pogoji, ki so strogo šibkejši od p-eliptičnosti. Presenetljivo je, da je ta rezultat veljaven na poljubnih odprtih podmnožicah $\Omega \subseteq \mathbb{R}^d$ in pod različnimi robnimi pogoji, vključno z Dirichletovimi, Neumannovimi in mešanimi [19, 41]. Glavni cilj te disertacije je raziskati, ali so novi strukturni pogoji, uvedeni za razširitev bilinearne vložitve na razrede operatorjev, opisane v (⋆), tudi zadostni za zagotovitev L^p-kontraktivnosti pripadajoče polgrupe. Na to vprašanje odgovarjamo pritrdilno: kot v neperturbiranem primeru, pogoji, šibkejši od tistih, potrebnih za dimenzijsko neodvisno bilinearno vložitev, še vedno zagotavljajo kontraktivnost. Natančneje, za operatorje v divergenčni formi s perturbacijami prvega reda dobimo zadosten pogoj za L^p-kontraktivnost, ki rahlo krepi klasični pogoj L^p-disipativnosti Cialdee in Maz'yaja [24], hkrati pa dopušča bolj splošne koeficiente in robne pogoje. Za Schrödingerjeve operaterje z negativnimi potenciali je pogoj, ki ga izpeljemo, skladen s klasičnimi rezultati, cf. Belyi–Semenov [10] in Perelmuter [74], pri čemer dobimo enak interval L^p-kontraktivnosti v vzorčnem primeru $A = I$.

Posledično, sledeč zamislim iz [19], kot stranski produkt bilinearne vložitve pokažemo tudi, da taki operatorji dopuščajo omejen H^∞ funkcijski račun in L^p maksimalno regularnost. To je mogoče, ker novi pogoji, ki posplošujejo p-eliptičnost, ohranjajo ključne invariančne lastnosti.

Nazadnje se na kratko dotaknemo izreka o trilinearni vložitvi, uvedenega v [22], ki razširja bilinearni okvir na multilinearni kontekst, ki vključuje trojice operatorjev v divergenčni formi in Hölderjeve eksponente. Posebna značilnost trilinearnega kontekstna je izguba simetrije, prisotne v bilinearnih primerih: en sam eksponent ne določa več ostalih. Ker je p-eliptičnost intrinzično povezana s konjugacijo, ta izguba simetrije naredi trilinearno teorijo bistveno bolj zahtevno od bilinearne. Medtem ko obstoječi rezultati temeljijo na različnih predpostavkah o matrikah, odvisno od tega, ali je $\Omega = \mathbb{R}^d$ ali poljubna odprta množica, ta disertacija naredi prvi korak k združenemu pristopu. Poleg tega v zadnjem poglavju razvijemo izpopolnjen dokaz bilinearne vložitve, zasnovan tako, da utrdi pot k enotnemu trilinearnemu rezultatu.

1 Operatorji drugega reda v divergenčni formi

Naj bo $A \in \mathcal{A}(\Omega)$. Fiksirajmo zaprto podmnožico $\mathscr{V}$ prostora Soboljeva $W^{1,2}(\Omega)$, ki vsebuje $W_0^{1,2}(\Omega)$. Označimo z $\mathscr{L} = \mathscr{L}_2^{A,\mathscr{V}}$ neomejen operator na $L^2(\Omega)$, pripadajoč gosto definirani, akretivni zvezni seskvilinearni formi

$$\mathfrak{a}_{A,\mathscr{V}}(u,v) \quad = \int_\Omega \langle A\nabla u, \nabla v\rangle_{\mathbb{C}^d}\,, \qquad \mathrm{D}(\mathfrak{a}_{A,\mathscr{V}}) = \mathscr{V}.$$

Natančneje,

$$\mathrm{D}(\mathscr{L}) := \Big\{u \in \mathrm{D}(\mathfrak{a}) : \exists w \in L^2(\Omega) :\ \mathfrak{a}(u,v) = \langle w, v\rangle_{L^2(\Omega)}\ \ \forall v \in \mathrm{D}(\mathfrak{a})\Big\}$$

in

$$\langle \mathscr{L}u, v\rangle_{L^2(\Omega)} = \int_\Omega \langle A\nabla u, \nabla v\rangle\,, \quad \forall u \in \mathrm{D}(\mathscr{L}), \quad \forall v \in \mathrm{D}(\mathfrak{a})\,. \tag{2}$$

Eliptičnost matrike A implicira, da $-\mathscr{L}$ generira krepko zvezno polgrupo na $L^2(\Omega)$

$$T_t = T_t^{A,\mathscr{V}}, \quad t > 0,$$

ki je analitična in kontraktivna v stožcu $\mathbf{S}_{\pi/2-\omega}$, za neki kot $\omega \in [0, \pi/2)$ [72].

Formalno je operator $\mathscr{L}$ podan z izrazom

$$\mathscr{L}u = -\operatorname{div}(A\nabla u).$$

1.1 Robni pogoji

Tukaj opisujemo nekatere razrede zaprtih podmnožic $\mathscr{V}$ prostora $W^{1,2}(\Omega)$, ki vsebujejo $W_0^{1,2}(\Omega)$ in izpolnjujejo dodatne pogoje, ki jih bomo uporabljali v delu. Razlaga sledi [21].

Rečemo, da je prostor $\mathscr{V} \subseteq W^{1,2}(\Omega)$ *invarianten* glede na:

- funkcijo $p : \mathbb{C} \to \mathbb{C}$, če $u \in \mathscr{V}$ implicira $p(u) := p \circ u \in \mathscr{V}$;
- družino $\mathscr{P}$ funkcij $\mathbb{C} \to \mathbb{C}$, če je invarianten glede na vse $p \in \mathscr{P}$.

Definirajmo funkciji $P, T : \mathbb{C} \to \mathbb{C}$ kot

$$P(\zeta) = \begin{cases} \zeta; & |\zeta| \leqslant 1, \\ \zeta/|\zeta|; & |\zeta| \geqslant 1, \end{cases} \qquad T(\zeta) = (\operatorname{Re}\zeta)_+.$$

Naj bo $\mathscr{V}$ zaprta podmnožica $W^{1,2}(\Omega)$, ki vsebuje $W_0^{1,2}(\Omega)$. Obravnavajmo pogoja

$$\mathscr{V} \text{ je invarianten glede na funkcijo } P, \tag{3}$$

$$\mathscr{V} \text{ je invarianten glede na funkcijo } T, \tag{4}$$

in predpostavimo bodisi samo (3) bodisi oba pogoja (3) in (4), odvisno od konteksta. Na splošno (3) ne implicira (4); glej [72, Example 4.3.2].

