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A multiscale approach to visco-electro-elastic complex materials: Asymptotic homogenization versus high-frequency continualization schemes

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ABSTRACT

This work deals with the study of the viscous effects on the electro-mechanical behavior of periodic materials paving the way for remarkable applications in many scientific fields. In a three-dimensional context, the constitutive equations that describe a visco-electro-elastic periodic material are recast into the complex frequency space via the two-sided Laplace transform yielding a Stroh-like formulation. Then, the governing equations are manipulated by means of the Floquet–Bloch transform to study the wave propagation and the characteristic equation, which is generalized for a periodic visco-electro-elastic laminate, is obtained and solved to achieve its frequency complex spectra. Thereafter, an asymptotic homogenization method and a continualization scheme, which is based on the kernel developed as a Padé approximant, are detailed to identify equivalent non-local visco-electro-elastic continua. Finally, the homogenized frequency band structures are compared with the heterogeneous one to validate the proposed models.

1. Introduction

The extensive application of piezoelectric devices in different industrial fields, such as aerospace, automotive, marine, aeronautics, and biomedical, has fueled extensive research on their physical properties. Piezoelectric composites, particularly, show time-dependent properties that are remarkable at high temperatures and have a consequent damping effect that industrial applications can exploit. Complex elastic and dielectric constants exist due to mechanical and electrical losses in the material and numerous studies exist on the topic (Cordero, Craciun, & Verardi, 2003; Ferry & Myers, 1961; Gerhardt, Lee, & Nowick, 1987; Lakes, 2017; Lay & Whitmore, 1971; Nowick, 2012). These mechanical and dielectric losses have been proven to greatly influence complex piezoelectric constants, whose imaginary parts are not fundamentally related to losses and whose investigation still deserves attention (Alemany et al., 1995; Holland, 1967; Meixner & Reik, 1959; Mueser & Schmitt, 1972; Müser & Petersson, 1971; Nowick & Heller, 1965). The existence of complex piezoelectric constants is often ascribable to the heterogeneity of the material at the microstructural level, as happens in bones (Bur, 1976; Maeda & Fukada, 1982; Mohammadkhah, Marinkovic, Zehn, & Checa, 2019), piezoelectric ceramics/polymer matrix composites (Badapanda et al., 2013; Furukawa, Fujino, & Fukada, 1976; Furukawa & Fukada, 1977; Sherit, Wiederick, & Mukherjee, 1997), and PVDF polymers (Furukawa, Ishida, & Fukada, 1979; Roh, Varadan, & Varadan,

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2002; Tashiro, Kobayashi, Tadokoro, & Fukada, 1980). Piezoelectric relaxation, represented by complex moduli, takes place in the composite material, and modeling efforts are still needed to accurately predict the coupled overall behavior of heterogeneous visco-electro-elastic solids in terms of microstructural features. Existing studies on the effective properties of visco-electro-elastic composites have been conducted by (Jiang & Batra, 2001) exploiting the Mori–Tanaka mean-field approach, by (Li & Dunn, 2001) for bi-phase piezoelectric composites in the case of lossless inclusions and lossy matrix, by (Aldraihem, Baz, & Al-Saud, 2007) for composites having shunted piezoelectric inclusions. The overall behavior of visco-electro-elastic composites has been recently investigated by (Azrar, Bakkali, & Aljinaidi, 2014) using the correspondence principle and the Mori–Tanaka micromechanical mean-field approach.

In this framework, the present work presents a rigorous investigation of the effective dynamic behavior of coupled visco-electro-elastic periodic heterogeneous materials employing two different non-local homogenization methods, namely an asymptotic homogenization approach and a high-frequency continualization scheme. Asymptotic homogenization, variational-asymptotic approaches, as well as further analytical identification approaches are powerful multiscale techniques to study heterogeneous microstructured materials whose macroscopic performance is affected by phenomena occurring at the microstructural level (Bakhvalov & Panasenko, 1984; Bensoussan, Lions, & Papanicolaou, 1978; Kamotski & Smyshlyaev, 2019; Molotkov, 2000; Smyshlyaev & Cherednichenko, 2000). The asymptotic homogenization method is grounded on a sharp scale separation existing in multiscale systems and a power series representation of the fields with respect to an appropriate ordering parameter providing the macroscale system with a set of partial differential equations whose coefficients encode the role of the microstructure. It is nowadays extensively adopted to accurately investigate multifield phenomena such as piezoelectricity (Andrianov, Kolpakov, & Faella, 2024; Chen, Chatzigeorgiou, Meraghni, & Javili, 2022; Chen, Yvonnet, Yao, & Park, 2021; De Bellis, Bacigalupo, & Zavarise, 2019; Fantoni, Bacigalupo, & Paggi, 2017; Gao, He, Liu, Lü, & Wang, 2025; Guinovart, Chaki, & Guinovart-Díaz, 2024; Guinovart-Sanjuán, Mohapatra, Rodríguez-Ramos, Espinosa-Almeyda, & Rodríguez-Bermúdez, 2023; He, Liu, & Chen, 2023; Lee, Zhang, & Gu, 2023; Mawassy, Reda, Ganghoffer, Eremeyev, & Lakiss, 2021; Muhafera, Kosta, Torrent, Pernas-Salomón, & Shmuel, 2022), thermo-diffusivity (Bacigalupo, Morini, & Piccolroaz, 2016; Fantoni & Bacigalupo, 2020; Salvadori, Bosco, & Grazioli, 2014; Vazic, Abali, & Newell, 2023), thermo-elasticity (Préve, Bacigalupo, & Paggi, 2021; Rudoy & Sazhenkov, 2024; Yang, Lei, Gao, & Li, 2015; Zhi, Raju, Tay, & Tan, 2021), electro-magneto-elasticity (Böhm, Hudobivnik, Marino, & Wriggers, 2021; Borcea & Bruno, 2001; Christofi, Hadjiloizi, Kalamkarov, & Georgiades, 2022; Tang & Yu, 2008), viscoelasticity (Bengtsson, Mousavi, Afshar, & Gamstedt, 2023; Del Toro, Bacigalupo, & Paggi, 2019; Ferreira, Lopes, Leiderman, Aragão, & Pereira, 2023; He & Pindera, 2021; Hirsekorn, Marcin, & Godon, 2022; Li, Zheng, & Li, 2024), and hygro-mechanics (Bosco, Claessens, & Suiker, 2020; Pal & Ramaswamy, 2023; Vega et al., 2022), to cite a few.

The term continualization is usually adopted when discrete Lagrangian systems are identified with homogeneous continua (Askes & Metrikine, 2005). Specifically, such systems are made of connected individual units, each defined by mass, periodically arranged by following specific geometric configurations and subject to translations and rotations. Within this framework, the continualization is used to denote the translation of a heterogeneous laminate into an equivalent homogeneous one. Moreover, an enhanced continualization procedure has been proposed to address the one-dimensional discrete models (Bacigalupo & Gambarotta, 2019, 2021). The enhanced technique differs from the previously proposed standard one in the choice of the regularization kernel appearing in the down-scaling laws. Such a choice, for which the kernel shows polar singularities at the boundary of the first Brillouin zone, guarantees an accurate approximation of the material frequency spectrum throughout the entire first Brillouin zone (Bacigalupo, Badino, Diana, & Gambarotta, 2024; Del Toro, De Bellis and Bacigalupo, 2024; Fantoni, Bacigalupo, & Gambarotta, 2024), thus avoiding dynamic instabilities affecting the previous continualization schemes. Based on a proper manipulation of the integral kernel, the high-frequency continualization schemes enable to identify either integral-type or gradient-type non-local equivalent continuum models and the corresponding governing field equations characterized by the integral-type inertial term.

The described procedures are capable of capturing the size effects arising from the underlying microstructure and the wave dispersion phenomenon that occurs in discrete periodic materials (Challamel, Aydogdu, & Elishakoff, 2018; Challamel, Kocsis, & Wang, 2015; Hache, Challamel, Elishakoff, & Wang, 2017; Liu, Chen, & Xu, 1999; Zhang, Wang, Challamel, & Elishakoff, 2014).

Within the framework of the existing literature on the dynamic characterization of periodically microstructured materials with visco-electro-elastic phases, the present work aims to introduce and compare two distinct methodologies. The first is an asymptotic homogenization procedure, designed to approximate the band structure of periodic materials and to identify equivalent gradient-type continua. The second is a continualization approach for laminates, which allows for a convergent approximation of the entire Floquet–Bloch spectrum. The two methods are critically compared to highlight the distinctive features and potential of each approach.

The paper is organized as follows. In Section 2, the field equations that characterize a periodic microstructured heterogeneous visco-electro-elastic material are presented. By applying a two-sided Laplace transform in time, the problem is reformulated in a Stroh-like formalism, and the application of a multidimensional Floquet–Bloch transform recasts the problem in the wavevector space. The analysis is then specialized to the case of a laminate periodic material, where the free propagation of Bloch waves perpendicular to the material layering is addressed using the transfer matrix method. This approach allows for the determination of the roots of a palindromic characteristic polynomial, from which the analytical complex frequency spectrum of the material is obtained. In Section 3, the asymptotic homogenization method is described. Specifically, the down-scaling laws are formulated in terms of the macro fields and perturbation functions, which are obtained as solutions of the cell problems. The substitution of these down-scaling laws into the field equations leads to the average field equations of infinite order which, if properly treated through a perturbative procedure, yield the governing equations of a higher-order gradient-type visco-electro-elastic continuum. A procedure for asymptotically approximating the complex frequency spectrum of the periodic visco-electro-elastic material is then detailed, with

dispersion curves provided for both first- and second-order continua. By employing a generalized macro-homogeneity condition in the Laplace space, the overall constitutive and inertial constants are derived in closed form for a first-order visco-electro-elastic continuum model. In Section 4, a continualization scheme based on Padé approximation is detailed, resulting in either integro-differential or gradient-type governing equations for the equivalent non-local continuum. In Section 5, two illustrative applications are analyzed to assess the effectiveness of the asymptotic homogenization and continualization methods. The first application focuses on the free wave propagation problem in laminate materials. The local and non-local overall constitutive and inertial tensors are derived analytically. The corresponding spectra are also obtained in closed form using the transfer matrix method and compared with the complex frequency spectra resulting from the continualized and homogenized models. In the second application, to verify the reliability of the homogenization method, numerical results for a biphasic fiber-reinforced composite – comprising a periodic array of fibers embedded in a soft matrix – are considered. In this case, the micro-mechanical response is computed numerically and compared with the homogenized response, demonstrating the method's capability to characterize materials with complex topologies. Section 6 contains the concluding remarks and outlines possible directions for future developments.

2. Model formulation for visco-electro-elastic periodic materials

Within the framework of the small displacement theory, a visco-electro-elastic periodic material, which displays time-dependent properties, is defined (Li & Dunn, 2001). Introducing a reference system $\{O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the periodic cell $\mathcal{A} = [0, \varepsilon]^3$ is set with dimension ε , characterized by the orthogonal periodicity vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$, where $\mathbf{v}_i = d_i \mathbf{e}_i = \varepsilon \mathbf{e}_i$, for $i = 1, 2, 3$. The material domain is defined by the arrangement of the cell $\mathcal{A}$ along the directions of $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$. The constitutive equations involve the second-order stress tensor $\boldsymbol{\sigma}(\mathbf{x}, t) = \sigma_{ij} \mathbf{e}_{ij}$ and the electric displacement $\mathbf{D}(\mathbf{x}, t) = D_i \mathbf{e}_i$ that are displayed in terms of the hereditary integrals as

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \int_{-\infty}^t \left(\mathbb{G}^m(\mathbf{x}, t - \tau) \dot{\boldsymbol{\varepsilon}}(\mathbf{x}, \tau) - \mathbf{p}^m(\mathbf{x}, t - \tau) \dot{\mathbf{E}}(\mathbf{x}, \tau) \right) d\tau, \quad (1a)$$

$$\mathbf{D}(\mathbf{x}, t) = \int_{-\infty}^t \left(\tilde{\mathbf{p}}^m(\mathbf{x}, t - \tau) \dot{\boldsymbol{\varepsilon}}(\mathbf{x}, \tau) + \boldsymbol{\eta}^m(\mathbf{x}, t - \tau) \dot{\mathbf{E}}(\mathbf{x}, \tau) \right) d\tau, \quad (1b)$$

where $\mathbf{x} = x_j \mathbf{e}_j$ is the position vector of a material point, t is the time, $\mathbb{G}^m(\mathbf{x}, t) = G_{ijkl}^m \mathbf{e}_{ijkl}$ is the fourth-order micro elasticity relaxation tensor with major and minor symmetries, $\boldsymbol{\varepsilon}(\mathbf{x}, t) = \varepsilon_{ij} \mathbf{e}_{ij}$ is the second-order strain tensor, $\mathbf{p}^m(\mathbf{x}, t) = p_{ijh}^m \mathbf{e}_{ijh}$ is the third-order piezoelectric tensor with minor symmetries in j and h , whereas the base for a generic tensor of order n is defined as $\mathbf{e}_{i\dots j} = \mathbf{e}_i \otimes \dots \otimes \mathbf{e}_j$. Moreover, $\mathbf{E}(\mathbf{x}, t) = E_i \mathbf{e}_i$ is the electric field, and $\boldsymbol{\eta}^m(\mathbf{x}, t) = \eta_{ij}^m \mathbf{e}_{ij}$ is the second-order dielectric permittivity tensor with major symmetry. The superimposed dot is the derivative with respect to time.

It must be observed that the constitutive Eqs. (1a)–(1b) are characterized by integro-differential relations with causal kernels. The stress field $\boldsymbol{\sigma}(\mathbf{x}, t)$ and the electric displacement field $\mathbf{D}(\mathbf{x}, t)$ obey the balance equations

