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Properties of Born Approximations for Calculating the Effects of Motion on the Electromagnetic Field

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Received: 13 November 2025 | **Revised:** 5 June 2026 | **Accepted:** 24 June 2026

ABSTRACT

The non-relativistic motion of media induces very weak effects on electromagnetic fields. To compute them in an efficient and reliable way a new approach, based on Born approximations, was recently proposed. Several unknown properties of the approximations are deduced in this work. It is shown that they are features of the analyzed approximations and do not depend on specific problems. In particular, they hold true for one-dimensional, two-dimensional, or three-dimensional problems involving moving media, regardless of any non-uniform translational or rotational motion of matter. The deduced properties are related to the reliability of the approximations and to the possibility to exploit them to find the results of interest in efficient ways. In the final part of the paper, the deduced properties are independently verified by deriving closed-form solutions for simple classes of problems involving media in motion. These findings also demonstrate the practical relevance of the results.

1 | Introduction

Electromagnetic boundary value problems involving media in motion are very important in astrophysics, space science and engineering because they appear in a wide range of practical and theoretical contexts. For instance, they are essential for understanding how electromagnetic waves interact with moving materials, such as plasmas [1, 2], fluids [3, 4], bulk solids [5], solids [6–8], or machine components [9] (chapter 17).

In the presence of moving media also electromagnetic inverse scattering problems are of interest [10–12]. They aim at the reconstruction of velocity fields and possibly other properties of materials.

For both direct and inverse problems, reliably computing the effects of motion on electromagnetic fields is crucial [12–15]. In [16], we introduced a new approach, based on Born's classical approximation for solving electromagnetic scattering problems, which provides very reliable estimates of the indicated effects, when these are particularly weak and therefore difficult to evaluate [16].

In this paper, we firstly deduce general and previously unknown properties of the Born approximations of the effects of motion on the electromagnetic field. These properties improve both our understanding and the accurate modeling of such effects. In particular, it is shown that for small values of the largest normalized speed, the Born approximation of order i determines effects of motion whose norms are controlled by the i-th power of largest normalized speed itself. As a consequence, the Born approximation of order j, for $j > i$, cannot provide any improvement on the first i dominant terms of the estimate of the effects of motion. Some of the above results were actually introduced in [16] for specific problems formulated in two-dimensional domains involving cylinders in uniform, axial motion, illuminated by time-harmonic uniform plane waves propagating in the plane orthogonal to the cylinders' axes. However, in this manuscript we not only prove the results introduced in our previous paper, but we also show that all these properties are intrinsic features of the Born approximations of the effects of motion. This means that they apply to all electromagnetic boundary value problems with time-invariant motion, regardless of the problem properties (such as radiation-scattering or guided wave problems), the type of sources or illuminating fields, the dimensionality of the domain (one-, two-, or three-dimensional), or the nature of the motion, whether uniform or non-uniform, translational or rotational.

Subsequently, a condition is introduced regarding the effects of motion on the electromagnetic field. While it places some limitations on the effects allowed by the formulations of the problems of interest, it does not significantly reduce the possible applications of our analysis. It is then shown that for problems satisfying the above condition the i-th order Born approximation of the effects of motion provides exact results, for small values of the largest normalized speed, for the first i dominant terms, those controlled by the first i powers of the largest normalized speed itself. When this condition is satisfied, the Born approximations not only exhibit the properties described above, but also display the correct behavior, thereby ensuring their reliability for small normalized speeds. Moreover, it is shown that Born approximations are generally not reliable for estimating the effects of motion on electromagnetic fields when the aforementioned weak condition is not satisfied.

As a byproduct, for problems that satisfy the weak condition, one can determine in advance the appropriate order of the Born approximation needed for a given goal: if the effects of motion of order i are of interest then just the approximation of i-th order should be computed. The computation should not be stopped to the j-th order, for $j < i$, and it should not be extended to the k-th order, for $k > i$. As a matter of fact, in the former case, the results would not have the required quality, whereas in the latter the same quality could be obtained with less effort.

As an additional outcome, we provide a formal theoretical derivation for a property previously observed numerically within the specific framework of [16], but not yet formally conceptualized as an intrinsic property of Born approximations. We show that this property is not limited to those specific configurations, but represents a general feature applicable to most classes of problems of interest. In particular, for velocity distributions belonging to vector spaces of dimension one, the effects of motion on electromagnetic fields need to be computed only a few times. Reliable approximations for distributions proportional to those previously considered can then be obtained by simply multiplying the solutions by scaling factors. This result can be used to save CPU time by skipping numerous challenging simulations. It holds true if the considered problem satisfies the condition discussed above.

The final parts of the paper are devoted to the analysis of problems allowing an independent verification of the deduced properties. Due to the well known difficulties encountered by most methods in calculating reliable approximations of the effects of motion [16], the verification could not rely on numerical methods and we had to focus on problems which could be solved analytically [17, 18]. In particular, we considered one-dimensional and two-dimensional problems in order to verify our results in a simple, meaningful, and independent manner. The verification and the obtained analytical and numerical results confirm that the conditions and properties analyzed and deduced in this work have practical relevance.

The paper is organized as follows. Section 2 defines the problems of interest in detail. The most significant and largely unknown properties of the Born approximations of the effects of motion on electromagnetic fields are deduced in Sections 3 and 4. Section 5 establishes the reliability of Born approximations for problems satisfying a simple and weak condition. Section 6 introduces a strategy to reduce the computational burden, showing how multiple Born approximations can be obtained by simply multiplying a single calculated solution by scaling factors. The validation of our theory is discussed in Section 7 through simple yet meaningful problems. Finally, Section 8 presents several numerical results that confirm our deductions, highlighting the unique properties of this new methodology and the most effective ways to exploit them in practice.

2 | Definition of the Problem of Interest

In this paper, we are interested in electromagnetic problems that involve moving media under time-harmonic excitation. Translational or rotational movements may be considered in the problem of interest: the only restriction is that the velocity field, which describes the movement of all materials and is denoted by $\mathbf{v}$, is time-invariant. This assumption, together with the time invariance of the constitutive parameters and the sinusoidal time dependence of the sources, allow us to work in the frequency domain [9, 19]. The factor $e^{j\omega t}$ is ubiquitous and is suppressed.

These problems were studied in several papers of the authors. The main references for this section are [13], for cylinders moving in the axial direction, and [14], for rotating axisymmetric objects.

Our problems will be formulated in open, bounded, and connected domains $\Omega \in \mathbb{R}^n$, $n = 1, 2, 3$, having Lipschitz continuous, stationary boundaries denoted by Γ. To take into account the possible presence of different materials, we assume that Ω can be decomposed into m subdomains, denoted Ω_i, $i \in M = \{1, \dots, m\}$. M_0 is the set of indices identifying subdomains occupied by media at rest; M_β is the same but for moving media. The interior of the union of the closures of Ω_i, for $i \in M_\beta$, will be denoted by $\Omega_\beta = (\cup_{i \in M_\beta} \overline{\Omega}_i)^\circ$.

In any subdomain Ω_i, $i \in M_0$, we will always consider media which are linear, isotropic, time-invariant, characterized by regular complex-valued fields $\varepsilon_r(\mathbf{r})$ and $\mu_r(\mathbf{r})$. In any Ω_i, $i \in M_\beta$, the features are the same apart from the fact that the constitutive parameters $\varepsilon_r(\mathbf{r})$ and $\mu_r(\mathbf{r})$ are real-valued fields. This limitation is present in all our studies on electromagnetic problems involving moving media from the very beginning. The reasons for this constraint can be found, for example, in [20]. We will not try to overcome it in the present paper.

In any Ω_i, $i \in M_\beta$, the velocity field $\mathbf{v}(\mathbf{r}) = |\mathbf{v}|\hat{\mathbf{v}} = v\hat{\mathbf{v}}$ is regular and almost everywhere different from zero. Moreover, since empty space can always be considered at rest [20], in the same subdomains the constitutive parameters of the media at rest are assumed to be different from those of vacuum. Finally, we assume that these regular fields are restrictions to Ω_i of fields defined in larger open sets containing $\overline{\Omega}_i$.

It is very well known [21, 22] that a linear and isotropic medium is perceived as a bianisotropic material when it is moving. Its constitutive relations can be written in the following form [9, 21]:

$$\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E} + \varepsilon_0\varepsilon_r\alpha v^2[(\hat{\mathbf{v}}\times\mathbf{E})\times\hat{\mathbf{v}}] + \alpha v(\hat{\mathbf{v}}\times\mathbf{H}), \tag{1}$$

$$\mathbf{B} = \mu_0\mu_r\mathbf{H} + \mu_0\mu_r\alpha v^2[(\hat{\mathbf{v}}\times\mathbf{H})\times\hat{\mathbf{v}}] - \alpha v(\hat{\mathbf{v}}\times\mathbf{E}), \tag{2}$$

where

$$\alpha = \frac{\mu_r\varepsilon_r - 1}{c_0^2 - \mu_r\varepsilon_r v^2} = \frac{n^2-1}{c_0^2(1-n^2\beta^2)}, \tag{3}$$

being $\beta = \frac{v}{c_0}$, a real-valued field in any Ω_i, $i \in M_\beta$. c_0 is the speed of light in vacuum and β is a positive quantity since v is the magnitude of the velocity field. For the bounds given by the special theory of relativity we can assume that in Ω_i, for $i \in M_\beta$, we have $\beta < 1$ and $\beta < \frac{1}{n}$. Since we will be particularly interested in small values of speed, without loss of generality we assume that we can find $C_{\beta M}$:

$$0 \le \beta \le C_{\beta M} < min\left(1, \frac{1}{n}\right) \tag{4}$$

in Ω_i, for any $i \in M_\beta$. Then $\mathbf{v}$ is not only piecewise regular, but it is also essentially bounded [23]. Moreover, α is a well defined field in Ω_i, for any $i \in M_\beta$, and $|\alpha|$ is there bounded, since n is bounded for physical reasons.

For our next developments, we recast the above constitutive relations in terms of the usual constitutive parameters of bianisotropic media [24]:

$$\mathbf{D} = \varepsilon^v\mathbf{E} + \xi\mathbf{H}, \tag{5}$$

$$\mathbf{B} = \zeta\mathbf{E} + \mu^v\mathbf{H}. \tag{6}$$

We introduced the superscript v to avoid misunderstandings about constitutive parameters: $\varepsilon = \varepsilon_0\varepsilon_r$ and $\mu = \mu_0\mu_r$ are scalar fields and refer to those of media at rest; ε^v, ξ, ζ and μ^v are tensor fields defining the electromagnetic properties of moving media. In the following, according to [16], we will often consider ε^0 and μ^0, which are the tensor fields obtained by multiplying the constitutive parameters of media at rest and the identity tensor I: $\varepsilon^0 = \varepsilon_0\varepsilon_r I$ and $\mu^0 = \mu_0\mu_r I$.

From Equations (1), (2), (5) and (6) one easily deduces

$$\varepsilon^v\mathbf{E} = \varepsilon_0\varepsilon_r\mathbf{E} + \varepsilon_0\varepsilon_r\alpha v^2[(\hat{\mathbf{v}}\times\mathbf{E})\times\hat{\mathbf{v}}], \tag{7}$$

$$\xi\mathbf{H} = \alpha v(\hat{\mathbf{v}}\times\mathbf{H}), \tag{8}$$

$$\zeta\mathbf{E} = -\alpha v(\hat{\mathbf{v}}\times\mathbf{E}), \tag{9}$$

$$\mu^v\mathbf{H} = \mu_0\mu_r\mathbf{H} + \mu_0\mu_r\alpha v^2[(\hat{\mathbf{v}}\times\mathbf{H})\times\hat{\mathbf{v}}]. \tag{10}$$

To derive the mathematical formulations of the problems of interest we first introduce the following additional notations. Let $(L^2(\Omega))^n$ be the usual Hilbert space of square-integrable vector fields on Ω with values in $\mathbb{C}^n$, $n = 2, 3$, [23] and with scalar product given by $(\mathbf{u},\mathbf{w})_{0,\Omega} = \int_\Omega \mathbf{w}^*\cdot\mathbf{u}\,d\Omega$, where $\mathbf{w}^*$ denotes the conjugate transpose of the column vector $\mathbf{w}$. $\|\|_{0,\Omega}$ will denote the corresponding norm. The space where we will seek the fields $\mathbf{E}$ and $\mathbf{H}$ is U, a subspace $(L^2(\Omega))^3$. It is defined, together with its scalar product, $(,)_U$, and norm, $\|\|_U$, in [13], for cylinders moving in the axial direction, and [14], for rotating axisymmetric objects. We will not distinguish between the two cases because our results will apply to both of them. However, in case we need to provide details, we will always do that by referring to the case of three-dimensional domains. For example, from [14] we deduce $U = \{\mathbf{w}\in(L^2(\Omega))^3 |\, \nabla\times\mathbf{w}\in(L^2(\Omega))^3 \text{ and } \mathbf{w}\times\mathbf{n}\in L_t^2(\Gamma)\}$, where $L_t^2(\Gamma) = \{\mathbf{w}\in(L^2(\Gamma))^3 | \mathbf{w}\cdot\mathbf{n} = 0 \text{ almost everywhere on } \Gamma\}$. Accordingly, $(\mathbf{u},\mathbf{w})_U = (\mathbf{u},\mathbf{w})_{0,\Omega} + (\nabla\times\mathbf{u},\nabla\times\mathbf{w})_{0,\Omega} + (\mathbf{u}\times\mathbf{n},\mathbf{w}\times\mathbf{n})_{0,\Omega}$ and $\|\mathbf{w}\|_U = ((\mathbf{w},\mathbf{w})_U)^{1/2}$. The corresponding details for other cases can be usually obtained with some trivial changes. They can be easily deduced by comparing the details in [13] and [14].