Znano je (glej [72, Propositions 4.4&4.11], [23, Section 2.1] in [21, Appendix A]), da pogoja (3) in (4) veljata v naslednjih pomembnih primerih:

(a) $\mathscr{V} = W_0^{1,2}(\Omega)$,

(b) $\mathscr{V} = W^{1,2}(\Omega)$,

(c) $\mathscr{V} = \widetilde{W_D}^{1,2}(\Omega)$, zaprtje v $W^{1,2}(\Omega)$ množice $\{u \in W^{1,2}(\Omega) : \operatorname{dist}(\operatorname{supp} u, D) > 0\}$, kjer je D (morda prazna) zaprta podmnožica $\partial\Omega$,

(d) $\mathscr{V} = W_D^{1,2}(\Omega)$, zaprtje v $W^{1,2}(\Omega)$ množice $\{u_{|\Omega} : u \in C_c^\infty(\mathbb{R}^d \setminus D)\}$, kjer je D (morda prazna) zaprta podmnožica $\partial\Omega$.

Ko $\mathscr{V}$ spada v kateregakoli od posebnih primerov (a)-(d), rečemo, da je operator $\mathscr{L}$ podvržen (a) *Dirichletovim*, (b) *Neumannovim*, (c) *mešanim* ali (d) *dobrim mešanim robnim pogojem*.

Kadar bomo delali s pari prostorov $(\mathscr{V}, \mathscr{W})$, jih bomo morali iz zgoraj navedenih tipov izbirati preudarno. Nekatere kombinacije ne bodo dovoljene. Naslednja predpostavka je uvedena, ker [19, Lemma 19] morda ne velja, če $(\mathscr{V}, \mathscr{W})$ ne izpolnjujeta tega pogoja. Posebej, če je en prostor tipa (d), mora biti drugi prostor ali iste vrste ali tiste, opisane v (a). Formulirali in dokazali bomo bilinearni vložitveni izrek pod naslednjim dodatnim pogojem za $\mathscr{V}$ in $\mathscr{W}$, opisanim v (a)-(d). Splošnega primera, kjer so $\mathscr{V}$ in $\mathscr{W}$ poljubne kombinacije tipov iz (a)-(d), ni mogoče obravnavati z metodo toplotnega toka, razvito v [19].

Privzetek BE. Rečemo, da par $(\mathscr{V}, \mathscr{W})$ izpolnjuje Privzetek BE, če velja eden od naslednjih pogojev:

- $\mathscr{V}$ in $\mathscr{W}$ spadata v kateregakoli od posebnih primerov (a)-(c), ali
- $\mathscr{V}$ in $\mathscr{W}$ sta tipa, opisanega v (a) ali (d).

Kombinacije, dovoljene po Privzetku BE, so povzete v Tabeli 1.

Opomba 1. Ta predpostavka velja le za bilinearni vložitveni izrek in jo je treba dodati k formulacijam ustreznih izrekov v [19, 21, 78]. Poudarjamo, da vsi rezultati v [19, 78], ki izhajajo iz bilinearne vložitve—kot so omejenost H^∞-funkcijskega računa ali L^p maksimalna regularnost—ostajajo nespremenjeni, saj sledijo iz uporabe bilinearne vložitve za operatorju sam in njegov adjungirani operator, ki izpolnjujeta enake robne pogoje. Dejstvo, da je ta predpostavka potrebna za formulacijo [19, Lemma 19], pa ne pomeni, da je potrebna za bilinearni vložitveni izrek; vseeno je treba vpeljati nov pristop.

Prostor, opisan v (c), je nov v kontekstu bilinearnih vložitev na poljubnih domenah. Vpeljan je, da omogoča kombinacije mešanih in Neumannovih robnih pogojev, ki bi sicer bili prepovedani po Privzetku BE.

	$W_0^{1,2}(\Omega)$	$W_D^{1,2}(\Omega)$	$\widetilde{W_D}^{1,2}(\Omega)$	$W^{1,2}(\Omega)$
$W_0^{1,2}(\Omega)$	✓	✓	✓	✓
$W_D^{1,2}(\Omega)$	✓	✓	×	×
$\widetilde{W_D}^{1,2}(\Omega)$	✓	×	✓	✓
$W^{1,2}(\Omega)$	✓	×	✓	✓

Table 1: Privzetek BE: dovoljene in prepovedane kombinacije prostorov Soboljeva; simbol ✓ označuje dovoljeno kombinacijo, × pa prepovedano.

1.2 p-eliptičnost

Pojem *p-eliptičnosti* igra osrednjo vlogo v celotni disertaciji, saj določa strukturni pogoj za koeficiente, ki leži v ozadju vseh glavnih rezultatov.

Pogoj p-eliptičnosti sta vpeljala Carbonaro in Dragičević v okviru brezdimenzijskih bilinearnih vložitev [20] ter neodvisno od njiju Dindoš in Pipher pri preučevanju robnih problemov [30]. Lahko ga obravnavamo kot algebrajsko krepitev standardnega pogoja eliptičnosti, ki se ujema s standardnim, ko je $p = 2$, in ponuja naravno interpolacijo med realnimi in kompleksnimi eliptičnimi matrikami.