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{x}, t) + \mathbf{b}(\mathbf{x}, t) = \rho^m(\mathbf{x}) \dot{\mathbf{u}}(\mathbf{x}, t), \quad (2a)$$

$$\nabla \cdot \mathbf{D}(\mathbf{x}, t) = \rho_e(\mathbf{x}, t), \quad (2b)$$

where $\mathbf{b}(\mathbf{x}, t)$ are the body forces, $\rho_e(\mathbf{x}, t)$ are the free charge densities, $\rho^m(\mathbf{x})$ is the material density and $\mathbf{u}(\mathbf{x}, t) = u_i \mathbf{e}_i$ is the micro displacement (Lenglet, Hladky-Hennion, & Debus, 2003; Wilm, Ballandras, Laude, & Pastureaud, 2002). The time-dependent tensors appearing in Eqs. (1a)–(1b) satisfy the condition $g^m(\mathbf{x} + \mathbf{v}_i, t) = g^m(\mathbf{x}, t)$, $i = 1, 2, 3, \forall \mathbf{x} \in \mathcal{A}$ since the material is $\mathcal{A}$ -periodic. In the following, it is assumed that the electric field is expressed in terms of the micro electrical potential field as $\mathbf{E}(\mathbf{x}, t) = -\nabla \phi(\mathbf{x}, t)$ and the dependence on the variables is omitted for the sake of readability. The problem (1a)–(2b) can be reformulated in a Stroh-like formalism by applying the two-sided Laplace transform in time, herein defined for a generic function $g(\mathbf{x}, t)$ $\mathcal{L}[g(\mathbf{x}, t)] := \hat{g}(\mathbf{x}, s)$, as

$$\nabla \cdot \hat{\boldsymbol{\sigma}} + \hat{\mathbf{b}} - \rho^m s^2 \hat{\mathbf{u}} = 0, \quad (3a)$$

$$\nabla \cdot \hat{\mathbf{D}} - \hat{\rho}_e = 0, \quad (3b)$$

$$\hat{\boldsymbol{\sigma}} - \hat{\mathbb{C}}^m \nabla \hat{\mathbf{u}} - \hat{\mathbf{e}}^m \nabla \hat{\phi} = 0, \quad (3c)$$

$$\hat{\mathbf{D}} - \hat{\mathbf{e}}^m \nabla \hat{\mathbf{u}} + \hat{\boldsymbol{\beta}}^m \nabla \hat{\phi} = 0, \quad (3d)$$

where the s -dependent transformed tensors are introduced as $\hat{\mathbb{C}}^m = s \hat{\mathbb{G}}^m$, $\hat{\mathbf{e}}^m = s \hat{\mathbf{p}}^m$, $\hat{\mathbf{e}}^m = s \hat{\mathbf{p}}^m$, $\hat{\boldsymbol{\beta}}^m = s \hat{\boldsymbol{\eta}}^m$. The field equations of a visco-electro-elastic heterogeneous body, expressed in the physical domain and in the complex frequency space, are reformulated using a standard notation as

$$\nabla \cdot (\hat{\mathbb{C}}^m \nabla \hat{\mathbf{u}} + \hat{\mathbf{e}}^m \nabla \hat{\phi}) + \hat{\mathbf{b}} - \rho^m s^2 \hat{\mathbf{u}} = 0, \quad (4a)$$

$$\nabla \cdot (\hat{\mathbf{e}}^m \nabla \hat{\mathbf{u}} - \hat{\boldsymbol{\beta}}^m \nabla \hat{\phi}) - \hat{\rho}_e = 0, \quad (4b)$$

which are expressed in terms of the transformed micro-displacement $\hat{\mathbf{u}}$ and the transformed micro electrical potential field $\hat{\phi}$. Focusing on the analysis of the wave propagation within the visco-electro-elastic periodic material and evoking the multidimensional Floquet–Bloch transform in the wavevector space for a generic function $\hat{g}(\mathbf{x}, s)$ as $F_B[\hat{g}(\mathbf{x}, s)] := \hat{g}(\mathbf{x}, \mathbf{k}, s)$, the Eqs. (3a)–(3d) can be reshaped as

$$(\nabla \cdot \hat{\boldsymbol{\sigma}} + \iota \hat{\boldsymbol{\sigma}} \cdot \mathbf{k}) + \hat{\mathbf{b}} - \rho^m s^2 \hat{\mathbf{u}} = 0, \quad (5a)$$

$$\nabla \cdot \hat{\mathbf{D}} + i\mathbf{k} \cdot \hat{\mathbf{D}} - \hat{\rho}_e = 0, \quad (5b)$$

$$\hat{\sigma} - \hat{\mathbb{C}}^m(\nabla \hat{\mathbf{u}} + i\mathbf{k} \otimes \hat{\mathbf{u}}) - \hat{\epsilon}^m(\nabla \hat{\phi} + i\mathbf{k} \hat{\phi}) = 0, \quad (5c)$$

$$\hat{\mathbf{D}} - \hat{\epsilon}^m(\nabla \hat{\mathbf{u}} + i\mathbf{k} \otimes \hat{\mathbf{u}}) + \hat{\beta}^m(\nabla \hat{\phi} + i\mathbf{k} \hat{\phi}) = 0, \quad (5d)$$

bearing in mind the $\mathcal{A}$ -periodicity of the constitutive tensors.

2.1. Free Bloch waves propagation for laminates

We consider a periodic visco-electro-elastic laminate that is characterized by a periodic cell made of $\mathcal{M}$ layers having total thickness S and arranged along the x_δ -direction (with $\delta = 1$ or 2 or 3). To deal with the free wave propagation occurring on the x_δ -direction that is perpendicular to the material layering with $\mathbf{k} = k_\delta \mathbf{e}_\delta$ and for ceramics poled in the x_δ direction, it is convenient to reshape the Eqs. (5a)–(5d) as

$$(\hat{\sigma}_{i\delta,\delta} + ik_\delta \hat{\sigma}_{i\delta}) + \hat{b}_i - \rho^m s^2 \hat{u}_i = 0, \quad (6a)$$

$$\hat{\mathbf{D}}_{\delta,\delta} + ik_\delta \hat{\mathbf{D}}_\delta - \hat{\rho}_e = 0, \quad (6b)$$

$$\hat{\sigma}_{i\delta} - \hat{\mathbb{C}}_{i\delta k\delta}^m(\hat{u}_{k,\delta} + ik_\delta \hat{u}_k) - \hat{\epsilon}_{i\delta\delta}^m(\hat{\phi}_{,\delta} + ik_\delta \hat{\phi}) = 0, \quad (6c)$$

$$\hat{\mathbf{D}}_\delta - \hat{\epsilon}_{k\delta\delta}^m(\hat{u}_{k,\delta} + ik_\delta \hat{u}_k) + \hat{\beta}_{\delta\delta}^m(\hat{\phi}_{,\delta} + ik_\delta \hat{\phi}) = 0, \quad (6d)$$

where the symbol $_{,\delta}$ stands for the partial derivative with respect to the variable x_δ . The field equations that characterize the single layer are the same as (6a)–(6d) with constant coefficients, therefore we can find the first-order system

$$\mathbf{H} \hat{\mathbf{r}}_{,\delta} + \mathbf{K} \hat{\mathbf{r}} = \mathbf{0}, \quad (7)$$

where the 8×1 vector $\hat{\mathbf{r}} = (\hat{\mathbf{r}}^{(L)} \ \hat{\mathbf{r}}^{(T)})^T$ collects the Bloch amplitudes in the wavevector domain for the longitudinal and for the transverse Bloch wave propagation problems as $\hat{\mathbf{r}}^{(L)} = (\hat{u}_\delta \ \hat{\phi} \ \hat{\sigma}_{\delta\delta} \ \hat{\mathbf{D}}_\delta)^T$ and $\hat{\mathbf{r}}^{(T)} = (\hat{u}_\beta \ \hat{u}_\gamma \ \hat{\sigma}_{\beta\delta} \ \hat{\sigma}_{\gamma\delta})^T$. Moreover, the non-zero components of the 8×8 matrices $\mathbf{H}$ and $\mathbf{K}$ are detailed in the section A of Supplementary Material of this paper. The transfer matrix method can assess the relationship between the vectors $\hat{\mathbf{r}}_m^+$ and $\hat{\mathbf{r}}_m^-$ appraised at the upper and the lower boundaries of the m th layer with thickness s_m as

$$\hat{\mathbf{r}}_m^+ = \exp(-\mathbf{H}^{-1} \mathbf{K} s_m) \hat{\mathbf{r}}_m^- = \tilde{\mathbf{T}}_m(k_\delta, s) \hat{\mathbf{r}}_m^-, \quad (8)$$

where the matrix $\mathbf{H}$ is invertible. Then, through the equivalence, we ascertain the transfer matrix $\tilde{\mathbf{T}}_m(k_\delta, s)$ associated to the m th layer ($m = 1, \dots, \mathcal{M}$), which is recast as

$$\tilde{\mathbf{T}}_m(k_\delta, s) = \mathbf{T}_m(s) \exp(-ik_\delta s_m), \quad (9)$$

by proving that $\mathbf{T}_m(s)$ counts on the complex frequency (Del Toro, Bacigalupo, Lepidi, & Mazzino, 2022). The relation (8) holds for the entire periodic cell as

$$\hat{\mathbf{r}}_{\mathcal{M}}^+ = \mathbf{T}(s) \exp(-ik_\delta S) \hat{\mathbf{r}}_1^-, \quad (10)$$

where $\mathbf{T}(s)$ is the transfer matrix governing the periodic cell. The periodicity condition involving the Bloch amplitudes vector yields the eigenproblem

$$(\mathbf{T}(s) - \exp(ik_\delta S) \mathbf{I}) \hat{\mathbf{r}}_1^- = (\mathbf{T}(s) - \varphi \mathbf{I}) \hat{\mathbf{r}}_1^- = \mathbf{0}, \quad (11)$$

where φ is the Floquet multiplier. The eigenvalues of the transfer matrix $\mathbf{T}(s)$ correspond to the solutions of the characteristic polynomial

$$\mathcal{V}^{het}(\varphi, s) = \det(\mathbf{T}(s) - \varphi \mathbf{I}) = \sum_{j \in \mathbb{N}} \varphi^j I_{8-j}(s), \quad (12)$$

with $I_8(s) = I_0(s) = 1$, $I_7(s) = I_1(s)$, $I_6(s) = I_2(s)$, $I_5(s) = I_3(s)$ because the polynomial (12) is palindromic. The invariants in (12) correspond to

$$I_1(s) = -\text{tr}(\mathbf{T}(s)), \quad (13a)$$

$$I_2(s) = -\frac{1}{2} \text{tr}(\mathbf{T}(s)^2) + \frac{1}{2} (\text{tr}(\mathbf{T}(s)))^2, \quad (13b)$$

$$I_3(s) = -\frac{1}{3} \text{tr}(\mathbf{T}(s)^3) + \frac{1}{2} \text{tr}(\mathbf{T}(s)) \text{tr}(\mathbf{T}(s)^2) - \frac{1}{6} (\text{tr}(\mathbf{T}(s)))^3, \quad (13c)$$

$$I_4(s) = -\frac{1}{4} \text{tr}(\mathbf{T}(s)^4) + \frac{1}{4} (\text{tr}(\mathbf{T}(s)))^2 \text{tr}(\mathbf{T}(s)^2) + \frac{1}{3} \text{tr}(\mathbf{T}(s)) \text{tr}(\mathbf{T}(s)^3) + \frac{1}{8} (\text{tr}(\mathbf{T}(s)^2))^2 + \frac{1}{24} (\text{tr}(\mathbf{T}(s)))^4. \quad (13d)$$

The transformation $\chi = \varphi + \frac{1}{\varphi}$ is useful to re-map the eighth degree polynomial (12) into the corresponding fourth degree polynomial

$$\mathcal{V}^{het}(\chi, s) = \chi^4 + I_1(s) \chi^3 + (I_2(s) - 4) \chi^2 + (I_3(s) - 3I_1(s)) \chi + (I_4(s) - 2I_2(s) + 2), \quad (14)$$

whose roots can be analytically determined. The resolution of the characteristic polynomials (12) or (14) permits to extrapolate the frequency complex spectrum of a periodic visco-electro-elastic laminate material by assessing the real and the imaginary parts of k_δ as

$$k_\delta = \text{Re}(k_\delta) + i\text{Im}(k_\delta) = \frac{\ln(\varphi)}{iS} = \frac{\arg(\text{Re}(\varphi) + i\text{Im}(\varphi))}{S} - i \frac{\ln(\text{Re}(\varphi)^2 + \text{Im}(\varphi)^2)}{2S}. \quad (15)$$

3. Asymptotic homogenization: Identification of equivalent continua and spectral approximation

An asymptotic homogenization technique is detailed here, based on the separation of slow (macroscopic) and fast (microscopic) variables. Specifically, the microfields are expressed in terms of perturbation functions, and the infinite-order average field equations are derived, with the technical details reported in the Supplementary Materials. In Section 3.1, the free wave propagation in a homogenized visco-electro-elastic material is analyzed by properly addressing the infinite-order average field equations. In Section 3.2, the homogenized constitutive tensors are obtained by applying a generalized macro-homogeneity condition.

The transformed micro-fields $\hat{u}$ and $\hat{\phi}$ are written as asymptotic expansions with respect to the parameter ε as

$$\hat{u}_h\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, s\right) = \sum_{l \in \mathbb{N}} \varepsilon^l \hat{u}_h^{(l)}\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, s\right), \quad \hat{\phi}\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, s\right) = \sum_{l \in \mathbb{N}} \varepsilon^l \hat{\phi}^{(l)}\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, s\right). \quad (16)$$

Therefore, the generalized total derivative of the function $\hat{h}\left(\mathbf{x}, \xi = \frac{\mathbf{x}}{\varepsilon}\right)$ is defined with respect to the variable x_k as

$$\frac{D}{Dx_k} \hat{h}\left(\mathbf{x}, \xi = \frac{\mathbf{x}}{\varepsilon}\right) = \left(\frac{\partial \hat{h}}{\partial x_k} + \frac{1}{\varepsilon} \hat{h}_{,k} \right) \Big|_{\xi = \frac{\mathbf{x}}{\varepsilon}}, \quad (17)$$

where $\mathbf{x}$ plays the role of the macroscopic coordinate (or slow variable), $\xi \in Q = [0, 1]^3$ is the microscopic coordinate (or fast variable) defined on the unit cell Q and the symbol $_{,k}$ indicates the generalized partial derivative with respect to ξ . First, by plugging the asymptotic expansions (16) and exploiting the definition in (17) into the transformed fields at the microscale (4a)–(4b), these fields are replaced accordingly. Then, by collecting the terms with the same power of ε , the recursive differential problems are obtained as

$$\varepsilon^{-2} f_i^{(0)}(\mathbf{x}, s) + \varepsilon^{-1} f_i^{(1)}(\mathbf{x}, s) + \varepsilon^0 f_i^{(2)}(\mathbf{x}, s) + \dots + \varepsilon^l f_i^{(l+2)}(\mathbf{x}, s) + \dots + \hat{b}_i(\mathbf{x}, s) = 0, \quad (18a)$$

$$\varepsilon^{-2} g^{(0)}(\mathbf{x}, s) + \varepsilon^{-1} g^{(1)}(\mathbf{x}, s) + \varepsilon^0 g^{(2)}(\mathbf{x}, s) + \dots + \varepsilon^l g^{(l+2)}(\mathbf{x}, s) + \dots - \hat{p}_e(\mathbf{x}, s) = 0, \quad (18b)$$

where $f_i^{(\gamma)}$ and $g^{(\gamma)}$ identify the source terms related to the problems at the order $\varepsilon^{\gamma-2}$, $\gamma = 0, 1, \dots, l+2$, which rely on the slow variable in the space domain. The solvability condition of the problems for ε^{-2} in the class of the Q -periodic functions leads to the solutions

$$\hat{u}_h^{(0)}(\mathbf{x}, \xi, s) = \hat{U}_h(\mathbf{x}, s), \quad \hat{\phi}^{(0)}(\mathbf{x}, \xi, s) = \hat{\Phi}(\mathbf{x}, s), \quad (19)$$

that in the space domain is affected by the slow variable. The procedure to derive the solutions of the problems (18a)–(18b) is reported on the Section B of the Supplementary Material of the present paper. The functions (19) allow to determine the solutions related to the problems for the order ε^{-1} as

$$\hat{u}_h^{(1)}(\mathbf{x}, \xi, s) = N_{hpq_1}^{(1)}(\xi) \frac{\partial \hat{U}_p(\mathbf{x})}{\partial x_{q_1}} + \tilde{N}_{hq_1}^{(1)}(\xi) \frac{\partial \hat{\Phi}(\mathbf{x}, s)}{\partial x_{q_1}}, \quad (20a)$$

$$\hat{\phi}^{(1)}(\mathbf{x}, \xi, s) = W_{q_1}^{(1)}(\xi) \frac{\partial \hat{\Phi}(\mathbf{x}, s)}{\partial x_{q_1}} + \tilde{W}_{pq_1}^{(1)}(\xi) \frac{\partial \hat{U}_p(\mathbf{x}, s)}{\partial x_{q_1}}, \quad (20b)$$

where $N_{hpq_1}^{(1)}$, $\tilde{N}_{hq_1}^{(1)}$, $W_{q_1}^{(1)}$ and $\tilde{W}_{pq_1}^{(1)}$ are the perturbation functions at the first-order that are influenced by the variable ξ in the space domain. Similarly, the functions (19)–(20a)–(20b) enable to achieve the solutions for the order ε^0 as

$$\hat{u}_h^{(2)}(\mathbf{x}, \xi, s) = N_{hpq_1q_2}^{(2)}(\xi) \frac{\partial^2 \hat{U}_p(\mathbf{x})}{\partial x_{q_1} \partial x_{q_2}} + \tilde{N}_{hq_1q_2}^{(2)}(\xi) \frac{\partial^2 \hat{\Phi}(\mathbf{x}, s)}{\partial x_{q_1} \partial x_{q_2}} + N_{hp}^{(2,2)}(\xi) s^2 \hat{U}_p(\mathbf{x}, s), \quad (21a)$$

$$\hat{\phi}^{(2)}(\mathbf{x}, \xi, s) = W_{q_1q_2}^{(2)}(\xi) \frac{\partial^2 \hat{\Phi}(\mathbf{x}, s)}{\partial x_{q_1} \partial x_{q_2}} + \tilde{W}_{pq_1q_2}^{(2)}(\xi) \frac{\partial^2 \hat{U}_p(\mathbf{x}, s)}{\partial x_{q_1} \partial x_{q_2}}, \quad (21b)$$

where $N_{hpq_1q_2}^{(2)}$, $\tilde{N}_{hq_1q_2}^{(2)}$, $N_{hp}^{(2,2)}$, $W_{q_1q_2}^{(2)}$ and $\tilde{W}_{pq_1q_2}^{(2)}$ are the perturbation functions corresponding to the second order. The perturbation functions have zero mean over Q and satisfy the normalization condition $\langle \cdot \rangle = \frac{1}{|Q|} \int_Q \cdot d\xi = 0$, where $|Q|$ denotes the size of the unit cell, which results to be unitary if the cell is cubic. Furthermore, the perturbation functions are the solutions of the cell problems, whose formulation is provided in Section C of the Supplementary Material of this paper.