One can then formulate the following time-harmonic electromagnetic boundary value problems in the presence of moving media [14] (Problem 1; see also Problem 1 of [13] or problem (1) of [16]):

Problem 1. Under the indicated hypotheses on the domain, its subdomains, their boundaries and on the media involved, given $\omega > 0$, $\mathbf{J}_e, \mathbf{J}_m \in (L^2(\Omega))^3$, and $\mathbf{f}_R \in L_t^2(\Gamma)$, find $\mathbf{E}, \mathbf{H} \in U$, satisfying

$$\begin{cases} \nabla\times\mathbf{H} - j\omega\varepsilon^v\mathbf{E} - j\omega\xi\mathbf{H} = \mathbf{J}_e & \text{in } \Omega, \\ \nabla\times\mathbf{E} + j\omega\zeta\mathbf{E} + j\omega\mu^v\mathbf{H} = -\mathbf{J}_m & \text{in } \Omega, \\ (\mathbf{H}\times\mathbf{n}) - Y(\mathbf{n}\times(\mathbf{E}\times\mathbf{n})) = \mathbf{f}_R & \text{on } \Gamma, \end{cases} \tag{11}$$

where $\mathbf{J}_e$ and $\mathbf{J}_m$ are the electric and magnetic current densities, Y denotes the admittance operator [9] (p. 598), and $\mathbf{f}_R$ is the source term of inhomogeneous admittance boundary conditions.

For the solutions of the same problems in the presence of media at rest we will use the symbol $(\mathbf{E}_0, \mathbf{H}_0)$ [16]. In this case, the formulation is the same provided that $\mathbf{E}$ is replaced by $\mathbf{E}_0$, $\mathbf{H}$ by $\mathbf{H}_0$, ε^v by ε^0, μ^v by μ^0, and ξ and ζ by 0.

The effects of motion on the electromagnetic field is then given by $(\mathbf{E}-\mathbf{E}_0)$, $(\mathbf{H}-\mathbf{H}_0)$. Such effects are known to be small, for non-relativistic velocities [16]. This is the reason why it is difficult to compute them. However, in [16] we proposed a new approach to get reliable approximations of these effects, for non-relativistic velocities. The focus of this manuscript is to deduce several additional, significant properties of these approximations.

From Problem 1 and the corresponding formulation for $\mathbf{E}_0$ and $\mathbf{H}_0$ we easily deduce that $(\mathbf{E}-\mathbf{E}_0)$ and $(\mathbf{H}-\mathbf{H}_0)$ can be calculated by solving the following problem (problem (5) of [16]):

Problem 2. Under the indicated hypotheses on the domain, its subdomains, their boundaries and on the media involved, given $\omega > 0$, $\mathbf{J}_{e,eq} \in (L^2(\Omega))^3$ and $\mathbf{J}_{m,eq} \in (L^2(\Omega))^3$, find $(\mathbf{E}-\mathbf{E}_0)$, $(\mathbf{H}-$

$\mathbf{H}_0) \in U$, satisfying

$$\begin{cases} \nabla \times (\mathbf{H} - \mathbf{H}_0) - j\omega\varepsilon^0(\mathbf{E} - \mathbf{E}_0) = \mathbf{J}_{e,eq} & \text{in } \Omega, \\ \nabla \times (\mathbf{E} - \mathbf{E}_0) + j\omega\mu^0(\mathbf{H} - \mathbf{H}_0) = -\mathbf{J}_{m,eq} & \text{in } \Omega, \\ (\mathbf{H} - \mathbf{H}_0) \times \mathbf{n} - Y\,(\mathbf{n} \times (\mathbf{E} - \mathbf{E}_0) \times \mathbf{n}) = 0 & \text{on } \Gamma, \end{cases} \tag{12}$$

where the equivalent electric and magnetic current densities are given by:

$$\mathbf{J}_{e,eq} = j\omega(\varepsilon^{v} - \varepsilon^0)\mathbf{E} + j\omega\xi\mathbf{H}, \tag{13}$$

$$\mathbf{J}_{m,eq} = j\omega(\mu^{v} - \mu^0)\mathbf{H} + j\omega\zeta\mathbf{E}. \tag{14}$$

They are equal to zero outside Ω_β since, by definition, $(\varepsilon^{v} - \varepsilon^0)$, ξ, $(\mu^{v} - \mu^0)$ and ζ are trivial tensor fields outside the subdomain occupied by moving media. Moreover, the considered hypotheses on the constitutive parameters and on the velocity field guarantee that all entries of $(\varepsilon^{v} - \varepsilon^0)$, ξ, $(\mu^{v} - \mu^0)$ and ζ are bounded. Then, Equations (13) and (14) transform square-integrable vector fields, $\mathbf{E}$ and $\mathbf{H}$, into square-integrable equivalent sources, $\mathbf{J}_{e,eq}$ and $\mathbf{J}_{m,eq}$.

Problem 2 concerns electromagnetic radiation from prescribed sources in media at rest. It is known to be well posed [23] under very weak assumptions on the domain Ω and its subdomains Ω_i, on their boundaries Γ and $\partial\Omega_i$, on the constitutive parameters ε^0 and μ^0 and on the admittance operator Y. This means that it has a unique solution and that we can find $C_{eje}, C_{ejm}, C_{hje}, C_{hjm} \in \mathbb{R}$, independent of $\mathbf{J}_{e,eq}$ and $\mathbf{J}_{m,eq}$, such that

$$\|(\mathbf{E} - \mathbf{E}_0)\|_U \leq C_{eje}\|\mathbf{J}_{e,eq}\|_{0,\Omega} + C_{ejm}\|\mathbf{J}_{m,eq}\|_{0,\Omega}, \tag{15}$$

and

$$\|(\mathbf{H} - \mathbf{H}_0)\|_U \leq C_{hje}\|\mathbf{J}_{e,eq}\|_{0,\Omega} + C_{hjm}\|\mathbf{J}_{m,eq}\|_{0,\Omega}. \tag{16}$$

Since we are interested in the effects of motion on the electromagnetic field, we can consider different velocity fields for a given problem. For this reason, we will refer to problems with fixed domain, subdomains, boundaries, ω, ε^0, μ^0, Y, but with velocity fields that may vary from one case to another, as a specific class of problems of interest. Within the class of problems defined above, we further consider a subclass in which the velocity fields belong to a subspace V and satisfy additional properties beyond those introduced just below Equation (4).

By taking account that the effects of motion in Problem 2 are hidden in $\mathbf{J}_{e,eq}$ and $\mathbf{J}_{m,eq}$, the real numbers C_{eje}, C_{ejm}, C_{hje} and C_{hjm} are completely independent of the velocity field $\mathbf{v}$. Therefore, they are constant for any specific class of problems of interest and can change only if one considers different classes of problems.

The new methodology introduced in [16] to obtain reliable approximations of the effects of motion on the electromagnetic field defines good approximations of the equivalent current densities. Due to the linearity and the well posedness of Problem 2, the solutions obtained by using good approximations of the sources provide good estimates of the effects of interest.

3 | Definition of Sequences of Approximate Equivalent Sources and of Born Approximations

In [16] we proposed to apply the classical Born approximation [25] (p. 328) to determine the effects of non-relativistic motion on the electromagnetic field.

In particular, for any incident field the first-order Born approximation of $((\mathbf{E} - \mathbf{E}_0), (\mathbf{H} - \mathbf{H}_0))$, denoted by $((\mathbf{E} - \mathbf{E}_0)_{ba1}, (\mathbf{H} - \mathbf{H}_0)_{ba1})$, is obtained by solving Problem 2 when the equivalent sources are approximated by (see Equations 8 and 9 of [16]):

$$\mathbf{J}_{e,eq,a1} = j\omega(\varepsilon^{v} - \varepsilon^0)\mathbf{E}_0 + j\omega\xi\mathbf{H}_0, \tag{17}$$

$$\mathbf{J}_{m,eq,a1} = j\omega(\mu^{v} - \mu^0)\mathbf{H}_0 + j\omega\zeta\mathbf{E}_0. \tag{18}$$

The basic idea of the new approach, as mentioned above, is to exploit the weakness of the effects of motion on the electromagnetic field, for velocity fields of practical interest. Since such effects, that is $(\mathbf{E} - \mathbf{E}_0)$ and $(\mathbf{H} - \mathbf{H}_0)$, are small under the indicated condition, we can use $\mathbf{E}_0$ and $\mathbf{H}_0$ as good approximations of $\mathbf{E}$ and $\mathbf{H}$. As a byproduct, by comparing Equations (13), (14), (17) and (18), $\mathbf{J}_{e,eq,a1}$ is a good approximation of $\mathbf{J}_{e,eq}$ and $\mathbf{J}_{m,eq,a1}$ is a good approximation of $\mathbf{J}_{m,eq}$.

According to [16], we can extend our definitions to introduce higher-order Born approximations. To this end let us define in an explicit way the zero-order approximations of $\mathbf{E}$ and $\mathbf{H}$ as follows:

$$\mathbf{E}_{a0} = \mathbf{E}_0, \tag{19}$$

$$\mathbf{H}_{a0} = \mathbf{H}_0. \tag{20}$$

Since $((\mathbf{E} - \mathbf{E}_0)_{ba1}, (\mathbf{H} - \mathbf{H}_0)_{ba1})$, approximates $((\mathbf{E} - \mathbf{E}_0), (\mathbf{H} - \mathbf{H}_0))$, better approximations of $\mathbf{E} = \mathbf{E}_0 + (\mathbf{E} - \mathbf{E}_0)$ and $\mathbf{H} = \mathbf{H}_0 + (\mathbf{H} - \mathbf{H}_0)$ are given by:

$$\mathbf{E}_{a1} = (\mathbf{E} - \mathbf{E}_0)_{ba1} + \mathbf{E}_{a0}, \tag{21}$$

$$\mathbf{H}_{a1} = (\mathbf{H} - \mathbf{H}_0)_{ba1} + \mathbf{H}_{a0}. \tag{22}$$

From these better approximation of the fields $\mathbf{E}$ and $\mathbf{H}$, we can deduce better approximations of the equivalents sources, $\mathbf{J}_{e,eq,a2}$ and $\mathbf{J}_{m,eq,a2}$, by using Equations (17) and (18) with $(\mathbf{E}_0, \mathbf{H}_0)$, that is, $(\mathbf{E}_{a0}, \mathbf{H}_{a0})$, replaced by $(\mathbf{E}_{a1}, \mathbf{H}_{a1})$.

By solving Problem 2 with equivalent sources given by $\mathbf{J}_{e,eq,a2}$ and $\mathbf{J}_{m,eq,a2}$ we get the second-order Born approximation of $((\mathbf{E} - \mathbf{E}_0), (\mathbf{H} - \mathbf{H}_0))$, which is denoted by $((\mathbf{E} - \mathbf{E}_0)_{ba2}, (\mathbf{H} - \mathbf{H}_0)_{ba2})$.

We have now defined the first two elements of the sequences $\left\{(\mathbf{J}_{e,eq,ai}, \mathbf{J}_{m,eq,ai})\right\}_{i=1,2,\ldots}$, $\left\{((\mathbf{E} - \mathbf{E}_0)_{bai}, (\mathbf{H} - \mathbf{H}_0)_{bai})\right\}_{i=1,2,\ldots}$, and $\left\{(\mathbf{E}_{ai}, \mathbf{H}_{ai})\right\}_{i=0,1,\ldots}$. All other elements of the sequences can be obtained by generalizing the previous definitions as follows:

$$\mathbf{J}_{e,eq,ai} = j\omega(\varepsilon^{v} - \varepsilon^0)\mathbf{E}_{am} + j\omega\xi\mathbf{H}_{am}, \tag{23}$$

$$\mathbf{J}_{m,eq,ai} = j\omega(\mu^{v} - \mu^0)\mathbf{H}_{am} + j\omega\zeta\mathbf{E}_{am}, \tag{24}$$

$$\mathbf{E}_{ai} = (\mathbf{E}-\mathbf{E}_0)_{bai} + \mathbf{E}_{a0}, \tag{25}$$

$$\mathbf{H}_{ai} = (\mathbf{H}-\mathbf{H}_0)_{bai} + \mathbf{H}_{a0}, \tag{26}$$

for $i \geq 1$, $m = i-1$, where the i-th order Born approximation of the effects of motion, that is $((\mathbf{E}-\mathbf{E}_0)_{bai}, (\mathbf{H}-\mathbf{H}_0)_{bai})$, is found by solving Problem 2 with equivalent sources given by $\mathbf{J}_{e,eq,ai}$ and $\mathbf{J}_{m,eq,ai}$.

4 | Basic Properties of Born Approximations

In this section, we prove several properties of the Born approximations. These properties clearly demonstrate how the quality of the approximations depends on integer powers of the largest normalized speed.