Podrobnejši pregled geneze p-eliptičnosti je podan v [22, Section 1.2]. Njeno zasnovo sta Carbonaro in Dragičević predstavila v [20] po letih postopnega izpopolnjevanja *metode toplotnega toka* (*heat-flow method*), povezane s specifično *Bellmanovo funkcijo* $\mathcal{Q}$, ki sta jo prvič uvedla Nazarov in Treil [67] in je bila nadalje razvijana skozi vrsto del na področju bilinearnh vložitev in spektralnih množiteljev [16–18, 34–36]. V [36] je podrobna analiza $\mathcal{Q}$ razkrila prefinjene konveksne lastnosti, ki so se izkazale za ključne pri dokazovanju izreka o bilinearni vložitvi. Ti izsledki so nakazali, da $\mathcal{Q}$ morda poseduje globlje strukturne konveksne lastnosti, kar je bilo kasnejo potrjeno in sistematično analizirano v nadaljnjih delih [16–18], s kulminacijo v [20]. V tem kontekstu se je pozornost postopno premaknila od same Bellmanove funkcije $\mathcal{Q}$ h konveksnosti njenih elementarnih komponent — natančneje, potenčnih funkcij $|\zeta|^p$ — obravnavanih glede

na matriko A. To je vodilo k uvedbi posplošenega pojma konveksnosti, pridružene matrikam koeficientov, ki se ujema s klasično konveksnostjo, ko je A identiteta. Pogoj p-eliptičnosti se je pojavil prav iz tega okvira: lahko ga interpretiramo kot enakomerno konveksnost $|\zeta|^p$ glede na A, kasneje ponovno formulirano v algebrajski obliki, prikazani spodaj v (5).

Pred razvojem splošne teorije je bila konveksnost glede na A raziskana v številnih posebnih konfiguracijah: ko je A identiteta [34, 36, 67], realna akretivna [35], oblike $e^{i\phi}I_d$ [17] ali oblike $e^{i\phi}B$ z B realno, konstantno in s simetričnim delom, ki je pozitivno definiten [18]. Uvedba teh kompleksnih rotacij v [17] je bila motivirana z novo idejo preučevanja bilinearnih vložitev v kompleksnem času, namesto omejitve na realni čas, kot v prejšnjih delih. Ta vpogled je odprl pot nadaljnjemu razvoju teorije v smeri eliptičnih operatorjev v divergenčni formi s kompleksnimi koeficienti [20].

Sedaj predstavimo natančno definicijo, kot sta jo vpeljala Carbonaro in Dragičević v [20]. Naj bo $A \in \mathcal{A}(\Omega)$ in $p \in (1,\infty)$. Rečemo, da je A *p-eliptična*, če je $\Delta_p(A) > 0$, kjer

$$\Delta_p(A) := \operatorname*{ess\,inf}_{x\in\Omega} \min_{|\xi|=1} \operatorname{Re} \left\langle A(x)\xi, \xi + |1-2/p|\overline{\xi} \right\rangle_{\mathbb{C}^d}.$$

Ekvivalentno, A je p-eliptična, če obstaja $C = C(A,p) > 0$, tako da za skoraj vsak $x \in \Omega$ velja

$$\operatorname{Re} \left\langle A(x)\xi, \xi + |1-2/p|\overline{\xi} \right\rangle_{\mathbb{C}^d} \geqslant C|\xi|^2, \quad \forall \xi \in \mathbb{C}^d. \tag{5}$$

Iz definicije sledi, da je Δ_p invarianten glede na konjugacijo p, kar pomeni, da $\Delta_p(A) = \Delta_{p'}(A)$, saj $|1-2/p| = |1-2/p'|$. Primer $p = 2$ predstavlja standardni pogoj eliptičnosti, saj je $\Delta_2(A) = \lambda(A)$, kar ilustrira splošnejšo naravo p-eliptičnosti. Po drugi strani, za fiksen $p \neq 2$ je razred kompleksnih p-eliptičnih matrik strogo manjši od razreda vseh eliptičnih matrik. Zato pogoj p-eliptičnosti ne le posplošuje klasično eliptičnost, temveč jo tudi krepi. Natančneje, če označimo z $\mathcal{A}_p(\Omega)$ razred vseh p-eliptičnih matrik na Ω, družina $\{\mathcal{A}_p : p \in [2,\infty)\}$ tvori padajočo verigo matrik, tako da

$$\begin{aligned} \{\text{eliptične matrike na } \Omega\} &= \mathcal{A}_2(\Omega), \\ \{\text{realne eliptične matrike na } \Omega\} &= \bigcap_{p\in[2,\infty)} \mathcal{A}_p(\Omega). \end{aligned}$$

Tako pojem p-eliptičnosti predstavlja zvezen prehod med realnimi in kompleksnimi eliptičnimi matrikami. Poleg tega zagotavlja ocene regularnosti za lokalne rešitve $\mathscr{L}u = 0$ s kompleksnimi koeficienti [30], ki nadomeščajo De Giorgi-Nash-Moserjevo teorijo, ki je na voljo le v realnem primeru.

Ena izmed prvih temeljnih posledic p-eliptičnosti zadeva L^p-obnašanje polgrupe, ki jo generira $-\mathscr{L}$. Naslednji izrek, avtorjev Carbonara in Dragičevića, to povezavo natančno določa.

Izrek 2. ([19]) *Naj bo $p > 1$, $A \in \mathcal{A}_p(\Omega)$. Naj $\mathscr{V}$ zadošča (3). Potem obstaja $\vartheta = \vartheta(p,A) > 0$, tako da iz $|1-2/r| \leqslant |1-2/p|$ sledi, da je $\{T_z : z \in \mathbf{S}_\vartheta\}$ analitična in kontraktivna v $L^r(\Omega)$.*

Pojem p-eliptičnosti je bil sprva uveden za določitev optimalnega pogoja na koeficientih, ki bi zagotavlja brezdimenzijsko bilinearno vložitev, s čimer se posploši primer realnih koeficientov, dokazan v [35].

Naslednji izrek o bilinearni vložitvi, prvotno dokazan v [20] za $\Omega = \mathbb{R}^d$ in nato v [19] za poljubno odprto podmnožico Ω iz $\mathbb{R}^d$, predstavlja temelj teorije in služi kot glavno izhodišče za razširitve, razvite v tej disertaciji.

Izrek 3. ([20],[19]) *Naj bosta $\mathscr{V}$ in $\mathscr{W}$ dve zaprti podmnožici $W^{1,2}(\Omega)$, ki zadoščata Privzetku BE. Naj bo $p > 1$, $q = p/(p-1)$ in $A, B \in \mathcal{A}(\Omega)$. Privzemimo, da sta A, B p-eliptični. Potem obstaja $C > 0$, tako da velja*

$$\int_0^\infty \int_\Omega \left|\nabla T_t^{A,\mathscr{V}} f(x)\right| \left|\nabla T_t^{B,\mathscr{W}} g(x)\right| \,\mathrm{d}x\,\mathrm{d}t \leqslant C\|f\|_p\|g\|_q,$$

za vse $f, g \in (L^p \cap L^q)(\Omega)$.