As a result, the asymptotic expansions (16) can be formulated in terms of the perturbations functions by leading to the down-scaling relation as

$$\hat{u}_h\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, s\right) = \left\{ \hat{U}_h(\mathbf{x}, s) + \varepsilon \left(N_{hpq_1}^{(1)}(\xi) \frac{\partial \hat{U}_p(\mathbf{x}, s)}{\partial x_{q_1}} + \tilde{N}_{hq_1}^{(1)}(\xi) \frac{\partial \hat{\Phi}(\mathbf{x}, s)}{\partial x_{q_1}} \right) + \right. \quad (22a)$$

$$\begin{aligned}
& + \varepsilon^2 \left(N_{hpq_1q_2}^{(2)}(\xi) \frac{\partial^2 \hat{U}_p(\mathbf{x}, s)}{\partial x_{q_1} \partial x_{q_2}} + \tilde{N}_{hq_1q_2}^{(2)}(\xi) \frac{\partial^2 \hat{\Phi}(\mathbf{x}, s)}{\partial x_{q_1} \partial x_{q_2}} + N_{hp}^{(2,2)}(\xi) s^2 \hat{U}_p(\mathbf{x}, s) \right) + O(\varepsilon^3) \Big|_{\xi=\frac{\mathbf{x}}{\varepsilon}}, \\
\hat{\phi}\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, s\right) & = \left\{ \hat{\Phi}(\mathbf{x}, s) + \varepsilon \left(W_{q_1}^{(1)}(\xi) \frac{\partial \hat{\Phi}(\mathbf{x}, s)}{\partial x_{q_1}} + \tilde{W}_{pq_1}^{(1)}(\xi) \frac{\partial \hat{U}_p(\mathbf{x}, s)}{\partial x_{q_1}} \right) + \right. \\
& \left. + \varepsilon^2 \left(W_{q_1q_2}^{(2)}(\xi) \frac{\partial^2 \hat{\Phi}(\mathbf{x}, s)}{\partial x_{q_1} \partial x_{q_2}} + \tilde{W}_{pq_1q_2}^{(2)}(\xi) \frac{\partial^2 \hat{U}_p(\mathbf{x}, s)}{\partial x_{q_1} \partial x_{q_2}} \right) + O(\varepsilon^3) \right\} \Big|_{\xi=\frac{\mathbf{x}}{\varepsilon}},
\end{aligned} \tag{22b}$$

where the transformed micro-fields $\hat{u}_h$ and $\hat{\phi}$ are related to the transformed macro-fields, $\hat{U}_h$ and $\hat{\Phi}$, and their gradients. The macrofields can be defined as the average of the respective transformed micro-fields on the unit cell Q as

$$\hat{U}_h(\mathbf{x}, s) = \left\langle \hat{u}_h\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon} + \boldsymbol{\zeta}, s\right) \right\rangle, \quad \hat{\Phi}(\mathbf{x}, s) = \left\langle \hat{\phi}\left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon} + \boldsymbol{\zeta}, s\right) \right\rangle, \tag{23}$$

where $\boldsymbol{\zeta} \in Q$ is the translation variable with $\varepsilon \boldsymbol{\zeta} \in \mathcal{A}$, which determines the translation of the heterogeneous domain with respect to $\mathcal{L}$ -periodic source terms. When we replace the down scaling relations (22a)–(22b) into the Eqs. (4a)–(4b) and we gather the similar powers of ε , we achieve the average field equations of infinite order as

$$\begin{aligned}
& n_{ipq_1q_2}^{(2)} \frac{\partial^2 \hat{U}_p}{\partial x_{q_1} \partial x_{q_2}} - n_{ip}^{(2,2)} s^2 \hat{U}_p + \tilde{n}_{iq_1q_2}^{(2)} \frac{\partial^2 \hat{\Phi}}{\partial x_{q_1} \partial x_{q_2}} + \sum_{n \in \mathbb{N}_{\geq 0}^{\leq 1}} \varepsilon^{n+1} \left(\sum_{|q|=n+3} n_{ipq}^{(n+3)} \frac{\partial^{n+3} \hat{U}_p}{\partial x_q} + n_{ipq}^{(n+3,2)} s^2 \frac{\partial^{n+1} \hat{U}_p}{\partial x_q} \right) + \\
& + \sum_{n \in \mathbb{N}_{\geq 0}^{\leq 1}} \varepsilon^{n+1} \left(\sum_{|q|=n+3} \tilde{n}_{iq}^{(n+3)} \frac{\partial^{n+3} \hat{\Phi}}{\partial x_q} - \sum_{|q|=n+1} \tilde{n}_{iq}^{(n+3,2)} s^2 \frac{\partial^{n+1} \hat{\Phi}}{\partial x_q} \right) - \varepsilon^2 s^4 n_{ip}^{(4,4)} \hat{U}_p + \hat{b}_i + O(\varepsilon^3) = 0,
\end{aligned} \tag{24a}$$

$$\begin{aligned}
& w_{iq_1q_2}^{(2)} \frac{\partial^2 \hat{\Phi}}{\partial x_{q_1} \partial x_{q_2}} - \tilde{w}_{pq_1q_2}^{(2)} \frac{\partial^2 \hat{U}_p}{\partial x_{q_1} \partial x_{q_2}} + \sum_{n \in \mathbb{N}_{\geq 0}^{\leq 1}} \left(\varepsilon^{n+1} \sum_{|q|=n+3} w_q^{(n+3)} \frac{\partial^{n+3} \hat{\Phi}}{\partial x_q} - \sum_{|q|=n+3} \tilde{w}_{pq}^{(n+3)} \frac{\partial^{n+3} \hat{U}_p}{\partial x_q} \right) + \\
& + \sum_{n \in \mathbb{N}_{\geq 0}^{\leq 1}} \varepsilon^{n+1} \sum_{|q|=n+1} \tilde{w}_{pq}^{(n+3,2)} s^2 \frac{\partial^{n+1} \hat{U}_p}{\partial x_q} + \varepsilon^2 s^2 w_q^{(2,2)} \frac{\partial^2 \hat{\Phi}}{\partial x_q} - \hat{p}_e + O(\varepsilon^3) = 0,
\end{aligned} \tag{24b}$$

where the coefficients of the gradients are the known terms provided by the corresponding cell problems that can be found in the Section C of the Supplementary Material.

The truncation of Eqs. (24a)–(24b) must be performed with appropriate order to maintain the hyperbolicity of the governing equations. This aspect is important for characterizing thermodynamically consistent higher-order equivalent visco-electro-elastic continua, accordingly with Bacigalupo and Gambarotta (2021) and Filimonov (1996). Moreover, a formal resolution of Eqs. (24a)–(24b) is achieved through a perturbative procedure that is outlined in Section D of the Supplementary Material. If the retrieved system of equations is truncated at the zeroth order of ε , then the field equations reduce to those governing a first-order homogeneous continuum. Conversely, the recursive macro differential problems obtained at higher orders exhibit the same structure as those characterizing a first-order continuum, with the source terms accounting for the nonlocalities affecting Eqs. (24a)–(24b). It must be remarked that the identification of higher-order models can be realized by exploiting other techniques including variational-asymptotic, energy-based and computational approaches as well as other analytical techniques of identification (Bacca, Bigoni, Dal Corso, & Veber, 2013; Bacigalupo, Paggi, Dal Corso, & Bigoni, 2018; Barceló-Mercader, Codony, Mocci, & Arias, 2024; Del Toro et al., 2019; Filimonov, 1996; Smyshlyaev & Cherednichenko, 2000; Tran, Monchiet, & Bonnet, 2012).

The resolution of these equations with vanishing source terms is addressed in Section 3.1, where an asymptotic procedure is presented to characterize the dispersion properties of visco-electro-elastic materials for several approximation orders. In Section 3.2, an identification strategy is introduced, based on a generalized macro-homogeneity condition. Here, the effective constitutive and inertial properties are derived from the previously established down-scaling relations, enabling the full characterization of the equivalent continuum at the macroscale.

3.1. Asymptotic approximation of the frequency complex spectrum

In order to approximate the complex spectrum of a periodic visco-electro-elastic material, the spatial Fourier transform with a complex argument, F , is applied to Eqs. (24a)–(24b), which describe the free propagation of Bloch waves. The system is truncated at order ε^0 , which is assumed to correspond to the spectrum of the equivalent first-order homogenized material (Del Toro, De Bellis, Vasta & Bacigalupo, 2024; Filimonov, 1996). Specifically, the Fourier transform in space, applied to a generic function $\hat{g}(\mathbf{x}, s)$, defined as

$$F[\hat{g}(\mathbf{x}, s)] = \int_{\mathbb{R}^3} \hat{g}(\mathbf{x}, s) \exp(-i\mathbf{k} \cdot \mathbf{x}) d\mathbf{x} := \check{g}(\mathbf{k}, s), \quad \mathbf{k} \in \mathbb{C}^3, \tag{25}$$

remodels the Eqs. (24a)–(24b) truncated at ε^0 in terms of the following 2×2 block matrix as

$$\begin{pmatrix} n^{(2)} : \mathbf{k}^{\otimes 2} + n^{(2,2)} s^2 & \tilde{n}^{(2)} : \mathbf{k}^{\otimes 2} \\ \tilde{w}^{(2)} : \mathbf{k}^{\otimes 2} & w^{(2)} : \mathbf{k}^{\otimes 2} \end{pmatrix} \begin{pmatrix} \check{U}(\mathbf{k}, s) \\ \check{\Phi}(\mathbf{k}, s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{26}$$

where the symbol $:$ stands double contraction and $\mathbf{k}^{\otimes i} = \mathbf{k} \otimes \mathbf{k} \dots \otimes \mathbf{k}$. Moreover, the tensors involved in (26) are expressed as

$$\mathbf{n}^{(2)} = n_{ipq_1q_2}^{(2)} \mathbf{e}_{ipq_1q_2}, \quad \mathbf{n}^{(2,2)} = n_{ip}^{(2,2)} \mathbf{e}_{ip}, \quad \tilde{\mathbf{n}}^{(2)} = \tilde{n}_{iq_1q_2}^{(2)} \mathbf{e}_{iq_1q_2}, \quad \tilde{\mathbf{w}}^{(2)} = \tilde{w}_{pq_1q_2}^{(2)} \mathbf{e}_{pq_1q_2}, \quad \mathbf{w}^{(2)} = w_{q_1q_2}^{(2)} \mathbf{e}_{q_1q_2}. \quad (27)$$

Proceeding with the analysis of homogeneous plane waves recasts the system (26) into

$$\left\{ v^2 \begin{pmatrix} \mathbf{n}^{(2)} : \mathbf{z}^{\otimes 2} & \tilde{\mathbf{n}}^{(2)} : \mathbf{z}^{\otimes 2} \\ \tilde{\mathbf{w}}^{(2)} : \mathbf{z}^{\otimes 2} & \mathbf{w}^{(2)} : \mathbf{z}^{\otimes 2} \end{pmatrix} - \begin{pmatrix} \mathbf{n}^{(2,2)} \omega^2 & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} \right\} \begin{pmatrix} \check{\mathbf{U}} \\ \check{\Phi} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}, \quad (28)$$

where the Laplace variable, namely the complex frequency $s \in \mathbb{C}$, as well as the wave vector $\mathbf{k} \in \mathbb{C}^3$ are mapped into the angular frequency $\omega \in \mathbb{R}$ and into the wave number $v \in \mathbb{C}$ associated with the generic direction of propagation $\mathbf{z} \in \mathbb{R}^3$, respectively, by means of the relations $s = i\omega$ and $\mathbf{k} = v\mathbf{z}$. The system (28) can be suitably summarized as

$$(\mathbf{C}^{(0,0)} + v^2 \mathbf{C}^{(2,0)}) \check{\mathbf{r}} = \mathbf{0}, \quad (29)$$

whereas the linearization of (29) produces the following standard eigenproblem

$$(\mathbf{G} - v\mathbf{I}) \check{\mathbf{p}} = \mathbf{0}, \quad (30)$$

in terms of the eigenvalue v and the 8×1 eigenvector $\check{\mathbf{p}} = (v\check{\mathbf{U}} \quad v\check{\Phi} \quad \check{\mathbf{U}} \quad \check{\Phi})^T$. The reader can find the expression of the block matrix $\mathbf{G}$ in the Section E of the Supplementary material. The zeroes of the characteristic polynomial

$$\mathcal{V}_0^{hom}(v, \omega) = \det(\mathbf{G} - v\mathbf{I}), \quad (31)$$

provide the dispersion curves related to the first-order continuum, which can be derived either as the intersection of the hyper-surfaces $\text{Re}(\mathcal{V}_0^{hom}) = 0$ and $\text{Im}(\mathcal{V}_0^{hom}) = 0$ or by parameterizing the roots of (31) with respect to ω . On the other hand, to achieve an approximation at the second-order of the Floquet–Bloch spectrum concerning with a visco-electro-elastic material with a periodic microstructure, we have to truncate the Eqs. (24a)–(24b) at ε^2 leading to

$$\left\{ \begin{pmatrix} \mathbf{n}^{(2)} : \mathbf{k}^{\otimes 2} + \mathbf{n}^{(2,2)} s^2 & \tilde{\mathbf{n}}^{(2)} : \mathbf{k}^{\otimes 2} \\ \tilde{\mathbf{w}}^{(2)} : \mathbf{k}^{\otimes 2} & \mathbf{w}^{(2)} : \mathbf{k}^{\otimes 2} \end{pmatrix} + \varepsilon \begin{pmatrix} i(s^2 \mathbf{n}^{(3,2)} \cdot \mathbf{k} - \mathbf{n}^{(3)} : \cdot \mathbf{k}^{\otimes 3}) & -i(s^2 \tilde{\mathbf{n}}^{(3,2)} \cdot \mathbf{k} + \tilde{\mathbf{n}}^{(3)} : \cdot \mathbf{k}^{\otimes 3}) \\ i(s^2 \tilde{\mathbf{w}}^{(3,2)} \cdot \mathbf{k} + \tilde{\mathbf{w}}^{(3)} : \cdot \mathbf{k}^{\otimes 3}) & i\mathbf{w}^{(3)} : \cdot \mathbf{k}^{\otimes 3} \end{pmatrix} \right. \\ \left. + \varepsilon^2 \begin{pmatrix} -s^2 \mathbf{n}^{(4,2)} : \mathbf{k}^{\otimes 2} + \mathbf{n}^{(4)} : \cdot \mathbf{k}^{\otimes 4} - s^2 \mathbf{n}^{(4,4)} & -s^2 \tilde{\mathbf{n}}^{(4,2)} : \mathbf{k}^{\otimes 2} + \tilde{\mathbf{n}}^{(4)} : \cdot \mathbf{k}^{\otimes 4} \\ -(s^2 \tilde{\mathbf{w}}^{(4,2)} : \mathbf{k}^{\otimes 2} + \tilde{\mathbf{w}}^{(4)} : \cdot \mathbf{k}^{\otimes 4}) & \mathbf{w}^{(4)} : \cdot \mathbf{k}^{\otimes 4} - s^2 \mathbf{w}^{(2,2)} : \mathbf{k}^{\otimes 2} \end{pmatrix} \right\} \begin{pmatrix} \check{\mathbf{U}}(\mathbf{k}, s) \\ \check{\Phi}(\mathbf{k}, s) \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}, \quad (32)$$