With this aim, let us firstly analyze the terms involving the tensors $(\varepsilon^v - \varepsilon^0)$ and $(\mu^v - \mu^0)$. For any $i \in M_\beta$, from Equations (7) and (10), we get:

$$(\varepsilon^v - \varepsilon^0)\mathbf{A} = \varepsilon_0\varepsilon_r\mathbf{A} + \varepsilon_0\varepsilon_r\alpha v^2[(\hat{\mathbf{v}}\times\mathbf{A})\times\hat{\mathbf{v}}] - \varepsilon_0\varepsilon_r\mathbf{A} = \varepsilon_0\varepsilon_r\alpha v^2[(\hat{\mathbf{v}}\times\mathbf{A})\times\hat{\mathbf{v}}], \tag{27}$$

$$(\mu^v - \mu^0)\mathbf{A} = \mu_0\mu_r\mathbf{A} + \mu_0\mu_r\alpha v^2[(\hat{\mathbf{v}}\times\mathbf{A})\times\hat{\mathbf{v}}] - \mu_0\mu_r\mathbf{A} = \mu_0\mu_r\alpha v^2[(\hat{\mathbf{v}}\times\mathbf{A})\times\hat{\mathbf{v}}]. \tag{28}$$

In particular, for any $\mathbf{r} \in \Omega_i$, $i \in M_\beta$, we have $|(\hat{\mathbf{v}}\times\mathbf{A})\times\hat{\mathbf{v}}| \leq |\mathbf{A}|$, since $\hat{\mathbf{v}}$ is a unit vector. Then, $[(\hat{\mathbf{v}}\times\mathbf{A})\times\hat{\mathbf{v}}] \in (L^2(\Omega_i))^3$ for any $\mathbf{A} \in (L^2(\Omega_i))^3$ and $\|(\hat{\mathbf{v}}\times\mathbf{A})\times\hat{\mathbf{v}}\|_{0,\Omega_i} \leq \|\mathbf{A}\|_{0,\Omega_i}$.

But for any $i \in M_\beta$ the scalar fields ε_r and μ_r are restrictions to Ω_i of regular fields defined in open sets larger than $\overline{\Omega}_i$. Then, for the extreme-value theorem [26] (p. 150) we can find m_{ei} and m_{mi}, such that $|\varepsilon_r| \leq m_{ei}$ and $|\mu_r| \leq m_{mi}$, $i \in M_\beta$. As a consequence, n is bounded for any $i \in M_\beta$ and this implies that $|\alpha|$ is bounded, too, in the same subdomains, under the condition reported in Equation (4). Let $\delta = \alpha c_0^2$. $|\delta|$ is bounded exactly as $|\alpha|$ and $\alpha v^2 = \delta\beta^2$. Let m_{di}: $|\delta| \leq m_{di}$. Finally, let $S_{\beta i} \leq C_{\beta M} < 1$ be an upper bound for $|\beta| = \beta$, for any $i \in M_\beta$.

With these hypotheses we then deduce, for any $i \in M_\beta$:

$$\begin{aligned}
&\|(\varepsilon^v - \varepsilon^0)\mathbf{A}\|^2_{0,\Omega_i} \\
&= \int_{\Omega_i} \left|(\varepsilon^v - \varepsilon^0)\mathbf{A}\right|^2 dV = \int_{\Omega_i} \left|\varepsilon_0\varepsilon_r\alpha v^2[(\hat{\mathbf{v}}\times\mathbf{A})\times\hat{\mathbf{v}}]\right|^2 dV \\
&= \varepsilon_0^2 \int_{\Omega_i} \varepsilon_r^2\alpha^2 v^4 \left|(\hat{\mathbf{v}}\times\mathbf{A})\times\hat{\mathbf{v}}\right|^2 dV \leq \varepsilon_0^2 m_{ei}^2 m_{di}^2 S_{\beta i}^4 \int_{\Omega_i} |(\hat{\mathbf{v}}\times\mathbf{A})\times\hat{\mathbf{v}}|^2 dV \\
&= \varepsilon_0^2 m_{ei}^2 m_{di}^2 S_{\beta i}^4 \|(\hat{\mathbf{v}}\times\mathbf{A})\times\hat{\mathbf{v}}\|^2_{0,\Omega_i} \leq \varepsilon_0^2 m_{ei}^2 m_{di}^2 S_{\beta i}^4 \|\mathbf{A}\|^2_{0,\Omega_i}
\end{aligned} \tag{29}$$

so that $\|(\varepsilon^v - \varepsilon^0)\mathbf{A}\|_{0,\Omega_i} \leq \varepsilon_0 m_{ei} m_{di} S_{\beta i}^2 \|\mathbf{A}\|_{0,\Omega_i}$. In an analogous way we get $\|(\mu^v - \mu^0)\mathbf{A}\|_{0,\Omega_i} \leq \mu_0 m_{mi} m_{di} S_{\beta i}^2 \|\mathbf{A}\|_{0,\Omega_i}$.

Let m_e, m_m, m_d, S_β the largest of the corresponding values for all $i \in M_\beta$. We obtain that the previous inequalities hold for all $i \in M_\beta$, if m_{ei}, m_{di}, $S_{\beta i}$ and m_{mi} are replaced, respectively, by m_e, m_d, S_β and m_m. Then we get:

$$\|(\varepsilon^v - \varepsilon^0)\mathbf{A}\|_{0,\Omega_\beta} \leq \varepsilon_0 m_e m_d S_\beta^2 \|\mathbf{A}\|_{0,\Omega_\beta}, \tag{30}$$

$$\|(\mu^v - \mu^0)\mathbf{A}\|_{0,\Omega_\beta} \leq \mu_0 m_m m_d S_\beta^2 \|\mathbf{A}\|_{0,\Omega_\beta}. \tag{31}$$

By the same token, from Equations (8) and (9) we deduce:

$$\begin{aligned}
\|\xi\mathbf{A}\|_{0,\Omega_\beta} &= \|\zeta\mathbf{A}\|_{0,\Omega_\beta} = \|\alpha v(\hat{\mathbf{v}}\times\mathbf{A})\|_{0,\Omega_\beta} \\
&= \|\frac{1}{c_0}\delta\beta(\hat{\mathbf{v}}\times\mathbf{A})\|_{0,\Omega_\beta} \leq \frac{1}{c_0} m_d S_\beta \|\mathbf{A}\|_{0,\Omega_\beta}.
\end{aligned} \tag{32}$$

With these inequalities we examine again the equations defining equivalent electric and magnetic current densities in our methodology. We consider, in particular, Equations (13) and (14), which define the exact equivalent sources, but the same considerations apply to Equations (23) and (24), for the approximate ones. We deduce:

$$\begin{aligned}
\|\mathbf{J}_{e,eq}\|_{0,\Omega} &= \|\mathbf{J}_{e,eq}\|_{0,\Omega_\beta} = \|j\omega(\varepsilon^v - \varepsilon^0)\mathbf{E} + j\omega\xi\mathbf{H}\|_{0,\Omega_\beta} \\
&\leq \omega\|(\varepsilon^v - \varepsilon^0)\mathbf{E}\|_{0,\Omega_\beta} + \omega\|\xi\mathbf{H}\|_{0,\Omega_\beta} \\
&\leq \omega\varepsilon_0 m_e m_d S_\beta^2 \|\mathbf{E}\|_{0,\Omega_\beta} + \frac{\omega}{c_0} m_d S_\beta \|\mathbf{H}\|_{0,\Omega_\beta}.
\end{aligned} \tag{33}$$

In an analogous way we get:

$$\begin{aligned}
\|\mathbf{J}_{m,eq}\|_{0,\Omega} &= \|\mathbf{J}_{m,eq}\|_{0,\Omega_\beta} = \|j\omega(\mu^v - \mu^0)\mathbf{H} + j\omega\zeta\mathbf{E}\|_{0,\Omega_\beta} \\
&\leq \omega\|(\mu^v - \mu^0)\mathbf{H}\|_{0,\Omega_\beta} + \omega\|\zeta\mathbf{E}\|_{0,\Omega_\beta} \\
&\leq \omega\mu_0 m_m m_d S_\beta^2 \|\mathbf{H}\|_{0,\Omega_\beta} + \frac{\omega}{c_0} m_d S_\beta \|\mathbf{E}\|_{0,\Omega_\beta},
\end{aligned} \tag{34}$$

$$\|\mathbf{J}_{e,eq,a1}\|_{0,\Omega} = \|\mathbf{J}_{e,eq,a1}\|_{0,\Omega_\beta} \leq \omega\varepsilon_0 m_e m_d S_\beta^2 \|\mathbf{E}_0\|_{0,\Omega_\beta} + \frac{\omega}{c_0} m_d S_\beta \|\mathbf{H}_0\|_{0,\Omega_\beta}, \tag{35}$$

$$\|\mathbf{J}_{m,eq,a1}\|_{0,\Omega} = \|\mathbf{J}_{m,eq,a1}\|_{0,\Omega_\beta} \leq \omega\mu_0 m_m m_d S_\beta^2 \|\mathbf{H}_0\|_{0,\Omega_\beta} + \frac{\omega}{c_0} m_d S_\beta \|\mathbf{E}_0\|_{0,\Omega_\beta}. \tag{36}$$

By the continuous dependence of the solutions of Problem 2 on the sources (see Equations 15 and 16) and the very definition of the first-order Born approximation we get

$$\|(\mathbf{E}-\mathbf{E}_0)_{ba1}\|_U \leq C_{eje}\|\mathbf{J}_{e,eq,a1}\|_{0,\Omega} + C_{ejm}\|\mathbf{J}_{m,eq,a1}\|_{0,\Omega}. \tag{37}$$

Then, by using inequalities (35) and (36)

$$\begin{aligned}
&\|(\mathbf{E}-\mathbf{E}_0)_{ba1}\|_U \\
&\leq C_{eje}\left(\omega\varepsilon_0 m_e m_d S_\beta^2 \|\mathbf{E}_0\|_{0,\Omega_\beta} + \frac{\omega}{c_0} m_d S_\beta \|\mathbf{H}_0\|_{0,\Omega_\beta}\right) \\
&\quad + C_{ejm}\left(\omega\mu_0 m_m m_d S_\beta^2 \|\mathbf{H}_0\|_{0,\Omega_\beta} + \frac{\omega}{c_0} m_d S_\beta \|\mathbf{E}_0\|_{0,\Omega_\beta}\right) \\
&= \frac{\omega}{c_0} m_d S_\beta \left(C_{eje}\|\mathbf{H}_0\|_{0,\Omega_\beta} + C_{ejm}\|\mathbf{E}_0\|_{0,\Omega_\beta}\right) \\
&\quad + \omega m_d S_\beta^2 \left(C_{eje}\varepsilon_0 m_e \|\mathbf{E}_0\|_{0,\Omega_\beta} + C_{ejm}\mu_0 m_m \|\mathbf{H}_0\|_{0,\Omega_\beta}\right).
\end{aligned} \tag{38}$$

Similarly

$$\begin{aligned}
&\|(\mathbf{H}-\mathbf{H}_0)_{ba1}\|_U \\
&\le \frac{\omega}{c_0} m_d S_\beta \Big(C_{hje}\|\mathbf{H}_0\|_{0,\Omega_\beta} + C_{hjm}\|\mathbf{E}_0\|_{0,\Omega_\beta} \Big) \\
&\quad + \omega m_d S_\beta^2 \Big(C_{hje}\varepsilon_0 m_e \|\mathbf{E}_0\|_{0,\Omega_\beta} + C_{hjm}\mu_0 m_m \|\mathbf{H}_0\|_{0,\Omega_\beta} \Big).
\end{aligned} \tag{39}$$

Under the condition in Equation (4), which is valid for all practical velocity fields, the norms $\|\mathbf{J}_{e,eq,a1}\|_{0,\Omega}$, $\|\mathbf{J}_{m,eq,a1}\|_{0,\Omega}$, $\|(\mathbf{E}-\mathbf{E}_0)_{ba1}\|_U$ and $\|(\mathbf{H}-\mathbf{H}_0)_{ba1}\|_U$ are bounded by expressions depending on $\mathbf{v}$ solely via S_β. These expressions are always composed of two addends, proportional to S_β and S_β^2, respectively.

So far, the focus has been on the properties of the first-order Born approximation. To deduce analogous features of higher-order approximations, we consider, for $i \ge 1$:

$$\begin{aligned}
&\mathbf{J}_{e,eq,a(i+1)} - \mathbf{J}_{e,eq,ai} \\
&= j\omega(\varepsilon^v - \varepsilon^0)\mathbf{E}_{ai} + j\omega\xi\mathbf{H}_{ai} - j\omega(\varepsilon^v - \varepsilon^0)\mathbf{E}_{a(i-1)} + j\omega\xi\mathbf{H}_{a(i-1)} \\
&= j\omega(\varepsilon^v - \varepsilon^0)(\mathbf{E}_{ai} - \mathbf{E}_{a(i-1)}) + j\omega\xi(\mathbf{H}_{ai} - \mathbf{H}_{a(i-1)}),
\end{aligned} \tag{40}$$

$$\begin{aligned}
\mathbf{J}_{m,eq,a(i+1)} - \mathbf{J}_{m,eq,ai} &= j\omega(\mu^v - \mu^0)(\mathbf{H}_{ai} - \mathbf{H}_{a(i-1)}) \\
&\quad + j\omega\zeta(\mathbf{E}_{ai} - \mathbf{E}_{a(i-1)}).
\end{aligned} \tag{41}$$

In the above equations, one needs $\mathbf{E}_{ai} - \mathbf{E}_{a(i-1)}$ and $\mathbf{H}_{ai} - \mathbf{H}_{a(i-1)}$ for $i \ge 1$. They are provided by Equations (21) and (22) when $i = 1$. From Equations (25) and (26) we obtain, for $i > 1$:

$$\begin{aligned}
\mathbf{E}_{ai} - \mathbf{E}_{a(i-1)} &= (\mathbf{E}-\mathbf{E}_0)_{bai} + \mathbf{E}_{a0} - (\mathbf{E}-\mathbf{E}_0)_{ba(i-1)} - \mathbf{E}_{a0} \\
&= (\mathbf{E}-\mathbf{E}_0)_{bai} - (\mathbf{E}-\mathbf{E}_0)_{ba(i-1)},
\end{aligned} \tag{42}$$

$$\mathbf{H}_{ai} - \mathbf{H}_{a(i-1)} = (\mathbf{H}-\mathbf{H}_0)_{bai} - (\mathbf{H}-\mathbf{H}_0)_{ba(i-1)}. \tag{43}$$

Since $\mathbf{E}_{ai} - \mathbf{E}_{a(i-1)} = (\mathbf{E}-\mathbf{E}_0)_{ba1}$ and $\mathbf{H}_{ai} - \mathbf{H}_{a(i-1)} = (\mathbf{H}-\mathbf{H}_0)_{ba1}$, using Equations (38) and (39) we deduce that their $\|\|_U$ norms are bounded above by the sum of terms having S_β and S_β^2 as common factors.