V [21] je bil zgornji izrek razširjen na Schrödingerjeve operatorje z nenegativnimi potenciali.

Glavni cilj te disertacije je posplošitev tako Izreka 2 kot Izreka 3 na operatorje, ki pripadajo razredom, opisanim v (⋆).

2 Perturbacije prvega reda

Začnemo z obravnavo prvega razreda operatorjev v (⋆). Naj bo $A \in \mathcal{A}(\Omega)$, $b, c \in L^\infty(\Omega, \mathbb{C}^d)$, in naj bo $V \in L^1_{\mathrm{loc}}(\Omega)$ nenegativen. Tako kot v neperturbiranem primeru je operator natančno definiran preko seskvilinearne forme. Fiksirajmo zaprto podmnožico $\mathscr{V}$ Sobolevovega prostora $W^{1,2}(\Omega)$, ki vsebuje $W_0^{1,2}(\Omega)$, in obravnavajmo seskvilinearno formo $\mathfrak{a} = \mathfrak{a}_{A,b,c,V,\mathscr{V}}$, definirano z

$$\begin{aligned} \mathrm{D}(\mathfrak{a}) &= \left\{u \in \mathscr{V} : \int_\Omega V|u|^2 < \infty\right\}, \\ \mathfrak{a}(u,v) &= \int_\Omega \langle A\nabla u, \nabla v\rangle_{\mathbb{C}^d} + \left\langle \nabla u, \overline{b}\right\rangle_{\mathbb{C}^d} \overline{v} + u\,\langle c, \nabla v\rangle_{\mathbb{C}^d} + Vu\overline{v}. \end{aligned}$$

Da bi zagotovili, da pripadajoči operator generira krepko zvezno, analitično in kontraktivno polgrupo na $L^2(\Omega)$, moramo vpeljati pogoj, ki igra vlogo eliptičnosti za perturbirane operatorje. V ta namen označimo z $\mathcal{B}(\Omega)$ razred vseh $\mathscr{A} = (A, b, c, V) \in \mathcal{A}(\Omega) \times (L^\infty(\Omega, \mathbb{C}^d))^2 \times L^1_{\mathrm{loc}}(\Omega, \mathbb{R}_+)$, za katere obstajata $\mu \in (0, 1]$ in $M > 0$, tako da za skoraj vsak $x \in \Omega$ velja

$$\mathrm{Re}\,\langle A(x)\xi, \xi\rangle + \mathrm{Re}\left\langle \overline{b}(x) + c(x), \xi\right\rangle + V(x) \geqslant \mu(|\xi|^2 + V(x)), \quad \forall \xi \in \mathbb{C}^d, \tag{6}$$

$$|\overline{b}(x) - c(x)| \leqslant M\sqrt{V(x)}. \tag{7}$$

Za vsak $\mathscr{A} \in \mathcal{B}(\Omega)$ označimo z $\mathscr{L} = \mathscr{L}_2^{\mathscr{A},\mathscr{V}}$ neomejeni operator na $L^2(\Omega)$, povezan s seskvilinearno formo $\mathfrak{a}$. Formalno je $\mathscr{L}$ podan z izrazom

$$\mathscr{L}u = -\mathrm{div}\,(A\nabla u) + \left\langle \nabla u, \overline{b}\right\rangle - \mathrm{div}\,(uc) + Vu.$$

Razred $\mathcal{B}(\Omega)$ se ujema z $\mathcal{A}(\Omega)$ v neperturbiranem primeru. Za vsak $\mathscr{A} \in \mathcal{B}(\Omega)$ je operator $-\mathscr{L}$ generator krepko zvezne polgrupe na $L^2(\Omega)$

$$T_t = T_t^{\mathscr{A},\mathscr{V}}, \quad t > 0,$$

ki je analitična in kontraktivna v stožcu $\mathbf{S}_{\pi/2-\omega}$, za neki kot $\omega \in [0, \pi/2)$.

2.1 Nehomogena p-eliptičnost: razred $\mathcal{B}_p(\Omega)$

Sedaj uvedemo pojem *neenakomerne* p-eliptičnosti, ki posploši klasično p-eliptičnost na perturbacije prvega reda in vključuje potenciale.

Naj bo $p > 1$ in $\mathscr{A} = (A, b, c, V)$ kot zgoraj. Označimo s $q = p/(p-1)$ konjugirani eksponent k p. Definiramo

$$\mathcal{J}_p(\xi) = \xi + (p-2)\mathrm{Re}\,\xi, \qquad \xi \in \mathbb{C}^d,$$

in uvedemo funkcijo $\Gamma_p^{\mathscr{A}} = \Gamma_p^{A,b,c,V} : \Omega \times \mathbb{C}^d \to \mathbb{R}$ z

$$\Gamma_p^{\mathscr{A}}(x,\xi) = \mathrm{Re}\,\langle A(x)\xi, \mathcal{J}_p\xi\rangle_{\mathbb{C}^d} + \mathrm{Re}\,\left\langle \overline{b}(x) + (\mathcal{J}_p c)(x), \xi\right\rangle_{\mathbb{C}^d} + V(x).$$

Označimo z $\mathcal{S}_p(\Omega)$ razred vseh $\mathscr{A} = (A, b, c, V) \in \mathcal{A}_p(\Omega) \times (L^\infty(\Omega, \mathbb{C}^d))^2 \times L^1_{\mathrm{loc}}(\Omega, \mathbb{R}_+)$, za katere obstaja $\mu > 0$ tako da za skoraj vsak $x \in \Omega$ velja

$$\Gamma_p^{\mathscr{A}}(x,\xi) \geqslant \mu(|\xi|^2 + V(x)), \quad \forall \xi \in \mathbb{C}^d,$$

in velja pogoj (7).