where the symbols $\cdot$ and $::$ represent the third and the fourth order contraction. The tensors involved in (32) are defined as

$$\begin{aligned} \mathbf{n}^{(3)} &= n_{ipq_1q_2q_3}^{(3)} \mathbf{e}_{ipq_1q_2q_3}, & \tilde{\mathbf{n}}^{(3)} &= \tilde{n}_{iq_1q_2q_3}^{(3)} \mathbf{e}_{iq_1q_2q_3}, \\ \mathbf{w}^{(3)} &= w_{q_1q_2q_3}^{(3)} \mathbf{e}_{q_1q_2q_3}, & \tilde{\mathbf{w}}^{(3)} &= \tilde{w}_{pq_1q_2q_3}^{(3)} \mathbf{e}_{pq_1q_2q_3}, \\ \mathbf{n}^{(3,2)} &= n_{ipq_1}^{(3,2)} \mathbf{e}_{ipq_1}, & \tilde{\mathbf{n}}^{(3,2)} &= \tilde{n}_{iq_1}^{(3,2)} \mathbf{e}_{iq_1}, \\ \mathbf{w}^{(2,2)} &= w_{q_1q_2}^{(2,2)} \mathbf{e}_{q_1q_2}, & \tilde{\mathbf{w}}^{(3,2)} &= \tilde{w}_{pq_1}^{(3,2)} \mathbf{e}_{pq_1}, \\ \mathbf{n}^{(4)} &= n_{ipq_1q_2q_3q_4}^{(4)} \mathbf{e}_{ipq_1q_2q_3q_4}, & \tilde{\mathbf{n}}^{(4)} &= \tilde{n}_{iq_1q_2q_3q_4}^{(4)} \mathbf{e}_{iq_1q_2q_3q_4}, \\ \mathbf{w}^{(4)} &= w_{q_1q_2q_3q_4}^{(4)} \mathbf{e}_{q_1q_2q_3q_4}, & \tilde{\mathbf{w}}^{(4)} &= \tilde{w}_{pq_1q_2q_3q_4}^{(4)} \mathbf{e}_{pq_1q_2q_3q_4}, \\ \mathbf{n}^{(4,2)} &= n_{ipq_1q_2}^{(4,2)} \mathbf{e}_{ipq_1q_2}, & \tilde{\mathbf{n}}^{(4,2)} &= \tilde{n}_{iq_1q_2}^{(4,2)} \mathbf{e}_{iq_1q_2}, \\ \mathbf{n}^{(4,4)} &= n_{ip}^{(4,4)} \mathbf{e}_{ip}, & \tilde{\mathbf{w}}^{(4,2)} &= \tilde{w}_{pq_1q_2}^{(3,4)} \mathbf{e}_{pq_1q_2}. \end{aligned} \quad (33)$$

If we deal with homogeneous plane waves and we analyze the spatial damping then (32) rephrases as

$$(\mathbf{C}^{(0)} + v\mathbf{C}^{(1)} + v^2\mathbf{C}^{(2)} + v^3\mathbf{C}^{(3)} + v^4\mathbf{C}^{(4)}) \check{\mathbf{r}} = \mathbf{0}, \quad (34)$$

equipped with

$$\mathbf{C}^{(0)} = \mathbf{C}^{(0,0)} + \varepsilon^2 \mathbf{C}^{(0,2)}, \quad (35a)$$

$$\mathbf{C}^{(1)} = \varepsilon \mathbf{C}^{(1,1)}, \quad (35b)$$

$$\mathbf{C}^{(2)} = \mathbf{C}^{(2,0)} + \varepsilon^2 \mathbf{C}^{(2,2)}, \quad (35c)$$

$$\mathbf{C}^{(3)} = \varepsilon \mathbf{C}^{(3,1)}, \quad (35d)$$

$$\mathbf{C}^{(4)} = \varepsilon^2 \mathbf{C}^{(4,2)}, \quad (35e)$$

where the matrices in (34) are reported on the Section E of the Supplementary Material and $\check{\mathbf{r}} = (\check{\mathbf{U}} \quad \check{\Phi})^T$. By linearizing (34), we obtain the eigenproblem

$$(\mathbf{N} - \varphi \mathbf{I}) \check{\mathbf{p}} = \mathbf{0}, \quad (36)$$

which depends on the eigenvalue $\varphi = \varepsilon^2 v$ and the 16×1 eigenvector $\check{\mathbf{p}} = (v^3 \check{\mathbf{U}} \quad v^3 \check{\Phi} \quad \dots \quad \check{\mathbf{U}} \quad \check{\Phi})^T$. The reader can find the matrix $\mathbf{N}$ in the Section E of the Supplementary Material. As previously pointed out, the resolution of the characteristic polynomial associated

to the eigenproblem (36) as

$$\mathcal{V}_2^{hom}(\varphi, \omega) = \det(\mathbf{N} - \varphi \mathbf{I}), \quad (37)$$

discloses the dispersion curves of a second-order continuum, which can be also provided by intersection of the hyper-surfaces $\Re e(\mathcal{V}_2^{hom}) = 0$ and $\Im m(\mathcal{V}_2^{hom}) = 0$. There is point in emphasizing that the zeroth-order and second-order approximations in ε of the frequency complex spectra stemmed from the asymptotic homogenization scheme are valid within the low-frequency and long-wave regime. Additionally, as the approximation order increases, the results improve considerably, leading to better spectrum approximations and extending their applicability to the short-wave regime but at low frequency limit.

3.2. Identification of the first-order continuum via energetic method

An energetic method based on a generalized macro-homogeneity condition in the Laplace space is herein presented to identify the overall visco-electro-elastic moduli associated with a first-order homogenized continuum.

Within the cluster $\mathcal{P}$ of periodic cells, which can be determined for instance by the periodicity of the source terms in the micromechanical governing equations, we account for the energy-like functional in the Laplace domain with respect to the transformed energy-like density $\hat{\psi}_m(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, s)$, which is defined as the sum of the transformed kinetic energy density and the transformed electric enthalpy density at the microscale as

$$\begin{aligned} \hat{\psi}_m &= \int_{\mathcal{P}} \hat{\psi}_m(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, s) d\mathbf{x} = \\ &= \int_{\mathcal{P}} \left(\frac{1}{2} \rho^m s^2 \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} - \frac{1}{2} \nabla \hat{\mathbf{u}} : (\hat{\mathbb{C}}^m \nabla \hat{\mathbf{u}}) - \nabla \hat{\mathbf{u}} : (\hat{\mathbb{C}}^m \nabla \hat{\phi}) + \frac{1}{2} \nabla \hat{\phi} \cdot (\hat{\beta}^m \nabla \hat{\phi}) + \hat{\mathbf{b}} \cdot \hat{\mathbf{u}} - \hat{\rho}_e \hat{\phi} \right) d\mathbf{x}. \end{aligned} \quad (38)$$

We have to observe that the transformed energy-like density $\hat{\psi}_m$ is subject to the translation variable $\zeta \in Q$, being as $\hat{\psi}_m^\zeta$. Therefore, the transformed energy-like functional $\hat{\psi}_m^\zeta$ is written as

$$\hat{\psi}_m^\zeta = \int_{\mathcal{P}} \hat{\psi}_m(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon} + \zeta, s) d\mathbf{x} = \int_{\mathcal{P}} \hat{\psi}_m^\zeta(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, s) d\mathbf{x}, \quad (39)$$

since the relation $\hat{\psi}_m(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon} + \zeta, s) = \hat{\psi}_m^\zeta(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}, s)$ holds and affects all the transformed micro constitutive tensors and the inertial term, appearing in Eq. (38). Hence, the transformed energy-like functional $\hat{\psi}_m^\zeta$ has to be averaged with respect to all the realizations in order to eliminate its ζ -dependence and so the generalized transformed mean energy-like functional at the microscale is introduced as

$$\hat{\psi}_m \doteq \langle \hat{\psi}_m^\zeta \rangle = \int_{\mathcal{P}} \left\langle \frac{1}{2} \rho^m s^2 \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} - \frac{1}{2} \nabla \hat{\mathbf{u}} : (\hat{\mathbb{C}}^m \nabla \hat{\mathbf{u}}) - \nabla \hat{\mathbf{u}} : (\hat{\mathbb{C}}^m \nabla \hat{\phi}) + \frac{1}{2} \nabla \hat{\phi} \cdot (\hat{\beta}^m \nabla \hat{\phi}) + \hat{\mathbf{b}} \cdot \hat{\mathbf{u}} - \hat{\rho}_e \hat{\phi} \right\rangle d\mathbf{x}. \quad (40)$$

Replacing the down-scaling relations (22a)–(22b) into the generalized transformed mean energy-like functional at the microscale (40) truncated at the order zero of ε yields

$$\begin{aligned} \hat{\psi}_m^0 &= \frac{1}{2} \langle \rho^m \rangle s^2 \int_{\mathcal{P}} \hat{\mathbf{U}}_i \hat{\mathbf{U}}_i d\mathbf{x} + \int_{\mathcal{P}} \hat{\mathbf{b}}_i \hat{\mathbf{U}}_i d\mathbf{x} - \int_{\mathcal{P}} \hat{\rho}_e \hat{\Phi} d\mathbf{x} + \\ &- \frac{1}{2} \left[\langle \hat{\mathbb{C}}_{ijhk}^{(1)} B_{ijpq_1}^{(1)} B_{hksr_1}^{(1)} \rangle + \langle 2\hat{\mathbb{C}}_{ijh}^{(1)} B_{ijpq_1}^{(1)} R_{sr_1h}^{(1)} \rangle - \langle \hat{\beta}_{ij}^{(1)} R_{sr_1j}^{(1)} R_{pq_1i}^{(1)} \rangle \right] \int_{\mathcal{P}} \frac{\partial \hat{\mathbf{U}}_p}{\partial x_{q_1}} \frac{\partial \hat{\mathbf{U}}_s}{\partial x_{r_1}} d\mathbf{x} + \\ &- \left[\langle \hat{\mathbb{C}}_{ijhk}^{(1)} B_{ijpq_1}^{(1)} T_{hr_1k}^{(1)} \rangle \langle \hat{\mathbb{C}}_{ijh}^{(1)} B_{ijpq_1}^{(1)} P_{hr_1}^{(1)} \rangle + \langle \hat{\mathbb{C}}_{ijh}^{(1)} T_{ir_1j}^{(1)} R_{pq_1h}^{(1)} \rangle - \langle \hat{\beta}_{ij}^{(1)} R_{pq_1j}^{(1)} P_{ir_1}^{(1)} \rangle \right] \int_{\mathcal{P}} \frac{\partial \hat{\mathbf{U}}_p}{\partial x_{q_1}} \frac{\partial \hat{\Phi}}{\partial x_{r_1}} d\mathbf{x} + \\ &- \frac{1}{2} \left[\langle \hat{\beta}_{ij}^{(1)} P_{jq_1}^{(1)} P_{ir_1}^{(1)} \rangle - \langle \hat{\mathbb{C}}_{ijhk}^{(1)} T_{iq_1j}^{(1)} T_{hr_1k}^{(1)} \rangle - 2\langle \hat{\mathbb{C}}_{ijh}^{(1)} T_{iq_1j}^{(1)} P_{hr_1}^{(1)} \rangle \right] \int_{\mathcal{P}} \frac{\partial \hat{\Phi}}{\partial x_{q_1}} \frac{\partial \hat{\Phi}}{\partial x_{r_1}} d\mathbf{x}, \end{aligned} \quad (41)$$

where the localization tensors are introduced as $B_{ijpq_1}^{(1)} = (\delta_{ip} \delta_{jq_1} + N_{ipq_1,j}^{(1)})$, $B_{hksr_1}^{(1)} = (\delta_{hs} \delta_{kr_1} + N_{hksr_1,k}^{(1)})$, $P_{hr_1}^{(1)} = (\delta_{hr_1} + W_{r_1,h}^{(1)})$, $T_{hr_1k}^{(1)} = \tilde{N}_{hr_1,k}^{(1)}$, $R_{sr_1h}^{(1)} = \tilde{W}_{sr_1,h}^{(1)}$ and the apex in the symbol $\hat{\psi}_m^0$ means that only the coefficients of ε^0 are kept. The generalized transformed mean energy-like functional at the macroscale is provided as

$$\begin{aligned} \hat{\psi}_M &= \int_{\mathcal{P}} \hat{\psi}_M(\mathbf{x}, s) d\mathbf{x} = \frac{1}{2} \rho s^2 \int_{\mathcal{P}} \hat{\mathbf{U}}_i \hat{\mathbf{U}}_i d\mathbf{x} + \int_{\mathcal{P}} \hat{\mathbf{b}}_i \hat{\mathbf{U}}_i d\mathbf{x} - \int_{\mathcal{P}} \hat{\rho}_e \hat{\Phi} d\mathbf{x} + \\ &- \frac{1}{2} \hat{\mathbb{C}}_{pq_1sr_1} \int_{\mathcal{P}} \frac{\partial \hat{\mathbf{U}}_p}{\partial x_{q_1}} \frac{\partial \hat{\mathbf{U}}_s}{\partial x_{r_1}} d\mathbf{x} - \hat{\mathbb{C}}_{pq_1r_1} \int_{\mathcal{P}} \frac{\partial \hat{\mathbf{U}}_p}{\partial x_{q_1}} \frac{\partial \hat{\Phi}}{\partial x_{r_1}} d\mathbf{x} + \frac{1}{2} \hat{\beta}_{q_1r_1} \int_{\mathcal{P}} \frac{\partial \hat{\Phi}}{\partial x_{q_1}} \frac{\partial \hat{\Phi}}{\partial x_{r_1}} d\mathbf{x}. \end{aligned} \quad (42)$$

The imposition of the generalized macro-homogeneity condition in the Laplace space, $\hat{\psi}_m^0 \doteq \hat{\psi}_M$, enables to identify the components of the transformed constitutive tensors and the inertial term at the macroscale related to the homogenized continuum at the first-order as

$$\rho = \langle \rho^m \rangle, \quad (43a)$$

$$\hat{\mathbb{C}}_{pq_1sr_1} = \langle \hat{\mathbb{C}}_{ijhk}^{(1)} B_{ijpq_1}^{(1)} B_{hksr_1}^{(1)} \rangle + \langle 2\hat{\mathbb{C}}_{ijh}^{(1)} B_{ijpq_1}^{(1)} R_{sr_1h}^{(1)} \rangle - \langle \hat{\beta}_{ij}^{(1)} R_{pq_1i}^{(1)} R_{sr_1j}^{(1)} \rangle, \quad (43b)$$

$$\hat{e}_{pq_1r_1} = \langle \hat{C}_{ijhk}^m B_{ijpq_1}^{(1)} T_{hr_1k}^{(1)} \rangle + \langle \hat{e}_{ijh}^m B_{ijpq_1}^{(1)} P_{hr_1}^{(1)} \rangle + \langle \hat{e}_{ijh}^m T_{ir_1j}^{(1)} R_{pq_1h}^{(1)} \rangle - \langle \hat{\beta}_{ij}^m R_{pq_1j}^{(1)} P_{ir_1}^{(1)} \rangle, \quad (43c)$$

$$\hat{\beta}_{q_1r_1} = \langle \hat{\beta}_{ij}^m P_{jq_1}^{(1)} P_{ir_1}^{(1)} \rangle - \langle \hat{C}_{ijhk}^m T_{iq_1j}^{(1)} T_{hr_1k}^{(1)} \rangle - 2 \langle \hat{e}_{ijh}^m T_{iq_1j}^{(1)} P_{hr_1}^{(1)} \rangle, \quad (43d)$$

in terms of localization tensors that depend on perturbation functions associated with the micromechanical properties of the periodic material. It must be observed that dispersion curves deriving from the identification of the first-order continuum (Cauchy type) coincide with those assessed by the first-order continuum model in Section 3.1 via a proper truncation at the order zero of the eigenproblem at the macroscale and after appropriate algebraic manipulations of the tensors, see for details [De Bellis et al. \(2019\)](#). The asymptotic homogenization method when combined with a properly defined energy-based identification procedure is a remarkable tool that allows the approximations of complex spectra in the long-wave and low-frequency regime, which is suitable for periodic materials having complex microstructures. Moreover, the asymptotic homogenization method for the identification of gradient-type continua, such as the first-order one, is significant for characterizing the dynamic response of heterogeneous materials by means of a synthetic yet precise description, when the principle of scale separation is ensured.