If we use these bounds in Equations (40) and (41) for $i = 1$, and exploit the properties of $(\varepsilon^v - \varepsilon^0)$, ξ, $(\mu^v - \mu^0)$ and ζ, as we did above, we obtain that the $\|\|_{0,\Omega}$ norms of $\mathbf{J}_{e,eq,a2} - \mathbf{J}_{e,eq,a1}$ and $\mathbf{J}_{m,eq,a2} - \mathbf{J}_{m,eq,a1}$ are bounded from above by the sum of terms proportional to S_β^2, S_β^3 and S_β^4, respectively. Once again, by the linearity and the well posedness of Problem 2, we deduce that the same type of bounds apply to the $\|\|_U$ norms of $(\mathbf{E}-\mathbf{E}_0)_{ba2} - (\mathbf{E}-\mathbf{E}_0)_{ba1}$, $(\mathbf{H}-\mathbf{H}_0)_{ba2} - (\mathbf{H}-\mathbf{H}_0)_{ba1}$, $\mathbf{E}_{a2} - \mathbf{E}_{a1}$ and $\mathbf{H}_{a2} - \mathbf{H}_{a1}$.

By the same token, for any $i \ge 2$, the $\|\|_{0,\Omega}$ norms of $\mathbf{J}_{e,eq,a(i+1)} - \mathbf{J}_{e,eq,ai}$ and $\mathbf{J}_{m,eq,a(i+1)} - \mathbf{J}_{m,eq,ai}$ and the $\|\|_U$ norms of $(\mathbf{E}-\mathbf{E}_0)_{ba(i+1)} - (\mathbf{E}-\mathbf{E}_0)_{bai}$, $(\mathbf{H}-\mathbf{H}_0)_{ba(i+1)} - (\mathbf{H}-\mathbf{H}_0)_{bai}$, $\mathbf{E}_{a(i+1)} - \mathbf{E}_{ai}$ and $\mathbf{H}_{a(i+1)} - \mathbf{H}_{ai}$ are bounded above by sums of terms having $S_\beta^{i+1}, S_\beta^{i+2}, \ldots, S_\beta^{2(i+1)}$ as common factors. Combining this observation with the previous case ($i = 1$), we conclude that the same type of bounds holds for any $i \ge 1$.

Since we can always write, for $i \ge 2$:

$$(\mathbf{E}-\mathbf{E}_0)_{bai} = (\mathbf{E}-\mathbf{E}_0)_{ba1} + \sum_{n=1}^{i-1}((\mathbf{E}-\mathbf{E}_0)_{ba(n+1)} - (\mathbf{E}-\mathbf{E}_0)_{ban}), \tag{44}$$

$$(\mathbf{H}-\mathbf{H}_0)_{bai} = (\mathbf{H}-\mathbf{H}_0)_{ba1} + \sum_{n=1}^{i-1}((\mathbf{H}-\mathbf{H}_0)_{ba(n+1)} - (\mathbf{H}-\mathbf{H}_0)_{ban}), \tag{45}$$

we can think of the i-th order Born approximation of the effects of motion as a superposition of effects: to the first-order approximation we add $i-1$ successive updates. The first addend presents terms whose $\|\|_U$ norms are proportional to S_β and S_β^2, whereas the $\|\|_U$ norms of n-th update, for $1 \le n \le i-1$, contain terms proportional to $S_\beta^{n+1}, S_\beta^{n+2}, \ldots, S_\beta^{2(n+1)}$.

The technical analysis presented in this section provides the formal justification for one of the broad claims advanced in the Introduction. Without this framework, extrapolating the specialized findings of [16] to radiation or guided wave scenarios, complex source configurations, arbitrary dimensions, and multiple objects with uniform or non-uniform, translational or rotational motions would remain speculative. What could previously only be conjectured has now been rigorously proven to be an intrinsic feature of Born approximations.

5 | Reliability of Born Approximations

This section discusses the conditions under which Born approximations are reliable.

As observed in Section 4, the $\|\|_{0,\Omega}$ norm of the first-order approximate equivalent sources and, consequently, the $\|\|_U$ norm of the first-order Born approximations of the effects of motion on the electromagnetic field are bounded above by the sum of two terms, one controlled by S_β and the other by S_β^2. This is due to the properties of the tensor fields $\varepsilon^v - \varepsilon^0$, $\mu^v - \mu^0$, ξ and ζ and to the independence of $\mathbf{E}_0$ and $\mathbf{H}_0$ from $\mathbf{v}$.

The same conclusion cannot be obtained in the same way for the corresponding norms of the exact equivalent sources, $\|\mathbf{J}_{e,eq}\|_{0,\Omega}$ and $\|\mathbf{J}_{m,eq}\|_{0,\Omega}$, and, then, for the U norm of the effects of motion, $\|\mathbf{E}-\mathbf{E}_0\|_U$ and $\|\mathbf{H}-\mathbf{H}_0\|_U$, because the fields $\mathbf{E}$ and $\mathbf{H}$ which appear in inequalities (33) and (34) depend on $\mathbf{v}$.

However, if for a specific subclass of problems of interest, referring to a subspace of velocity fields V, and a given incident field we know that

$$\begin{aligned}
&\text{we can find } C_E > 0, C_H > 0, C_b \in (0,1): \\
&\|\mathbf{E}\|_{0,\Omega_\beta} \le C_E,\ \|\mathbf{H}\|_{0,\Omega_\beta} \le C_H,\ \forall \mathbf{v} \in V : S_\beta \le C_b,
\end{aligned} \tag{46}$$

we obtain for $\mathbf{J}_{e,eq}$, $\mathbf{J}_{m,eq}$, $\mathbf{E}-\mathbf{E}_0$ and $\mathbf{H}-\mathbf{H}_0$ the same control we have deduced in Section 4 for $\mathbf{J}_{e,eq,a1}$, $\mathbf{J}_{m,eq,a1}$, $(\mathbf{E}-\mathbf{E}_0)_{ba1}$ and $(\mathbf{H}-\mathbf{H}_0)_{ba1}$, at least for all $\mathbf{v}$ fields in V such that $S_\beta \le C_b$. In particular, analogously to Equations (38) and (39), we can find $C_{E_1} > 0, C_{E_2} > 0, C_{H_1} > 0, C_{H_2} > 0, C_b \in (0,1)$:

$$\|\mathbf{E} - \mathbf{E}_0\|_U \leq C_{E_1} S_\beta + C_{E_2} S_\beta^2, \tag{47}$$

$$\|\mathbf{H} - \mathbf{H}_0\|_U \leq C_{H_1} S_\beta + C_{H_2} S_\beta^2, \ \forall \mathbf{v} \in V : S_\beta \leq C_b.$$

The last two inequalities provide a type of continuous dependence of the effects of motion on the velocity field since S_β is a norm for essentially bounded velocity fields.

We have deduced that condition (46) implies condition (47).

On the contrary, for any $\mathbf{A} \in U$ we have $\|\mathbf{A}\|_{0,\Omega_\beta} \leq \|\mathbf{A}\|_{0,\Omega} \leq \|\mathbf{A}\|_U$. Then, for $\mathbf{A} = \mathbf{E}$ or $\mathbf{A} = \mathbf{H}$,

$$\|\mathbf{A}\|_{0,\Omega_\beta} = \|\mathbf{A} - \mathbf{A}_0 + \mathbf{A}_0\|_{0,\Omega_\beta} \leq \|\mathbf{A} - \mathbf{A}_0\|_{0,\Omega_\beta} + \|\mathbf{A}_0\|_{0,\Omega_\beta} \leq \|\mathbf{A} - \mathbf{A}_0\|_U + \|\mathbf{A}_0\|_{0,\Omega_\beta}. \tag{48}$$

Therefore condition (47) implies the validity of condition (46) since $\mathbf{E}_0$ and $\mathbf{H}_0$ are by definition independent of $\mathbf{v}$.

We can then conclude that (46) is necessary and sufficient for condition (47). Condition (46) is expected to hold true for any subclass of problems of practical interest, for any reasonable subspace V, and for any incident field, since to violate it there should be a sequence of $\mathbf{v}_i \in V$ whose S_{β_i} goes to zero for $i \to \infty$ such that $\|\mathbf{E}\|_{0,\Omega_\beta}$ or $\|\mathbf{H}\|_{0,\Omega_\beta}$ diverge for $i \to \infty$.

It is important to note that, despite condition (46) is not overly challenging, no one has proved it for any V and any incident field, to the best of authors' knowledge. For example, in [13] and [14] it was shown that for any problem of interest, without significant limitations in terms of constitutive parameters, velocity and incident fields, one can find $\mathbf{E}$, $\mathbf{H}$, $\mathbf{E}_0$ and $\mathbf{H}_0$, with $\|\,\|_U$ norms of the indicated fields controlled by the $(L^2(\Omega))^3$ and $L_t^2(\Gamma)$ norms of the sources $\mathbf{J}_e$, $\mathbf{J}_m$ and $\mathbf{f}_R$, where $\mathbf{f}_R$ is the known term of admittance boundary conditions (see the third equation of 12). However, the constants involved in the control of the $\|\,\|_U$ norm of $\mathbf{E}$ and $\mathbf{H}$ depend in general on the specific $\mathbf{v}$ field and no result guarantees that such constants do not diverge as S_β goes to zero.

Let us suppose that we know in advance that condition (46) holds true for a specific subclass of problems of interest, with $\mathbf{v} \in V$, and a specific incident field. From the previous deductions we know that condition (47) is satisfied as well. But by their very definitions (see Equations 13 and 14, 19, 20, 23 and 24), we get, for $i \geq 1$ and $m = i - 1$

$$\mathbf{J}_{e,eq} - \mathbf{J}_{e,eq,ai} = j\omega\left(\varepsilon^v - \varepsilon^0\right)(\mathbf{E} - \mathbf{E}_{am}) + j\omega\xi\,(\mathbf{H} - \mathbf{H}_{am}), \tag{49}$$

and

$$\mathbf{J}_{m,eq} - \mathbf{J}_{m,eq,ai} = j\omega\left(\mu^v - \mu^0\right)(\mathbf{H} - \mathbf{H}_{am}) + j\omega\zeta\,(\mathbf{E} - \mathbf{E}_{am}). \tag{50}$$

Moreover, by the usual properties of Problem 2, for $i \geq 1$

$$(\mathbf{E} - \mathbf{E}_0) - (\mathbf{E} - \mathbf{E}_0)_{bai} = \mathbf{E} - ((\mathbf{E} - \mathbf{E}_0)_{bai} + \mathbf{E}_0)) = \mathbf{E} - \mathbf{E}_{ai} \tag{51}$$

and

$$(\mathbf{H} - \mathbf{H}_0) - (\mathbf{H} - \mathbf{H}_0)_{bai} = \mathbf{H} - ((\mathbf{H} - \mathbf{H}_0)_{bai} + \mathbf{H}_0)) = \mathbf{H} - \mathbf{H}_{ai}. \tag{52}$$

Then, using condition (47) and applying the inequalities used repeatedly above, we deduce that $\|\mathbf{J}_{e,eq} - \mathbf{J}_{e,eq,a1}\|_{0,\Omega}$, $\|\mathbf{J}_{m,eq} - \mathbf{J}_{m,eq,a1}\|_{0,\Omega}$, $\|(\mathbf{E} - \mathbf{E}_0) - (\mathbf{E} - \mathbf{E}_0)_{ba1}\|_U$ and $\|(\mathbf{H} - \mathbf{H}_0) - (\mathbf{H} - \mathbf{H}_0)_{ba1}\|_U$ are controlled at least by S_β^2, for all $\mathbf{v} \in V$: $S_\beta \leq C_b$.

It is now trivial to proceed iteratively for higher-order Born approximations. We can then conclude that if it is known that condition (46) holds true for a specific subclass of problems of interest and for a given incident field, then the $\|\,\|_U$ norms of $(\mathbf{E} - \mathbf{E}_0) - (\mathbf{E} - \mathbf{E}_0)_{bai}$ and $(\mathbf{H} - \mathbf{H}_0) - (\mathbf{H} - \mathbf{H}_0)_{bai}$ are controlled by S_β^{i+1}. Consequently, under the indicated conditions, the i-th order Born approximation of the effects of motion provides exact results for the first i dominant terms, those controlled by $S_\beta^1, S_\beta^2, \ldots, S_\beta^i$. Moreover, this implies that to approximate the effect of motion up to the order of S_β^i, for small values of S_β and under condition (46), it is necessary and sufficient to find Born approximations up to the order i.