Opazimo:

- $(A, b, c, V) \in \mathcal{S}_2(\Omega)$ natanko tedaj, ko veljata (6) in (7), torej $\mathcal{S}_2(\Omega) = \mathcal{B}(\Omega)$.
- Ko je $b = c = 0$, se ta nov pogoj ujema s p-eliptičnostjo, kar pomeni, da $(A, 0, 0, V) \in \mathcal{S}_p(\Omega)$ natanko tedaj, ko $A \in \mathcal{A}_p(\Omega)$.

Končno definiramo razred

$$\mathcal{B}_p(\Omega) = \mathcal{S}_p(\Omega) \cap \mathcal{S}_q(\Omega),$$

ki se prav tako ujema z $\mathcal{A}_p(\Omega)$, ko je $b = c = 0$, kar sledi iz invariance p-eliptičnosti glede na konjugacijo. Ujema se tudi z $\mathcal{B}(\Omega)$, ko je $p = 2$.

Motivacija za uvedbo $\mathcal{B}_p$ poleg $\mathcal{S}_p$ je bila pridobiti razred, ki poleg generalizacije p-eliptičnosti ohranja naslednje lastnosti, ki jih imajo razredi $\mathcal{A}_p$:

(i) invariantnost glede na konjugacijo, tj. $\mathcal{A}_p(\Omega) = \mathcal{A}_q(\Omega)$;

(ii) padanje glede na p, tj. $\{\mathcal{A}_p(\Omega) : p \in [2, \infty)\}$ tvori padajočo verigo matrik;

(iii) invariantnost glede na adjungiranje, tj. $A \in \mathcal{A}_p(\Omega) \iff A^* \in \mathcal{A}_q(\Omega)$.

Razred $\mathcal{S}_p$ zadošča lastnosti, analogni (iii), ne pa (i) in (ii). Po definiciji je $\mathcal{B}_p$ invarianten glede na konjugacijo, prav tako pa je padajoč glede na p. Zato se razred $\mathcal{B}_p$ zdi bolj primerna posplošitev p-eliptičnosti. Poleg tega se stroga pozitivnost Γ_p ohrani tudi pri majhnih kompleksnih rotacijah koeficientov.

Strukturne lastnosti, ki jih ohranja razred $\mathcal{B}_p(\Omega)$, nakazujejo, da zagotavlja naravno ogrodje za posplošitev lastnosti polgrupe (Izrek 2) in izreka o bilinearni vložitvi (Izrek 3) na operatorje s perturbacijami prvega reda. Natančneje, $\mathcal{B}_p(\Omega)$ predstavlja pravi pogoj, pod katerim ti rezultati še vedno veljajo v tem splošnem okviru, kot prikazujeta naslednja izreka:

Izrek 4. *Naj bo $r \leqslant 2 \leqslant p$ tako, da $\mathscr{A} \in (\mathcal{S}_r \cap \mathcal{S}_p)(\Omega)$. Naj $\mathscr{V}$ zadošča (3). Potem obstaja $\vartheta = \vartheta(p, r, \mathscr{A}) > 0$, tako da je $\{T_z : z \in \mathbf{S}_\vartheta\}$ analitična in kontraktivna v $L^s(\Omega)$ za vse $s \in [r, p]$.*

Izrek 5. *Naj bosta $\mathscr{V}$ in $\mathscr{W}$ dve zaprti podmnožici $W^{1,2}(\Omega)$, ki zadoščata Privzetku BE. Naj bo $p > 1$, $q = p/(p-1)$ in $\mathscr{A}, \mathscr{B} \in \mathcal{B}(\Omega)$. Predpostavimo, da $\mathscr{A}, \mathscr{B} \in \mathcal{B}_p(\Omega)$. Potem obstaja $C > 0$, tako da velja*

$$\int_0^\infty \int_\Omega \sqrt{|\nabla T_t^{\mathscr{A},\mathscr{V}} f|^2 + V|T_t^{\mathscr{A},\mathscr{V}} f|^2}\,\sqrt{|\nabla T_t^{\mathscr{B},\mathscr{W}} g|^2 + W|T_t^{\mathscr{B},\mathscr{W}} g|^2} \leqslant C\|f\|_p\|g\|_q,$$

za vse $f, g \in (L^p \cap L^q)(\Omega)$.

3 Negativni potenciali

Sedaj obravnavamo drugi razred operatorjev v (⋆). Naj bo $A \in \Omega$ in $V \in L^1_{\mathrm{loc}}(\Omega)$. Osredotočamo se na določen razred potencialov.

Definicija 6. Naj bo $\mathscr{V}$ zaprt podprostor $W^{1,2}(\Omega)$, ki vsebuje $W^{1,2}_0(\Omega)$. Realna lokalno integrabilna funkcija V se imenuje *krepko subkritični potencial*, če obstajata $\alpha \geqslant 0$ in $\sigma \in [0,1)$, tako da

$$\int_\Omega V_-|v|^2 \leqslant \alpha \int_\Omega |\nabla v|^2 + \sigma \int_\Omega V_+|v|^2, \qquad \forall v \in \mathscr{V}. \tag{8}$$

Označimo z $\mathcal{P}(\Omega, \mathscr{V})$ razred vseh krepko subkritičnih potencialov za $\mathscr{V}$.

Tako kot v prejšnjih poglavjih operator definiramo preko seskvilinearnih form. Imejmo seskvilinearno formo $\mathfrak{a} = \mathfrak{a}_{A,V,\mathscr{V}}$, definirani kot

$$\mathrm{D}(\mathfrak{a}) = \left\{ u \in \mathscr{V} : \int_\Omega V_+|u|^2 < \infty \right\},$$
$$\mathfrak{a}(u,v) = \int_\Omega \langle A\nabla u, \nabla v\rangle_{\mathbb{C}^d} + Vu\overline{v}.$$

Ponovno potrebujemo pogoj, ki igra vlogo eliptičnosti. Označimo z $\mathcal{AP}(\Omega, \mathscr{V})$ razred vseh $(A,V) \in \mathcal{A}(\Omega) \times \mathcal{P}(\Omega, \mathscr{V})$, za katere velja

$$A - \alpha I_d \in \mathcal{A}(\Omega). \tag{9}$$

Za vsak $(A,V) \in \mathcal{AP}(\Omega)$ označimo z $\mathscr{L} = \mathscr{L}_2^{A,V,\mathscr{V}}$ neomejen operator na $L^2(\Omega)$, povezan s seskvilinearno formo $\mathfrak{a}$. Formalno je $\mathscr{L}$ podan z izrazom

$$\mathscr{L}u = -\mathrm{div}\,(A\nabla u) + Vu.$$

Razred $\mathcal{AP}(\Omega, \mathscr{V})$ se v neperturbiranem primeru ujema z $\mathcal{A}(\Omega)$. Za vsak $(A,V) \in \mathcal{AP}(\Omega)$ je operator $-\mathscr{L}$ generator krepko zvezne polgrupe na $L^2(\Omega)$

$$T_t = T_t^{A,V,\mathscr{V}}, \quad t > 0,$$

ki je analitična in kontraktivna v stožcu $\mathbf{S}_{\pi/2-\omega}$, za neki kot $\omega \in [0, \pi/2)$.