4. Continualization scheme: Identification of equivalent continua models and spectral approximation in laminates

A continualization scheme, based on a kernel developed using Padé approximants, is here discussed to provide the governing equations of a multi-field generalized gradient-type non-local visco-electro-elastic continuum model capable to synthetically and accurately characterize the dispersive properties of the laminate materials.

Assuming a system of equally spaced interfaces stacked along the x_δ -direction, the relation (10) may be written as the difference equation

$$\mathbf{r}_{l+1} = \mathbf{T}(s) \mathbf{r}_l, \quad (44)$$

where l and $l+1$ are two successive interfaces and $\mathbf{r}_l$ is the vector that collects the unknown nodal fields, namely the displacement, the electrical potential, the electric displacement and the stress. Hence, we can employ the two-sided $\mathcal{Z}$ -transform to Eq. (44), which for a discrete function g_i is defined as $\mathcal{Z}[g_i] = \sum_{i \in \mathbb{Z}} g_i(h) z^{-i}$, $z \in \mathbb{C}$, with $\mathcal{Z}[g_{i+1}] = z \mathcal{Z}[g_i]$. Re-mapping the number $z \in \mathbb{C}$ as the complex exponential $z = \exp(ik_\delta S)$, $k_\delta \in \mathbb{C}$, and by imposing the equivalence between the $\mathcal{Z}$ -transform of the fields vector $\mathbf{r}_l$ and the $\mathcal{F}$ -transform of the vector $\hat{\mathbf{R}}(x_\delta, s)$ at the continuum as

$$\hat{\mathbf{r}}(k_\delta, s) = \mathcal{Z}[\mathbf{r}_l]|_{z=\exp(ik_\delta S)} \doteq \mathcal{F}[\hat{\mathbf{R}}(x_\delta, s)] = \hat{\mathbf{R}}(k_\delta, s), \quad (45)$$

we can establish the eigenproblem

$$(\mathbf{T}(s) - \exp(ik_\delta S) \mathbf{I}) \hat{\mathbf{R}}(k_\delta, s) = \mathbf{0} \quad (46)$$

that depends on the wavenumber k_δ and the angular frequency s . It is possible to derive the integro-differential equation that characterizes a periodic visco-electro-elastic laminate as

$$\mathcal{L}^{-1}[\mathcal{F}^{-1}[\exp(ik_\delta S) \mathcal{F}[\mathcal{L}[\mathbf{R}(x_\delta, t)]]] - \mathcal{L}^{-1}[\mathbf{T}(s)] * \mathbf{R}(x_\delta, t)] = \mathbf{0}, \quad (47)$$

where $\mathcal{F}^{-1}$ is the inverse-spatial Fourier transform with complex argument, $\mathcal{L}^{-1}$ is the inverse-time Laplace transform and, in such a case, the convolution symbol in time is also considered as a simple contraction of index. Hereafter, the kernel $\mathcal{K}(k_\delta) := \exp(ik_\delta S)$ is recast with the Padé approximant of degree $[\mathcal{N}, \mathcal{N}]$, with $\mathcal{N} \in \mathbb{N}$, as

$$K(k_\alpha) := \frac{p(k_\alpha)}{q(k_\alpha)} = \frac{\sum_{n=0}^{\mathcal{N}} a_n (ik_\alpha S)^n}{\sum_{n=0}^{\mathcal{N}} b_n (ik_\alpha S)^n},$$

where the polynomials $p(k_\alpha)$ and $q(k_\alpha)$ have the same degree $\mathcal{N}$ and their coefficients are defined as

$$a_n = \frac{(2\mathcal{N} - r)! \mathcal{N}! (-1)^r}{(2\mathcal{N})! r! (\mathcal{N} - r)!}, \quad b_n = \frac{(2\mathcal{N} - r)! \mathcal{N}!}{(2\mathcal{N})! r! (\mathcal{N} - r)!}$$

with $r = 0 \dots \mathcal{N}$ (see for details [Cuyt, 2006](#)) yielding the accomplishment of the governing equation of a multi-field generalized gradient-type non-local continuum at order $\mathcal{N}$ in a Stroh-like formalism as

$$\sum_{n \in \mathcal{N}} S^{2n-1} a_{2n-1} \frac{\partial^{2n-1} \mathbf{R}(x_\alpha, t)}{\partial x_\alpha^{2n-1}} = \sum_{n \in \mathcal{N}} S^{2n-1} b_{2n-1} \mathcal{F}_t^{-1}[\tilde{\mathbf{T}}(\omega)] * \frac{\partial^{2n-1} \mathbf{R}(x_\alpha, t)}{\partial x_\alpha^{2n-1}}, \quad (48)$$

with $\tilde{\mathbf{T}}(\omega) = \mathbf{T}(\omega) - \mathbf{I}$, $s = i\omega$, and $\mathcal{N}$ is a proper bounded subset of $\mathbb{N}$. The frequency complex spectrum of a periodic visco-electro-elastic laminate is determined through the solutions of the following characteristic polynomial in the wavenumber and angular frequency domain

$$\mathcal{V}_p^{hom}(k_\alpha, \omega) = \det \left(\sum_{n \in \mathcal{N}} b_{2n-1} (ik_\alpha S)^{2n-1} \tilde{\mathbf{T}}(\omega) - \sum_{n \in \mathcal{N}} a_{2n-1} (ik_\alpha S)^{2n-1} \mathbf{I} \right), \quad (49)$$

where the complex-valued wave number k_α can be determined in terms of the real-valued angular frequency ω and through a suitable selection of physically feasible solutions according to the procedure proposed in [Del Toro, De Bellis, and Bacigalupo \(2023\)](#). It is important to remark that the resolution of the polynomial (49) is capable of approximating the whole spectrum in a convergent

way. It is observed that the frequency complex spectrum characterizes the propagation of spatially damped dispersive Bloch waves. Leveraging on an alternative fashion, the dispersion curves are delivered by intersection of the hyper-surfaces $\mathcal{R}e(\mathcal{V}_p^{hom}) = 0$ and $\mathcal{I}m(\mathcal{V}_p^{hom}) = 0$.

Additionally, if the matrix $\bar{T}(\omega)$ in (49) is rewritten as a properly truncated Taylor series with respect to ω , the governing equation of a multi-field generalized gradient-type non-local continuum model at order $\mathcal{N}$ for low-frequency, in the physical and in the time domain, outlines as

$$\left(\sum_{n \in \mathcal{N}} b_{2n-1} S^{2n-1} \sum_{n \in \mathcal{N}} \frac{\partial^{2n-1} \bar{T}(\omega)}{\partial \omega^{2n-1}} \right) \bigg|_{\omega=0} \frac{1}{t^{2n-1} (2n-1)!} \frac{\partial^{4n-2} \mathbf{R}(x_\alpha, t)}{\partial x_\alpha^{2n-1} \partial t^{2n-1}} - \sum_{n \in \mathcal{N}} a_{2n-1} S^{2n-1} \frac{\partial^{2n-1} \mathbf{R}(x_\alpha, t)}{\partial x_\alpha^{2n-1}} = \mathbf{0}. \quad (50)$$

Then, a low-frequency approximation of the complex spectrum can be achieved through the resolution of the following characteristic polynomial within the wavenumber and angular frequency domain

$$\mathcal{V}_p^{hom(0)}(k_\alpha, \omega) = \det \left(\sum_{n \in \mathcal{N}} b_{2n-1} (ik_\alpha S)^{2n-1} \sum_{n \in \mathcal{N}} \frac{\partial^{2n-1} \bar{T}(\omega)}{\partial \omega^{2n-1}} \right) \bigg|_{\omega=0} \frac{\omega^{2n-1}}{(2n-1)!} - \sum_{n \in \mathcal{N}} a_{2n-1} (ik_\alpha S)^{2n-1} \mathbf{I} \right). \quad (51)$$

The solution of the polynomial (51) can be approached considering the wave number k_α as a complex-valued eigenvalue parameterized on the real-valued angular frequency ω , or alternatively, by treating the angular frequency as the eigenvalue parameterized on the real-valued wave number. Therefore, two cases can be distinguished: the spectrum derived by parameterizing on ω describes the propagation of spatially damped dispersive Bloch waves at low frequencies; the spectrum obtained by parameterizing on the real-valued wave number characterizes the propagation modes of low-frequency Bloch waves traveling through the material. By relying on an alternative analytical strategy, a low-frequency approximation of the complex spectrum can be also determined by intersection of the hyper-surfaces $\mathcal{R}e(\mathcal{V}_p^{hom(0)}) = 0$ and $\mathcal{I}m(\mathcal{V}_p^{hom(0)}) = 0$. It is important to emphasize that the spectrum derived from the polynomial (51) represents a low-frequency approximation of the band structure of the laminate material. Similar approximations to the ones discussed could be achieved using the asymptotic homogenization method as illustrated in 3.1.

An envelope-type continualized model for short-wave approximation can be achieved by approximating the kernel $\mathcal{K}(k_\alpha)$ as a truncated Taylor series or Padé approximants centered in $k_\alpha d = \pi$ with d the dimension of the periodic cell. In such a case, it has been shown in Del Toro et al. (2023) that the spectrum derived with the envelope-type continualization scheme converges completely to the heterogeneous one along a fixed direction.

In summary, the continualization scheme herein proposed permits a convergent approximation of the spectrum regarding laminate materials. Specifically, as the order of approximation increases, the accuracy of the results improves significantly. It is worth emphasizing that, compared to the schemes consisting of approximating the integral kernel as truncated Taylor series that are also convergent, the Padé approximants result to have a faster rate of convergence, providing very good results.

5. Illustrative examples

In this section, two examples are discussed by using the asymptotic homogenization and continualization schemes highlighted in Section 3 and the transfer matrix method exposed in section (2.1). Specifically, in section (4.1), we focus on visco-electro-elastic laminate periodic material, whose spectrum is analytically determined with the transfer matrix method (2.1), and compared with the frequency complex spectra obtained with the homogenization and continualization schemes in order to test the reliability of the methods. In section (4.2), we concentrate on the numerical micro-mechanical solution of a biphasic fiber-reinforced composite, consisting of a periodic arrangement of fibers embedded within a soft matrix. This configuration serves as a representative example to demonstrate the potential of the asymptotic homogenization model for the constitutive characterization of composite materials with complex microstructures.

5.1. Visco-electro-elastic biphasic laminate

In the first example, the in-plane laminate is made of the periodic cell, having size ϵ , which is characterized by two phases that are the Lead Zirconate Titanate (PZT-5H) and the epoxy, both considered as visco-electro-elastic materials (Li & Dunn, 2001; Yang et al., 2005). Within the mathematical framework discussed in the Section 2.1, the layers are supposed to be arranged along the x_2 -direction with the poling occurring along the same direction. Moreover, the tensors defined in (1a)–(1b) are written in terms of the truncated Prony series as

$$\mathbf{g}(\mathbf{x}, t) = \mathbf{g}^{(\infty)}(\mathbf{x}) \left(1 + \gamma \exp \left(-\frac{t}{\tau_r(\mathbf{x})} \right) \right), \quad (52)$$

where $\mathbf{g}^{(\infty)}$ stands for the longterm material response, γ represents the viscosity ratio of the relaxation function and $\tau_r(\mathbf{x})$ is the relaxation time at the microscale. Performing the two-sided Laplace transform to (52) results in

$$\hat{\mathbf{h}} = s \hat{\mathbf{g}} = \mathbf{g}^{(\infty)} \frac{\tau_r s (1 + \gamma) + 1}{\tau_r s + 1}. \quad (53)$$

Furthermore, the dimensionless parameters and variables are recalled as $r_c = \frac{\hat{G}_{2222}^{(2)}}{\hat{G}_{2222}^{(1)}}$, $r_\rho = \frac{\rho^{(2)}}{\rho^{(1)}}$, $\frac{s^{(1)}}{s^{(2)}} = 1$, $r_e = \frac{\hat{p}_{222}^{(2)}}{\hat{p}_{222}^{(1)}}$, $r_\beta = \frac{\hat{\eta}_{22}^{(2)}}{\hat{\eta}_{22}^{(1)}}$, $\bar{s} = \frac{s}{s_{ref}}$, $\bar{\tau}_r^{(\zeta)} = \tau_r^{(\zeta)} s_{ref}$, ($\zeta = 1, 2$), $\bar{x}_2 = \frac{x_2}{\epsilon}$, $\bar{u}_2 = \frac{\hat{u}_2}{\epsilon}$, $\bar{\Phi} = \frac{\hat{\Phi}}{\Phi_{ref}}$ and $\bar{k}_2 = k_2 \epsilon$ with $s_{ref} = \frac{1}{\epsilon} \sqrt{\frac{\hat{G}_{2222}^{(1)}}{\rho^{(1)}}}$, $\Phi_{ref} = \epsilon \frac{\hat{G}_{2222}^{(1)}}{\hat{p}_{222}^{(1)}}$, where the apex (1) refers to the piezoelectric phase, (2) to the viscoelastic phase, $s^{(1)}$, $s^{(2)}$ are the thicknesses of the layers, and γ is the same for both phases.