On the other hand, from Equations (44) and (45) and the comments below them, we deduce the following negative result for our approach: no Born approximations can provide satisfactory results if the effects of motion on the electromagnetic field have $\|\,\|_U$ norms which are not controlled by S_β. Fortunately, according to our previous analysis, for this to happen it is necessary that, for the specific subclass of problems of interest and the specific incident field, condition (46) does not hold true.

6 | Born Approximations Deduced by Multiplication With Scaling Factors

In many cases, Born approximations can be deduced directly from previously obtained results, without the need for additional simulations. To this end, we consider a class of problems of interest with a given incident field. Consider, moreover, a specific velocity field $\overline{\mathbf{v}}$ having all properties reported in Section 2. In particular, we can find $C_{\beta M}$ such that $\frac{n|\overline{\mathbf{v}}|}{c_0} \leq C_{\beta M} < 1$ in Ω_i, for any $i \in M_\beta$. Let $\overline{V}$ be the vector space of velocity fields of dimension one generated by $\overline{\mathbf{v}}$, that is $\overline{V} = \{\mathbf{v} : \mathbf{v} = a\overline{\mathbf{v}}, a \in \mathbb{R}\}$. For this subspace of velocity fields some of our developments of Section 4 can be written more explicitly in terms of the parameter a.

To achieve this result, we consider again the expressions of the first-order approximate equivalent sources, given by Equations (17) and (18). Taking account that $v = |a||\overline{\mathbf{v}}|$ and that $\mathbf{v} = v\hat{\mathbf{v}} = a\overline{\mathbf{v}} = a|\overline{\mathbf{v}}|\hat{\overline{\mathbf{v}}}$, for any $\mathbf{v} \in \overline{V}$, by using Equations (8) and (27) we get:

$$\begin{aligned}\mathbf{J}_{e,eq,a1} &= j\omega\xi\mathbf{H}_0 + j\omega(\varepsilon^v - \varepsilon^0)\mathbf{E}_0 \\ &= j\omega\alpha v(\hat{\mathbf{v}} \times \mathbf{H}_0) + j\omega\varepsilon_0\varepsilon_r\alpha v^2[(\hat{\mathbf{v}} \times \mathbf{E}_0) \times \hat{\mathbf{v}}] \\ &= j\omega\alpha a|\overline{\mathbf{v}}|(\hat{\overline{\mathbf{v}}} \times \mathbf{H}_0) + j\omega\varepsilon_0\varepsilon_r\alpha a^2|\overline{\mathbf{v}}|^2\left[\left(\hat{\overline{\mathbf{v}}} \times \mathbf{E}_0\right) \times \hat{\overline{\mathbf{v}}}\right].\end{aligned} \tag{53}$$

In this section, results for $\mathbf{J}_{e,eq,a1}$, $(\mathbf{E} - \mathbf{E}_0)_{ba1}$, $\mathbf{J}_{e,eq,a2} - \mathbf{J}_{e,eq,a1}$ and $(\mathbf{E} - \mathbf{E}_0)_{ba2} - (\mathbf{E} - \mathbf{E}_0)_{ba1}$ are derived. Analogous results hold for $\mathbf{J}_{m,eq,a1}$, $(\mathbf{H} - \mathbf{H}_0)_{ba1}$, $\mathbf{J}_{m,eq,a2} - \mathbf{J}_{m,eq,a1}$ and $(\mathbf{H} - \mathbf{H}_0)_{ba2} - (\mathbf{H} - \mathbf{H}_0)_{ba1}$, respectively, and are omitted for brevity.

In the above equation, all quantities on the right-hand side are independent of a, apart from the terms a, a^2 and α. The latter is defined by Equation (3). We have:

$$\alpha = \frac{n^2-1}{c_0^2 - n^2 v^2} = \frac{n^2-1}{c_0^2}\frac{1}{1-a^2(\frac{n|\overline{\mathbf{v}}|}{c_0})^2}. \tag{54}$$

α is an even and analytic function of a, for $a : |a| < \frac{1}{C_{\beta M}}$, in Ω_i, for any $i \in M_\beta$. Its second-order McLaurin polynomial with Peano form of the remainder can be easily calculated and is given by:

$$\alpha = \frac{n^2-1}{c_0^2} + \frac{n^2(n^2-1)|\overline{\mathbf{v}}|^2}{c_0^4}a^2 + h_2(a;\mathbf{r})a^2, \tag{55}$$

where $\lim_{a\to 0} h_2(a;\mathbf{r}) = 0, \forall \mathbf{r} \in \Omega_i, i \in M_\beta$.

If we replace the right-hand side of Equation (55) in Equation (53) we obtain:

$$\begin{aligned}\mathbf{J}_{e,eq,a1} = {} & j\omega|\overline{\mathbf{v}}|(\hat{\overline{\mathbf{v}}}\times\mathbf{H}_0)\left(\frac{n^2-1}{c_0^2}\right)a \\ & + j\omega\varepsilon_0\varepsilon_r|\overline{\mathbf{v}}|^2\left[\left(\hat{\overline{\mathbf{v}}}\times\mathbf{E}_0\right)\times\hat{\overline{\mathbf{v}}}\right]\left(\frac{n^2-1}{c_0^2}\right)a^2 \\ & + j\omega|\overline{\mathbf{v}}|(\hat{\overline{\mathbf{v}}}\times\mathbf{H}_0)\left(\frac{n^2(n^2-1)|\overline{\mathbf{v}}|^2}{c_0^4} + h_2(a;\mathbf{r})\right)a^3 \\ & + j\omega\varepsilon_0\varepsilon_r|\overline{\mathbf{v}}|^2\left[\left(\hat{\overline{\mathbf{v}}}\times\mathbf{E}_0\right)\times\hat{\overline{\mathbf{v}}}\right]\left(\frac{n^2(n^2-1)|\overline{\mathbf{v}}|^2}{c_0^4} + h_2(a;\mathbf{r})\right)a^4.\end{aligned} \tag{56}$$

Due to the linearity of Problem 2, for the incident field and the subclass of problems considered in this section, we can think of the first order Born approximation as the superposition of four terms:

$$(\mathbf{E}-\mathbf{E}_0)_{ba1} = \sum_{j=1}^{4}(\mathbf{E}-\mathbf{E}_0)_{ba1,j}\, a^j, \tag{57}$$

where $(\mathbf{E}-\mathbf{E}_0)_{ba1,1}$ is completely independent of a and is determined by the first addends in the right-hand side of Equation (56) and in the corresponding equation for $\mathbf{J}_{m,eq,a1}$, both divided by a. The same considerations applies to $(\mathbf{E}-\mathbf{E}_0)_{ba1,2}$ provided that the second addends divided by a^2 are considered. $(\mathbf{E}-\mathbf{E}_0)_{ba1,j}$, for $j = 3, 4$, are deduced in an analogous way but have a weak dependence on a due to the presence of the $h_2(a;\mathbf{r})$ in the corresponding source terms.

Then, for a given quality of approximation required, we can find $C_1 < \frac{1}{C_{\beta M}}$ such that for all $a : 0 < |a| < C_1$ we have:

$$\begin{aligned}\|(\mathbf{E}-\mathbf{E}_0)_{ba1,j}a^{j-1}\|_U &= \|(\mathbf{E}-\mathbf{E}_0)_{ba1,j}\|_U|a|^{j-1} \\ &<< \|(\mathbf{E}-\mathbf{E}_0)_{ba1,1}\|_U, \quad j = 2,3,4.\end{aligned} \tag{58}$$

Consequently, within the required quality of approximation, we obtain:

$$(\mathbf{E}-\mathbf{E}_0)_{ba1} \simeq (\mathbf{E}-\mathbf{E}_0)_{ba1,1}\, a, \tag{59}$$

for all $a : 0 < |a| < C_1$.

This means that we can compute first-order Born approximations of the effects of motion on the electromagnetic field for all values of a in the indicated range by simply computing it for a particular value of a. In particular, denote it by a_1, $0 < |a_1| < C_1$. Then for any $a_2 : |a_2| < C_1$, the first-order Born approximation is obtained from the former one, achieved for $a = a_1$, by multiplying it by $\frac{a_2}{a_1}$. Moreover, if condition (46) holds true for the incident field and for the subclass of problems considered in this section, our previous results provide clear physical implications: the effects of motion on the electromagnetic field can be approximated for an infinite set of values of a by computing a single first-order Born approximation.

Let us now suppose we are interested in some components of the effects of motion on the electromagnetic field, in some subdomain $\Omega_s \subset \Omega$. This could be the case, for example, in inverse problems or if we want to measure the effects of motion with some directional antenna.

It may occur that $((\mathbf{E}-\mathbf{E}_0)_{ba1,1}, (\mathbf{H}-\mathbf{H}_0)_{ba1,1})$ vanishes for certain components or within the subdomain of interest. This can happen in various models as well as in corresponding realistic scenarios. One example arises in the presence of axially moving cylinders when evaluating the effects of motion on the dominant components of the incident field [16]. Another example occurs when slabs move with translational velocity parallel to their interfaces. In these cases, the situation becomes even more critical, because no first-order effects occur outside the moving slab for any electromagnetic field component [18].

Although not strictly necessary, we assume the presence of a second-order effect and show how our previous considerations can be extended to such cases. This assumption is motivated by the fact that we are not aware of any situations in which both first- and second-order effects are absent while a third-order effect is present.

For this extension we cannot rely anymore on the first-order Born approximation because it is not reliable for second order effects. However, if condition (46) is satisfied, we can rely on the second-order Born approximation. To find it as an update of the first-order one, we consider Equations (40), (21) and (22):

$$\mathbf{J}_{e,eq,a2} - \mathbf{J}_{e,eq,a1} = j\omega\xi(\mathbf{H}-\mathbf{H}_0)_{ba1} + j\omega(\varepsilon^v - \varepsilon^0)(\mathbf{E}-\mathbf{E}_0)_{ba1}, \tag{60}$$

This equation has the same form of (53). The difference is that $\mathbf{E}_0$ and $\mathbf{H}_0$ have been replaced, respectively, by $(\mathbf{E}-\mathbf{E}_0)_{ba1}$ and $(\mathbf{H}-\mathbf{H}_0)_{ba1}$.

Taking account of our previous procedure and of Equation (59) we deduce that we can find $C_2 < \frac{1}{C_{\beta M}}$ such that for all $a : 0 < |a| < C_2$:

$$\mathbf{J}_{e,eq,a2} - \mathbf{J}_{e,eq,a1} \simeq j\omega|\overline{\mathbf{v}}|(\hat{\overline{\mathbf{v}}}\times(\mathbf{H}-\mathbf{H}_0)_{ba1,1})\left(\frac{n^2-1}{c_0^2}\right)a^2, \tag{61}$$

where all terms with a factor a^j, $j \geq 3$, have been neglected.

Finally, by solving Problem 2 with these sources we obtain:

$$(\mathbf{E}-\mathbf{E}_0)_{ba2} - (\mathbf{E}-\mathbf{E}_0)_{ba1} \propto a^2, \tag{62}$$

within the required quality of approximation, for all $a : 0 < |a| < C_2$.

This means that for the incident field and the subclass of problems considered in this section, we can find the components of the effects of motion in the subdomain $\Omega_s \subset \Omega$ of interest, where no first-order effects of a are present, by simply computing them once, for example, for $a = a_1 : 0 < |a_1| < C_2$. For any $a_2 : |a_2| < C_2$ we can find the same quantities by multiplying them by $\frac{a_2^2}{a_1^2}$.

It is worth noting that our conclusions regarding the behavior of motion-induced effects for small values of a are consistent with the numerical evidence reported in our previous works [16, 27]. For instance, the results obtained using the second-order Born approximation, shown in figures 2 and 3 of [16], clearly exhibit a scaling proportional to $\frac{a_2^2}{a_1^2} = 10^{-6}$ as β varies from 10^{-5} to 10^{-8}. However, in those earlier studies, this scaling behavior was observed only numerically, for specific problems. Those results were obtained for the axial component of the effects of motion on the electric field, for cylinders undergoing uniform axial motion and illuminated by a TM-polarized, time-harmonic, uniform plane wave propagating in the plane orthogonal to the cylinders' axes. No consideration was given to formulating such features as intrinsic properties of Born approximations. The present work fills this gap by establishing these properties in their full generality, demonstrating that they are not problem-specific but apply broadly to all scenarios of interest under the stated weak hypotheses.

Finally, to implement the proposed shortcuts in practice, which can save a substantial number of simulations and significant CPU time, or alternatively many measurements, a small number of preliminary computational or instrumental assessments may be required. These estimates provide information on the order of magnitude of the dominant effect of a and on the negligibility of higher-order effects for the components and subdomains of interest.

With this, some important properties of the Born approximation of the effects of motion on the electromagnetic field have been deduced. In the next section we will verify, through a completely separate method, based on the analytical calculation of $(\mathbf{E}_0, \mathbf{H}_0)$ and $(\mathbf{E}, \mathbf{H})$, that the results predicted by our theory hold true for simple one-dimensional problems.