3.1 Perturbirana p-eliptičnost: razred $\mathcal{AP}_p(\Omega, \mathscr{V})$

V duhu prejšnjega poglavja želimo uvesti nov pogoj na koeficiente operatorja $\mathscr{L}^{A,V,\mathscr{V}}$, ki bi posplošil p-eliptičnost in igral analogno vlogo v kontekstu bilinearnih vložitev in kontraktivnosti polgrup.

Naj bosta $p > 1$ in q njegov konjuirani eksponent, tj. $1/p + 1/q = 1$. Naj bo $A \in \mathcal{A}_p(\Omega)$, $\mathscr{V}$ zaprt podprostor $W^{1,2}(\Omega)$, ki vsebuje $W^{1,2}_0(\Omega)$, in $V \in \mathcal{P}(\Omega, \mathscr{V})$. Pravimo, da

- $(A,V) \in \mathcal{AP}_p(\Omega, \mathscr{V})$, če
$$\Delta_p \left(A - \alpha \frac{pq}{4} I_d \right) > 0, \tag{10}$$
tj. če je $A - \alpha(pq)/4\, I_d$ p-eliptičen.

Ko je $V_- = 0$, jasno $V \in \mathcal{P}(\Omega, \mathscr{V})$ za vse zaprte podprostore $\mathscr{V}$ iz $W^{1,2}(\Omega)$ in $\alpha = 0$. V tem primeru se (10) ujema s p-eliptičnostjo, tj.

$$A \in \mathcal{A}_p(\Omega) \iff (A, V) \in \mathcal{AP}_p(\Omega, \mathscr{V}) \text{ za vse } \mathscr{V}.$$

Poleg tega razred $\mathcal{AP}_p$ ohranja veliko lastnosti, ki jih posedujejo razredi $\mathcal{A}_p$, kot so invarianca glede na konjugacijo, padanje s p ter invarianca glede na adjungiranje.

Na koncu opazimo, da se (10) ujema z (9) za $p = 2$, tj. $\mathcal{AP}(\Omega, \mathscr{V}) = \mathcal{AP}_2(\Omega, \mathscr{V})$.

Razred $\mathcal{AP}_p$ se je izkazal za primerno posplošitev p-eliptičnosti, kar nam omogoča razširiti tako Izrek 2 kot Izrek 3. Glede L^p-kontraktivnosti polgrupe prav tako zahtevamo, da $\mathscr{V}$ zadošča (4), ko $V_- \neq 0$. To zagotavlja, da ustrezne funkcije pripadajo $\mathscr{V}$, kar je potrebno za uporabo subkritične neenakosti (8).

Izrek 7. *Naj $\mathscr{V}$ zadošča (3) in (4). Naj bo $p > 1$ in $(A, V) \in \mathcal{AP}_p(\Omega, \mathscr{V})$. Potem obstaja $\vartheta = \vartheta(p, A, V, \mathscr{V}) > 0$, tako da če $|1 - 2/r| \leqslant |1 - 2/p|$, tedaj je $\{T_z : z \in \mathbf{S}_\vartheta\}$ analitična in kontraktivna v $L^r(\Omega)$.*

Če je $V_- = 0$, isti sklep velja pod blažjimi pogoji na $\mathscr{V}$, tj. ko $\mathscr{V}$ zadošča zgolj (3).

Izrek 8. *Naj $(\mathscr{V}, \mathscr{W})$ zadošča Privzetku BE. Naj bo $p > 1$. Naj bo q njegov konjugirani eksponent, tj. $1/p + 1/q = 1$. Privzemimo, da $(A, V) \in \mathcal{AP}_p(\Omega, \mathscr{V})$ in $(B, W) \in \mathcal{AP}_p(\Omega, \mathscr{W})$. Potem obstaja $C > 0$, tako da*

$$\int_0^\infty \int_\Omega \sqrt{\left|\nabla T_t^{A,V,\mathscr{V}} f\right|^2 + \left|V\right| \left|T_t^{A,V,\mathscr{V}} f\right|^2} \sqrt{\left|\nabla T_t^{B,W,\mathscr{W}} g\right|^2 + \left|W\right| \left|T_t^{B,W,\mathscr{W}} g\right|^2} \leqslant C \|f\|_p \|g\|_q.$$

za vse $f, g \in (L^p \cap L^q)(\Omega)$.

4 Metoda toplotnega toka

Ta razdelek je namenjen predstavitvi glavne strategije za dokazovanje izrekov o bilinearni vložitvi, ki je splošno znana pod imenom *metoda toplotnega toka*.

Naj bo $p \geqslant 2$ in naj q označuje njegov konjugirani eksponent. Naj bosta $\mathscr{A}$ in $\mathscr{B}$ matriki koeficientov dveh operatorjev v divergenčni formi ter naj bosta $\mathscr{V}$ in $\mathscr{W}$ zaprta podprostora, ki zadoščata Privzetku BE. Za funkciji $f, g \in (L^p \cap L^q)(\Omega)$ je osrednja ideja preučevati monotonost toka

$$\mathcal{E}(t) = \int_\Omega \mathcal{Q}(T_t^{\mathscr{A},\mathscr{V}} f, T_t^{\mathscr{B},\mathscr{W}} g),$$

ki je povezan s posebno *Bellmanovo funkcijo* $\mathcal{Q}$, ki sta jo uvedla Nazarov in Treil [67]. Tukaj uporabljamo poenostavljeno različico, predstavljeno v [35], ki vključuje le dve spremenljivki:

$$\mathcal{Q}(\zeta, \eta) = |\zeta|^p + |\eta|^q + \delta \begin{cases} |\zeta|^2 |\eta|^{2-q}, & |\zeta|^p \leqslant |\eta|^q, \\ (2/p)\,|\zeta|^p + (2/q - 1)\,|\eta|^q, & |\zeta|^p \geqslant |\eta|^q, \end{cases} \tag{11}$$

kjer je $\zeta, \eta \in \mathbb{C}$ in je $\delta > 0$ pozitiven parameter.