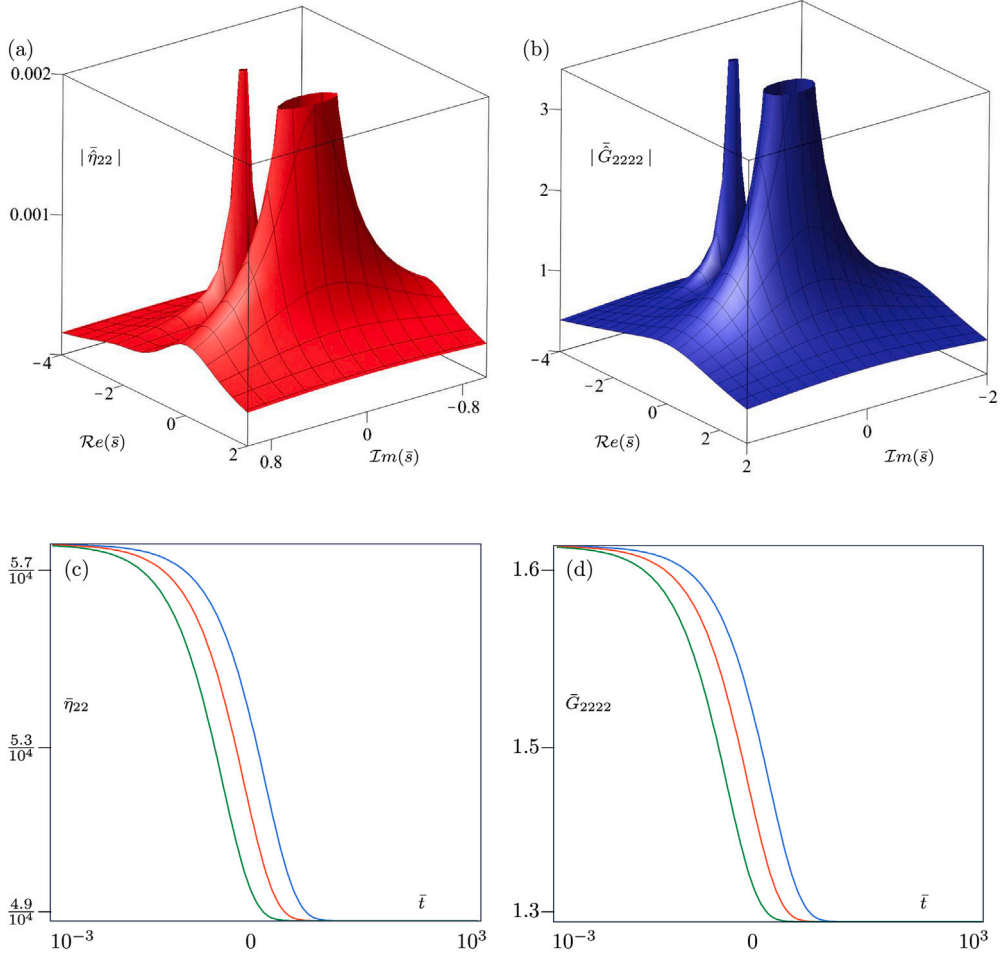


Fig. 1. Magnitude of the dimensionless transformed constitutive tensor components vs. the real part and the imaginary part of the dimensionless complex angular frequency $\bar{s}$: $|\tilde{\eta}_{22}|$ (a) and $|\tilde{G}_{2222}|$ (b). Dimensionless constitutive tensor components vs. the dimensionless time $\bar{t}$ by varying the dimensionless relaxation time as $\bar{\tau}_r^{(\zeta)} = 1$ (blue), $\bar{\tau}_r^{(\zeta)} = \frac{1}{2}$ (red) and $\bar{\tau}_r^{(\zeta)} = \frac{1}{4}$ (green): $\tilde{\eta}_{22}$ (c) and $\tilde{G}_{2222}$ (d) for $r_c = 2$, $r_\rho = 2$, $r_e = \frac{1}{100}$, $r_\beta = \frac{1}{1000}$ and $\gamma = \frac{1}{5}$.

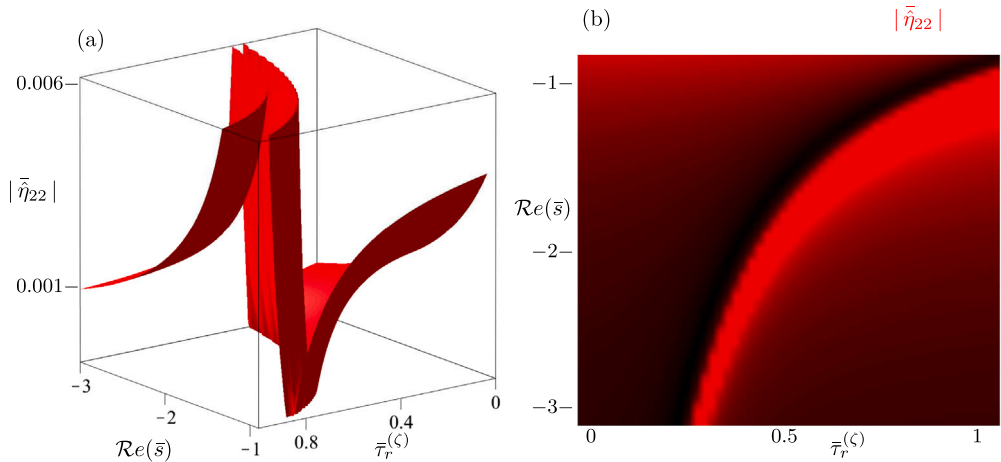


Fig. 2. Magnitude of the dimensionless transformed constitutive tensor component $|\tilde{\eta}_{22}|$ by varying the real part of the dimensionless complex angular frequency and the dimensionless relaxation time: 3D structure (a) and 2D structure (b) for $r_c = 2$, $r_\rho = 2$, $r_e = \frac{1}{100}$, $r_\beta = \frac{1}{1000}$, $\gamma = \frac{1}{2}$.

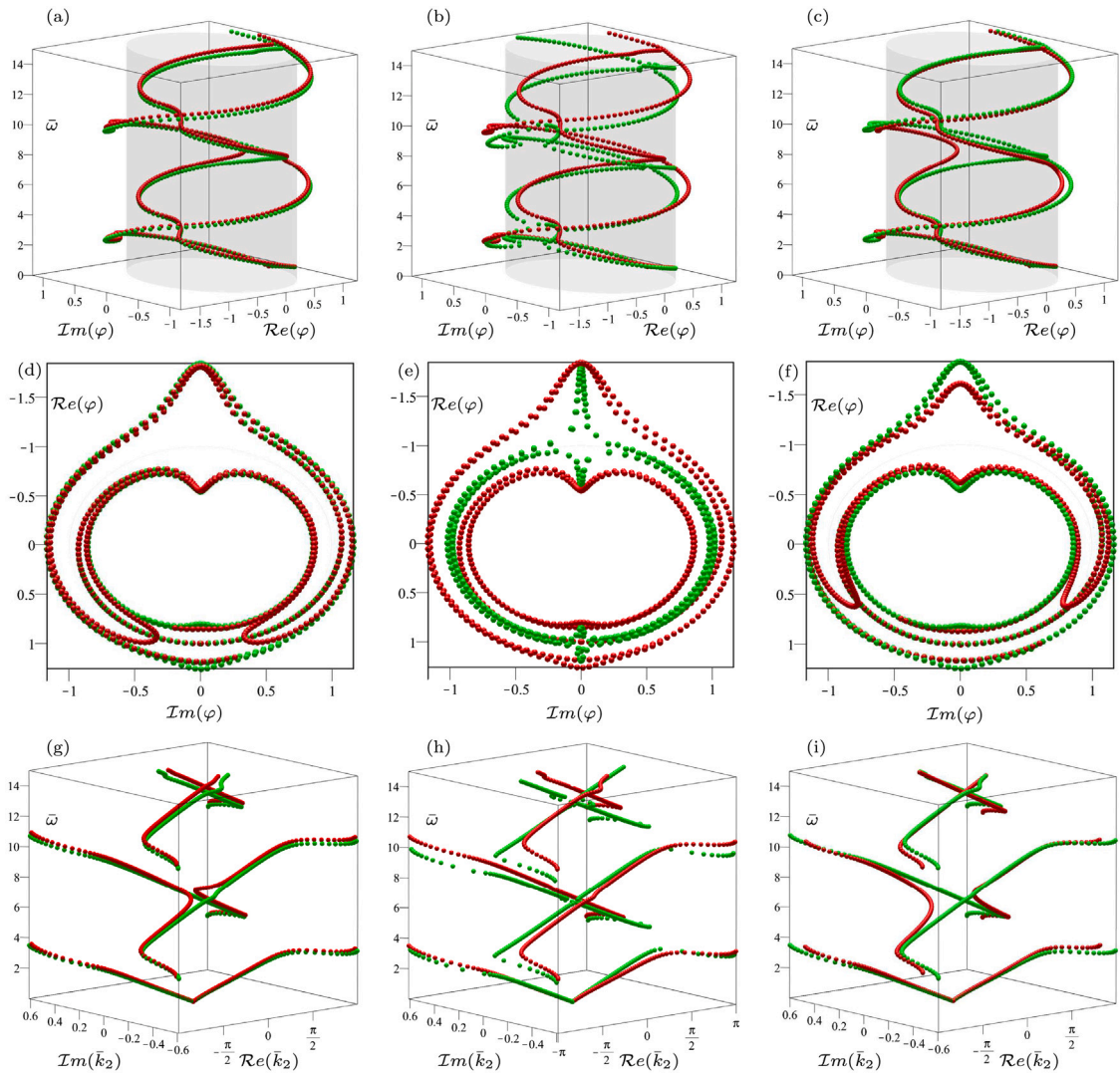


Fig. 3. Real part and imaginary part of the Floquet multipliers φ versus the dimensionless frequency $\bar{\omega}$ by varying r_e as $r_e = \frac{1}{10}$ (red), $r_e = \frac{1}{50}$ (green)(a), γ as $\gamma = \frac{1}{5}$ (red), $\gamma = \frac{1}{40}$ (green)(b) and r_β for $r_\beta = 2$ (red) and $r_\beta = \frac{1}{1000}$ (green)(c). Two-dimensional views of the real part and the imaginary part of the Floquet multipliers φ in the complex plane by changing r_e (d), γ (e), and r_β (f) for fixed not-null constitutive parameters $r_c = 2$, $r_\rho = 2$ and $\bar{\tau}_r^{(c)} = \frac{1}{2}$. Complex three-dimensional spectra and band structure obtained by changing r_e (g), γ (h), and r_β (i).

In the following, we analyze the behavior of some dimensionless transformed tensor components obtained with the asymptotic homogenization method outlined in (3), then we study the influence of the piezoelectric and the dielectric effect by deriving analytically the frequency complex spectra related to the visco-electro-elastic laminate periodic material via the transfer matrix method proposed in (2.1) and, finally, in order to verify the reliability of the schemes described in (3) and (4), we compare the frequency complex spectra obtained by means of the homogenization and continualization with those derived analytically via the transfer matrix method.

In Fig. 1, there is a representation of the magnitude of the dimensionless transformed constitutive tensor components $|\tilde{\eta}_{22}| = \left| s_{ref} \frac{\hat{G}_{2222}^{(2)} \hat{\rho}_{22}^{(2)}}{(\hat{\rho}_{22}^{(2)})^2} \right|$ (Fig. 1-(a)) and $|\tilde{G}_{2222}| = \left| s_{ref} \frac{\hat{G}_{2222}^{(2)}}{\hat{\rho}^{(2)}} \right|$ (Fig. 1-(b)) with respect to the real and imaginary parts of the complex frequency $\bar{s} = \mathcal{Re}(\bar{s}) + i\mathcal{Im}(\bar{s})$, where their poles can be distinguished.

The components of the transformed constitutive tensors $\hat{C}_{ijhk} = s\hat{G}_{ijhk}$ and $\hat{\beta}_{ij} = s\hat{\eta}_{ij}$ are related to the known coefficients in (24a)–(24b) by means of the relations $n_{ipq_1q_2}^{(2)} = \frac{1}{2}(\hat{C}_{pq_1iq_2} + \hat{C}_{pq_2iq_1})$ and $w_{q_1q_2}^{(2)} = \hat{\beta}_{q_1q_2}$ (Fantoni et al., 2017). The behavior of the dimensionless constitutive tensor components $\tilde{\eta}_{22} = \mathcal{L}^{-1}(\tilde{\eta}_{22})$ is analyzed in Fig. 1-(c) and that of $\tilde{G}_{2222} = \mathcal{L}^{-1}(\tilde{G}_{2222})$ in Fig. 1-(d) with respect to dimensionless time $\bar{t} = t s_{ref}$ in the logarithmic scale. They are derived by increasing the dimensionless relaxation time $\bar{\tau}_r^{(c)}$ as $\bar{\tau}_r^{(c)} = 1$ (blue), $\bar{\tau}_r^{(c)} = \frac{1}{2}$ (red) and $\bar{\tau}_r^{(c)} = \frac{1}{4}$ (green).

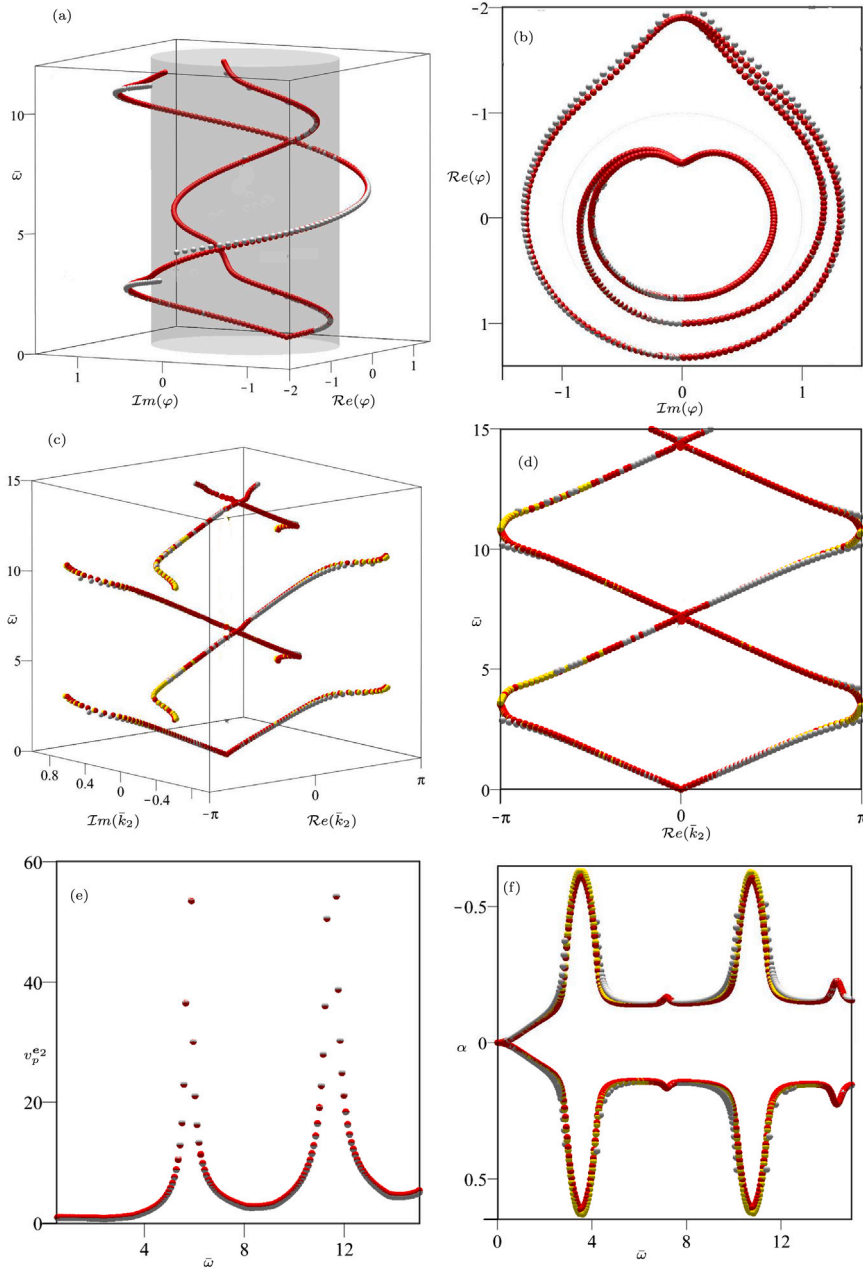


Fig. 4. Comparison between the heterogeneous (red) Floquet multiplier φ vs. those obtained with the continualization scheme for $\mathcal{N} = [2, 2]$ (grey) and $\mathcal{N} = [3, 3]$ (yellow) (a), view in the $(\text{Re}(\varphi), \text{Im}(\varphi))$ -plane (b). Heterogeneous (red) complex spectrum vs. those derived from the continualization scheme (yellow and grey) (c), view in the $(\text{Re}(\bar{k}_2), \text{Im}(\bar{k}_2))$ -plane (d), phase velocity (e) and attenuation factor (f) with $r_c = 2$, $r_p = 2$, $r_e = \frac{1}{100}$, $r_\beta = \frac{1}{1000}$, $\bar{\epsilon}_r^{(c)} = \frac{1}{2}$ and $\gamma = \frac{1}{5}$.