7 | Verification of Condition (46) for Simple Problems

In this section we consider simple one-dimensional problems in which a homogeneous slab of thickness d, occupying the region $\{(x, y, z) \in \mathbb{R}^3 : x \in (0, d)\}$, is illuminated by an electromagnetic, uniform, monochromatic, plane wave propagating as a progressive wave along the x axis. The slab can be at rest or in motion, with uniform velocity $\mathbf{v} = v_z \hat{\mathbf{z}}$, $v_z \in \mathbb{R}$. $\beta = \frac{v_z}{c_0}$ and $S_\beta = |\beta|$.

In these subclasses of problems we have just three media: medium 1 is in the half space $x < 0$ while medium 3 occupies the half space $x > d$. These media are assumed to be linear, isotropic, time-invariant and characterized by effective constitutive parameters $\varepsilon_i, \mu_i \in \mathbb{C}$, $i = 1, 3$. The slab is made of medium 2, a lossless material having $\varepsilon_2, \mu_2 \in \mathbb{R}$ when it is at rest.

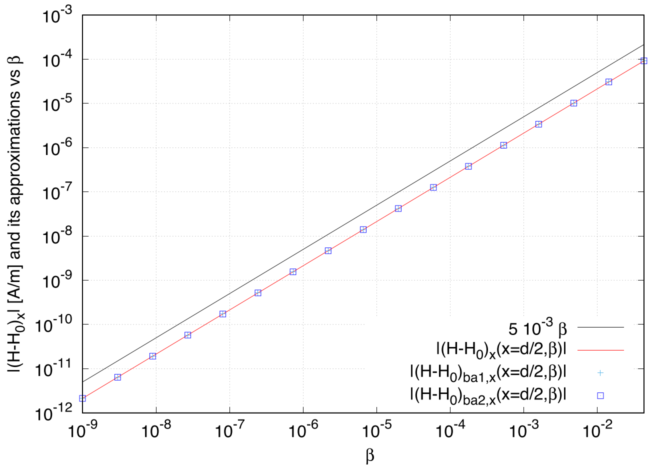

FIGURE 1 | Behavior of the magnitude of the x components of $\mathbf{H} - \mathbf{H}_0$ and $(\mathbf{H} - \mathbf{H}_0)_{baj}$, $j = 1, 2$, in the middle of the slab, for several values of β.

The simple geometry for these classes of problems is shown in figure 1 of [18]. These problems were analyzed in some details in that paper. In particular, it is well known that the general expressions of the cartesian components of $\mathbf{E}_0$ and $\mathbf{H}_0$, when the slab is at rest, are given by:

$$E_{0x} = 0 = H_{0x}, \quad \forall x \in \mathbb{R}, \tag{63}$$

$$E_{0y} = A_{0piy} e^{-jk_{i0}x} + A_{0riy} e^{jk_{i0}x}, \text{ in medium } i, i = 1, 2, 3, \tag{64}$$

$$H_{0z} = Y_{i0} A_{0piy} e^{-jk_{i0}x} - Y_{i0} A_{0riy} e^{jk_{i0}x}, \text{ in medium } i, i = 1, 2, 3, \tag{65}$$

$$E_{0z} = A_{0piz} e^{-jk_{i0}x} + A_{0riz} e^{jk_{i0}x}, \text{ in medium } i, i = 1, 2, 3, \tag{66}$$

$$H_{0y} = -Y_{i0} A_{0piz} e^{-jk_{i0}x} + Y_{i0} A_{0riz} e^{jk_{i0}x}, \text{ in medium } i, i = 1, 2, 3, \tag{67}$$

where $k_{i0} = \omega\sqrt{\mu_i \varepsilon_i} = \frac{\omega}{c_0} n_i$ and $Y_{i0} = \sqrt{\frac{\varepsilon_i}{\mu_i}} = \frac{k_{i0}}{\omega\mu_i} = \frac{\omega\varepsilon_i}{k_{i0}}$. Moreover, the complex coefficients A_{0p1y} and A_{0p1z} are defined by the incident field and $A_{0r3y} = 0 = A_{0r3z}$, since there is no regressive wave in medium 3. Finally, the unknown complex coefficients A_{0r1y}, $A_{0r1z} A_{0p2y}$, A_{0p2z}, A_{0r2y}, A_{0r2z}, A_{0p3y} and A_{0p3z} can be found by enforcing the continuity of the y and z components of $\mathbf{E}_0$ and $\mathbf{H}_0$ at the interfaces at $x = 0$ and $x = d$. The resulting linear systems of four equations for the two linear polarizations of the incident field are reported in equation (9) of [18].

Analogously, when the slab is moving in the z direction one finds [18] the following expressions for the cartesian components of $\mathbf{E}$ and $\mathbf{H}$:

$$E_y = A_{piy} e^{-jk_i x} + A_{riy} e^{jk_i x}, \text{ in medium } i, i = 1, 2, 3, \tag{68}$$

$$H_z = Y_{iy} A_{piy} e^{-jk_i x} - Y_{iy} A_{riy} e^{jk_i x}, \text{ in medium } i, i = 1, 2, 3, \tag{69}$$

$$E_z = A_{piz} e^{-jk_i x} + A_{riz} e^{jk_i x}, \text{ in medium } i, i = 1, 2, 3, \tag{70}$$

$$H_y = -Y_{iz} A_{piz} e^{-jk_i x} + Y_{iz} A_{riz} e^{jk_i x}, \text{ in medium } i, i = 1,2,3, \quad (71)$$

$$E_x = 0 = H_x, \text{ in media 1 and 3}, \quad (72)$$

$$E_x = \frac{\alpha v_z}{\varepsilon_2(1+\alpha v_z^2)} H_y, \text{ in medium 2}, \quad (73)$$

$$H_x = -\frac{\alpha v_z}{\mu_2(1+\alpha v_z^2)} E_y, \text{ in medium 2}, \quad (74)$$

where $k_i = k_{i0}$, $Y_{iy} = Y_{i0} = Y_{iz}$, for $i = 1, 3$, and [18]

$$k_2 = \frac{\omega}{c_0}\sqrt{\frac{n_2^2-\beta^2}{1-\beta^2}}, \quad Y_{2y} = \frac{k_2}{\omega\mu_2} > 0, \quad Y_{2z} = \frac{\omega\varepsilon_2}{k_2} > 0. \quad (75)$$

For the same reasons as when the slab is at rest, the complex coefficients A_{p1y} and A_{p1z} are defined by the incident field and $A_{r3y} = 0 = A_{r3z}$, due to the lack of the regressive wave in medium 3. The unknown complex coefficients A_{r1y}, A_{r1z}, A_{p2y}, A_{p2z}, A_{r2y}, A_{r2z}, A_{p3y} and A_{p3z} can be found, as in the previous case, by enforcing the continuity of the y and z components of $\mathbf{E}$ and $\mathbf{H}$ at the interfaces at $x = 0$ and $x = d$. This is done in [18] (see, in particular, equation 24 of that paper).

Equation (45) of [18] implies that the differences $A_{pi\alpha} - A_{0pi\alpha}$ and $A_{rj\alpha} - A_{0rj\alpha}$, $i = 2,3$, $j = 1,2$, $\alpha = y, z$, are controlled by a factor β^2.

Moveover, in medium 2, for all $x \in (0, d)$, by Equations (68), (70), (64) and (66), we deduce

$$\begin{aligned} E_\alpha - E_{0\alpha} &= A_{p2\alpha} e^{-jk_2 x} + A_{r2\alpha} e^{jk_2 x} - A_{0p2\alpha} e^{-jk_{20} x} - A_{0r2\alpha} e^{jk_{20} x} \\ &= (A_{p2\alpha} e^{-jk_2 x} - A_{0p2\alpha} e^{-jk_{20} x}) + (A_{r2\alpha} e^{jk_2 x} - A_{0r2\alpha} e^{jk_{20} x}), \quad \alpha = y, z. \end{aligned} \quad (76)$$

Analogously, by Equations (71), (69), (67) and (65) in medium 2 we get

$$\begin{aligned} &H_\gamma - H_{0\gamma} \\ &= \pm Y_{2\alpha} A_{p2\alpha} e^{-jk_2 x} \mp Y_{2\alpha} A_{r2\alpha} e^{jk_2 x} \mp Y_{20} A_{0p2\alpha} e^{-jk_{20} x} \pm Y_{20} A_{0r2\alpha} e^{jk_{20} x} \\ &\pm (Y_{2\alpha} A_{p2\alpha} e^{-jk_2 x} - Y_{20} A_{0p2\alpha} e^{-jk_{20} x}) \mp (Y_{2\alpha} A_{r2\alpha} e^{jk_2 x} - Y_{20} A_{0r2\alpha} e^{jk_{20} x}). \end{aligned} \quad (77)$$

In the above expression, $\alpha = y, z$; $\gamma = z$ when $\alpha = y$ and $\gamma = y$ when $\alpha = z$; finally, the upper (respectively, lower) signs must be used when $\alpha = y$ (respectively, $\alpha = z$), according to [18].

In the right-hand sides of Equation (76) we have differences of coefficients multiplied by exponentials. But we have already observed that the differences of coefficients are proportional to β^2, like the difference $Y_{2\alpha} - Y_{20}$ in Equation (37) of [18]. There is, then, a perfect parallelism between the terms on the right-hand sides of Equation (76) and that computed by equation (39) of [18], if d is replaced by x and $x \in (0, d)$. The presence of the admittances in Equation (77) does not change the conclusion because the difference $Y_{2\alpha} - Y_{20}$ is proportional to β^2.

From the indicated parallelism, taking account of the result provided by equation (39) of [18], we deduce that we can find a positive constant C such that

$$|E_\alpha - E_{0\alpha}| < C\beta^2, \quad |H_\alpha - H_{0\alpha}| < C\beta^2, \quad \alpha = y, z, \quad (78)$$

for all $x \in (0, d)$, for small values of β.

Since $E_{0\alpha}$ and $H_{0\alpha}$, $\alpha = y, z$, are independent of β, we conclude that for any fixed subclass of one-dimensional problems E_α and H_α, $\alpha = y, z$, are bounded inside the slab for small values of β. Moreover, for the same subclass of problems, we have $\varepsilon_2 > 0$ and $\mu_2 > 0$. Finally, $1 + \alpha v_z^2 = \frac{1-\beta^2}{1-n_2^2\beta^2}$ and, under the condition reported in Equation (4), it is $\geq C > 0$. Then, taking account of the boundedness of α under the same condition, by using Equations (73) and (74) we get that E_x and H_x are bounded, too, inside the slab. Therefore, we have proved that, for any subclass of simple problems of interest and for any incident field of the type considered, condition (46) holds true. We then expect to be able to verify all properties of the Born approximations of the effects of motion on the electromagnetic field, for all possible subclasses of one-dimensional problems of the type considered, with uniform velocity fields and orthogonal incidence of the illuminating plane wave. Finally, the above deductions highlight that condition (51) is not overly abstract, as there are practical applications where it holds. This also suggests that it is reasonable to try to prove its validity for other problems.

8 | Numerical Results

The most relevant numerical investigations for our findings are those designed to challenge the deduced properties. To minimize the risk of failing to detect discrepancies between theoretical predictions and numerical results, we focus on problems admitting analytical solutions. For these reasons, in this section we will first consider a specific subclass of the one-dimensional problems introduced in Section 7. It is defined by the following data: $d = 1$ m, $\varepsilon_{r1} = 1$, $\mu_{r1} = 1$, $\varepsilon_{r2} = 2$, $\mu_{r2} = 1$, $\varepsilon_{r3} = 1.5 - j4\ 10^{-2}$, $\mu_{r3} = 1$. As for the incident field, we consider $f = 1$ GHz and a linear polarization along the y axis. In particular, using the symbols introduced in Section 7, the coefficients defining the incident field are $A_{0piy} = 1 = A_{piy}$ and $A_{0piz} = 0 = A_{piz}$. Finally, all the considered β values will belong to the range $[10^{-9}, 4.3\ 10^{-1}]$. The corresponding range of speed values is $[0.3, 12.9\ 10^6]$ m/s. Most engineering applications deal with speed values smaller than 1000 m/s ($\beta \simeq 3.3\ 10^{-6}$). However, in our simulations we will consider a significantly extended range in order to assess the performance of the Born approximations under more challenging conditions.

For the considered polarization, the x and z components of $\mathbf{E}$, $\mathbf{E}_{ai}$, $i \geq 0$, $(\mathbf{E} - \mathbf{E}_0)_{baj}$, $j \geq 1$, and the y components of $\mathbf{H}$, $\mathbf{H}_{ai}$, $i \geq 0$, $(\mathbf{H} - \mathbf{H}_0)_{baj}$, $j \geq 1$ are trivial in all media. Moreover, also all x components of $\mathbf{H}$, $\mathbf{H}_{ai}$, $i \geq 0$, $(\mathbf{H} - \mathbf{H}_0)_{baj}$, $j \geq 1$ are trivial in media 1 and 3.

For the indicated reasons, the results of interest will be provided in terms of the remaining components, where they are different from zero.

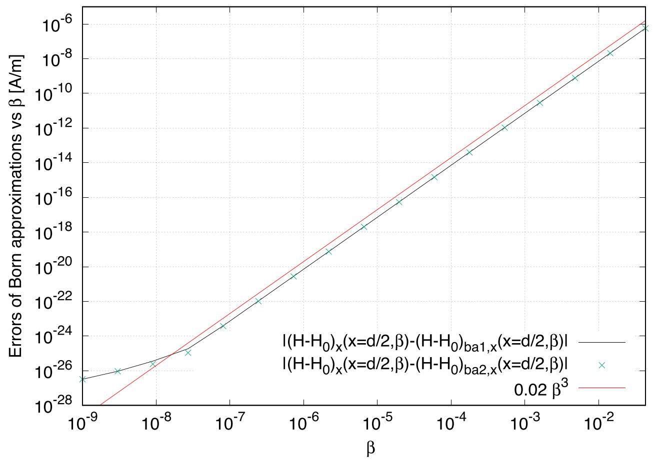

FIGURE 2 | Magnitude of the error made by Born approximations to estimate the effects of motion on the x component of the magnetic field, in the middle of the slab, for several values of β.