Iz zgornjih ocen Bellmanove funkcije $\mathcal{Q}$ formalno dobimo naslednjo *zgornjo oceno*:

$$-\int_0^\infty \mathcal{E}'(t)\, \mathrm{d}t \leqslant \mathcal{E}(0) = \int_\Omega \mathcal{Q}(f, g) \lesssim \|f\|_p^p + \|g\|_q^q.$$

Posledično se dokaz bilinearne vložitvene neenakosti zreducira na dokaz ustrezne *spodnje ocene* za $-\mathcal{E}'(t)$.

V neperturbiranem primeru, to je v odsotnosti členov prvega reda ter potencialov, se analiza te spodnje ocene izkaže za tesno povezano s konveksnostjo koeficientnih matrik. Natančneje, v [20] sta avtorja uvedla pojem konveksnosti glede na pare kompleksnih matrik, imenovan tudi (A, B)–konveksnost. S tem, ko sta dokazala, da Bellmanova funkcija $\mathcal{Q}$ zadošča tej posplošeni konveksnostni lastnosti glede na p–eliptične matrike [20, Izrek 5.2], sta uspela dokazati želeno spodnjo oceno in s tem tudi izrek o bilinearni vložitvi.

Delo na poljubnih odprtih množicah Ω, namesto na celotnem prostoru $\mathbb{R}^d$, prinaša znatne dodatne težave, saj mnoge ugodne lastnosti operatorjev in pripadajočih polgrup ne veljajo več. Ena glavnih težav, obravnavanih v [19], je bila utemeljitev integracije po delih, ki je potrebna za izpeljavo spodnje ocene. Ta problem je bil rešen z aproksimacijo Bellmanove funkcije $\mathcal{Q}$ z ustreznim zaporedjem funkcij, z dokazom spodnje ocene za aproksimirajoče funkcije in nato z dokazom izvorne ocene z limitnim argumentom, ki ključno uporablja Fatoujev izrek.

Na koncu želimo na kratko omeniti primer nenegativnih potencialov, obravnavan v [21]. Ko je metoda toplotnega toka enkrat vzpostavljena v neperturbiranem primeru, glavna dodatna težava izhaja iz morebitne neomejenosti potenciala. Ta problem se reši tako, da se bilinearni vložitveni izrek najprej dokaže za omejene potenciale, kjer je metoda toplotnega toka enostavnejša, nato pa se splošni primer izpelje z aproksimacijo poljubnih nenegativnih potencialov z njihovimi odrezki.

Za dokaz bilinearnih vložitvenih izrekov v kontekstih, obravnavanih v tej disertaciji, prilagodimo in razširimo zgoraj opisano metodo toplotnega toka.

V prisotnosti členov prvega reda in nenegativnih potencialov dodatni členi nižjega reda zahtevajo bistveno izpopolnitev izvornega pristopa. Njihova prisotnost nas je vodila k uvedbi novega pojma posplošene konveksnosti, ki ni več formuliran zgolj glede na matrike, temveč vključuje tudi vektorska polja in potenciale. Razred $\mathcal{B}_p$, uveden prej, v tem splošnejšem okviru igra vlogo p–eliptičnosti. Zagotavlja namreč, da Bellmanova funkcija $\mathcal{Q}$ izkazuje ustrezne konveksnostne lastnosti glede na pare koeficientov iz tega razreda.

Vključitev členov prvega reda pa dokaz spodnje ocene za $-\mathcal{E}'(t)$ tehnično oteži. Tako kot v [19] se tudi tukaj zanašamo na aproksimirajoče zaporedje Bellmanovih funkcij, da utemeljimo potrebno integracijo po delih. Glavna težava se pojavi v limitnem postopku, kjer želimo iz aproksimirajočega zaporedja ponovno pridobiti izvorno oceno. V obravnavanem okviru neposredna uporaba Fatoujevega izreka ni več zagotovljena zaradi nehomogenosti problema. Za premagovanje te težave sprva predpostavimo, da so koeficienti gladki. Pod to dodatno predpostavko lahko z uporabo t.i. notranje eliptične regularnosti utemeljimo limitni postopek z uporabo Lebesgueovega izreka o dominirani konvergenci, namesto da bi se zanašali zgolj na Fatoujev izrek. Ko je bilinearna vložitvena neenakost dokazana za gladke koeficiente, splošni primer sledi z nadaljnjim aproksimacijskim argumentom, s čimer je dokaz zaključen.

Prilagoditev metode toplotnega toka primeru operatorjev z negativnimi potenciali je enako občutljiva. Prisotnost netrivialnega negativnega dela potenciala onemogoča pridobitev želene spodnje ocene za $-\mathcal{E}'(t)$ zgolj na podlagi p–eliptičnosti matričnih koeficientov. Zato je potrebna dodatna kontrola. Ta težava se reši z uvedbo novega razreda $\mathcal{AP}_p$, ki omogoča močnejše konveksnostne ocene za Bellmanovo funkcijo $\mathcal{Q}$, dovolj robustne, da kompenzirajo prispevek negativnega dela potenciala. Ena glavnih težav pri uporabi metode toplotnega toka v tem okviru je rigorozna utemeljitev uporabe subkritične neenakosti (8) za določene funkcije v, ki so naravno povezane z Bellmanovo funkcijo. Zlasti je treba preveriti, da takšne funkcije pripadajo prostoru $\mathscr{V}$, da lahko uporabimo (8), ter utemeljiti veljavnost ustreznega verižnega pravila za izračun ∇v. Poleg tega delo s potenciali, ki zadoščajo subkritični neenakosti (8), onemogoča uporabo argumenta z odrezovanjem, uporabljenega v [21], saj ta razred potencialov ni stabilen glede na odrezovanje pozitivnega dela. Posledično je treba bilinearno vložitveno neenakost dokazati neposredno za potenciale, katerih pozitivni del je lahko neomejen, kar zahteva bolj prefinjeno analizo.