In Fig. 2, the magnitude of the dimensionless transformed constitutive tensor component $|\tilde{\eta}_{22}|$ is reported by considering a three-dimensional (Fig. 2-(a)) and a two-dimensional view (Fig. 2-(b)) by varying the dimensionless real part of the complex frequency $\text{Re}(\bar{s})$ and the dimensionless relaxation time, where the position of the poles is observed by considering the dimensionless imaginary part of the complex frequency $\text{Im}(\bar{s})$ as null.

In order to investigate the Bloch waves that propagate along the direction orthogonal to the material layering with $k_1 = 0$, it is point to remark that they are mechanics waves whose piezoelectric and dielectric couplings may influence and modify the dispersive properties. This circumstance can be observed in Fig. 3 that illustrates the real part and the imaginary part of the Floquet multiplier φ , which is the solutions of the characteristic polynomial (12), in terms of the dimensionless frequency $\bar{\omega}$ by varying the ratio between the piezoelectric couplings r_e as $r_e = \frac{1}{10}$ (red) and $r_e = \frac{1}{50}$ (green) (a), the parameter γ as $\gamma = \frac{1}{5}$ (red) and $\gamma = \frac{1}{40}$

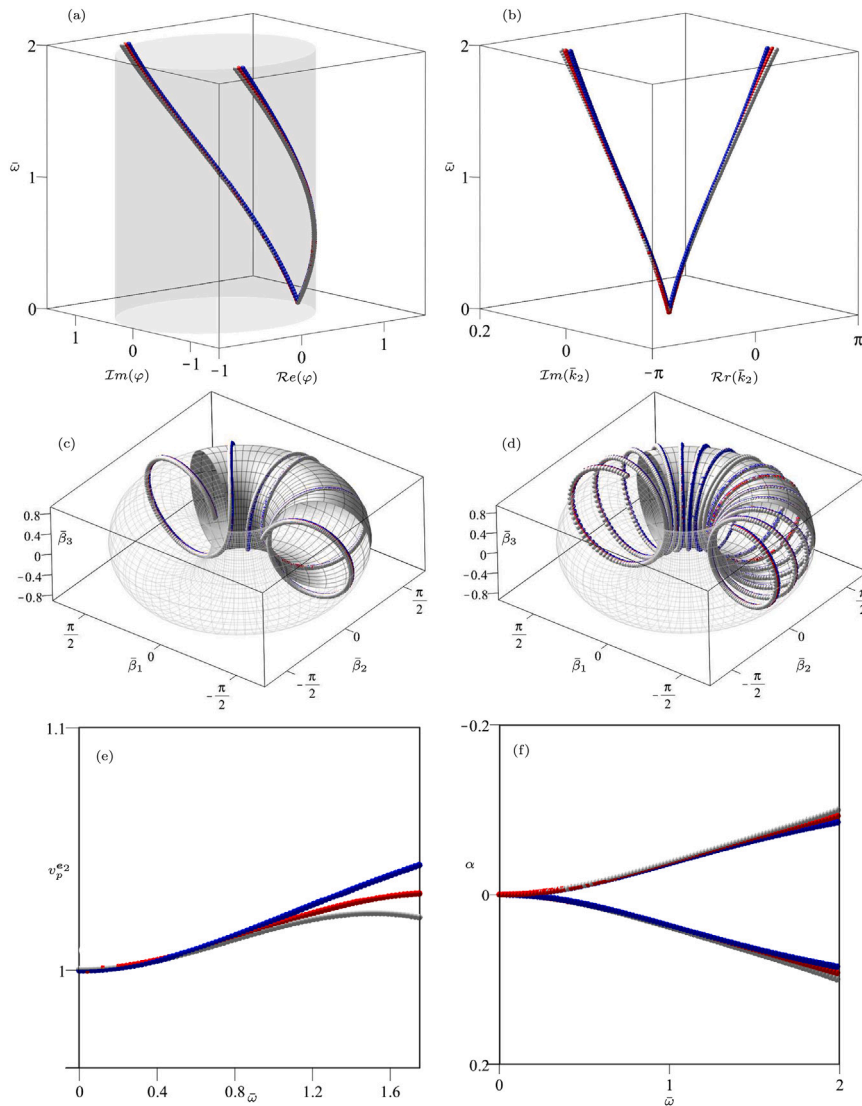


Fig. 5. Heterogeneous (red) Floquet multiplier φ of low frequencies vs. those obtained with the continualization scheme for $\mathcal{N} = [2, 2]$ (grey) and the second order homogenization method (blue) (a). Heterogeneous (red) complex spectrum vs. those derived from the continualization scheme (grey) and the second order homogenization method (blue) (b) and their representation on toroidal coordinates by increasing the dimensionless time $\bar{t}$ as: $\bar{t} = 10$ (c) and $\bar{t} = 50$ (d). Phase velocity (e) and attenuation factor (f) with $r_c = 2$, $r_p = 2$, $r_e = \frac{1}{100}$, $r_\beta = \frac{1}{100}$, $\bar{\epsilon}_r^{(c)} = \frac{1}{2}$ and $\gamma = \frac{1}{2}$.

(green)(b) and the ratio between the dielectric permittivities r_β as $r_\beta = 2$ (red) and $r_\beta = \frac{1}{1000}$ (green)(c). Moreover, the real part and the imaginary part of the Floquet multipliers φ are represented in the complex plane by changing r_e (d), γ (e) and r_β (f). Finally, the complex three-dimensional spectra and band structure are shown and obtained by solving the relation (15) by changing the parameters r_e (g), γ (h) and r_β (i) for the same values as the previous figure. In each diagram two stop bands can be distinguished and it can be observed that there is a slight influence on the elastic waves by varying the piezoelectric, the dielectric ratios and the parameter γ .

To assess the agreement between the high-frequency continualization (4) and the transfer matrix methods 2.1, which analytically provides the frequency complex spectrum for laminate periodic materials, Fig. 4 highlights a comparison between the methods. Fig. 4-(a) represents the behavior between the heterogeneous Floquet multipliers (red) and those derived with the continualization scheme by increasing the truncation order $\mathcal{N}$ as $\mathcal{N} = [2, 2]$ (grey) and $\mathcal{N} = [3, 3]$ (yellow) in Eq. (49) and inverting the relation $k_2 = \ln(\varphi)/(iS)$. In Fig. 4-(b), the Floquet multipliers obtained analytically (red) and via the continualization scheme (grey/yellow) are represented in the $(Re(\varphi), Im(\varphi))$ -plane. In both Figures, the heterogeneous (red) Floquet multipliers and the one obtained from the continualization scheme by setting $\mathcal{N} = [3, 3]$ (yellow) are coincident. Fig. 4-(c) illustrates the three-dimensional heterogeneous spectra (red) and the ones derived from Eq. (49) by setting the truncation order $\mathcal{N}$ at $\mathcal{N} = [2, 2]$ (grey) and $\mathcal{N} = [3, 3]$ (yellow). Fig. 4-(d) proposed the same spectra (yellow/ grey) and the heterogeneous one (red) within the $(Re(k_2), \bar{\omega})$ -plane by observing that,

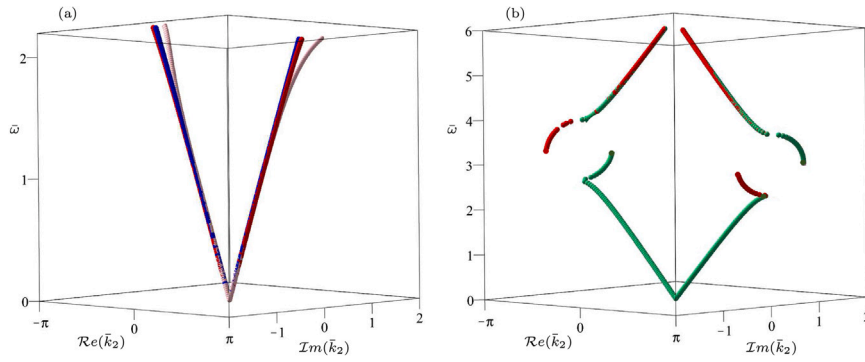


Fig. 6. Heterogeneous (red) complex spectrum vs. those derived with the continualization scheme by setting the kernel as Padé approximant and the entries of transfer matrix as a truncated Taylor series for $\mathcal{N} = 3$ (grey) and the second order homogenization method (blue) (a). Heterogeneous (red) complex spectrum vs. the one derived via the continualization scheme by setting the kernel as Padé approximant and the entries of transfer matrix as a truncated Taylor series for $\mathcal{N} = 40$ (green) (b).

as the truncation order of Eq. (49) increases, the continualization scheme represents an excellent tool to approximate the entire complex spectrum for high values of the frequency. Fig. 4-(e) shows the phase velocity $v_p^{e_2} = \frac{\tilde{\omega}}{\text{Re}(\tilde{k}_2)}$ specialized when the wave travels in the direction e_2 and related to the continualization scheme, by increasing the order $\mathcal{N}$ as $\mathcal{N} = [2, 2]$ (grey), $\mathcal{N} = [3, 3]$ (yellow), and the transfer matrix method (red), whereas Fig. 4-(f) determines the attenuation factor α defined as $\alpha = -\tilde{\omega} \text{Im}(\frac{\tilde{k}_2}{\tilde{\omega}})$ concerning with the continualization scheme (grey and yellow) and the transfer matrix method (red) by setting the dimensionless parameters as $r_C = 2$, $r_\rho = 2$, $r_e = \frac{1}{100}$, $r_\beta = \frac{1}{100}$, $\tilde{\tau}_r^{(\zeta)} = \frac{1}{2}$ and $\gamma = \frac{1}{2}$ (Carcione, 2007).

To validate the asymptotic homogenization (3), the continualization schemes (4), for layered periodic materials, with the transfer matrix method 2.1, which allows to analytically derive the frequency complex spectrum of the materials in the low frequency and in the long wave regime, Fig. 5 offers a comparison between the three methods. Specifically, Fig. 5-(a) illustrates the Floquet multiplier (red) obtained analytically by solving the polynomial (14) and those derived from the continualization scheme with $\mathcal{N} = [2, 2]$ (grey) and from the second-order homogenization method (blue). The latter is computed by solving Eq. (37) and inverting the relation $k_2 = \frac{\ln(\varphi)}{iS}$. Fig. 5-(b) illustrates the behavior of the three-dimensional frequency spectrum obtained analytically by solving the relation (15) (red) and those that stem from Eq. (49) by setting $\mathcal{M} = [2, 2]$ (grey) and the second-order homogenization method (blue) by solving Eq. (37). Fig. 5-(c) proposes an alternative representation of the heterogeneous spectrum (red) and those that derive from the continualization (grey) and the homogenization methods (blue) by means of the toroidal coordinates by decreasing the time variable as $\tilde{t} = 10$ (c) and $\tilde{t} = 50$ (d). The relations between the toroidal and the cartesian coordinates are

$$\tilde{\beta}_1 = \frac{\sinh(|e^{i\tilde{k}_2}|) \cos(\arg(e^{i\tilde{k}_2}))}{\cosh(|e^{i\tilde{k}_2}|) - \cos(\arg(e^{i\tilde{\omega}\tilde{t}}))}, \quad \tilde{\beta}_2 = \frac{\sinh(|e^{i\tilde{k}_2}|) \sin(\arg(e^{i\tilde{k}_2}))}{\cosh(|e^{i\tilde{k}_2}|) - \cos(\arg(e^{i\tilde{\omega}\tilde{t}}))}, \quad \tilde{\beta}_3 = \frac{\sin(\arg(e^{i\tilde{\omega}\tilde{t}}))}{\cosh(|e^{i\tilde{k}_2}|) - \cos(\arg(e^{i\tilde{\omega}\tilde{t}}))}.$$

Fig. 5-(e) shows the phase velocity $v_p^{e_2}$ related to the homogenization (blue), the continualization (grey) and the transfer matrix methods (red), whereas Fig. 5-(f) highlights the attenuation factor α concerning with the three methods. Fig. 6-(a) illustrates the analytical spectra (red) together with the approximations derived from both the homogenization (blue) and the continualization scheme (grey), based on the resolution of the polynomial (51) with $\mathcal{N} = 3$. As the truncation order in (51) increases, the approximate spectra (green) show convergence, overlapping with the heterogeneous spectra in the chosen angular frequency range, as it appears in Fig. 6-(b). It is worth noting that continualization is a high-frequency scheme, permitting the approximation of the entire complex spectrum, in the case of laminate materials. On the other hand, the asymptotic homogenization method is an excellent low-frequency method that is versatile, not only for laminate materials, but also for more complex structures.

5.2. Visco-electro-elastic fiber-reinforced composite

The in-plane periodic composite under consideration consists of an epoxy matrix, phase (2), embedding a cluster of out of plane polarized PZT-5H fibers with a circular cross-section, periodically distributed, phase (1). Both the matrix and the fibers are treated as visco-electro-elastic materials. In particular, the components of the constitutive tensors related to the longterm material response have been taken in agreement with Li and Dunn (2001) and Yang et al. (2005). The relaxation times involved in the relaxation functions, modeled as a truncated Prony series, see (52) and (53), are $\tau_r^{(1)} = 1/100$ s and $\tau_r^{(1)}/\tau_r^{(2)} = 1/100$, while the viscosity ratios of the relaxation functions are detailed in what follows. Perfectly bonded phases and plane stress-electric displacement assumptions are assumed. The homogenized constitutive properties of the visco-electro-elastic material are analyzed with reference to the periodic cell, shown in the lower part of Fig. 7(a). The periodic cell is defined as $\mathcal{A} = [0, \varepsilon]^2$, where ε denotes the characteristic dimension of the cell. The periodic structure is described by two orthogonal periodicity vectors, ν_1 and ν_2 , which define the regular spatial arrangement of the fibers within the matrix. We assume the ratio $r_i/\varepsilon = 5/100$, where r_i is the radius of the inclusion.

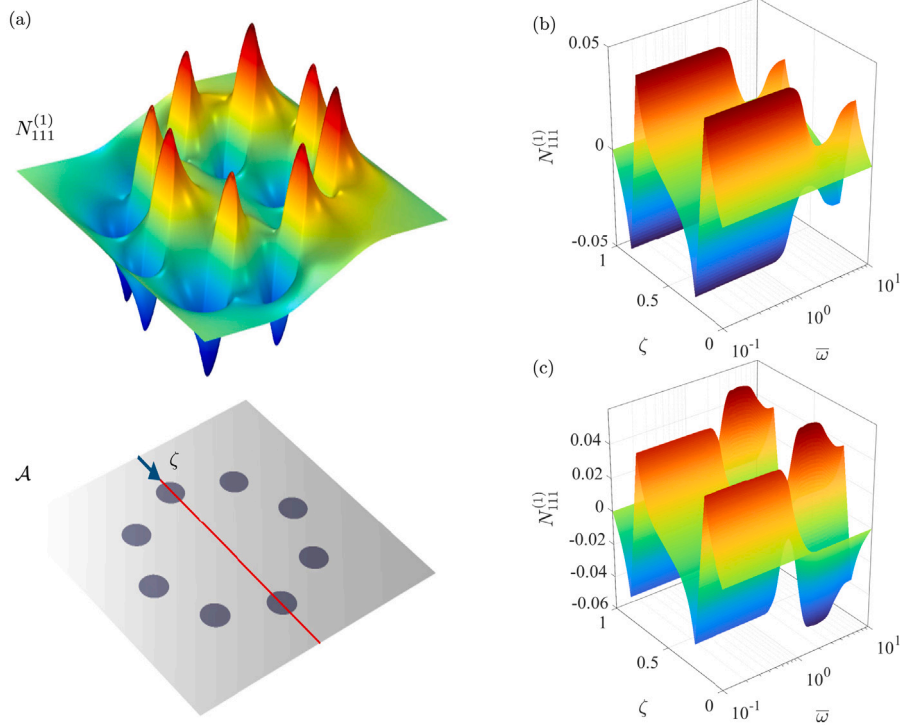


Fig. 7. (a) Schematic of the visco-electric-elastic Periodic Cell together with first order perturbation function $N_{111}^{(1)}$; (b) Contour plot of the perturbation function along ζ as $\bar{\omega}$ varies on a logarithmic scale, for $\gamma^{(2)} = 10$ and $\gamma^{(1)} = 1$; (c) Contour plot of the perturbation function along ζ as $\bar{\omega}$ varies on a logarithmic scale, for $\gamma^{(2)} = 100$ and $\gamma^{(1)} = 1$.