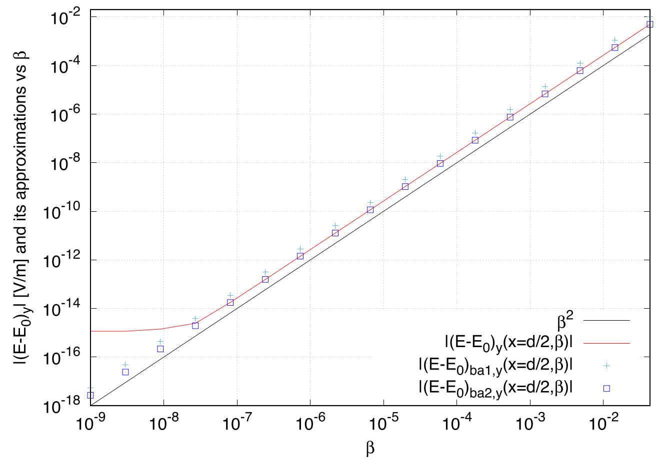

FIGURE 3 | Behavior of the magnitude of the y components of $\mathbf{E}-\mathbf{E}_0$ and $(\mathbf{E}-\mathbf{E}_0)_{baj}$, $j=1,2$, in the middle of the slab, for several values of β.

Let us firstly consider the x components of $\mathbf{H}-\mathbf{H}_0$ and $(\mathbf{H}-\mathbf{H}_0)_{baj}$, $j=1,2$, in the middle of the slab, for different values of β. The effects of motion on the magnitude of this component of the difference field and on its first and second-order Born approximations are shown in Figure 1. According to our results of Section 7, about the validity of condition (46) for the subclasses of problems at hand, and of Sections 4 and 5, such effects are expected to be at most of the first order. Moreover, a first order effect is for sure present. This is because, by Equation (78), neglecting second order terms, we get $E_y \simeq E_{a0y} \neq 0$, and by Equations (74), (63) and (3) we obtain $H_x = H_x - H_{a0x} = -\frac{\alpha v_z}{\mu_2(1+\alpha v_z^2)}E_y \simeq -\frac{\alpha v_z}{\mu_2(1+\alpha v_z^2)}E_{a0y} \simeq -\frac{n_2^2-1}{\mu_2 c_0}\beta E_{a0y}$. The figure shows that this is the case for a huge range of β values. The plot of the function $5\,10^{-3}\,\beta$ is shown as a reference for first order effects. The first and second-order Born approximations are very good and are almost indistinguishable, once again as predicted by the results reported in Section 5 for first-order effects.

Figure 2 shows the behaviors of the errors of the Born approximations of orders one and two in estimating the effects of motion on the component under consideration. The second-order approximation is slightly better than the one obtained with the first-order one, but by negligible amounts.

More significant is the fact that the error is third-order in β, as it can be easily deduced by a comparison with the plot of the function $0.02\,\beta^3$. This confirms, on the one hand, the results of Section 5 which shows that $(\mathbf{H}-\mathbf{H}_0)-(\mathbf{H}-\mathbf{H}_0)_{ba2}$ is controlled by S_β^3 and, on the other hand, it does not contradict the fact that $(\mathbf{H}-\mathbf{H}_0)-(\mathbf{H}-\mathbf{H}_0)_{ba1}$ is controlled at least by a factor S_β^2.

Finally, in the same figure we note the anomalies for very small values of β. In particular, the problem for $\beta < 3\,10^{-8}$ is due to the unreliability of calculating the difference of fields with double-precision operations: the presence of the additional factor β^2 makes the result about 10^{15} smaller than the two quantities involved.

The magnitudes of the effects of motion on the x-component of the magnetic field shown in the previous figures are evaluated at the center of the slab, for $x = d/2$, but completely similar results are obtained for any $x \in (0, d)$. We have verified this although we do not show any other data here.

We now focus on the effects of motion on the y component of the electric field.

From Equation (78) we deduce that $|(\mathbf{E}-\mathbf{E}_0)_y|$ is of the second order. Moreover, our results of Sections 5 and 7 guarantee that second or higher-order Born approximations are able to reliably approximate these small effects of motion. At the same time, the same quality of the approximation is not ensured by the first-order Born approximation. Figure 3 shows that all these predictions are correct. As a matter of fact, the approximations of the effects of motion provided by the first-order and the second-order Born approximations are not anymore the same in this case: in particular, the second-order is reliable while the first-order is not.

Figure 3 shows also that the standard estimate of the effects of motion, given by $\mathbf{E}-\mathbf{E}_0$, becomes unreliable for $\beta < 3\,10^{-8}$, dispite the extreme reliability of the calculations of the analytical solutions, $\mathbf{E}$ and $\mathbf{E}_0$. As explained above, this is due to the unreliability of double-precision arithmetic for computing effects as small as β^2. Instead, the Born approximations of the effects of motion do not suffer of the same drawback because such small effects are computed as the primary unknown fields of the formulations [16].

It is interesting to observe that the same problem was not present on the plot of $H_x - H_{0x}$ reported in Figure 1, which present just a first order effect. Instead, it was present in all plots of the errors made by Born approximations in Figure 2, which are third order but obtained by difference of quantities which are of the first order; the difference of orders, which determines the unreliability of double-precision arithmetics, is then equal to two.

The difference of reliability between first-order and second-order Born approximations is shown more clearly in Figure 4, where the plots of the errors made by Born approximations to estimate

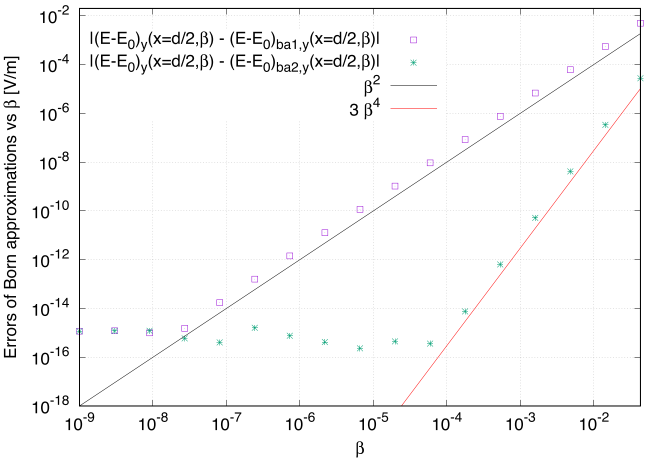

FIGURE 4 | Magnitude of the error made by Born approximations to estimate the effects of motion on the y component of the electric field, in the middle of the slab, for several values of β.

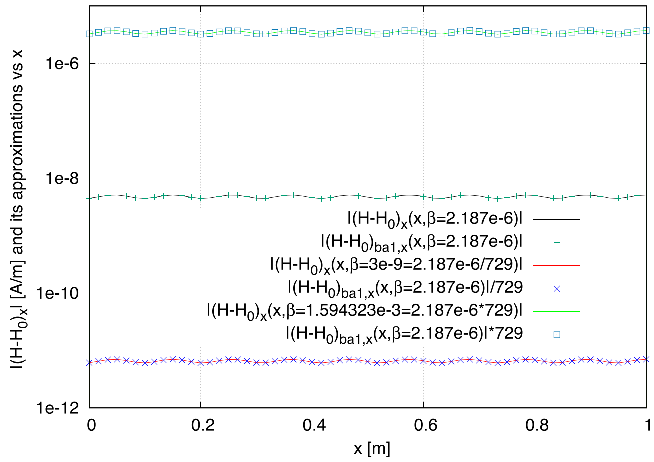

FIGURE 5 | Behavior of the magnitude of the x component of $\mathbf{H}-\mathbf{H}_0$ inside the slab, for three β values spanning more than five decades: $\beta_1 = \frac{\beta_2}{729}$, $\beta_2 = 2.187\,10^{-6}$ and $\beta_3 = \beta_2 \cdot 729$. The magnitude of the same component of $(\mathbf{H}-\mathbf{H}_0)_{ba1}$ for $\beta = \beta_2$ is reported as well. Born approximations are not computed for other values of β: the approximations for $\beta = \beta_1$ and $\beta = \beta_3$ are deduced by multiplying or dividing the computed approximation by 729.

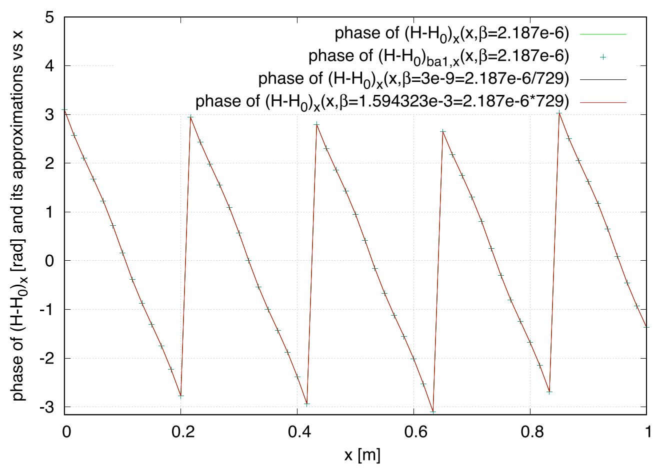

FIGURE 6 | Behavior of the phase of the x components of $\mathbf{H}-\mathbf{H}_0$ inside the slab, for three β values spanning more than five decades: $\beta_1 = \frac{\beta_2}{729}$, $\beta_2 = 2.187\,10^{-6}$ and $\beta_3 = \beta_2 \cdot 729$. The phase of the same component of $(\mathbf{H}-\mathbf{H}_0)_{ba1}$ for $\beta = \beta_2$ provides excellent results and the same conclusion holds for any approximation deduced by multiplying or dividing the solution by a real and positive number.

the effects of motion on the y component of the electric field, in the middle of the slab, for several values of β, are presented. In particular, by comparing the plots of Figures 3 and 4 one can observe that the error made by the first-order Born approximation for any value of β is as large as the quantity it has to approximate. The error of the second-order Born approximation is instead much smaller, being of the same order as β^4. The irregular results around 10^{-15} are due to errors of double precision arithmetic and are not related to the quality of Born approximations, as explained above.

To conclude this first part of the analysis of the numerical results, it is worth noting that for the problems considered, the effects of motion on the electromagnetic field outside the slab are always of the second order [18]. Since it is not possible to know a priori, for more general problems, the order of the effects of motion on the components of the electromagnetic field of interest, in practice it seems appropriate to proceed in any case with the calculation of the second-order approximation. It is, in fact, the best approximation for calculating second-order effects, as it was shown in Section 5.

The second part of this section is devoted to the numerical analysis of the quality of the approximations obtained by scaling factors.

For the simple subclass of one-dimensional problems we have considered, there is no need to follow the cautious advice given in the second last paragraph of Section 6. As a matter of fact, we know in advance from our previous work that the effects of motion on the x component of the magnetic field are of the first order and that they are not trivial just inside the slab. On the contrary, the effects on the y component of the electric field and on the z component of the magnetic field are of the second order and are present in the whole space.

For the indicated reasons, in Figures 5 and 6 we show, respectively, the behavior of the magnitude and of the phase of $(\mathbf{H}-\mathbf{H}_0)_x$ in the slab, for the following values of β: $\beta_1 = \frac{\beta_2}{729} = 3\,10^{-9}$, $\beta_2 = 2.187\,10^{-6}$ and $\beta_3 = \beta_2 \cdot 729 = 1.594323\,10^{-3}$. In the same figures we report the corresponding first-order Born approximation, $(\mathbf{H}-\mathbf{H}_0)_{ba1,x}$, which was computed just for $\beta = \beta_2$. The effects of motion on the x component of the magnetic field for $\beta = \beta_1$ and $\beta = \beta_3$ were instead directly deduced, without any additional simulation, by multiplying or dividing the computed approximation by 729. The quality of the obtained approximations for all values of β is impressive. In particular, Figure 5 shows that the magnitudes of the approximations obtained by a single simulation overlap those of $(\mathbf{H}-\mathbf{H}_0)_x$ for the corresponding β value. As for the phases, in Figure 6 they are all perfectly coincident. The plots of the phase of the approximations obtained

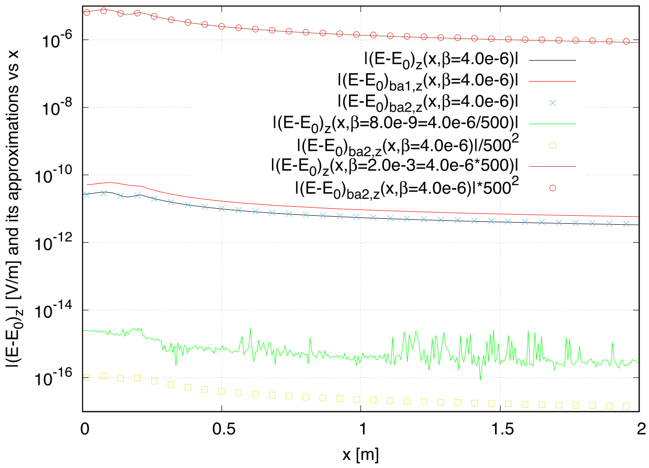

FIGURE 7 | Behavior of the magnitude of the z components of $\mathbf{E} - \mathbf{E}_0$ along the x axis, for three β values spanning more than five decades: $\beta_1 = \frac{\beta_2}{500}$, $\beta_2 = 4\,10^{-6}$ and $\beta_3 = \beta_2 \cdot 500$. The magnitude of the same component of $(\mathbf{E} - \mathbf{E}_0)_{ba1}$ for $\beta = \beta_2$ is unreliable. The second-order Born approximation is instead very good. It is computed just for $\beta = \beta_2$. The approximations for $\beta = \beta_1$ and $\beta = \beta_3$ are instead deduced by multiplying or dividing the computed approximation by 500^2.

by multiplying or dividing by 729 are not reported because any multiplication or division by a real and positive number does not affect the phase.