5 Uporaba: H^∞–funkcijski račun in L^p–maksimalna regularnost

Bilinearne vložitvene ocene za operatorje v divergenčni formi predstavljajo močno orodje za izpeljavo več temeljnih L^p–lastnosti takšnih operatorjev, zlasti omejenosti H^∞–funkcijskega računa in L^p–maksimalne parabolične regularnosti. V neperturbiranem primeru sta Carbonaro in Dragičević pokazala, da operatorji s p–eliptičnimi matrikami koeficientov dopuščajo omejen H^∞–funkcijski račun na L^p, kar kot posledico pomeni tudi maksimalno regularnost.

Osnovni mehanizem temelji na splošnem rezultatu Cowlinga, Dousta, McIntosha in Yagija [26], ki pravi, da sektorialni operator $\mathscr{L}$ na Banachovem prostoru $\mathcal{X}$ ima omejen H^∞–funkcijski račun, če zadošča ustrezni bilinearni oceni oblike

$$\int_0^\infty \left|\langle \mathscr{L} e^{-te^{\pm i\vartheta}\mathscr{L}} f, g\rangle\right| \, \mathrm{d}t \lesssim \|f\|_{\mathcal{X}}\|g\|_{\mathcal{X}^*}, \tag{12}$$

za nek $\vartheta > 0$ in za vse $f \in \mathcal{X}$, $g \in \mathcal{X}^*$. Omejen H^∞–funkcijski račun nato vodi do L^p–maksimalne regularnosti s pomočjo izreka Dore–Venni [32, 79].

V primeru operatorjev v divergenčni formi brez členov nižjega reda pristop v [19] temelji na obvladovanju leve strani (12) z bilinearno integralsko oceno istega tipa kot v izreku o bilinearni vložitvi, povezano s parom koeficientov $(e^{\pm i\vartheta}A, e^{\mp i\vartheta}A^*)$. Takšna redukcija je mogoča zaradi invariantnosti p–eliptičnosti glede na adjungiranje in majhne kompleksne rotacije.

Ker sta rezultata iz [26] in izrek Dore–Venni povsem funkcionalno-analitične narave in temeljita zgolj na sektorialnosti operatorja, se ta strategija naravno razširi na razrede operatorjev, obravnavane v tej disertaciji. Zlasti posplošeni strukturni pogoji, uvedeni tukaj (razreda $\mathcal{B}_p$ in $\mathcal{AP}_p$), ohranjajo invariantnostne lastnosti, potrebne za uporabo bilinearnih vložitev pri kompleksnih rotacijah in adjungiranih operatorjih. Kot posledico dobimo naslednji rezultat.

Posledica 9. *Privzemimo, da $\mathscr{V}$ spada v enega izmed posebnih primerov (a)–(d) v Razdelku 1.1. Naj bo $p > 1$ in naj $\mathscr{A}$ pripada bodisi razredu $\mathcal{B}_p(\Omega)$ bodisi razredu $\mathcal{AP}_p(\Omega, \mathscr{V})$. Naj bo $-\mathscr{L}_p$ generator polgrup $(T_t)_{t>0}$ na $L^p(\Omega)$. Tedaj $\mathscr{L}_p$ ima omejen holomorfen funkcijski račun s kotom $\vartheta < \pi/2$. Posledično ima $\mathscr{L}_p$ parabolično maksimalno regularnost.*

6 Proti trilinearni vložitvi

Zaključno poglavje disertacije je posvečeno predstavitvi novega dokaza Izreka 3.

V [22] je bila vzpostavljena trilinearna vložitev za operatorje v divergenčni formi (brez členov nižjega reda), pri čemer so avtorji ločili dva primera: $\Omega = \mathbb{R}^d$ in Ω je poljubna odprta podmnožica $\mathbb{R}^d$. V prvem primeru je bil rezultat dosežen z adaptacijo *heat-flow* metode iz [20], v drugem primeru pa so se avtorji opirali na aproksimacijske tehnike iz [19]. Vendar pa, za razliko od bilinearnega primera—kjer je bilo mogoče uporabiti obe metodi pod istimi strukturnimi predpostavkami na koeficiente matrik—trilinearni primer za poljubne odprte množice Ω zahteva strožje pogoje na koeficiente, da bi bilo mogoče uporabiti aproksimacijski postopek iz [19].

Končni cilj je premostiti to vrzel z enovitim dokazom trilinearne vložitve, ki se izogne dodatnim predpostavkam. Ključna ideja je najprej ponovno preučiti bilinearni primer in razviti novo tehniko za dokaz bilinearne vložitve, ki jo nato prilagodimo in razširimo na trilinearni kontekst.

Bistvena zamisel je spremeniti aproksimacijsko zaporedje, uporabljeno v metodi toplotnega toka. Naj bo $\{\varphi_\nu\}_\nu$ zglajevalno jedro. Približno povedano, konstrukcija v [19] poteka tako, da se $\mathcal{Q} * \varphi_\nu$ pomnoži z gladkimi funkcijami z omejenim nosilcem, ki so enake 1 na čedalje večjih podmnožicah, in se nato doda skrbno oblikovan korekcijski člen, ki povrne (A, B)–konveksnost. To nam da družino (A, B)–konveksnih aproksimacij, za katere je mogoče uporabiti metodo toplotnga toka, in pri limitnem prehodu se lahko uporabi Fatoujev izrek.

Predstavimo novo aproksimacijsko zaporedje. Tako kot v [19] pomnožimo $\mathcal{Q} * \varphi_\nu$ z gladkimi funkcijami z omejenim nosilcem, ki so enake 1 na vedno večjih podmnožicah; vendar so v naši konstrukciji te podmnožice prilagojene specifični strukturi $\mathcal{Q}$ in ni dodan noben korekcijski člen. Ta natančnejša geometrijska prilagoditev omogoča neposredno delo z novim zaporedjem in prehod v limito z uporabo Lebesguevega izreka o dominirani konvergenci, namesto da bi se zanašali zgolj na Fatoujev izrek.