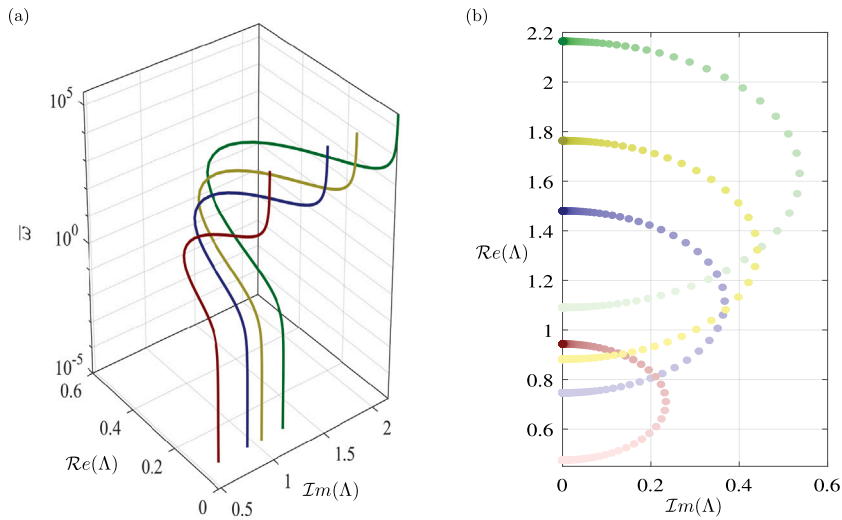


Fig. 8. Real and imaginary parts of the overall constitutive components Λ vs. $\bar{\omega}$ in a logarithmic scale. Λ corresponds to $\hat{C}_{1111}/\hat{C}_{1111}^{(2,\infty)} = \hat{C}_{2222}/\hat{C}_{2222}^{(2,\infty)}$, $\hat{C}_{1122}/\hat{C}_{1122}^{(2,\infty)}$, $\hat{C}_{1212}/\hat{C}_{1212}^{(2,\infty)}$ and $\hat{\beta}_{11}/\hat{\beta}_{11}^{(2,\infty)} = \hat{\beta}_{22}/\hat{\beta}_{22}^{(2,\infty)}$ represented by the blue, red, green, and yellow curves, respectively. (a) 3D plot; (b) 2D plot with $\bar{\omega}$ varying from light to dark in a logarithmic color scale.

In the upper part of Fig. 7(a), the contour plot of the perturbation function $N_{111}^{(1)}$ is schematically reported. The perturbation functions are Q -periodic and continuously differentiable along the cell boundaries, satisfying the zero mean condition within Q in accordance with the normalization constraints. Moreover, they become evanescent when the phases exhibit identical constitutive properties. It is evident that these functions are significantly influenced by the angular frequency, as the micromechanical constitutive properties depend on the Laplace-transformed variable s , which is mapped onto the real angular frequency, i.e., $s = i\omega$.

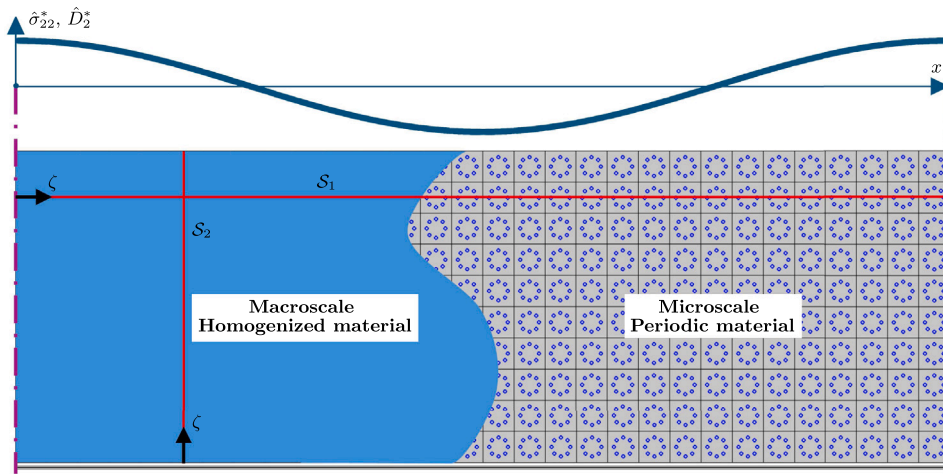


Fig. 9. Schematic representation of a strip subjected to harmonic periodic sources, illustrating both the heterogeneous medium and its homogenized counterpart.

As expected, this influence becomes increasingly pronounced as the angular frequency rises. In this regard, along the red section line in the periodic cell, the perturbation function is plotted in Fig. 7(b) and (c) as a function of the curvilinear abscissa ζ and the dimensionless angular frequency $\bar{\omega} = \omega/\omega_{ref}$ (with ω_{ref} taken as unitary), considering two different values of γ for the matrix. It clearly emerges that this parameter significantly influences the response, both qualitatively and quantitatively.

The first-order perturbation functions play a role in the global first-order constitutive tensors of the continuum, as stated in Eq. (42). Consequently, these tensors will depend on the angular frequency through both the perturbation functions and the micromechanical constitutive properties. It follows that in Fig. 8 the complex components of the global tensors in the Laplace-transformed space are shown. Specifically, Figure (a) presents the behavior of the real and imaginary parts of the dimensionless Λ component as a function of the dimensionless angular frequency. The non-vanishing components $\hat{C}_{1111}/\hat{C}_{1111}^{(2,\infty)} = \hat{C}_{2222}/\hat{C}_{2222}^{(2,\infty)}$, $\hat{C}_{1122}/\hat{C}_{1122}^{(2,\infty)}$, $\hat{C}_{1212}/\hat{C}_{1212}^{(2,\infty)}$ and $\hat{\beta}_{11}/\hat{\beta}_{11}^{(2,\infty)} = \hat{\beta}_{22}/\hat{\beta}_{22}^{(2,\infty)}$ are represented by blue, red, green, and yellow curves, respectively. Each component is divided by the corresponding value associated with the long-term material response of phase (2). As expected, a non-monotonic trend of the overall constants is observed, along with their asymptotic behavior as the angular frequency increases. In Figure (b), the real and imaginary parts of the overall components are plotted as parametric curves with respect to the angular frequency ω . The non-monotonic behavior is highlighted, along with the evolution of the real and imaginary parts, which are represented as parametric curves. The same colors used in Figure (a) are employed to identify the corresponding components, while the color gradually transitions from light to dark as ω increases on a logarithmic scale.

To evaluate the performance of the proposed homogenization procedure, an infinite strip of the two-phase periodic medium is considered as schematically reported in Fig. 9, constructed by assembling periodic cells along both the horizontal and vertical directions, as shown in Fig. 7(a). The specimen, with a height of 10ϵ and unit out-of-plane depth, is subjected to slow harmonic boundary conditions applied on the upper horizontal boundary with a period of 30ϵ . These conditions have non-vanishing components.

$$\hat{\sigma}_{22}^*(x_1) = \Xi_\sigma \cos\left(\frac{2\pi x_1}{30\epsilon}\right), \quad \hat{D}_2^*(x_1) = \Xi_D \cos\left(\frac{2\pi x_1}{30\epsilon}\right),$$

defined in the Laplace-transformed domain, which in the time domain correspond to a spatially periodic and impulsive forcing at the initial instant. The following boundary conditions are prescribed on the lower boundary: $\hat{\mathbf{u}} = \mathbf{0}$ and $\phi = 0$. Consistently with the determination of the homogenized constitutive properties, the plane stress and electric displacement assumptions are considered. The numerical results of the micro-mechanical model are compared with the corresponding ones obtained by using the first-order homogenized model. In Fig. 10 both numerical problems are solved assuming amplitudes $\Xi_\sigma/\hat{C}_{1111}^{(2,\infty)} = 8/51$ and $\Xi_D\epsilon/(\hat{\beta}_{22}^{(2,\infty)}\hat{\phi}_{ref}) = 7/(31\epsilon_0)$, where ϵ_0 is the vacuum permittivity and $\hat{\phi}_{ref}$ is considered unitary herein.

In Fig. 10(a), the dimensionless micro-mechanical displacement component $\hat{u}_2/\epsilon$ (dotted lines) and the corresponding macro-mechanical displacement component $\hat{U}_2/\epsilon$ (solid lines) are plotted against the dimensionless coordinate ζ/ϵ along the horizontal line S_1 . The comparison is shown for three values of the dimensionless angular frequency $\bar{\omega} = \omega/\omega_{ref}$: $\bar{\omega} = 10^{-4}$ (green), $\bar{\omega} = 1$ (red), and $\bar{\omega} \simeq 3$ (blue). Similarly, in Fig. 10(b), the dimensionless components $\hat{\phi}/\hat{\phi}_{ref}$ and $\hat{\Phi}/\hat{\phi}_{ref}$ are presented using the same line styles. Furthermore, in Fig. 10(c), an analogous comparison is carried out with reference to the horizontal displacement component. Finally, in Fig. 10(d), the functions u_2 are plotted along the section line S_2 . The results demonstrate a perfect agreement between the solutions obtained from the homogenized model and the micro-mechanical model. In particular, Fig. 9(c) highlights that the homogenized model effectively provides results that closely approximate the average of those computed using the micro-mechanical model, which exhibits pronounced fluctuations due to the characteristics of the microstructure. Furthermore, it is evident that the homogenized model is capable of accurately approximating the response of the micro-mechanical model even as the angular frequency varies, directly influencing the constitutive equations of the material at both scales.

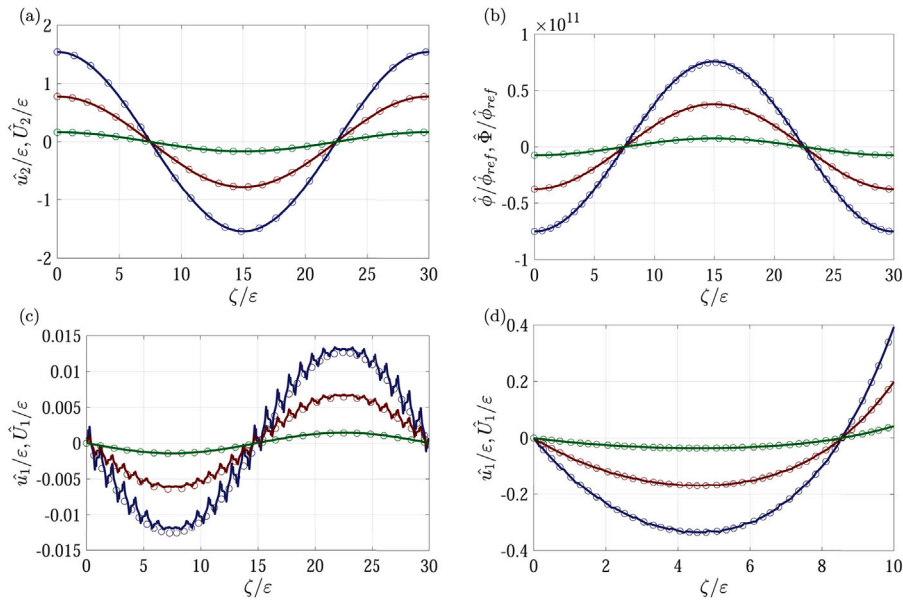


Fig. 10. Comparison of dimensionless micro-mechanical variables (dotted lines) and their corresponding macro-mechanical counterparts (solid lines) as a function of ζ/ϵ for three values of the dimensionless angular frequency $\bar{\omega} = \omega/\omega_{ref}$: $\bar{\omega} = 10^{-4}$ (green), $\bar{\omega} = 1$ (red), and $\bar{\omega} \approx 3$ (blue). (a) $\hat{u}_2/\epsilon, \hat{U}_2/\epsilon$ along S_1 ; (b) $\hat{\phi}/\hat{\phi}_{ref}, \hat{\Phi}/\hat{\Phi}_{ref}$ along S_1 ; (c) $\hat{u}_1/\epsilon, \hat{U}_1/\epsilon$ along S_1 ; (d) $S_1; \hat{u}_2/\epsilon, \hat{U}_2/\epsilon$ along S_2 .

6. Final remarks and future developments

The present work focuses on the multiscale modeling of microstructured visco-electro-elastic periodic materials, abundantly used in different industrial applications. The study of free propagation of Bloch waves inside the material is tackled by recasting the governing equations in a Stroh-like formalism through a two-sided Laplace transform and in the wave vector space by means of a multidimensional Floquet–Bloch transform. The transfer matrix method is also detailed for laminate materials in order to obtain the relative characteristic equation and the analytical solution for the roots of a fourth-order characteristic polynomial. Two methods are tailored to identify the equivalent non-local visco-electro-elastic continuum. On the one hand, the two-scale asymptotic homogenization technique, relying on a net length-scale separation between the macro and the micro scales, is proposed leading to the average field equations of infinite order, whose formal resolution can be dealt with using a perturbative procedure. In this way, the gradient-type governing equations characterizing an energetically consistent higher-order visco-electro-elastic medium can be obtained. A procedure to asymptotically approximate the dispersion features of the periodic medium is presented and detailed for a first- and a second-order continuum. Furthermore, a macro-homogeneity condition is imposed in order to provide the closed form of the overall constitutive and inertial constants for a first-order medium in terms of the complex angular frequency. On the other hand, a continualization scheme is described. It consists of evaluating the variables at the microscale at the edges of the periodic cell that represent the discrete fields. Then, they are transformed into the complex variable z domain, re-mapped into the wave number space, and forced to coincide with the Fourier transform of continuous counterparts that characterize the macro descriptors. Finally, the integro-differential and the multi-field gradient-type governing equations of a non-local visco-electro-elastic continuum are determined by rewriting the integral kernel as a Padé approximant.

The schemes herein derived are verified by considering first a visco-electro-elastic laminate material. Specifically, the frequency complex spectrum related to the laminate is analytically determined using the transfer matrix method and compared with those derived from the proposed schemes. Secondly, the numerical micro-mechanical response resulting from a biphasic fiber-reinforced composite is compared with homogenized one. Several insights may be disclosed: the asymptotic homogenization method is revealing to be precise in reproducing the low-frequency band structure of the laminate material. Furthermore, it proves to be accurate in modeling the constitutive behavior of composite materials with complex microstructures. Their qualitative and quantitative response is strongly influenced by the value of the angular frequency, and such a trend is rigorously predicted by the homogenized medium. The continualization scheme approximates in a precise manner the entire complex spectrum of a visco-electro-elastic laminate material, proving that the spectrum of the continuum model converges to the one of the heterogeneous laminate as the continualization order increases in the entire first Brillouin zone. This represents an important advantage over asymptotic and variational-asymptotic homogenization techniques, which could provide low-frequency approximations of the spectrum. On the other hand, the scheme is suitable only for laminate materials whereas the asymptotic and asymptotic-variational homogenization techniques are applicable to generic geometries.

As future developments, one can consider the exploration of a continualization schemes based on a multi-point Padé approximation of the integral kernel that could provide more accurate and efficient spectral approximations and the development of

generalized high-frequency continualization schemes tailored to materials with complex microstructures that could pave the way for new research and application.

CRedit authorship contribution statement

Rosaria Del Toro: Conceptualization, Methodology, Validation, Writing – original draft. **Francesca Fantoni:** Conceptualization, Methodology, Validation, Writing – original draft. **Maria Laura De Bellis:** Conceptualization, Methodology, Validation, Writing – original draft. **Andrea Bacigalupo:** Conceptualization, Methodology, Validation, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.ijengsci.2025.104331>.

Data availability

Data will be made available on request.

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