As it was explained above, for first order effects, like those shown if Figures 5 and 6, there is no need to compute Born approximations of the second-order. However, they are expected to be crucial, even for the properties under analysis, for second order effects. The previous numerical evaluations were carried out by considering a subclass of one-dimensional problems. Since the obtained results are general, in this final verification we will consider homogeneous cylinders with circular cross-section, undergoing axial motion, illuminated by a plane wave with TM polarization propagating in vacuum in a direction orthogonal to the cylinder axis. To be specific, the cylinder will have a radius equal to $\frac{\lambda_0}{\sqrt{2}}$, its axis coincident with the z-axis, $\varepsilon_{r2} = 2$, $\mu_{r2} = 1$, and the incident wave will propagate along the y-axis with a frequency of 1 GHz.

For this type of problem, motion-induced effects of second order appear on the z-component of the electric field [16]. $(\mathbf{E} - \mathbf{E}_0)_z$ can be calculated analytically, as in the previous 1D cases. Conversely, the Born approximations require the use of numerical methods. The results in Figure 7 were obtained using the finite element method, with a numerical investigation domain having a radius equal to 14 times that of the cylinder, discretized using meshes composed of triangular elements generated from 196 equally spaced circumferences [16].

In this case we used $\beta_1 = \frac{\beta_2}{500} = 8\,10^{-9}$, $\beta_2 = 4\,10^{-6}$ and $\beta_3 = \beta_2 \cdot 500 = 2\,10^{-3}$.

To remove any doubt, in Figure 7 we show also the behavior of the first-order Born approximation computed for $\beta = \beta_2$. As expected, it is affected by significant errors, of the order of 100%, and there is no interest in using a scaling factor on this bad approximation. Instead, the results are particularly satisfactory for second-order Born approximations: if one deduces approximations for $\beta = \beta_3$ by multiplying by 500^2 the unique approximation calculated for $\beta = \beta_2$, one gets very precise results in terms of magnitude. Figures 7 shows a complete overlapping between the approximations obtained in this way and $(\mathbf{E} - \mathbf{E}_0)_z$. For $\beta = \beta_1$ the same considerations apply. Actually, according to our theory, we can be more confident on these results than on those obtained for $\beta = \beta_3$. The problem in this case is due to the lack of reliability of the traditional approach, for such small value of β. In particular, for a velocity of the order of 2.4 meters per second the traditional method to estimate the effects of motion gives erroneous results, with relative errors as high as 2500%!

Phase data are omitted because they are similar to those shown in Figure 6.

In conclusion, Figures 5, 6 and 7 show that the considered property holds true and that it can be very useful to reduce the number of simulations. This is of particular interest for complex problems, such as the one considered in the final numerical test, where Born approximations can only be computed using numerical simulators [16].

As a corollary, we observe that for a set of small values of velocity, which are in any case of interest in practical and industrial applications, we can compute reliable results not only by computing Born approximations for all values of interest, but also by multiplying by scaling factors the results obtained by Born approximations for a single or a few values.

9 | Conclusions

In this paper several unknown properties of the Born approximations of the effects of motion on the electromagnetic field are deduced. These properties are shown to be intrinsic features of the approximations themselves: they do not depend on specific problems and remain valid for one-dimensional, two-dimensional, or three-dimensional problems involving moving media, regardless of the type of time-independent velocity of matter. In particular, it is shown under which condition Born approximations of the effects of motion provide exact results, for small values of the largest normalized speed, for the first dominant terms controlled by the first powers of the largest normalized speed. The right order of the Born approximation for a specific goal is then identified. Moreover, a result is proved that can be used to save CPU time by skipping numerous challenging simulations: Born approximations can be computed a few times and then it is possible to deduce from them reliable approximations for velocity fields proportional to those considered in the previous computations. Finally, the deduced theoretical properties are proved to have practical relevance by showing that they hold for simple problems involving media in uniform motion that are homogeneous and isotropic in their rest frames. The same result has not been proven for more general situations involving three-dimensional non-uniform velocity fields with anisotropic media.

Nomenclature

$\mathbf{A}$	generic vector field
$A_{pi\gamma}, A_{ri\gamma}$	constants for the expansion of E_γ in medium i, $\gamma = y, z$
$A_{0pi\gamma}, A_{0ri\gamma}$	constants for the expansion of $E_{0\gamma}$ in medium i, $\gamma = y, z$
$\mathbf{B}$	magnetic flux density
$C_{\beta M}$, C_{eje}, C_{ejm}, C_{hje}, C_{hjm}, C_E, C_H, C_b, C_{E_1}, C_{E_2}, C_{H_1}, C_{H_2}, C_1, C_2	real non-negative constants
$\mathbf{D}$	electric displacement
$\mathbf{E}$	electric field
$\mathbf{E}_0$	electric field when all media are at rest
$(\mathbf{E}-\mathbf{E}_0, \mathbf{H}-\mathbf{H}_0)$	effects of motion on the electromagnetic field
$((\mathbf{E}-\mathbf{E}_0)_{bai}$, $(\mathbf{H}-\mathbf{H}_0)_{bai})$	Born approximation of order i of the effects of motion on the electromagnetic field; $i \geq 1$
$(\mathbf{E}_{ai}, \mathbf{H}_{ai})$	approximation of order i of $(\mathbf{E}, \mathbf{H})$; $i \geq 0$; for $i = 0$ $(\mathbf{E}_{a0}, \mathbf{H}_{a0}) = (\mathbf{E}_0, \mathbf{H}_0)$
$(\mathbf{E}-\mathbf{E}_0)_{ba1,j}$ j-th	term in the four-term expansion of $(\mathbf{E}-\mathbf{E}_0)_{ba1}$
$(\mathbf{E}-\mathbf{E}_0)_{bai,\gamma}$	component γ, $\gamma = x, y, z$, of $(\mathbf{E}-\mathbf{E}_0)_{bai}$, $i \geq 1$
$\mathbf{H}$	magnetic field
$\mathbf{H}_0$	magnetic field when all media are at rest
$(\mathbf{H}-\mathbf{H}_0)_{ba1,j}$	j-th term in the four-term expansion of $(\mathbf{H}-\mathbf{H}_0)_{ba1}$
$(\mathbf{H}-\mathbf{H}_0)_{bai,\gamma}$	component γ, $\gamma = x, y, z$, of $(\mathbf{H}-\mathbf{H}_0)_{bai}$, $i \geq 1$
I	identity tensor field
$\mathbf{J}_e$	electric current density
$\mathbf{J}_m$	magnetic current density
$\mathbf{J}_{e,eq}$	equivalent electric current density
$\mathbf{J}_{m,eq}$	equivalent magnetic current density
$\mathbf{J}_{e,eq,ai}$	approximate equivalent electric current density giving the i-th order Born approximation; $i \geq 1$
$\mathbf{J}_{m,eq,ai}$	approximate equivalent magnetic current density giving the i-th order Born approximation; $i \geq 1$
M	set of indices identifying subdomains
M_0	subset of indices identifying subdomains occupied by media at rest
M_β	subset of indices identifying subdomains occupied by moving media
$S_{\beta i} \leq C_{\beta M} < 1$	an upper bound for $\lvert\beta\rvert = \beta$, $i \in M_\beta$
V	subspace of velocity fields
$\overline{V}$	vector space of velocity fields of dimension one generated by $\overline{\mathbf{v}}$
Y	admittance operator involved in admittance boundary conditions
Y_{i0}	admittance of medium i at rest
Y_{iy}, Y_{iz}	admittances of medium i
a	real number such that $a\overline{\mathbf{v}} \in \overline{V}$
a_1, a_2	particular values of a
c_0	speed of light in vacuum
d	thickness of a slab
$\mathbf{f}_R$	source term of inhomogeneous admittance boundary conditions
$h_2(a; \mathbf{r})$	Peano form of the remainder of a second-order McLaurin polynomial
k_{i0}	wavenumber of medium i at rest
k_i	wavenumber of medium i
m_{ei} (respectively, m_{mi}, m_{di})	largest magnitude of $\lvert\varepsilon_r\rvert$ (respectively, $\lvert\mu_r\rvert$, $\lvert\delta\rvert$), for $\mathbf{r} \in \Omega_i$, $i \in M_\beta$
m_e (respectively, m_m, m_d, S_β)	largest value of m_{ei} (respectively, m_{mi}, m_{di}, $S_{\beta i}$), $i \in M_\beta$
n	refractive index
$\mathbf{n}$	outward unit normal vector
$\mathbf{r}$	a point in the domain
t	time coordinate
$\mathbf{v}$	velocity field
v	magnitude of $\mathbf{v}$
$\hat{\mathbf{v}}$	versor of the velocity field
$\overline{\mathbf{v}}$	a specific velocity field
$\hat{\overline{\mathbf{v}}}$	versor of the velocity field $\overline{\mathbf{v}}$
w_γ with $\gamma = x$, $y, z; \gamma$	component of the field $\mathbf{w}$
x, y, z	Cartesian coordinates
Γ	boundary of Ω
Ω	open, bounded, and connected domain of the problem
Ω_i	subdomains of Ω, $i \in M$; Ω_s another subdomain of Ω
Ω_β	the interior of the union of the closures of Ω_i, for $i \in M_\beta$
$\overline{\Omega}$	closure of Ω
$\partial\Omega_i$	boundary of Ω_i
α	field used to simplify some equations
$\beta = \frac{v}{c_0}$	normalized velocity

$\delta = \alpha c_0^2$	field used to simplify some equations
ε_r (μ_r)	relative permittivity (permeability)
ε_0 (μ_0)	vacuum permittivity (permeability)
$\varepsilon = \varepsilon_0 \varepsilon_r$ ($\mu = \mu_0 \mu_r$)	scalar permittivity (permeability) of media at rest
ε^v (μ^v)	tensor permittivity (permeability) of moving media
$\varepsilon^0 = \varepsilon_0 \varepsilon_r I$ ($\mu^0 = \mu_0 \mu_r I$)	tensor permittivity (permeability) of media at rest
ξ, ζ	tensor constitutive parameters of bianisotropic media
ω	angular frequency
$(L^2(\Omega))^n$	Hilbert space of square-integrable vector fields on Ω
$(\mathbf{u}, \mathbf{w})_{0,\Omega} = \int_\Omega \mathbf{w}^* \cdot \mathbf{u}\, d\Omega$	scalar product of fields in $(L^2(\Omega))^n$
$\mathbf{w}^*$	conjugate transpose of the column vector $\mathbf{w}$
$\Vert\mathbf{w}\Vert_{0,\Omega} = \left((\mathbf{w}, \mathbf{w})_{0,\Omega}\right)^{1/2}$	norm of a vector field in $(L^2(\Omega))^n$
$U = \{\mathbf{w} \in (L^2(\Omega))^3 \mid \nabla \times \mathbf{w} \in (L^2(\Omega))^3$ and $\mathbf{w} \times \mathbf{n} \in L_t^2(\Gamma)\}$	when $\Omega \in \mathbb{R}^3$; the vector space where we seek the solutions of interest
$L_t^2(\Gamma) = \{\mathbf{w} \in (L^2(\Gamma))^3 \mid \mathbf{w} \cdot \mathbf{n} = 0$ almost everywhere on $\Gamma\}$	when $\Omega \in \mathbb{R}^3$; the vector space of the tangential traces of $\mathbf{w} \in U$
$(\mathbf{u}, \mathbf{w})_U = (\mathbf{u}, \mathbf{w})_{0,\Omega} + (\nabla \times \mathbf{u}, \nabla \times \mathbf{w})_{0,\Omega} + (\mathbf{u} \times \mathbf{n}, \mathbf{w} \times \mathbf{n})_{0,\Omega}$ when	$\Omega \in \mathbb{R}^3$; the scalar product of fields in U
$\Vert\mathbf{w}\Vert_U = ((\mathbf{w}, \mathbf{w})_U)^{1/2}$	norm of a vector field in U

Author Contributions

Mirco Raffetto: conceptualization, data curation, formal analysis, investigation, methodology, project administration, resources, software, supervision, validation, visualization, writing – original draft, writing – review and editing. **Kirill Zeyde:** conceptualization, data curation, formal analysis, investigation, methodology, project administration, resources, software, supervision, validation, visualization, writing – original draft, writing – review and editing.

Acknowledgements

Open access publishing facilitated by Università degli Studi di Genova, as part of the Wiley - CRUI-CARE agreement.

Funding

The authors have nothing to report.

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The authors have nothing to report.

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